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Title:

Incomplete Financial Markets: Volatility and Transaction Costs

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Preface & Acknowledgements

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Contents

Chapter 1

Summary & Introduction

This thesis consists of four papers, two of which I have written jointly with Mich Tvede and two of which I have written on my own. Let me first try to explain the background for my results and my motivation for choosing the papers I have included. Instead of summarizing each paper individually, I will try to put them into a context I find natural.

This thesis has as its main subject the theory of *G*eneral *E*quilibrium (GE). GE has as its main object the study of markets, our behavior as traders and the impact on welfare. To start with the last word, *equilibrium*, we are concerned with behavior and prices that are invariant, in the sense that traders cannot improve upon their personal welfare by exchanging commodities on the market at the going prices. Implicitly this implies that non-equilibrium states are uninteresting and unimportant! As scientists we try to find systematic and persistent patterns, and non-equilibrium behavior must by definition be unsystematic and change as time passes by. The first word, *general* (as opposed to *partial*), emphasises that we are concerned with *several* markets - and not just a single market. One can, in some sense, say that we are concerned with a closed system, i.e., a system which does not receive impulses from external and unexplainable sources. All changes must be explained by modeled phenomena.

The majority of analyses on GE has thus far been focused on

- *Existence*, i.e., does stable states exist?
- *Determinacy*, i.e., if stable states exist, how many different stable state does our theory predict? and
- *Efficiency*, i.e., how desirable is the stable states?

These are of course important and interesting questions, and we shall address some of them in this thesis. However, one message I would like to promote is that general equilibrium analysis is more than these issues. I would like to emphasise that a GE analysis is essential in understanding the interrelation between the financial markets and the real economic decisions, such as consumption, savings and investments. Maybe too much energy has been put into the extension of results concerning the above mentioned categories, while other issues have been neglected.

My thesis concern general equilibrium with a special emphasis on the financial markets. Up until the late 1950's the financial markets were considered as just merely another market and encompassed into the theory of Arrow and Debreu (see [4] and [18]). They reduced the theory of financial markets to a one-shot game, where financial contracts were perfect contingencies on either commodities or income. This implied that the theory of financial markets fitted perfectly into the existing theory, and moreover, the efficiency issues were then solved: financial markets admitted efficient allocation of risk and savings decisions, and economic policy decisions were reduced to distributional considerations. These were just applications of the fundamental theorems of welfare economics and showed the advantages of the abstract approach. The consequence was also that either people could trade only initially, and then let the contingent contracts execute the subsequent trades, or they could transfer income using simple contracts and then trade on spot markets. Both methods produced the same result. A third result was then directly reusable, namely that such trades and prices constituting an equilibrium always existed, which guarantees that if the world approximately is as assumed in the model, then it makes sense to analyze such states. If the existence was not generically guaranteed, then it would be like King Arthur and his knights' search for the Holy Grail.

However, in the mid 1970's a branch of GE emerged, which showed that the issue of efficient risk allocation was not solved yet (see [35], [53] just to mention a few references). The point is that financial markets are not perfect in the sense that not all people are capable of obtaining full insurance, even if they can meet the repayments requested by the insurance company. Thus, people might be credit constrained. This can be attributed to asymmetric information which implies that lenders are reluctant to lend money to borrowers, since the borrower cannot credibly convenience the lender that he will be able to repay the loan. Examples of this are the well-known moral hazard and adverse selection problems. This problem can partially be solved by the use of collateral, but this requires a sufficiently developed legal system. Also, the existing model assumed that the cost of trading in itself was either negligible or non-existing. The branch was later referred to as GEI - *G*eneral *E*quilibrium of *I*ncomplete financial markets. Within GEI the issue of why asset markets are incomplete is in general *not* treated, but the existing assets are taken as part of the fundamentals of the economy, and not a variable that agents can influence. Assuming that all existing securities are essential[1], generically, the models cannot explain why some assets are not traded, i.e., why the market breaks down, and why others are. There are some different theories on financial innovation [2] and collateral [26] addressing this issue, just to mention some few. However, we will stay in the realm of GEI and take the financial structure as given. Also, GEI only considers perfectly competitive behaviour, i.e., the hypothesis that traders and the households act as if they them self have a neglect able effect on the aggregate terms and, thus, take the prices and behaviour of other agents as given. This is of course an abstraction in many cases, but nevertheless it provides a good framework of analysis since it provides us with a benchmark case. Moreover, we do not consider the case of asymmetric information and thus the prices' role as signals and aggregator of information.

Generally, the exchange of commodities between individuals occurs either due to differences in preferences, endowments, or beliefs. This also holds within the financial markets, where we often label these causes as insurance and saving. While many authors have emphasized the importance of informational costs and value (see e.g. [5]), i.e., the costs related to acquiring information relevant to forming expectations and verifications, we emphasise the importance of transaction costs, in particular on the financial markets. A fundamental contribution in the general equilibrium literature of transaction costs was by [33], showing the first general existence theorem in an economy with transaction costs. We shall analyse such economies and the consequences of transaction costs more closely. Moreover, we depart from [33] in the existence result by considering intermediation costs, i.e., costs that are not based upon any use of real resources in the process of transactions.

The financial markets are an important part of an economy and its ability to function properly. Whether it is insurance against fire, accidents, theft, unemployment, sickness or disability, or it is the provision of company's financial foundation - financial markets provide these services. Since these markets are so vital for the economy, it is important to understand and comprehend the functioning of these markets in order to determine the right policies and design of the institutions under which the financial markets operate.

We shall be concerned with two subjects:

- Implications of transaction costs on equilibrium states

- Volatility and incomplete financial markets

[1] Loosely speaking, a security is *essential* if its payoffs cannot be generated by any other combination of securities.

1.1 Implications of transaction costs on equilibrium states

First, we are concerned with the implications for the equilibrium states that emerge as we introduce transactions costs on the financial markets. This is the subject of the articles "*On Financial Equilibrium and Intermediation costs*"[2], Chapter 2 and "*The Existence of a Free Lunch, Equilibrium in Income Streams, and Intermediation Costs*", Chapter 4.

Transaction costs are the wedge between the revenue of the seller and the expenses of the buyer. A transaction cost which everybody knows of is the exchange rate gap, i.e., the difference in the buying- and selling-price of exchange agencies. Another example is the interest gap of banks, i.e., the gap between the loan- and the borrowing-rate. However, these are only the "visible" transaction costs. Also, the costs of obtaining the required information of assessing the value of an asset, and the actual cost of actually obtain the required portfolio, are examples of transaction costs. Again, we take the particular form of transaction costs as exogenous given. A particular form of transaction cost is merely a redistribution of income in that the real costs associated with the transaction is zero. An obvious question is then why such institutions might exist, since they do not perform any socially productive function. We do not address this question, but an obvious reason could be monopoly power, resulting either from scale effects in production or publicly supported licenses limiting the competition. More general transaction costs are associated with bringing the two parties to a deal together, i.e., matching of traders, and also payments to market makers, i.e., agents who guarantee a sufficient liquidity in the market.

Obviously, transaction costs make insurance and saving more expensive and must hence by that measure be bad. However, they may have an advantage, namely that they restore the existence property. It was shown (see [35]) that existence of equilibrium was not guaranteed when traders were allowed to go short, which in some situations could affect prices in such a manner that the asset market was no longer durable. Examples have shown that this is a robust property when derivative assets are included in the set of essential securities. The consequence of transaction costs is to reduce this tendency to sell short assets, and thereby to reduce the effect on prices of asset markets. This guarantees the existence of a competitive equilibrium. Also, we show that small changes in the transaction costs do not change the resulting equilibrium, as long as they are positive. Having showed existence of equilibrium states with transaction costs, we turn to the question of how these equilibrium states may look. An interesting question is whether free lunches, i.e., costless profits, are

[2]Published in Journal of Mathematical Economics, 2008.

consistent with competitive equilibrium states. In particular, we ask the following questions

- Can there exist unexploited, free lunches?, and
- Can free lunches be exploited in equilibrium?

Without transaction costs the answer is negative for both of these questions. However, when transaction costs are present and they are strictly convex, i.e., the marginal transaction cost increases with increasing trade, then the answer is affirmative to both questions. With imperfect competition the first is valid, while the second is invalid. With irrational traders both questions, again, are answered affirmative, however, our results are true even with rational traders. Our results indicate that it's necessary condition that traders are sufficiently diverse in their need for income transfers, and that simple, representative consumer economies cannot capture such phenomena. Moreover, the income distribution influence the likelihood of exploited free lunches, the more income inequality; the more likely it is that free lunches can be exploited in equilibrium.

1.2 Volatility and incomplete financial markets

Since the first testable models of asset pricing emerged, an ever lasting question has been whether the competitive hypothesis and no-arbitrage principle can explain the asset prices and predict trade in securities. Also, the securities markets have been almost exploding in size, both the number of securities and the volume of trade, also relatively to the total income of the society. It is difficult to understand this using the basic Arrow-Debreu model which should place the responsibility for this in the preferences of the households and their need for income transfers. However, we offer different explanations:

- the design of the securities and/or
- the liberalization of financial markets.

Since the early 80's the financial markets have experienced a huge increase in the different types of securities available to be traded. In particular, the securities referred to as derivative securities[3] have emerged and their usage has exploded. While most critics have focused on the complexness and opaqueness, and hence the

[3]They have been named so because their dividend is *derived* from other (primary) securities. Examples of such securities are options on stocks and CDO (Credit Derivative Options) used by banks to trade credit risk.

difficulties of correctly valuing the security, we do not address this problem regardless how relevant it might be. Alternatively, we suggest that, if the derivatives are essential in the risk dispersion, it is the feedback between the underlying prices and their risk characteristics which is problematic. Secondly, we argue that additional to the direct effect of liberalization on the financial markets making trades less expensive, the indirect effect of changing future equilibrium prices can, potentially, have a very large effect.

Basically, we have four important conclusions:

- Asset prices can jump without the underlying fundamentals changing significantly when the asset market is incomplete,
- Asset trading can be arbitrarily large, even with complete markets,
- Liberalization of asset markets can explain an arbitrary large increase in the asset trading, and
- The presence of derivative assets mitigates the above stated results.

The three last results are included in the paper "*Volatility of Security Trade and Incomplete Financial Markets*", Chapter 3, while the first is in "*Jumps in Asset Prices: A General Equilibrium Explanation*, Chapter 5. The first considers a Ramsay-model, with uncertainty and heterogeneous consumers, financial markets and intermediation costs in an infinite-horizon economy. The emphasis in this paper is on the volume of trade and its relation with changing transaction costs and security design. The latter paper considers a finite-horizon economy and analyses the asset prices. It shows that the prices with incomplete financial markets can be subject to discontinuous jumps as the fundamental characteristics change by infinitely small changes. We show that with complete markets these jumps cannot occur, and hence we conclude that this is a way of testing whether financial markets are incomplete. However, the test can only falsify that markets are complete, since incomplete markets are only a necessary condition to obtain the discontinuity. Moreover, our example shows a multitude of different equilibria and hence an indeterminacy of the resulting equilibrium states. This is obtained even with real assets, thus contrasting the general result in the case of a finite set of states of nature. The result suggests that there is a qualitative difference between Radner equilibria in the finite and in the continuum case, and hence that the finite case is a poor approximation of the continuum case. Our results further show that the occurrence of a sudden large drop in asset prices cannot *per se* indicate whether there has been a bubble or not. Also, the "thick tails" of asset price distributions observed in data can be explained by our result.

Thus, the two papers analyse the financial markets with different focuses, either volume of trade or asset prices.

1.3 Conclusion

One of the conclusions that I want to emphasise is that *general* equilibrium analysis matters, illustrated by the paper "*Volatility of Security Trade and Incomplete Financial Markets*". Here, one of the necessary conditions for the results is the interplay between markets, namely the feedback of income exchanges and relative commodity prices on the opportunities of risk diversion.

A main hypothesis in microeconomic theory is that the world is populated by diverse agents, having different and divergent interests, and that these conflicting interests are the engine of the phenomena that we observe in the world. The results of the paper "*The Existence of a Free Lunch, Equilibrium in Income Streams, and Intermediation Costs*", Chapter 4, illustrate how this diversity is needed and is automatically assumed away in models of representative households economies.

I hope that the conclusions of this dissertation can contribute to the growing literature which illustrates that general equilibrium theory does not solely consist of existence results. I think that several economists have abandoned the GE due to the large focus on these perhaps inferior questions.

I would like to emphasise that we have not made any welfare conclusion, in particular, we have not concluded that transaction costs are neither positive nor negative. Here we have to take two effects into account each point in two different directions. Firstly, transaction costs are inefficient in that they prevent trades of income which would otherwise take place, beneficial for each two parties. However, this cannot, in general, lead us to the conclusion that in the absence of transaction everybody is better of. Here the theory of second-best comes into the picture, since the case of no transaction costs is in general inefficient, and price effects might decrease someone's welfare. Second, transaction costs re-establish the existence of a competitive equilibrium, but in order to assess the benefits of this we need a theory of what is happening when no competitive equilibrium exists.

Moreover, we cannot conclude whether the existence of arbitrage opportunities and exploration of these are socially costly.

A central part of economic activity is not represented in the papers constituting this dissertation, and this is production, i.e., the transformation of commodities into other commodities taken into its most broad definition. There are several reasons why production has been left out. Firstly, we do not think the inclusion of production would change our qualitative results, since the mechanisms yielding our result are not mitigated by firms. Second, introducing firms requires that we choose an objective on which the firms take decision to maximize. This is in general not obvious how this objective should be formed when the financial markets are incomplete. When the security markets are incomplete firms not only transform commodities, but they can further transform income streams and hence their decisions affect society's ability to allocate risk. When the owners of a company differs in their evaluation of risk this role plants a seed of disagreement between owners which the markets cannot remove (for a more thorough treatment of the firm in GEI see [21]). An inclusion would thus require a decision on the objective of the firm which is undecided theoretically yet, without contributing significantly. Third, much of the literature on GEI has thus far ignored the firm.

An important question which we have left unanswered is the nature of transaction cost and thus also what properties it has. An important property is that as the volume of trade increases infinitely the transaction costs also go to infinity. This is probably not controversial. However, the convexity assumption which is fundamental in the result of chapter 4, is obviously controversial. It exclude the case of a fixed fee and constant marginal costs.

Chapter 2

On Financial Equilibrium and Intermediation Costs

Tobias Markeprand

Abstract. This paper studies the set of competitive equilibria in financial economies with intermediation costs. We consider an arbitrary dividend structure, which includes options and equity with limited liabilities. We show a general existence result and upper-hemi continuity of the equilibrium correspondence. Finally, we prove that when intermediation costs approach zero, unbounded volume of asset trades is a necessary and sufficient condition, provided that, there is no financial equilibrium without intermediation costs.

JEL classification:. D41; D52; D53;

Keywords:. Incomplete Markets, Intermediation Costs, General Equilibrium Theory

2.1 Introduction

This paper studies the properties of the set of competitive equilibria when intermediation costs are present. We allow the asset structure to include non-linear dependence on spot market prices. We show that every economy has a competitive equilibrium and that the equilibrium correspondence parameterised by the cost function and endowments is upper-hemi continuous. Further, when the intermediation costs go to zero and no equilibrium exists without intermediation costs in the economy, the asset trades are unbounded. The results are due to a boundary on asset trades induced by the intermediation cost, which is assumed to go to infinity when the volume of asset trades goes to infinity.

We consider a two-period model with S different states. In each period-state, spot trading of goods takes place and asset trading takes place in the first period before uncertainty is revealed. Trading in asset markets is costly due to intermediation costs, and the revenues from these trades are redistributed to the consumers[1]. This could be in proportion to some pre-specified fraction.

The global existence property arises from the presence of intermediation costs. Non-existence, when such costs are not present, arises from the discontinuity of demand for assets when commodity prices converge to prices for which the dividend matrix drops in rank. The presence of intermediation costs prevents the drop in rank from inducing discontinuity of the demand correspondence.

This paper extends the existence result in [52] to a more general asset structure and intermediation cost function. The result in Préchac concerns real assets that depend linearly on the commodity prices, and with dividends that are positive. However, this excludes important classes of assets like options, futures etc. We obtain the stronger result by imposing more restrictions on the cost function. In Préchac, intermediation cost depends on the *value* of the trades and not on the *volume* of trade. Hence, if the asset price is zero, the cost is zero. When an asset yields dividends with different signs in different states, the price might be zero in equilibrium. The existence result is then obtained by assuming that the dividends are strictly positive. The issue of continuity of the equilibrium correspondence is not addressed by Préchac.

Initially, existence results in financial economies with incomplete markets were established by [53]; here an exogenous boundary on asset trades was imposed. Later, [35] showed that this boundary assumption was essential for the existence result, providing an example in which no financial equilibrium existed when the boundary

[1]Therefore, it is an intermediation cost rather than a transaction cost. We prefer to distinguish between the two, since the latter is a real cost that arises in the exchange of commodities, while the former is a transfer of income between agents.

assumption was dropped. Both contributions considered an asset structure that depended linearly on commodity spot prices. In relation to these results, [24] showed that the non-existence of equilibrium is a non-generic property: a small perturbation of the economy will re-establish the existence of equilibrium. However, this result was due to the linearity of the dividends' dependence on spot market prices as shown in [51]. They provide an example involving options, which gives rise to a robust non-existence of financial equilibrium. As showed in [11], robust non-existence of competitive equilibrium can also arise in economies when preferences are not strictly convex.

It is from this perspective the existence result of this paper should be viewed. Préchac shows existence when asset structure is linearly dependent on spot market prices and positive dividends, while in this paper we only assume continuity. This comes at the expense of more restrictions on the cost function. However, our result allows asset structures with nominal and real assets including options.

Let us briefly suggest an interpretation of our results. If we consider the competitive equilibrium a reasonable description of the state of the economy, then non-existence of equilibrium implies that prices cannot coordinate the actions of the agents. With robust examples of non-existence, such situations occur with nonzero probability. The results of this paper imply that these problems disappear when trade on financial markets is costly, even though the revenue from such costs is transferred back to the agents. Intuitively, the upper hemi continuity implies that, even though the agents make small mistakes in their assessment of the characteristics of the economy, the error in the expectation of prices and allocation will be small.

The paper will proceed as follows: In section 2.2 we formalise the model and present the results. Section 2.3 contains the proofs of the results.

2.2 The Model

We consider a two-period model with H, a finite set of consumers, and S, states of nature, to be revealed in the second period. We denote by s a generic state and take, due to ease of notation, $s = 0$ as the first period. In each state and period there are L perishable goods and hence the commodity space is $\mathbb{R}^{L(S+1)}$. We denote by $P \subset \mathbb{R}_+^{L(S+1)}$ the space of spot market prices[2]. There are J assets represented by

[2]Notation: Given $x, y \in \mathbb{R}^n$, we write $x \geq y$ iff. $x_i \geq y_i$ for $i = 1, ..., n$, $x > y$ iff. $x \geq y$ and $x \neq y$. This further gives us the sets $\mathbb{R}_+^n = \{r \in \mathbb{R}^n \mid r \geq 0\}$ and $\mathbb{R}_{++}^n = \{r \in \mathbb{R}^n \mid r \gg 0\}$. We denote by $\delta_i = (0, .., 1, ..., 0) \in \mathbb{R}^n$ the vector in $\mathbb{R}^n$ which is 0 on all coordinates expect in the i'th coordinate, where it is 1. Given a normed vector space X, we denote by $\mathcal{O}_\epsilon(x)$ the open ball with radius $\epsilon > 0$ and centre in $x \in X$. The set of all subsets of a set X is denoted 2^X and if

a dividend structure $V\colon P \to L(J,S)$[3] representing the space of income transfers. A portfolio is denoted by $z \in \mathbb{R}^J$ while $q \in Q \subset \mathbb{R}^J$ denotes an asset price vector. Denote by (u_h, ω_h, X_h) a consumer with utility function u_h, initial endowment of commodities $\omega_h \in \mathbb{R}^{L(S+1)}$ and consumption possibilities set $X_h \subset \mathbb{R}^{L(S+1)}$. We denote by $\Omega \subset \mathbb{R}^{HL(S+1)}$ the set of endowments.

Consider a function $c\colon \mathbb{R}^J \times \mathbb{R}^J \to \mathbb{R}$ such that $c(q,z) \geq 0$ is the intermediation cost of obtaining the portfolio $z = (z_j)_{j=1}^J \in \mathbb{R}^J$ given asset prices $q = (q_j)_{j=1}^J \in \mathbb{R}^J$. We denote by $\mathcal{C}$ the set of intermediation cost functions, i.e., costs by trading in the asset market in excess of the linear cost given by the price per unit.

Given spot market prices $p = (p(s))_{s=0}^S \in P$, asset prices $q \in \mathbb{R}^J$ and a transfer $w \in \mathbb{R}_+$, the budget set $B_h(p,q,w)$ of consumer h is the set of commodity bundles $x = (x(s))_{s=0}^S \in X_h$ and portfolios $z \in \mathbb{R}^J$, such that

$$p(0) \cdot x(0) \leq p(0) \cdot \omega_h(0) - q \cdot z - c(q,z) + w$$

and for every $s = 1, ..., S$,

$$p(s) \cdot x(s) \leq p(s) \cdot \omega_h(s) + V_s(p) \cdot z$$

We denote by

$$\phi_h(p,q,w) = (\phi_h^x, \phi_h^z)(p,q,w) = \arg\sup\{u_h(x) \mid (x,z) \in B_h(p,q,w)\}$$

the demand correspondence of consumer h. We shall sometimes denote the demand correspondence of h by $\phi_h(p,q,w;\omega,c)$ when we want to emphasise the underlying economy $(\omega,c) \in \Omega \times \mathcal{C}$.

We now define our equilibrium of this economy $(\omega,c) \in \Omega \times \mathcal{C}$, taking the intermediation cost function as given. It is a pair of commodity and asset prices, an allocation of commodities and portfolios, which are maximal for each consumer, each market clears and the revenue from the asset trades are distributed to the consumers by lump-sum transfers:

Definition 1 (Competitive Equilibrium) *A* competitive equilibrium *of* (ω,c) *is a tuple* (p,q,x,z)*, such that there exists* $w = (w_h)_{h\in H}$ *satisfying the following conditions:*

1. $(x_h, z_h) \in \phi_h(p,q,w_h)$ *for every* $h \in H$

2. $\sum_{h\in H} x_h - \omega_h = 0$

$X = \mathbb{R}^N$, then we write 2^N. If $X = \prod_{i\in I} X_i$, then we denote by $\mathrm{proj}_i\colon X \to X_i := (x_i)_{i\in I} \mapsto x_i$, the projection map.

[3] $L(J,S)$ is the space of linear mappings $\mathbb{R}^J \to \mathbb{R}^S$.

3. $\sum_{h\in H} z_h = 0$

4. $\sum_{h\in H} w_h = \sum_{h\in H} c(q, z_h)$

Denote by $E(\omega, c)$ the set of such tuples (p, q, x, z), where there exists a competitive equilibrium.

Let $Z = \mathbb{R}^{L(S+1)} \times \mathbb{R}^J \times (\mathbb{R}^{L(S+1)} \times \mathbb{R}^J)^H$, then $E(\omega, c) \subset Z$ for every $(\omega, c) \in \Omega \times \mathcal{C}$ and we shall refer to $E\colon \Omega \times \mathcal{C} \to 2^Z$ as the *Equilibrium correspondence.*

Assume next that the following conditions are satisfied:
Assumptions. Given the economy $(H, \mathbb{R}^{L(S+1)}, (u_h, \omega_h, X_h)_{h\in H}, V, c)$ we assume that

1. $\omega_h \in \operatorname{int} X_h$ for every $h \in H$
2. $X_h = \mathbb{R}_+^{L(S+1)}$ is closed, convex and bounded from below
3. $u_h\colon X_h \to \mathbb{R}$ is continuous, quasi-concave and strictly monotone
4. $V\colon P \to L(J, S)$ is continuous
5. $c\colon Q \times \mathbb{R}^J \to \mathbb{R}_+$ is continuous and $c(q, 0) = 0$ for every $q \in Q$
6. if $\lambda > 1$ then $c(q, \lambda z) \geq c(q, z)$ for every $z \in \mathbb{R}^J$ for every $q \in Q$
7. for any $\lambda \in [0, 1]$ and $z, z' \in \mathbb{R}^J$ we have $c(q, \lambda z + (1-\lambda)z') \leq \lambda c(q, z) + (1-\lambda)c(q, z')$ for every $q \in Q$
8. $c(q, e^j\lambda) \to \infty$ whenever $\lambda \to \infty$ for every $q \in Q$ and $j = 1, ..., J$

We denote by $\mathcal{C}$ the subset of $C(\mathbb{R}^J \times Q)$ with the compact-open topology, which satisfies the assumptions[4]. We note that $\mathcal{C}$ is not complete since $0 \in \overline{\mathcal{C}} \setminus \mathcal{C}$. In particular, this implies that any convergent sequence in the subspace $\mathcal{C}$ must have a limit point that is different from the zero function. Hence, $\mathcal{C}$ is a convex cone pointed at 0, which is not closed.

Remark 1 *We note that continuity and strict monotonicity of preferences imply the following property: for every $x \in X_h$ there exists $\varepsilon > 0$ and $y(0) - x(0) \in \mathbb{R}_+^L$ such that for every $(y(s))_{s=1}^S$ with $\|(x(s) - y(s))_{s=1}^S\| < \varepsilon$ implies that $u(y) > u(x)$.*

Dividend structures satisfying the assumptions include the following types:

[4]A sequence $\{f_n\}$ converges to f in the compact-open topology if and only if for every $K \subset \mathbb{R}^J$ compact and every $\epsilon > 0$ there exists $N \geq 1$, such that for every $n \geq N$ we have that $\sup_{x\in K} |f_n(x) - f(x)| < \epsilon$. According to [46], this space is metrizable, separable and complete.

1. $V(p) = V(p')$ for every $p, p' \in P$.

2. Consider the matrix $R \in \mathbb{R}^{SL}$ such that the dividend matrix $V(p)$ is given by $V_s(p) = p(s) \cdot R_s = V_s(p(s))$, where $p(s) \in \mathbb{R}^L_+$ is the commodity price vector of state s, while $R_s \in \mathbb{R}^L$ is a commodity bundle.

3. As in the above example, but letting $\bar{p}(s) = (\frac{p_l(s)}{p_l(0)})_{l=1}^L$ (when $p \gg 0$), we have that $V_s(p) = \bar{p}(s) \cdot R_s$.

4. Considering an asset $r^1 = (r_s(p(s)))_{s=1}^S$ with $r(\cdot)$ continuous, then letting an asset having the dividend vector given by $r^2(p) = (\max\{k - r_s(p(s)), 0\})_{s=1}^S$ for some $k \geq 0$, we have that $V(p) = (r^1(p), r^2(p))$ satisfies the assumptions.

Thus, the dividend structure includes securities such as nominal assets (i.e., bonds and Arrow securities), contingent contracts on commodity bundles, options and equity contracts with limited liabilities.

The last assumptions concern the properties of the intermediation costs: we assume that the costs are non-negative, increasing and zero if the asset trades are zero. Further, if the volume of asset trades goes to infinity, the costs go to infinity. Finally, the costs behave continuously in both asset prices and trades. An example of a cost function that satisfies these assumptions is the following: Let $c_j, k_j > 0$ for $j = 1, ..., J$, then

$$c(q, z) = \alpha \sum_{j=1}^{J} c_j |q_j z_j|^n + \beta \sum_{j=1}^{J} k_j |z_j|^m$$

for every $n, m \in \mathbb{N}$ satisfies the assumptions when $\alpha, \beta > 0$. An example of a cost function that does *not* satisfy the assumptions is

$$c(q, z) = \begin{cases} 0 & z = 0 \\ \alpha q \cdot |z| + F & z \neq 0 \end{cases}$$

for some $\alpha, F > 0$, since it violates the continuity and convexity property in $z = 0$.[5]

Our main result is that under our maintained assumptions, every economy possesses a competitive equilibrium and the equilibrium correspondence is (upper-hemi) continuous:

Theorem 1 *For every $(\omega, c) \in \Omega \times \mathcal{C}$ we have $E(\omega, c) \neq \emptyset$.*

Proposition 2 *$E\colon \Omega \times \mathcal{C} \to 2^Z$ is upper-hemi continuous.*

The difficult part is to show that the intermediation costs induce an endogenous boundary on the optimal portfolio choice. We do this by showing that arbitrary small

[5]With $|z| = (|z_1|, ..., |z_J|)$.

changes in future income can be converted into infinite positive income in the present period. Thus, arbitrary small changes in future consumption, by continuity and strict monotonicity of preferences, we can compensate the consumer by increasing the present consumption.

Further, consider a sequence $c^n \in \mathcal{C}$ with $c^n \to 0$ and $E(\omega, 0) = \emptyset$. The non-existence of equilibrium must imply the need for income transfer when costs approach zero and hence a positive demand for assets - the discounting of future state income must differ across consumers. But then as the cost function is ϵ-distance from the zero function for $\epsilon > 0$ small enough, there exists some positive amount of trade and as ϵ tends to zero, the asset trades go to infinity.

Proposition 3 *Assume that $E(\omega, 0) = \emptyset$, $c^n \in \mathcal{C}$, $c^n \to 0$ and $(p^n, q^n, x^n, z^n) \in E(\omega, c^n)$, then for every $M \geq 0$ there exists $N \geq 1$ such that if $n \geq N$ then $\|z^n\| > M$.*

In the example of [51], these results imply, that when intermediation costs on asset trade are positive, there exists a competitive equilibrium. However, as this cost goes to zero, the volume of asset trade goes to infinity.

Remark 2 *We note that $E(\omega, 0) \subset E(\omega, \mathcal{C})$ for every $\omega \in \Omega$. Thus, it is possible to construct an intermediation cost function that satisfies the conditions of continuity, convexity etc., while still obtaining the competitive equilibrium without intermediation costs. The point is to tax only those asset trades whose volume exceeds the equilibrium asset trade. The inclusion holds for every $\omega \in \Omega$ by the convention $\emptyset \subset A$ for any set A.*

2.3 Proofs of the Main Theorems

Before giving the proofs, we define the sets used in this section. We denote by

$$D = \prod_{s=0}^{S} P_s \times Q = \prod_{s=0}^{S} \{p(s) \in \mathbb{R}^L_+ \mid \sum_{l=1}^{L} p_l(s) = 1\} \times \{q \in \mathbb{R}^J \mid \max_j |q_j| \leq 1\}$$

the price space and

$$M(\omega) = \{x \in \prod_{h \in H} X_h \mid \sum_{h \in H} x_h - \omega_h = 0\}$$

the set of market equilibria of commodity allocations. A summary of the sets is:

$\mathbb{R}^{L(S+1)}$	The commodity space
$\Omega \subset \mathbb{R}^{HL(S+1)}$	The set of endowments
$X = \prod_{h\in H} X_h$	The set of allocations
$M(\omega)$	The set of spot market clearing allocations
$D \subset \mathbb{R}^{L(S+1)} \times \mathbb{R}^J$	The set of normalised spot and asset prices
$Z = D \times X \times \mathbb{R}^{HJ}$	The set of prices, allocations and portfolios
$\mathcal{C} \subset C(Q \times \mathbb{R}^J)$	The set of intermediation cost functions

For every $n \in \mathbb{N}$, let $\mathcal{E}^n = (\omega, c)^n \in \Omega \times \mathcal{C}$ be the restricted economy with

$$\begin{aligned} B_h^n(p,q,w;\omega,c) &= B_h(p,q,w;\omega,c) \cap \operatorname{proj}_h M(\omega) \times [-n,n]^J \\ \phi_h^n(p,q,w;\omega,c) &= \arg\sup\{u_h(x_h) \mid (x_h,z_h) \in B_h^n(p,q,w;\omega,c)\}, \end{aligned}$$

being the restricted budget and demand correspondences of consumer $h \in H$.

2.3.1 Proof of Theorem 1

We now state and prove a theorem similar to [53], stating that every bounded economy contains an equilibrium.

Lemma 1 *Given $n \in \mathbb{N}$, assume that ϕ_h^n is upper-hemi continuous, non-empty and convex-valued for every $h \in H$. Then there exists $(p^n, q^n, x^n, z^n) \in E(\mathcal{E}^n)$.*

Proof:

Define μ_0 by

$$\mu_0(\bar{x},\bar{z}) = \arg\max_{(p,q)\in D} \sum_{h\in H} p(0)\cdot(\bar{x}_h(0) - \omega_h(0)) + q\cdot \bar{z}_h,$$

and let μ_1 be given by

$$\mu_1(\bar{x},\bar{z}) = \left(\arg\max_{(p,q)\in D} p(s)\cdot \sum_{h\in H} \bar{x}_h(s) - \omega_h(s)\right)_{s=1}^{S}.$$

Note that the correspondence is well-defined since the maxima used in μ_0 and μ_1 are independent. Let then μ be given by

$$\mu(\bar{x},\bar{z},\bar{p},\bar{q}) = \mu_0(\bar{x},\bar{z}) \times \mu_1(\bar{x},\bar{z}) \times \left(\prod_{h\in H} \phi_h^n(\bar{p},\bar{q},\bar{w}_h) - (\omega_h, 0)\right),$$

with $\bar{w}^h = \pi_h \sum_{k\in H} c(\bar{q},\bar{z}_k)$ for some $\pi_h > 0$ and $\sum_{h\in H}\pi_h = 1$.

Assume that $(\bar{p},\bar{q},\bar{x},\bar{z}) \in \mu(\bar{p},\bar{q},\bar{x},\bar{z})$, then we have that

$$\sum_{h\in H} \bar{p}(0)\cdot(\bar{x}_h(0)-\omega_h(0))+\bar{q}\cdot\bar{z}_h \geq \sum_{h\in H} p(0)\cdot(\bar{x}_h(0)-\omega_h(0))+q\cdot\bar{z}_h,$$

for every $(p(0),q) \in P_0 \times Q$ and using the budget constraints

$$\sum_{h\in H} \bar{p}(0)\cdot(\bar{x}_h(0)-\omega_h(0))+\bar{q}\cdot\bar{z}_h = 0, \tag{2.1}$$

which implies that

$$\sum_{h\in H} p(0)\cdot(\bar{x}_h(0)-\omega_h(0))+q\cdot\bar{z}_h \leq 0.$$

Since $\delta_{0l} \in D$ for all $l = 1,...,L$, we have that $\sum_{h\in H}\bar{x}_h(0)-\omega_h(0) \leq 0$ and $\sum_{h\in H}\bar{z}_h \leq 0$. Also, we have that $-\delta_j \in Q$ such that $\sum_{h\in H}\bar{z}_h \geq 0$, implying that $\sum_{h\in H}\bar{z}_h = 0$. Further, we have that

$$\sum_{h\in H} \bar{p}(s)\cdot(\bar{x}_h(s)-\omega_h(s)) \geq \sum_{h\in H} p(s)\cdot(\bar{x}_h(s)-\omega_h(s))$$

for every $p(s) \in P_s$, but then since $\bar{p}(s)\cdot(\bar{x}_h(s)-\omega_h(s)) = V_s(\bar{p})\cdot\bar{z}_h$ and $\sum_{h\in H}\bar{z}_h = 0$, we have that $\sum_{h\in H} p(s)\cdot(\bar{x}_h(s)-\omega_h(s)) \leq 0$. Again, this implies that $\sum_{h\in H}\bar{x}_h(s)-\omega_h(s) \leq 0$ for every $s = 1,...,S$. By monotonicity, we have that $\sum_{h\in H}\bar{x}_h(s)-\omega_h(s) = 0$ for every $s = 0,1,...,S$.

By standard arguments, the correspondence μ is upper-hemi continuous, non-empty, convex-valued mapping a non-empty, convex and compact subset of an Euclidean space into itself. Thus, according to [1] there exists some $\bar{e} \in \mu(\bar{e})$.

□

Remark 3 *Note that we only need $L(S+1)-(S+1)$ price variables to obtain an equilibrium, and not $L(S+1)-2$, which is standard when the asset structure is not real. Hence, we can restrict ourselves to price systems that have positive spot prices in each state. This prevents examples of budget correspondences which are not lower-hemi continuous as was the case with the example given in [28, section 4].*

Next, we show that there exists an equilibrium for the original economy, i.e., that the optimal asset trades must be bounded:

Lemma 2 *Let $(p^n,q^n,x^n,z^n) \in E(\mathcal{E}^n)$ be an equilibrium for every $n \in \mathbb{N}$, then there exists some $(p,q,x,z) \in E(\omega,c)$, which is a cluster point of the sequence $(p^n,q^n,x^n,z^n)_{n\in\mathbb{N}}$.*

Proof:

Consider a sequence of equilibria (p^n, q^n, x^n, z^n) with $w_h^n = \pi_h \sum_{h \in H} c(q^n, z_h^n)$, where $\pi \in \mathbb{R}^H_{++}$ and $\sum_{h \in H} \pi_h = 1$. Since $(p^n, q^n, x^n)_{n \in \mathbb{N}} \subset D \times M(\omega)$, there exists a convergent subsequence and by passing to this subsequence we can assume that $(p^n, q^n, x^n) \to (p, q, x) \in D \times M(\omega)$.

Assume that $\lim_{n\to\infty} \|z_1^n\| = \infty$. Then by market clearing we can assume that $\lim_{n\to\infty} \|z_2^n\| = \infty$. But since $\lim_{n\to\infty} |w_1^n - q^n \cdot z_1^n - c(q^n, z_1^n)| < \infty$ and $\lim_{n\to\infty} w_1^n = \infty$, we must have that $\lim_{n\to\infty} q^n \cdot z_1^n + c(q^n, z_1^n) = \infty$.

Given the sequence $(z_1^n)_{n \in \mathbb{N}}$, define $\tilde{z}_1^n = (1-\lambda) z_1^n$ for some $\lambda > 0$. Then for every $n \in \mathbb{N}$,

$$-q^n \cdot \tilde{z}_1^n - c(q^n, \tilde{z}_1^n) \geq -q^n \cdot z_1^n - c(q^n, z_1^n) + \lambda(q^n \cdot z_1^n + c(q^n, z_1^n)),$$

by convexity of $c(q, \cdot)$ and $c(q, 0) = 0$, and thus for every $\lambda > 0$,

$$\begin{aligned} \lim_{n\to\infty} w_1^n - q^n \cdot \tilde{z}_1^n - c(q^n, \tilde{z}_1^n) &\geq \lim_{n\to\infty} \lambda(q^n \cdot z_1^n + c(q^n, z_1^n)) + p(0) \cdot (x_1(0) - \omega_1(0)) \\ &= \infty. \end{aligned}$$

Thus, we have obtained a portfolio that yields infinitely (in the limit) more income in the first period. Let $(\tilde{x}_1^n)_{n \in \mathbb{N}} \subset X_1$ be a sequence satisfying $\tilde{x}_1^n(s) = (1-\lambda) x_1^n(s)$ for $s = 1, ..., S$ and $n \in \mathbb{N}$. Then $\lim_{\lambda \to 0} \tilde{x}_1^n(s) = x_1(s)$ for every $s = 1, ..., S$, $n \in \mathbb{N}$ and

$$p^n(s) \cdot (\tilde{x}_1^n(s) - \omega_1(s)) = V_s(p^n) \cdot \tilde{z}_1^n.$$

But $\tilde{x}^n(0)$ can be chosen arbitrarily for n large enough. By using the result in remark 1, we obtain the desired contradiction, since $u_1(\tilde{x}_1^n) > u_1(x_1^n)$.

Since the sequence (p^n, q^n, x^n, z^n) is bounded, we have that there exists some $m \in \mathbb{N}$, such that for every $h \in H$

$$\begin{aligned} &\arg\max\{u_h(x_h) \mid (x_h, z_h) \in B_h^m(p, q, w_h)\} \\ &= \arg\max\{u_h(x_h) \mid (x_h, z_h) \in B_h(p, q, w_h)\}, \end{aligned}$$

since $B_h^{n_2}() \subset B_h^{n_1}()$ for every $n_1 > n_2$.

Hence the sequence (p^n, q^n, x^n, z^n) must contain a convergent subsequence, but then the cluster point (p, q, x, z) of this sequence is an equilibrium of $\mathcal{E}$: the market clearing conditions are obviously satisfied and the constraint on the revenue from intermediation costs also. Thus, we need to show that $(x_h, z_h) \in \phi_h(p, q, w_h)$. Assume that $(x'_h, z'_h) \in B_h(p, q, w_h)$ and $u_h(x'_h) > u_h(x_h)$. Then given $(p^n, q^n, w_h^n) \to (p, q, w_h)$, there exists by lower-hemi continuity of $B_h(\cdot)$ a sequence $(\bar{x}_h^n, \bar{z}_h^n) \in B_h(p^n, q^n, w_h^n)$ such that $(\bar{x}_h^n, \bar{z}_h^n) \to (x'_h, z'_h)$; but then there exists some $N \geq 1$ such that for every $n \geq N$ we have that $u_h(\bar{x}_h^n) > u_h(x_h^n)$ by continuity of u_h, a contradiction.

□

In order to apply the maximum theorem, we need the following result that the budget correspondence B_h^n is continuous, non-empty and convex-valued for every consumer h and every $n \in \mathbb{N}$:

Lemma 3 *The correspondence $B_h^n(\omega, c) \colon D \times \mathbb{R}_+ \to 2^X \times 2^J$ given $(\omega, c) \in \Omega \times \mathcal{C}$ is continuous, non-empty and convex-valued for every $n \in \mathbb{N}$.*

In the proof we omit sub- and superscripts indices of the consumers h, and we show the lower-hemi continuity of the original correspondence B_h.

Proof:

Non-emptiness and convexity is obvious.

Upper-hemi continuity: Since $B^n(p, q, w)$ have values in a compact space, this is equivalent to the closed graph property. Consider a sequence $(p^m, q^m, w^m) \to (p, q, w)$, $(x^m, z^m) \in B^n(p^m, q^m, w^m)$ and $(x^m, z^m) \to (x, z)$. But then we have that

$$\begin{aligned} p^m(0) \cdot x^m(0) + q^m \cdot z^m + c(q^m, z^m) &\leq p^m(0) \cdot \omega(0) + w^m \\ p^m(s) \cdot x^m(s) &\leq p^m(s) \cdot \omega(s) + V_s(p^m) \cdot z^m \end{aligned}$$

for every $m \in \mathbb{N}$ and taking limits, these inequalities are preserved, i.e., $(x, z) \in B^n(q, p, w)$.

Lower-hemi continuity: Consider $(x, z) \in B(p, q, w)$ and a sequence $(p^m, q^m, w^m) \to (p, q, w)$. If $p(0) \cdot x(0) + q \cdot z + c(q, z) < p(0) \cdot \omega(0) + w$, then there exists some $N \geq 1$ such that for every $m \geq N$ we have that $p^m(0) \cdot x(0) + q^m \cdot z + c(q^m, z) < p^m(0) \cdot \omega(0) + w^m$. Thus consider the sequence

$$(x^m(0), z^m) = \begin{cases} (\omega, 0) & m \leq N \\ (x(0), z) & m > N \end{cases}$$

If on the other hand $p(0) \cdot x(0) + q \cdot z + c(q, z) = p(0) \cdot \omega(0) + w$, then since $p(0) \cdot \omega(0) + w > 0$, there exists $(x', 0) \in \mathbb{R}^L \times \mathbb{R}^J$ such that for some N we have that $m \geq N$ implies

$$p^m(0) \cdot x' < p^m(0) \cdot \omega(0) + w^m.$$

Since the map $(x(0), z) \mapsto p^m(0) \cdot x(0) + q^m \cdot z + c(q^m, z)$ is continuous, for every m there exists $(x^m(0), z^m)$ such that

$$p^m(0) \cdot x^m(0) + q^m \cdot z^m + c(q^m, z^m) = p^m(0) \cdot \omega(0) + w^m,$$

and $(x^m(0), z^m)_{m \in \mathbb{N}}$ can be chosen to converge to $(x(0), z)$.

By the $s = 1, ..., S$ budget constraints, we let

$$t^m(s) = \frac{p^m(s) \cdot \omega(s) + V_s(p^m) \cdot z^m}{p^m \cdot x(s)},$$

such that $\lim_{m\to\infty} t^m(s) = 1$ by continuity of $V(\cdot)$. Hence we have found a convergent sequence $(x^m, z^m) \in B(p^m, q^m, w^m)$ with $(x^m, z^m) \to (x, z) \in B(p, q, w)$.

$\square$

This concludes the proof of theorem 1: by lemma 1 there exists for every bounded economy market clearing prices, allocations and portfolios maximising utility. By lemma 2 there exists a boundary on allocations and portfolios due to intermediation costs. By lemma 3 the budget correspondence is continuous, non-empty and convex valued, which implies that the demand correspondence is upper-hemi continuous and convex-valued and thus the conditions of lemma 1 are satisfied and the result of theorem 1 is proved.

2.3.2 Proofs of Proposition 2 and 3

In order to apply the maximum theorem to the demand correspondence, we need to know the continuity property of the budget correspondence wrt. the space of cost functions: To obtain the result of upper-hemi continuity of ϕ_h, we use the maximum theorem (see [1]) extended to general topological spaces due to the fact that $\mathcal{C}$ is not a subspace of any Euclidean space.

Lemma 4 *Given $(p, q, w) \in D \times \mathbb{R}_+$, the correspondence $B_h(p, q, w; \omega, \cdot) \colon \mathcal{C} \to 2^X \times 2^J$ is continuous.*

In the following, we denote $B(c) = B(p, q, w; \omega, c)$ for every $(p, q, w) \in D \times \mathbb{R}_+$ and $(\omega, c) \in \Omega \times \mathcal{C}$, and omitting the prescript h.

Proof:

It is easy to see that B has a closed graph: since if $c^n \to c$ and $(q^n, z^n) \to (q, z)$ then $c^n(q^n, z^n) \to c(q, z)$ by uniform convergence[6]. Since $c^n \to c$, we have that if $(x, z) \in B(c)$, then $\lim_{n\to\infty} p(0) \cdot (x(0) - \omega(0)) + q \cdot z + c^n(q, z) - w = 0$ and hence for every $\epsilon > 0$ there exists $\delta > 0$, such that if $c' \in \mathcal{O}_\delta(c) \cap \mathcal{C}$ then $B(c') \subset \mathcal{O}_\epsilon(B(c))$.

Consider now an element $(x, z) \in B(c)$ and a sequence (c^n) such that $c^n \to c$. We look for a sequence $(x^n, z^n) \in B(c^n)$ such that $(x^n, z^n) \to (x, z)$. Take $(x^n, z^n) = \arg d((x, z), B(c^n))$[7]. But convergence in the compact-open topology implies pointwise convergence in every compact subset. Taking the compact set containing $(\omega, 0)$ and the sequence (x^n, z^n), we have that $d((x, z), B(c^n)) \to 0$ and hence $\lim_{n\to\infty}(x^n, z^n) = (x, z)$.

$\square$

[6]The evaluation map $(c, (q, z)) \mapsto c(q, z)$ is continuous.

[7]Where we have that $d(x, A) = \inf_{a\in A} \rho(x, a)$ in any metric space X with metric ρ and $A \subset X$ closed.

Note that as a corollary to the lemmatas 3 and 4, the correspondence $B\colon D \times \mathbb{R}_+ \times \Omega \times \mathcal{C} \to 2^X \times 2^J$ is continuous in the product topology of $D \times \mathbb{R}_+ \times \Omega \times \mathcal{C}$.

As a corollary to the proof in lemma 2, we have that the set of equilibria is bounded given any intermediation cost function.

Lemma 5 *Let $(\omega^n, c^n) \to (\omega, c)$ then $\bigcup_{n\in\mathbb{N}} E(\omega^n, c^n)$ is bounded.*

Proof:

Indeed, every $E(\omega^n, c^n)$ is bounded. By assuming that there exists z^n, such that $\|z^n\| \to \infty$ and replicating the argument in the proof of lemma 2, we obtain a contradiction.

□

The equilibrium set is bounded:

Lemma 6 *$E(\omega, c)$ is compact for every $(\omega, c) \in \Omega \times \mathcal{C}$.*

Proof:

Since $E(\omega, c)$ is bounded, we can restrict the demand correspondences without changing the set of equilibria. But then $\phi_h^m\colon D \times \mathbb{R}_+ \to 2^{L(S+1)} \times 2^J$ is upper-hemi continuous. But then if $(p^n, q^n, x^n, z^n) \to (p, q, x, z)$, $(p^n, q^n, x^n, z^n) \in E(\omega, c)$, we have that $(x_h^n, z_h^n) = \phi_h^m(p^n, q^n, w_h^n) \to \phi_h^m(p, q, w_h) = (x_h, z_h)$ and the market clearing conditions are satisfied. Thus $(p, q, x, z) \in E(\omega, c)$.

□

This induces that the equilibrium correspondence is upper-hemi continuous in the endowment space, since given any sequence the equilibrium set $\bigcup_{n\in\mathbb{N}} E(\omega_n, c^n)$ is contained in a compact subspace and hence the closed graph property is equivalent with upper-hemi continuity.

Proof: [Proof of Proposition 2]

Since $\{E(\omega^n, c^n)\}_{n\in\mathbb{N}}$ is contained in a bounded set of Z, there exists a convergent subsequence (p^n, q^n, x^n, z^n) of selections in $E(\omega^n, c^n)$, but then by upper-hemi continuity of $\phi_h(\cdot)$, we have that $(x_h, z_h) \in \phi_h(p, q, w; \omega, c)$ and hence $(p, q, x, z) \in E(\omega, c)$.

□

This gives us the result of proposition 2: by lemma 4 the budget correspondence is continuous in $\Omega \times \mathcal{C}$; hence by the Maximum Theorem the demand correspondence is upper-hemi continuous.

Next we prove proposition 3.

Proof: [Proof of Proposition 3]

Using the result of [53], we have that if $\|z\| \leq L$ for all z then $E(\omega, 0) \neq \emptyset$, thus implying that if $E(\omega, 0) = \emptyset$ then for any allocations of commodities and portfolios

(x,z) such that $\sum_{h\in H} x_h - \omega_h = 0$ and $\sum_{h\in H} z_h = 0$ and prices (p,q) with $(x_h, z_h) \in B_h(p,q,w_h;\omega_h,0)$, there exist (x',z') and h with $u_h(x'_h) > u_h(x_h)$ and $(x'_h, z'_h) \in B_h(p,q,w_h;\omega_h,0)$. Assume that there exist $M \geq 0$ such that $\sup_n \|z^n\| \leq M$; let $(p^n, q^n, x^n, z^n) \in E(\omega, c^n)$ be a convergent (sub)sequence with limit $(p,q,x,z) \in Z$, then we have that

$$\begin{aligned} 0 &= \sum_{h\in H} x_h - \omega_h \\ 0 &= \sum_{h\in H} z_h \\ (x_h, z_h) &\in B_h(p,q,w_h;\omega_h,0), \end{aligned}$$

(the last inclusion follows from the closed graph property of the budget set) but then there exists (x',z') and $h \in H$ such that $u_h(x'_h) > u_h(x_h)$ and $(x'_h, z'_h) \in B_h(p,q,w_h;\omega_h,0)$; then by continuity of u_h there exists some $\delta > 0$ such that for every $\bar{x}_h \in X_h$ with $\|\bar{x}_h - x'_h\| < \delta$ implies that $u_h(\bar{x}_h) > u_h(x_h)$. Since $c_n \to 0$, there exists some $N \geq 1$ such that for all $n \geq N$ we have that $B_h(p^n, q^n, w_h^n; \omega_h, c^n) \cap O_\delta(x'_h, z'_h) \neq \emptyset$, but then (p^n, q^n, x^n, z^n) cannot be an equilibrium since h can obtain higher utility and can therefore not maximise utility.

□

2.4 Final remarks

In this paper we have proved the existence of equilibrium in financial economies when intermediation costs are present. We have shown that the equilibrium correspondence mapping cost functions into prices, allocations and portfolios is upper-hemi continuous. Also we have shown that if the cost function converges to the zero function, then the portfolios of the agents must diverge and hence are unbounded.

The assumption of strict monotonicity is obviously very restrictive, and the proof also shows that a less restrictive condition could be imposed and our result would still hold. However, this would be less standard and would not provide considerably new insights.

We could extend the existence result to cost functions that are non-convex and semi-continuous when considering large economies. However, as the proof is constructed, this would require a condition that implied that the reduction in intermediation cost due to some small fraction would not be zero. It is easy to see that a two-part tariff with positive marginal costs would satisfy this condition.

In this paper we have considered intermediation costs on the net asset trade. We note however that the result is easily extendable to the case where intermediation costs are on sale and purchase of the same asset.

Chapter 3

Volatility of Security Trade and Incomplete Financial Markets

Tobias Markeprand & Mich Tvede

Abstract. In the recent decades the ratio between the volume of trade in securities and income has increased dramatically. We show that this increase can be explained in a general equilibrium model with incomplete financial markets and intermediation costs. The model is a stochastic version of a multi-agent Ramsey model with stationary fundamentals of an exchange economy. We show that the effect of financial liberalizations on the trading volume is potentially very large. Further, we show how the changes in the dividend structure can explain both an increase in the volume and volatility of trade. This can happen even though the risk dispersion offered by the market is unchanged. These results can only occur in economies where the financial market is (potentially) incomplete, whose dividends depend on endogenous variables such as e.g. prices.

JEL classification:. D41; D52; D53; G11

Keywords:. Incomplete markets, General Equilibrium, Volatility, Portfolio Choice.

3.1 Introduction

Empirical studies have shown that the volatility of stocks has been increasing during the period from mid 70's until today (see [57]). The volume of trade on NYSE has increased almost monotonically throughout the decade 1990-1999 by an annually instantaneous rate of 16 %, while the number of trades has increased by almost 22 %. In the same period the GDP of the US has increased annually by 2.7 %. And the same pattern is observed throughout a large part of the world. During this period the main trend of public policy of financial regulation has been that of deregulation, and hence has decreased the costs of trading securities. Here we show that the high volume of trade can be explained by a decrease in the intermediation costs. While this might not seem too surprising, we claim that the observed increase in trades is not only the *direct* effect of a decrease in the trading costs. We also need to take into account the *indirect* equilibrium effects (and thus basically we justify a general equilibrium approach). The indirect effect goes as follows: when intermediation costs decreases, asset and commodity prices change current, as well as future, prices. These changes in prices (and price expectations) result in changed securities' returns, such as options, stocks, etc. If these induced changes in the dividend reduce the risk dispersion of a current portfolio, the traders need to trade more intensive in order to obtain the same risk profile. In this paper, we show that this effect can be very large and, potentially, without boundary. Thus, high volume of trade and volatility is not necessarily due to irrational noise traders but can be explained in a perfectly rational model!

We consider the following model: Assume that the fundamentals are stationary Markov processes, where the fundamentals are the initial endowments, the security dividends and the consumer preferences. Formulated differently, the economy is subject to random shocks which are independent and invariant over time. No production takes place and the only economic activity is the exchange of a perishable good and of short-lived securities with a zero net supply. The dividend of each security can depend upon the realisation of future security prices. A departure from the traditional finance models, trading securities is subject to an intermediation cost. There is a finite set of households each with an infinite lifespan. The consumer's utility satisfies the independence axiom and the von-Neumann Morgenstern utility is bounded away from the boundary. Households are, in addition to the usual intertemporal budget constraint, subject to a transversality condition using their own present value vector. This prevents Ponzi-game schemes, i.e., financing debt with increasing debt into the indefinite future. In short, our model is a stationary version of [43] with the addition of intermediation costs and price dependent dividends.

Let us relate this paper to the more general discussion on financial volatility and efficiency of the financial markets. The main point of the market efficiency hypothesis is that the price of an asset equals the present value (using an appropriate discount factor) of the future dividends. More specifically, the discount rates are martingales, i.e., the expected value is equal to the current value. One might be more correct in labeling this hypothesis the market *information* efficiency, since it does not tell us anything about the (Pareto) efficiency of the allocation of real goods like savings and investment. It only implies that it is not possible to obtain any arbitrage on the financial markets, i.e., there is no free-lunch. We shall not challenge the efficient market hypothesis here. However, we do allow the financial markets to be incomplete, such that not all risk profiles are tradable on the financial markets.

Much of the finance literature has focused on the pricing of securities, rather than the volume of trade. However, let us briefly discuss some contributions on volatility of security prices which we find relevant. They are relevant in that their models are within the same framework as ours. Also, the volume of trade and price changes are not independent, empirically according to [39], [50] where volume and prices are positive correlated, and theoretically [56] where rational traders, due to informational asymmetries, use trading volumes to interfere information and hence volume affects prices. In this way prices and volume of trade are intimately related. In [3] incomplete financial markets, or more specifically, limited market participation is also related to excess volatility of asset prices. They attribute this to liquidity trade, i.e., a sudden need of cash and the sale of securities in order to meet such needs, and limited market participation. A model like [20] is used and this opens up for self-fulfilling expectations in the decision of participation on the security market, which is subject to a fixed entry cost: If investors expect small participation rates, future security markets will be sensitive to liquidity trade, which can discourage investors from paticipating on the security market. Instead they use the liquid cash as means of saving. [17] studies volatility of security prices and financial innovation. Their results point in two directions, depending on the nature of risk, more specifically whether there is aggregate risk or not. When there is no aggregate risk, completing the asset market will generically reduce the asset price volatility. While in the case of aggregate risk, reducing the degree of incompleteness *per se* is not necessarily associated with a volatility reduction. We take the financial structure as exogenous. These are only two examples and many more could have been added.

Let us try to justify our introduction of intermediation cost on asset trading. Presumably the best known example of an intermediation cost is known as bid-ask price spreads, i.e., the difference between the prices that financial institutions buys assets from costumers and the price that they sell the assets at. Every person who has travelled from one currency area to another knows that the exchange of money in itself cost money. Also, trading on a stock exchange is subject to a trading cost net of the price of a stock. Thus, they do exist! We defer further discussion to section 3.7.

The outline of the paper is as follows: in section 3.2 we formulate the model and introduce the relevant concepts. In section 3.3 we state our main results. In section 3.4 shows the model formulated in income streams. This formulation in income streams illustrates how examples of non-existence of equilibria can exist. The point is that the model is equivalently formulated as portfolio choices and asset prices as to income stream choices through the achievable income streams generated by portfolio choices. The section shows that non-existence of an equilibrium is indeed a sufficient condition but not a necessary condition. In section 3.5 we show the existence of an ergodic equilibrium in any economy with positive transaction costs. In section 3.6 we also give specific examples of economies and their equilibria. Using the terminology of income spaces we obtain a wide range of economies where the volume of trade is boundless as the intermediation costs goes to zero. Finally, in section 3.7 we conclude and comment on the results and assumptions.

3.2 The Model

Time goes from $t = 0$ to ∞, thus $t \in \mathbb{N}_0$. In each period there is a single perishable consumption good and k shot-lived purely financial securities, i.e., with a zero net supply. Let there be n agents each living infinitely. No production takes place and the only available resources are the initial endowments of the consumption good. Let Y be a finite set of shocks and $\pi\colon Y \to P(Y)$ a transition map, i.e., given a state y, $\pi(y)$ is a probability distribution on Y and $\pi(y, y') \geq 0$ is the probability that the next state is y' conditional on y. We assume wlog that for every $y' \in Y$ there exists $y \in Y$ such that $\pi(y, y') > 0$, i.e., no state is superfluous.

Agent i evaluates any consumption stream $x = (x_t)_{t \in \mathbb{N}_0}$ which is uniformly bounded, where x_t is a random variable with values in $\mathbb{R}_+$ and distribution μ, by using the real-valued map

$$U_i(x) = E[\sum_{t=0}^{\infty} \beta_i^t u_i(x_t)] = \int \sum \beta_i^i u_i(x_t)\ d\mu(x),$$

where $u_i\colon \mathbb{R}_+ \to \mathbb{R}$ is a von-Neumann-Morgenstern utility function and β_i is the discounting rate. For any T denote by

$$U_i^T(x) = E[\sum_{t=0}^{T} \beta_i^t u_i(x_t)],$$

the "cut-off"-utility or T-horizon utility of some consumption stream.

We take the consumption space to be the set of uniformly bounded $\mathbb{R}_+$-sequences, i.e., $X = \ell_\infty(Y) = \{(x_t)_{t\in\mathbb{N}_0} \mid \sup_t \|x_t\| < \infty\}$ where $x_t \colon Y^t \to \mathbb{R}_+$ is the consumption at date t contingent on the history $(y_\tau)_{\tau\leq t}$.

Assume that the assets are short-lived with return $D(y,q)\theta \in \mathbb{R}$ in state y if the portfolio $\theta \in \mathbb{R}^k$ is held from the previous to the current period and the current asset price vector is q. We assume that $D(y,q) \geq 0$ for every $y \in Y$, $q \geq 0$. Let

$$\Theta = \{(\theta_i)_{i=1}^n \in \mathbb{R}^{nk} \mid \sum_{i=1}^n \theta_i = 0\}$$

be the set of market clearing portfolio (spot) allocations. Given a portfolio θ, let $c(\theta) \in \mathbb{R}_+$ be the intermediation cost of this portfolio. We assume that $c(\cdot)$ is continuous, convex, $c(0) = 0$ and $\lim_{\|\theta\|\to\infty} c(\theta) = \infty$. Denote by $\mathcal{C}$ the set of intermediation cost functions endowed with the compact-open topology.

We take the portfolio space to be the set of uniformly bounded $\mathbb{R}^k$-sequences, i.e., $Z = \ell_\infty(\mathbb{R}^k) = \{(z_t)_{t\in\mathbb{N}_0} \mid \sup_t \|z_t\| < \infty\}$ where $z_t \colon Y^t \to \mathbb{R}^k$ is the consumption at date t contingent on the history $(y_\tau)_{\tau\leq t}$.

Note that the securities here are in zero net supply, and thus there is no *aggregate* accumulation of wealth in the economy. Thus, in a particular equilibrium path an individual can be rich, but this accumulation of wealth of a single individual must be offset by a equivalent accumulation of debt by other consumers.

The reason why we choose short-lived securities, is threefold: first, they suite the stationary environment well, and second any finite-lived security structure would be no obstacle only more cumbersome notation, and finally, as shown in [44] with zero net supply of securities the model permit speculative bubbles, i.e., the price of a security can deviate from the *fundamental value*. Also, it eases the notational difficulties.

Each agent has an endowment $e_i \colon Y \to \mathbb{R}_{++}$ of consumption goods and an initial endowment of assets $\xi_i \in \mathbb{R}^k$ such that $\sum_i \xi_i = 0$. Assume that $e_i(y) + D(y,q)\xi_i > 0$ for every $y \in Y$ and $q \geq 0$.

A *financial economy with intermediation costs* is characterized by the following information

$$\mathcal{E} = (\pi, D, c, (e_i, u_i, \beta_i, \xi_i)_{i=1}^n),$$

and sometimes just $\mathcal{E} = (e, D, c)$, when the remaining characteristics are fixed. Note that for any intermediation cost function $c \in \mathcal{C}$ and D the set of economies is parametrized by a finite dimensional vector space $\mathbb{R}^{I\#Y}$.

Assume that $\lim_{x\to 0} u_i(x) = -\infty$ and $\lim_{x\to\infty} u_i(x) < \infty$, i.e., the von-Neumann Morgenstern function is bounded from above and boundless from below. This assumption implies the existence of a "reservation" consumption level given $r > 0$,

$\underline{x}_i(r) > 0$ such that

$$u_i(\underline{x}_i(r)) + \frac{\beta_i}{1-\beta_i} u_i(r) \leq \frac{1}{1-\beta_i} \min_{y \in Y} \{u_i(e_i(y))\} \tag{3.1}$$

i.e., the consumer prefers the initial endowment in all periods to consuming $\underline{x}$ today and the maximal consumption r in all of the future periods. This property imposes a lower bound on consumption allocations which can arise in equilibrium.

The *endogenous state space* is

$$Z = \Theta \times \Theta \times \mathbb{R}^n_+ \times \mathbb{R}^k \times \mathbb{R}^n_+.$$

A generic element is denoted by $(\theta^-, \theta, x, q, w)$, where $\theta^- \in \Theta$ is the portfolio allocation from the previous period, $\theta \in \Theta$ is the current portfolio, $x \in \mathbb{R}^n_+$ is the allocation of consumption goods, $q \in \mathbb{R}^k$ is the asset prices and $w \in \mathbb{R}^n_+$ is a transfer to each agent redistributing the revenue from intermediation costs. Then the *state space* is

$$S = \left\{ (y, (\theta^-, \theta, x, q, w)) \in Y \times Z \mid \begin{array}{l} \sum_i x_i - e_i(y) = 0 \\ \sum_i c(\theta_i) - w_i = 0 \end{array} \right\}. \tag{3.2}$$

For an S-valued finite horizon stochastic process $(s_t)_{t=1}^T$ (on some unspecified probability space) we write $s_t = [y_t, (\theta_t^-, \theta_t, x_t, q_t, w_t)] = [y_t, z_t]$. We call such a process $(s_t)_{t=1}^T$ a *consistent* state process provided that, for alle $t < T$, the conditional distribution of y_{t+1} given $s_1, ..., s_t$ is almost surely $\pi(y_t)$, i.e., $\text{Prob}\{y_{t+1} = y \mid s_0, \ldots, s_t\} = \pi(y_t, y)$ for every $y \in Y$[1]. A sequence $(a_t)_{t \in \mathbb{N}_0}$ of random variables (on the same probability space on which (s_t) is defined) is *adapted* to $(s_t)_{t \in \mathbb{N}_0}$ if a_t is $(s_1, ..., s_t)$-measurable[2] for all $t \in \mathbb{N}_0$.

Henceforth, we shall write $(x, \theta) = (x_t, \theta_t)_{t \in \mathbb{N}_0}$, when there is an infinite sequence of random variables.

Definition 2 (Feasible Strategy) *Given a state process $(s_t)_{t \in \mathbb{N}_0}$, a* feasible strategy *for agent i is an (s_t)-adapted process (θ, x) satisfying, for all t,*

$$q_t \cdot \theta_t + x_t = e_i(y_t) + D(y_t, q_t)\theta_{t-1} - c(\theta_t) + w_{it}$$

almost everywhere with $\theta_0 = \xi_i$ and such that

$$\lim_{T \to \infty} E[\beta_i^T u_i'(x_{iT})(q_T \theta_{iT} + c(\theta_{iT}) - w_T) \mid s_1, \ldots, s_t] = 0.$$

almost every where for every $t = 0, 1, \ldots$.

[1] See e.g. [22], section 10.

[2] The σ-algebra generated by the functions $(s_1, ..., s_t)$ is finer than the σ-algebra generated by a_t.

The first condition is referred to as the *intertemporal budget constraint* while the second is referred to as the *transversality condition.* The transversality condition guarantees that the agent does not enter into a Ponzi-game scheme. This condition is not necessary in [23] since the portfolios are uniformly bounded by definition. This boundary implies that the transversality condition is automatically satisfied. The transversality condition implies that the sequence of portfolios must have a boundary on the growth of wealth. While the concept of transversality is maybe hard to accept due to its lack of economic content, however, as showed in [43] any equilibrium with transversality constraints is equivalent to an equilibrium with debt constraints. This last concept may be more acceptable from an empirical point of view. However, it does not fit well into the competitive framework and the anonymity of markets. The point of their result is that the Ponzi-game condition is self-imposed using the transversality condition, i.e., if I do not think anyone is willing to accept a Ponzi-game strategy then it is not optimal for me to engage into a Ponzi-game scheme.

Note further that imposing the transversality condition implies that the budget constraint could be transformed into a single with the requirement that income transfers should be in the market space. However, we have chosen the form above as this is a minimum of modification relative to [23].

Definition 3 (Optimal Strategy) *An* optimal strategy *given* (s_t) *for* i *is a feasible strategy* (θ, x) *such that, for any other feasible strategy* (θ', x')*, we have* $U_i(x) \geq U_i(x')$*.*

Let us next introduce some central concepts of dynamic systems.

Consider a Borel space S of states[3], referred to as the *state space.* Denote by $P(S)$ the set of measures on S. A *transition map* is a measurable set $J \subset S$ and a measurable map $\Pi \colon J \to P(J)$ such that $\Pi(s) \in P(J)$ for every $s \in J$. An *equilibrium correspondence* is a correspondence $G \colon S \to P(S)$, where for every $s \in S$ the set $G(s) \subset P(S)$ is the set of probability measures μ on S which are compatible with temporary equilibrium conditions. We also refer to G as an expectations correspondence, when it has a closed graph. A measurable set $J \subset S$ is *self-justified* (wrt. G) if $J \neq \emptyset$ and $G(s) \cap P(J) \neq \emptyset$ for every $s \in J$. A transition map (J, Π) is a (homogenous) *Markov equilibrium* (wrt G) if $\Pi(s) \in G(s)$ for all $s \in J$, i.e., if any initial state in J is mapped into an equilibrium measure on J.

Given a transition map (J, Π), a measure μ is *invariant* if for every measurable subset $A \subset J$ we have that

$$\mu(A) = \int_J \Pi(s, A) \; d\mu(s),$$

[3] A complete, separable, metric space equipped with the Borel σ-algebra

i.e., if the measure μ is induced by the transition Π and the initial distribution μ[4]. Given an invariant measure μ, a measurable subset $A_0 \subset J$ is *invariant* if

$$\mu(\{s \in A_0 \mid \Pi(s) \in P(A_0)\}) = 1,$$

i.e., if μ-almost everywhere a state in A_0 will be transformed into A_0 by Π[5]. A measure μ is *ergodic* if $\mu(A_0) \in \{0, 1\}$ for any invariant set A_0[6]. We shall refer to (J, Π, μ) as an *ergodic Markov equilibrium* of G.

Let $S = Y \times Z$, with Y and Z being Borel spaces. Given $\mu \in P(S)$, let $\operatorname{supp}\mu$ denote the support of μ. Denote by $P_F(Y \times Z)$ the set of probability measures $\mu \in P(Y \times Z)$ such that to each μ there exists a map $h_\mu\colon Y \to Z$ such that if $s \in \operatorname{supp}\mu$ there exists a $y \in Y$ such that $s = (y, h_\mu(y))$ μ-a.e., i.e., $\mu(\mathrm{Gr}(h_\mu)) = 1$ where $\mathrm{Gr}\, h_\mu = \{(y, h_\mu(y)) \in Y \times Z \mid y \in Y\}$ is the graph of h_μ.

A Markov equilibrium (J, Π) is *spotless* if $\Pi(s) \in P_F(Y \times Z)$ for all $s \in J$, i.e., if for any state the transition map has a one-to-one correspondence between the exogenous and endogenous variables. A Markov equilibrium (J, Π) is *conditionally spotless* if, for all $s \in J$, there is some $M \subset P_F(S) \cap G(s)$ and $\alpha \in P(M)$ such that $\Pi(s) = \int_M \nu \, d\alpha(\nu)$.

How do these definitions relate to our model?

Definition 4 (Equilibrium) *An* equilibrium *of $\mathcal{E}$ is a consistent S-valued state process $s = (s_t)$ with the property that, for all i, the strategy (θ_i, x_i) is optimal given s.*

This is the standard competitive equilibrium concept. It is encompassed into our framework as follows.

Definition 5 (Equilibrium transition) *An* equilibrium transition *of $\mathcal{E}$ is a pair (J, Π), where $J \subset S$ is measurable and $\Pi\colon J \to P(J)$ is a transition map with the properties:*

1. *for any $\bar{y} \in Y$, there is a point $[\bar{y}, (\theta^-, \theta, x, q, w)] \in J$, and*
2. *each time-homogeneous Markov J-valued process (s_t) with transition Π is an equilibrium process.*

The first condition states that any starting parameter $y \in Y$ is admissible. The second condition, that each state process generated by the transition map is an equilibrium, i.e., consistent and individually optimal.

[4] If we consider the map $\mu \mapsto K_\Pi[\mu] = \int \Pi(s, A)\, d\mu(s)$ from $P(J)$ into itself, then an invariant measure is a fixed point of this map.

[5] Note that $\{s \in A_0 \mid \Pi(s) \in P(A_0)\} = \Pi^{-1}(P(A_0)) \cap A_0$.

[6] The set of ergodic measures is the set of extreme points of the set of invariant measures. Thus, if Π is continuous by the Krein-Milman there exists ergodic measures. However, this is in general too strong a requirement since Π is "endogenous".

3.3 The Results

Firstly, we state the result that there exists an open set of endowments which has no competitive equilibrium without intermediation costs. The second result, is that there exists *stationary* equilibrium processes, moreover these can be ergodic, while this may involve "sun-spots", see the discussion in [23]. And thirdly we state results concerning conditions on the economy under which the volume of the equilibrium portfolios increases without boundary when the intermediation costs go to zero.

Proposition 4 *There exists an economy $\mathcal{E} = (e, D, 0)$ such that there exists no equilibrium. There exists an economy $\mathcal{E} = (e, D, 0)$ and an open neighbourhood Ω' of e such that any economy $\mathcal{E}' = (e', D, 0)$, $e' \in \Omega$ does not have an equilibrium.*

Proof:

Example 11 in section 3.6 shows the first statement.

It is easy to see that perturbating the endowments in this example preserves the non-existence of an equilibrium, and hence the second statement is proved.

□

A finite horizon example is given in [51].

Proposition 5 *Any economy $\mathcal{E} = (e, D, c)$ has a spotless equilibrium transition with intermediation costs $c > 0$.*

Proof: The proof is in section 3.5. □

Let us add some comments on this result. First of all, it is remarkable that a stationary equilibrium exists with heterogeneous agents. However, as pointed out in [23] the solution is to expand the state space, and include the asset portfolio from the previous period in the state space, while imposing consistency requirements in the construction of an equilibrium. Why should we care about the existence of a *stationary* equilibrium? One important argument is the coordination of expectations among agents. To quote [32]

> "an equilibrium which does not display minimal regularity through time - maybe stationarity - is unlikely to generate the coordination between agents that it assumes.".

A stronger result would be to show the existence of an ergodic equilibrium. An advantage compared to "just" a THME is that, in the latter, you can have several invariant sets and thus there exist more simple equilibrium transitions - vaguely one can say that every measurable subset of states in an ergodic equilibrium is essential.

If you take a measurable subset of positive measure then any equilibrium process has a positive probability of occurring in this subset.

The stationarity and the use of a dynamic system is essential in the interpretation of our results on volatility, while not the technical arguments.

Proposition 6 *Assume that $(e, D, 0)$ does not have an equilibrium and that $c^n \to 0$, for every $K \geq 0$ there exists an integer $n_0 \geq 0$ such that for every $n \geq n_0$, there exists an equilibrium of (e, D, c^n), with θ^n being the portfolio process, such that $\|\theta^n\|_\infty \geq K$.*

Proof: The proof is in section 3.5. □

Let us try to explain the intuition behind our result: Assume that without intermediation costs there exists no equilibrium. This non-existence must arise due to a lack of boundary on the security trades, since all other variables are bounded. Introducing intermediation costs implies that financing such portfolios with increased volume must imply that at least agent has no lower bound on the consumption of the perishable good, and she can increase her utility by not using such a portfolio. Thus, there exists an equilibrium with intermediation costs. Considering then an economy with no intermediation costs and with no equilibrium, and a sequence of equilibria with intermediation costs, such that the costs go to zero, then the volume of security trades must go to infinity, eventually.

Corollary 1 *There exists an open set $\Omega' \subset \Omega$ and a dividend D, such that any $(e, D, c) \in \Omega' \times \{D\} \times \{0\}$ if $c^n \to 0$ and $(x^n, \theta^n, q^n) \in E(e, c^n)$ for every n, then there exists an integer N such that $\|\theta^n\|_\infty \geq K$ for every $n \geq N$.*

To conclude, we have showed that without changing the fluctuations of the fundamentals in the economy, the volume of asset trade can increase infinitely when the intermediation costs goes to zero. Moreover, for a given asset structure the probability of having an economy with this property is strictly positive.

Another result is stated as follows:

Corollary 2 *Let $e \in \Omega$ be such that $E(e, 0) = \emptyset$, then for every $K \geq 0$ there exists an open neighbourhood U of e such that if $e' \in U$, $c_n \to 0$ and $(x^n, \theta^n, q^n) \in E(e', c_n)$ for every n, then there exists an integer N such that $\|\theta^n\|_\infty \geq K$ for every $n \geq N$.*

The point is that using proposition 6, we get boundless asset trade when the pure equilibrium does not exist, while economies close to this economy also have equilibria that are close to the equilibria of this economy, but then going arbitrarily close to this economy, a zero sequence of costs implies asset trades without boundary. This last result obviously extends the scope of our result, since even asset structures with linear dependence on prices have open sets with arbitrarily boundless asset trade.

Our results so far have shown that a sufficient condition for a boundless increase in the volume of trade in securities is that the pure economy does not have an equilibrium. However, this condition is not necessary as shown in section 3.4.

3.4 Income streams

Let us reformulate the model in income streams instead of portfolios and the asset market is characterized by a map from prices into the collection of subsets of income streams. Thus asset prices and dividends are "build" into the concept of feasible income streams. This concept should help us explain the non-existence of equilibrium processes.

It is easy to see that the model is equivalently formulated using either of the two: For any $D\colon \mathbb{R}^n \to \mathbb{R}^m$ define $\langle D\rangle = \{D\theta \mid \theta \in \mathbb{R}^n\}$ to be the range of D. It is then easy to see that for any two economies $\mathcal{E} = (e, D, 0)$ and $\mathcal{E}' = (e, D', 0)$, with $\langle D\rangle = \langle D'\rangle$, then (x, θ, q) is an equilibrium of $\mathcal{E}$ if and only if it is an equilibrium of $\mathcal{E}'$. Denote by (r_i) the corresponding income transfer processes. By the result of [43] if $D = D_0$ is a constant, i.e., we consider purely nominal assets, then there always exists an equilibrium price $q_0 \in Q$, see e.g. [7]. Assume moreover that[7] $r_{it} \in \operatorname{ri}\langle D_0\rangle$ for some i and t, i.e., that at least one trade in all of the states is carried through[8]. Next, consider a continuous map $D\colon Q \to \mathbb{R}^Y$ such that $\langle D(q_0)\rangle = \langle D_0\rangle$, such that if D is homogenous of degree one, then, generically, the set of equilibria is finite and locally unique. We claim that we can choose a path $\gamma\colon [0,1] \to D$ such that $\gamma(0) = D(q_0) = D_0$ and $\gamma(1) = D_1$, such that $\langle\gamma(t)\rangle = \langle D_0\rangle$ for every $t < 1$ and $\operatorname{rank} D_1 < \operatorname{rank} D_0$[9]. Then for any sequence of economies $\mathcal{E}_n = (e, \gamma(t_n), 0)$, $t_n = \frac{1}{1-n^{-1}}$ all has p_0 as equilibrium pricevector. This is because what matters for

[7]Where $\operatorname{ri} A$ is the relative interior of a set $A \subset \mathbb{R}^n$, i.e., the interior relative to the smallest affin space containing A.

[8]This condition resembles the condition of [7] which states that the matrix $D_0^{-1}(f(p, \omega_0))$ has full rank, which basically states that consumers disagree on the *residual* states over the number of assets.

[9]To see this, let $A\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map of full rank. Consider then the linear map $B_t\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $Ae_i = B_t e_i$ for $i < n$ and $B_t e_n = (1-t)Ae_n$. Then $\operatorname{rank} B_t = \operatorname{rank} A$ for every $t \neq 1$ and $\operatorname{rank} B_1 = \operatorname{rank} A - 1$. Alternatively, let $Be_i = Ae_i$ for $i < n$ and $Be_n = Ae_{n-1}$ and consider the map $C_t\colon \mathbb{R}^n \to \mathbb{R}^m$ with $C_t e_i = Ae_i$ for $i < n$ and $C_t e_n = (tB + (1-t)A)e_n$. We note that regardless of the choice of C_t above, $\det C_t = (1-t)\det A$ (since $\det B = 0$ and in the last case $\det C_t = t\det B + (1-t)\det A$) such that since $\det A \neq 0$, $\det C_t \neq 0$ if and only if $t \neq 1$. Let $r \in \langle A\rangle$, $r = A\theta$ $\theta = (\theta_i e_i) \in \mathbb{R}^n$, then $r = B_t\bar{\theta}$ with $\bar{\theta} = (\theta_{-n}, (1-t)^{-1}\theta_n)$. Also, $r = C_t\bar{\bar{\theta}}$ with $\bar{\bar{\theta}} = (\theta_{-n}, \theta_n - t(\theta_n - \theta_{n-1}))$. This implies that $\langle A\rangle = \langle B_t\rangle = \langle C_t\rangle$ for every $t \neq 1$.

each consumer is the income stream, and not the particular portfolio necessary to achieve the income stream. Assume then that q_0 is not a no-trade equilibrium, then we claim that any sequence of asset trades (θ_n) associated with the equilibrium q_0 of the sequence of economies $(\mathcal{E}_n)$ must satisfy that $\lim \|\theta_n\| = \infty$. Assume not, then letting $\bar{\theta}$ be a limit point we must have by the continuity of the dual map $(A, x) \mapsto Ax$ on the Banach space of linear maps that $r = \lim \gamma(t_n)\theta_n = D_1\bar{\theta}$. However, since r is an interior point of $\langle D_0 \rangle = \langle \gamma(t) \rangle$, $t < 1$ we cannot have that $r \in \langle D_1 \rangle$.

However, more is true. Assume that the economy is regular such that in a small neighbourhood of e there exists a continuous function $q(e)$ of equilibrium prices. Consider then for any e' close enough to e the path $\tilde{\gamma}$ such that $\tilde{\gamma}(s) = D(q(se + (1-s)e'))$ and paste the paths γ and $\tilde{\gamma}$. This implies that there exists an open set of economies parameterized by the endowments such that asset trades are above some given threshold $M < \infty$. This indeed implies that high volumes of asset trades is not a negligible phenomenon.

We claim that $\operatorname{var}(\theta_i^n) \to \infty$, generically. If this were not true, then we should have that $\operatorname{var}(\frac{1}{\mathrm{E}\,\theta_i^n}\theta_i^n) \to 0$. However, this would imply that the income stream is constant across states, which is not true in general.

To illustrate our point(s), consider the following example:

Example 7 *Let*

$$D_t = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ t & t & 1 \end{pmatrix}$$

be the asset structure for any $t \in [0, 1)$. Then for any income stream $v = (v_1, v_2, v_3)$ the corresponding portfolio is given by

$$\theta^t = \begin{pmatrix} v_1 \\ -v_1 + \frac{v_2 - v_3}{1-t} \\ \frac{v_3 - tv_2}{1-t} \end{pmatrix}.$$

But then $\lim_{t \to 1} \|\theta^t\| = \infty$ whenever $v_2 \neq v_3$.

In the example we obtain that $\operatorname{var}(\theta_3^t) = \frac{1}{(1-t)^2}\operatorname{var}(v_3 - tv_2)$, such that if $\operatorname{var}(v_3 - v_2) \neq 0$ then we have that $\lim_{t \to 1}\operatorname{var}(\theta_3^t) = \infty$. Note also that the "normalized" variance is $\frac{\operatorname{var}(\theta_3^t)}{\mathrm{E}(\theta_3^t)} = \frac{1}{1-t}\frac{\operatorname{var}(v_3 - tv_2)}{\mathrm{E}(v_3 - tv_2)}$, and thus that it is not just due to the increased level of trade but also in order to obtain a better risk dispersion. Actually, we have

$$\operatorname{var}(\theta^t) = \begin{pmatrix} \operatorname{var}\theta_1^t \\ \operatorname{var}\theta_2^t \\ \operatorname{var}\theta_3^t \end{pmatrix} = \begin{pmatrix} \operatorname{var} v_1 \\ \operatorname{var} v_1 + \frac{1}{(1-t)^2}\operatorname{var}(v_2 - v_3) - \frac{1}{1-t}\operatorname{cov}(v_1, v_2 - v_3) \\ \frac{1}{1-t}\operatorname{var}(v_3 - tv_2) \end{pmatrix},$$

where $\operatorname{cov}(s,t) = \mathrm{E}[(s - \mathrm{E}\,s)(t - \mathrm{E}\,t)]$ *is the covariance wrt the random variables* s, t. *Again, whenever* $\operatorname{var}(v_2 - v_3) = \operatorname{var}(v_2 - v_3) \neq 0$ *it holds that* $\lim_{t\to 1} \operatorname{var}(\theta_2^t) = \lim_{t\to 1} \operatorname{var}(\theta_3^t) = \infty$.

3.5 Existence of equilibrium transitions

In this section we show the proof of proposition 5 and 6. For that purpose we define a (not necessarily non-empty-valued) correspondence which is going to be our equilibrium correspondence or expectations correspondence:

Definition 6 *Given a state* $s_0 = [y_0, (\theta_0^-, \theta_0, x_0, q_0, w_0)] \in S$, *denote by* $g(s_0) \subset P(S)$ *the set of measures* $\mu \in P(S)$ *such that*

1. $\operatorname{supp}(\mu) = \operatorname{Gr}(h_{s_0})$ *for some function* $h_{s_0}\colon Y \to Z$
2. *the marginal of* μ *on* Y *(resp.* Θ^-*) is* $\mu_Y = \pi(y_0)$ *(resp.* $\mu_{\Theta^-} = \delta_{\theta_0}$*)*[10]
3. *for all* i, $(x_{0i}, x_{1i}, \theta_{i0})$ *solves*

$$\max u_i(x_{0i}) + \beta\, \mathrm{E}[u_i(x_{1i})]$$

subject to

$$\begin{aligned} q_0 \cdot \theta_{i0} + x_{i0} &= D(y_0, q_0)\theta_{i0}^- + e_i(y_0) - c(\theta_{i0}) + w_{i0} \\ q_1 \cdot \theta_{i1} + x_{i1} &= D(y_1, q_1)\theta_{i0} + e_i(y_1) - c(\theta_{i1}) + w_{i1} \end{aligned}$$

for any random variable $s_1 = [y_1, (\theta_1^-, \theta_1, x_1, q_1, w_1)]$ *with distribution* μ

4. $x_1 \geq \underline{x}$ *almost surely.*

Remark:

Note that there is no measurebility problem in the definition of g condition 3 since, by condition 1, the support of μ is finitely valued and hence the maximization problem is a finite dimensional Euclidean space. This implies given $\mu \in g(s_0)$ then for every random variable s_1 with distribution μ there exists multipliers $\lambda_i(s_1) \geq 0$, elements of $\mathbb{R}^{\#Y}$ for every consumer i, such that $(x_{i0}, x_{i1}, \theta_{i0})$ solves

$$\max u(x_0) + \beta \sum_y \mu(y, h_{s_0}(y)) u(x_1(y, h_{s_0}(y))) + \lambda_{i0}(s_1) k_0(s_0) + \lambda_{i1}(s_1)^T k_1(s_0, s_1)$$

and $k_1(s_0, s_1) \in \mathbb{R}^{\#Y}$. The functions k_0, k_1 being the budget constraints of condition 3.

[10]We denote by δ_x the Dirac-measure with unit-mass on x.

Since c is a convex function the set of subgradients $\partial c(x)$ is non-empty[11] (see [54]) and thus according to the Kuhn-Tucker theorem [54, p. 281, Theorem 28.3] we must have that

$$\begin{aligned} u'(x_0) + \lambda_0 &= 0 && (3.3) \\ \beta\mu(y)u'(x_1(y)) + \lambda_1^y &= 0 && (3.4) \\ \lambda_0(q_0 + \kappa(\theta_0)) + \sum_y \lambda_1^y D(y) &= 0 && (3.5) \end{aligned}$$

where $\kappa(\theta_0) \in \partial c(\theta_0)$ being the necessary and sufficient conditions. Thus, we could substitute the condition 3 of definition 6 with equations (3.3)-(3.5).

End of remark

Remark:

Second, note the following: if $g \neq \emptyset$ and J is an invariant set, then we obtain a map $H\colon J \times Y \to Z$ given by

$$H(s_0, y) = H((y_0, [\theta_0^-, \theta_0, x_0, q_0, w_0]), y) = h_{s_0}(y) \in Z$$

such that a realised equilibrium path (s_t) must satisfy that $s_{t+1} = (y_{t+1}, H(s_t, y_{t+1}))$ and hence $z_{t+1} = H(z_t, y_{t+1})$.

End of remark

Denote by $\eta(T)$ the set of T-horizon equilibria states, i.e., the set of T-horizon processes $(s_t)_{t=1}^T$ that are optimal to each consumer using the utility index U_i^T.

The following lemmas show how to construct an ergodic equilibrium: The first establish a lower bound on the consumption compatible with any finite equilibrium.

Lemma 7 *There exists $\bar{x} \in \mathbb{R}_{++}^n$, such that for all T and $(s_t)_{t=1}^T \in \eta(T)$, then for all i and t, $x_{it} \geq \bar{x}_i$ almost surely.*

Proof:

Follows immediately from the fact that the endowment is always feasible, and the property from equation (3.1) with $r_0 = \max_y \sum_i e_i(y) > 0$ and $\bar{x}_i = \underline{x}_i(r_0) > 0$.

$\square$

Next, we show that any equilibrium price sequence must be uniformly bounded.

Lemma 8 *There exists $\bar{q} \in \mathbb{R}_+^k$, such that, for any finite T, if $(s_t)_{t=1}^T \in \eta(T)$, then for all t we have $|q_t| \leq \bar{q}$ almost surely.*

[11] $x^\star \in \mathbb{R}^n$ is a *subgradient* of c at x_0 if $c(x_0) - c(x) \geq x^\star \cdot (x_0 - x)$ for every $x \in \mathbb{R}^n$, and $\partial c(x)$ is called the subdifferential of c at x. When c is differentiable the subgradients are unique and equals it's gradient

Proof:

Let $(\theta_{it}, x_{it})_t$ be a strategy in equilibrium of consumer i. Consider the following modification of the strategy: In the current period t, replace the final portfolio θ_{it} with $(1-\epsilon)\theta_{it}$, spending additional $\epsilon q_t \cdot \theta_{it}$ on consumption. In any period $\tau > t$, replace $(\theta_{i\tau}, x_{i\tau})$ with $(1-\epsilon)(\theta_{i\tau}, x_{i\tau} + \frac{\epsilon}{1-\epsilon} e_i)$. This strategy is still feasible, since both the intertemporal budget constraints, the intermediation costs are convex, and the transversality condition are satisfied.

This modification of the strategy leads to a loss of expected utility in future periods that is bounded, and this bound can be made uniform, such that it does not depend on the horizon T, and it goes to zero when $\epsilon \to 0$. Moreover, the loss is independent of asset prices.

Assume next that (q_t) is unbounded. Note, that for every t there exists some i such that $\epsilon q_t \cdot \theta_{it} > 0$. Then, consider this consumer and using the above described strategy, since $q_t \to \infty$, this increment can be taken to be bounded away from zero even though $\epsilon \to 0$. But the loss goes to zero, and hence the strategy could not be optimal in the first place. Thus, (q_t) must be bounded.

$\square$

We note that this boundary on securities prices is also valid even when $c = 0$.

We next show that the portfolios of any finite horizon equilibria are uniformly bounded:

Lemma 9 *There exists $\bar{\theta} \in \mathbb{R}^k_+$, such that, for any finite T, if $(s_t)_{t=1}^T \in \eta(T)$, then for all t we have $|\theta_{it}| \leq \bar{\theta}$ almost surely, for every i.*

Proof:

Assume that for every $\bar{\theta}$ there exists T_0 and $(s_t)_{t=0}^{T_0} \in \eta(T_0)$ such that $\|\theta_{it}\| > \bar{\theta}$ for some t. Thus, there exists a sequence $(T_n)_{n\in\mathbb{N}}$, $(t_n)_{n\in\mathbb{N}}$ and $(s_t^n) \in \eta(T_n)$ such that $\lim_{n\to\infty} \|\theta^n_{it_n}\| = \infty$ for some i. Since $\sum_i w^n_{it_n} - c(\theta^n_{it_n}) = 0$ this must imply that there exists j such that $\lim_n w^n_{jt_n} = \infty$. Note that we can assume that it is the same j since the number of consumers is finite. Consider then the alternative strategy $(\tilde{x}^n_{jt}, \tilde{\theta}^n_{jt})$ with $\tilde{\theta}^n_{jt} = (1-\varepsilon)\theta^n_{j,t}$ for every $t \neq t_n$ then the consumption $\tilde{x}_{jt_n}$ must satisfy that

$$\begin{aligned}
\tilde{x}^n_{jt_n} - x^n_{jt_n} &= \varepsilon q^n_{t_n} \cdot \theta^n_{jt_n} - \epsilon D(y_{t_n}, q^n_{t_n}) \cdot \theta^n_{i,t_n-1} - c(\tilde{\theta}^n_{jt_n}) + c(\theta^n_{jt_n}) \\
&\geq \varepsilon (q^n_{t_n} \cdot \theta^n_{jt_n} - D(y_{t_n}, q^n_{t_n}) \cdot \theta^n_{i,t_n-1} + c(\theta^n_{jt_n}))
\end{aligned}$$

and since $\lim_n q^n_{t_n} \cdot \theta^n_{jt_n} - D(y_{t_n}, q^n_{t_n}) + c(\theta^n_{jt_n}) = \infty$ because (x^n_{it}) is bounded and $\lim_n w^n_{jt_n} = \infty$ we must have that $\lim_n \tilde{x}^n_{jt_n} - x^n_{jt_n} = \infty$. Further, let $\tilde{x}^n_{j,t} = (1-\varepsilon)x^n_{j,t}$ for every $t \neq t_n$ which is feasible. This must imply that taking $\varepsilon > 0$ sufficiently small there exists some n_0 such that $(\tilde{\theta}^{n_0}_{jt_{n_0}}, \tilde{x}^{n_0}_{jt_{n_0}})$ is a better feasible trading plan than $(\theta^{n_0}_{jt_{n_0}}, x^{n_0}_{jt_{n_0}})$ which is a contradiction.

$\square$

Lemma 9 also induces a boundary $\bar{w} \in \mathbb{R}_+$ such that any equilibrium process must have that $w_t \leq \bar{w}$ for every t.

We next show that any partial sequence of a finite equilibrium is also a equilibrium in the corresponding partial economy.

Lemma 10 *For any $T \geq 2$, if $(s_t)_{t=1}^T \in \eta(T)$, then $(s_t)_{t=2}^T \in \eta(T-1)$.*

Proof:

This follows from the Bellman's principle of optimality and, in particular, from the separability of the utility.

$\square$

For any T, let

$$S_T = \{s \in S \mid \exists (s_t)_{t=1}^T \in \eta(T) : s_1 = s = [y, (\xi, \theta, x, q, w)]\},$$

i.e., the set of *initial* states in *some* T-horizon equilibrium. We next show that for every finite horizon, this set projected onto Y is the entire space:

Lemma 11 *For all finite T and all $y_0 \in Y$, there exists $[y_0, (\xi, \theta, x, q, w)] \in S_T$*

Proof:

This follows from [45] and the extension to multiple periods in appendix A.

$\square$

In particular, this implies that $S_T \neq \emptyset$ for every T.

Define next the compact subset K of S by

$$K \equiv \{[y, (\theta^-, \theta, x, q, w)] \in S \mid x \geq \underline{x} \wedge |q| \leq \bar{q} \wedge |\theta| \leq \overline{\theta} \wedge w \leq \bar{w}\},$$

where $\underline{x} \in \mathbb{R}^n_{++}$ is given by equation (3.1), $\bar{q} \in \mathbb{R}^k_{++}$ is obtained from lemma 8 and $\overline{\theta} \in \mathbb{R}^k_{++}$ (and $\bar{w} \in \mathbb{R}_{++}$) is obtained from lemma 9.

We define a sequence $(C_{0T})_{T \in \mathbb{N}_0}$ as follows: let $C_{00} = K$ and for every $T \geq 1$ let

$$C_{0,T} = \{s \in K \mid \exists \nu \in g(s) : \nu_\star(C_{0,T-1}) = 1\},$$

i.e., the set of states of equilibrium such that the inner measure[12] of $C_{0,T-1}$ is 1. The reason why we use the *inner* measure, instead of the measure itself, is that we do

[12]Given a probability space $(\Omega, \mathfrak{M}, \nu)$ the inner measure $\nu_\star$ of ν is given by $\nu_\star(A) = \sup_{B \in \mathfrak{M}, B \subset A} \nu(B)$ for every subset $A \subset \Omega$.

not know the measurability property of $C_{0,T-1}$. Moreover, we consider the double sequence $(C_{i,j})_{i,j\in\mathbb{N}_0}$ for $i,j \geq 1$ given by

$$C_{i,j} = \{s \in C_{0,j} \mid \exists s' \in C_{i-1,j+1}, \nu \in G(s') : s \in \operatorname{supp}\nu, \nu_\star(C_{0j}) = 1\}.$$

Thus, C_{ij} contains the states that would occur in period i of an "$(i+j+1)$-period equilibrium" in K running from period 0 to period $i+j$. We next show that S_T is contained in every $C_{0,T}$, and hence according to lemma 11, $C_{0T} \neq \emptyset$ for every $T \in \mathbb{N}_0$:

Lemma 12 *For every T we have $S_T \subset C_{0,T}$.*

Proof:

The proof is by induction on T. When $T = 1$ then $C_{0,T-1} = C_{00} = K$ and thus

$$C_{0,1} = \{s \in K \mid \exists \nu \in g(s) : \nu_\star(K) = 1\}$$

Obviously, if $(s_t) \in S_1$ then $s_1 = s \in K$. But then consider the measure $\nu = \delta_{s_1}$ and since $\delta_{s_1} \in g(s)$ and $\delta_{s_1}(K) = 1$, since $s_1 \in K$ we are done. Next, assume that $S_{T-1} \subset C_{0,T-1}$, we must then show that this implies that $S_T \subset C_{0,T}$. Thus, consider some element $s \in S_T$, then there exists (s_t) such that $(s_t) \in \eta(T+1)$ and $s_1 = s$, but then by lemma 11 we have that $(s_t) \in \eta(T)$, such that $s_1 \in S_{T-1}$ and thus $s_1 \in C_{0,T-1}$, by the induction hypothesis. But then $\delta_{s_1}(C_{0,T-1}) = 1$.

□

We now apply the following result:

Theorem 8 ([23], Theorem 1.2) *Suppose that g is a correspondence with closed graph such that $C_{0,j} \neq \emptyset$ for every $j = 0, 1, \ldots$. If there exists i_0 such that C_{i_0,j_0} has compact closure for some j_0 then $\bigcap_{j\in\mathbb{N}_0} \operatorname{cl} C_{i_0,j}$ is a self-justified set.*

Thus, since closedness is preserved under arbitrary intersections, the self-justified set is a closed subset of a compact set, and hence compact.

In particular, lemma 12 implies that $C_{0,T} \neq \emptyset$ for every T. Now, it is easy to see that g has closed graph, and thus by lemma 12 and theorem 8, we obtain a compact self-justified set.

The next proposition is also from [23]:

Proposition 9 ([23], Proposition 1.3) *Let Y be a finite set, and for $s \in S$ let*

$$\tilde{g}(s) = \{\nu \in g(s) \mid \nu \in P_F(Y \times Z)\} = g(s) \cap P_F(Y \times Z),$$

with $g(s) \subset P(S)$ being an expectations correspondence. Then

1. *$\tilde{g}$ is an expectations correspondence.*

2. *If $\tilde{g}$ has a compact self-justified set J, then $\tilde{g}$ has a spotless THME, and there is an ergodic Markov equilibrium for the correspondence $\hat{g}\colon J \to P(J) := s \mapsto \hat{g}(s) = \operatorname{cl}\operatorname{conv}\tilde{g}(s)$.*

3. *If g is convex-valued, then an ergodic Markov equilibrium for $\hat{g}$ is a conditionally spotless ergodic Markov equilibrium for g.*

Denote next $\hat{g}\colon S \to P(S)$ by $\hat{g}(s) = \operatorname{cl}\operatorname{conv} g(s)$. Using proposition 9 we obtain a spotless THME for g and there is an ergodic Markov equilibrium for $\hat{g}$. Denote this ergodic equilibrium (J, Π, μ), where $\Pi\colon J \to P(J)$ is the equilibrium transition map, J is the compact self-justified and invariant set, and μ is an invariant measure on J.

Proposition 10 *Given a state process $(s_t)_{t=1}^{\infty} = ([y_t, (\theta_t^-, \theta_t, x_t, q_t)])_{t=1}^{\infty}$ with transition function Π, for any agent i, the corresponding strategy $(\theta_{it}, x_{it})_{t=1}^{\infty}$ is optimal.*

Proof:

The proof follows closely [23, pp. 768, Proposition 3.2] and we shall not replicate the proof here. We only note that according to the equations 3.3-3.5 we obtain a sequence of subgradients $\eta(\theta) \in \partial c(\theta)$ however according to [54, Theorem 24.4] this sequence converges to a subgradient of the limit portfolio. Using this *sub*gradient instead of the gradient in [23] the proof goes through.

□

This ends the proof of proposition 5. We next turn to the proof of proposition 6.

Proof: [Proof of Proposition 6]

First we note that continuity of the budget correspondence in the transaction costs is showed as in [45] when we endow the space of cost functions with the compact-open topology.

The non-existence of an equilibrium without any transaction costs, i.e., $c = 0$, implies the following: Let (s_t) be a sequence which is S-valued, then there exists some i, a strategy (x'_{it}, θ'_{it}) and τ such that $(x'_{i\tau}, \theta'_{i\tau}) \neq (x_{i\tau}, \theta_{i\tau})$ which is feasible and $U_i(x'_i) > U_i(x_i)$.

Assume that the sequence of equilibrium process $((s_t^n)_{t\in\mathbb{N}_0})_{n\in\mathbb{N}_0}$ is bounded and let $(s_t^n)_{t\in\mathbb{N}_0}$ be a limit point. Using the continuity of U_i there exists an open neighbourhood U of (x'_{it}, θ'_{it}) such that every strategy in U is strictly preferred to (x_{it}, θ_{it}). Using the continuity of the budget there exists some N large enough such that

(x'_{it}, θ'_{it}) is feasible with transaction costs c_n, $n \geq N$. Again, using the continuity of the utility there exists N' such that $U_i(x'_i) > U_i(x^n_i)$ for every $n \geq N'$.

$\square$

3.6 Examples

Consider the following example showing that there exists an economy in which there exists no equilibrium when the intermediation cost is zero:

Example 11 *Let* $H = \{A, B\}$, *a single perishable commodity,* $Y = \{1, 2\}$ *and* $\pi(1) = \pi(2) = \frac{1}{2}$. *The endowments*

$$\begin{aligned} e_0^A &= e_0^B = (e_0(1), e_0(2)) = (1, 1) \\ (e_t^A, e_t^B) &= ((e_t^A(1), e_t^A(2)), (e_t^B(1), e_t^B(2)))) = ((\frac{4}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{4}{3})), \end{aligned}$$

and the von-Neumann Morgenstern utility functions

$$u_h(x) = \frac{x^{1-\gamma_h}}{1-\gamma_h}.$$

with subjective discount factor β_h. *Let*

$$D_t = \begin{pmatrix} v_1 & v_1 \\ 0 & v_2 \end{pmatrix}$$

be the asset structure. Assuming that $\beta_A = \beta_B$ *the allocation*

$$(x_t^A, x_t^B) = ((1, 1), (1, 1))$$

will be an equilibrium. The prices on the Arrow-securities will then be

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \beta u'(1) v_1 \\ \pi_2 \beta u'(1) v_2 \end{pmatrix}$$

Assume on the other hand that $D_t = (1, 1)$. *Then* $x_t^h = e_t^h$, $h = A, B$ *will be the equilibrium allocation with asset price*

$$q = \beta \frac{1}{2}(u'(e_1) + u'(e_2)).$$

Assume next that $v_2 = \max\{k - q_1, 0\}$ *and* $v_1 = 1$ *(denote* $\tilde{V}$ *this asset structure) which implies that*

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \beta u'(1) \\ \frac{1}{2}\beta u'(1) \max\{k - \beta u'(1), 0\} \end{pmatrix}.$$

Asset 2 is then an (call) option on asset 1. Then when $k > \beta u'(1)$ there exists no equilibrium: Assume that $(x_t^A, x_t^B) = ((1,1),(1,1))$ is an equilibrium allocation of $\tilde{V}$, but then $q_2 = v_2 = 0$ and $q_1 = \beta u'(1)$. Hence, if

$$\frac{1}{2}\beta(u'(e_1) + u'(e_2)) > k > \beta u'(1)$$

we have a contradiction. However, this is possible when $\frac{1}{2}u'(e_1) + \frac{1}{2}u'(e_2) > u'(1)$ but this holds when

$$u'(\pi_1 e_1 + \pi_2 e_2) < \pi_1 u'(e_1) + \pi_2 u'(e_2)$$

which is the case here. But $u'''(x) = \gamma(\gamma+1)x^{-\gamma-2} > 0$ and thus u' is convex the inequality holds.

This is also the case when $v(x) = -\frac{1}{a}e^{-ax}$.

The point of this example is that, if we consider a sequence of equilibria with decreasing intermediation costs and this sequence converges to this equilibrium, *then the volume of asset trades will go to infinity.*

Example 12 *Consider the following economy: $Y = \{1\}$, $L = 2$ with*

$$\begin{aligned}
e_0 &= \begin{pmatrix} e_0^A \\ e_0^B \end{pmatrix} = \begin{pmatrix} (\frac{4}{3}, 1) \\ (\frac{4}{3}, \frac{2}{3}) \end{pmatrix} \\
e_t &= \begin{pmatrix} e_t^A \\ e_t^B \end{pmatrix} = \begin{pmatrix} (\frac{2}{3}, \frac{4}{3}) \\ (\frac{4}{3}, \frac{2}{3}) \end{pmatrix} \\
u_h(x_1, x_2) &= \sum_l \eta_l^h \frac{x_l^{1-\gamma}}{1-\gamma}
\end{aligned}$$

and the discounting $\beta_A = \beta_B = \frac{99}{100}$. Let $\gamma = 2$ and

$$\eta = \begin{pmatrix} \eta^A \\ \eta^B \end{pmatrix} = \begin{pmatrix} (2,1) \\ (1,1) \end{pmatrix}$$

We consider different asset structures: one with complete and several with different degrees of incomplete markets. These gives us relative prices (i.e. $r_t \equiv \frac{p_{1t}}{p_{2t}}$):

$$\begin{aligned}
(r_0, r_t) &= (0.545, 1.394) \\
(r_0, r_t) &= (0.574, 1.382), \\
(r_0, r_1, r_t) &= (0.555, 1.422, 1.382), \\
(r_0, r_1, r_2, r_t) &= (0.55, 1.408, 1.408, 1.382).
\end{aligned}$$

The first are relative prices when income transfers are perfectly tradeable; the second where income transfers are not tradeable at all; the third where trade is possible *between the first and second period but not in the succeeding periods, etc. We see a pattern here: if $T \geq 1$ markets are open initially and $t > T$ is closed, then we obtain relative prices $r_0^T \in [0.574, 0.545]$, $r_t^T \in [1.395, 1.422]$ for every $t < T$, and $r_t^T = 1.382$ for every $t > T$.*

Whence introducing the following option

$$D_t(p) = \max\left\{k - \frac{p_{1t}}{p_{2t}}, 0\right\} \tag{3.6}$$

there exists no equilibrium with $k \in (1.382, 1.394)$.

Alternatively, let the endowments be

$$e_0 = \begin{pmatrix} e_0^A \\ e_0^B \end{pmatrix} = \begin{pmatrix} \left(\frac{4}{3}, \frac{4}{3}\right) \\ \left(\frac{2}{3}, \frac{2}{3}\right) \end{pmatrix}$$

Again, we consider two different asset structures: one with complete and one with incomplete markets. Similarly, relative prices (i.e. $r_t \equiv \frac{p_{1t}}{p_{2t}}$) are then

$$\begin{aligned} (r_0, r_t) &= (1.445, 0.361) \\ (r_0, r_t) &= (1.579, 0.355) \\ (r_0, r_1, r_t) &= (1.496, 0.374, 0.355) \\ (r_0, r_1, r_2, r_t) &= (1.47, 0.367, 0.367, 0.355). \end{aligned}$$

Again, introducing the option in equation 3.6 there exists no equilibrium with $k \in (0.355, 0.361)$. We note that as a function of the number of open markets, the relative price of period 1 jumps up going from autarky to trade between period 0 and 1. Henceforth, the relative price decreases monotonically towards the complete market relative price, while still being above the autarky relative price.

Here, since there exists no equilibrium without intermediation cost, and any *equilibrium sequence with decreasing intermediation costs must have a boundless volume of asset trades.*

3.7 Conclusion and comments

Empirical observations indicate that the number and volume of trades of stocks have increased during the period 1990-2000, and that is has, by far, exceeded the increase in income (GDP). Thus, it is difficult to believe that the increased income can account for this remarkable increment. This paper suggests that the increased volume of trade can be explained by a decrease in the intermediation or transaction cost of trading on the financial markets. While this might not seem too surprising, we claim that the observed increase in volume of trade is not only the direct effect of a decrease in the trading costs. We suggest an additional contributing element: the indirect effect of the reduction in the intermediation costs. We show that when intermediation costs decrease this has an effect on asset and commodity prices, now as well as in the future. This change in prices results in changes in the price-dependent securities' dividends, such as options, stocks, etc. Now, if these induced changes in the dividend reduce the risk dispersion of a current portfolio, the traders need to trade more intensively in order to obtain the same risk profile. In this paper we show that this effect can be very large and, in extreme cases, without boundary.

Let us again try to justify the introduction of intermediation costs: we argue that even in the case of fixed fee there is an increasing intermediation/transaction cost in the amount of securities that you have to trade. However, we admit that the story does not fit very well with the competitive framework that we consider in this paper. The justification is based upon market microstructure theory and/or search models. They both take their starting point in the actual situation in which a trader finds him self: the actual trading process. But as long as the competitive framework is considered as an approximation to reality, we believe that the story is not that far from reality. Assume that given an initial portfolio and some expected average asset price, you wish to change your portfolio. Now, this desired change must be carried about using some form of trading strategy using a trading place, like a stock exchange. Now, all significant changes in portfolios in a market affect the trading price that you obtain: if you sell in a thin market you will end up with a low price, while buying in a thin market increases you buying price. Thus, you will have to choose some appropriate strategy to obtain the most profitable trading prices. The market game is then given by a strategy, specifying when to announce your buying and selling orders and how large volumes each buying/selling order should amount to. The payoff is the expected revenue from the trades carried out given the strategies of every trader. The point is that the larger the volume of trade a trader wants to carry through is to his disadvantages, and thus that the marginal cost increases as the volume of the desired portfolio change increases, and that in the limit this cost is infinity. The search theory would take the starting point that in order to make a trade the buyer and the seller literally have to find each other. However, we shall not pursue this any further.

As can be easily realized our result does not depend on the fact that we consider intermediation costs instead of transaction costs. Also, adding more spot commodities does not alter our results. It also seems robust to the assumption of stationarity. A more severe problem is that we take the security structure and intermediation cost function as exogenous. Ideally, these variables should be endogenous in a genuine general equilibrium model.

Chapter 4

The Existence of a Free Lunch, Equilibrium in Income Streams, and Transaction Costs

Tobias Markeprand

Abstract. We show that in the presence of strictly convex transaction costs, an arbitrage opportunity is consistent with equilibrium with rational agents. We show that there exist equilibria where arbitrage opportunities exist but are not exploited. Further, we show that equilibria exist where arbitrage opportunities exist and are exploited. This may undermine the importance of arbitrage in the theory of financial pricing based solely on the no arbitrage principle.

JEL classification:. D11, D51, G12, G14.

Keywords:. Incomplete Markets, General Equilibrium, Transaction Costs, Free Lunches.

4.1 Introduction

As [58] argues, the concept of no-arbitrage, which includes the law of one price or the absence of a free lunch, in its most general interpretation is fundamental in the concept of competitive equilibrium and in the way economists think. The basic content of the no-arbitrage principle is that you cannot gain something for free - there is no free lunch. And as [41] writes

> "In any economic equilibrium, it should not be possible to purchase at zero cost a bundle of goods that will strictly increase some agent's utility."

Also, this principle is the foundation upon which almost all of finance theory, empirically and theoretically, is based. We show that in the case that intermediation costs, or transactions costs, are present the no-arbitrage principle does not necessarily hold within the framework of a general equilibrium model with perfect competition and symmetric information. We show that agents with small demand for risk dispersion can benefit from small marginal intermediation costs and exploit the constant marginal asset prices, to exploit an arbitrage opportunity and obtain a positive income stream in each state, the current as well as the future states. We show that there exist equilibria where arbitrage opportunities are exploited and equilibria where they are not exploited. The key point when preferences are strictly monotone, is that the costless gains are bounded, as in [14][1] where it is called *Limited arbitrage*.

Often the argument for no-arbitrage goes as follows: Consider the price of gold, then the absence of arbitrage possibilities implies that the price must be the same in New York and in Paris; if not, e.g., if the price is 1 in New York and 0.9 in Paris, then one could buy an ounce of gold in Paris and sell it in New York gaining a profit of 0.1. A modification of this is that the prices can differ by the costs of transportation. However, this is *not* a violation of the no free lunch principle, since the price difference cannot be exploited to gain a riskless positive profit. The approach adopted in this paper assumes that the marginal transportation cost is not linear and increases when the quantity of gold is large enough; Hence only small quantities of gold can exploit the arbitrage opportunity, and explains why not all of the gold in Paris is shipped to New York.

Yet another illustration of the no-arbitrage argument, is the joke of a professor of economics who refuse the existence of a $ 100 bill lying on the ground, due to the fact that then someone would have picked it up! The examples in this paper show that this is in general false. The lesson learned from this paper is that you cannot

[1] We stress that the results of [14] and [15] are based upon a false prove, see the discussion [48] and [49].

conclude that there is not any certain gain to obtain - just because nobody else has earned it!

The existence of arbitrage opportunities in equilibrium has been showed before. However, this has been in the framework of market games, i.e., with imperfect competition, see e.g. [40]. The main argument is that arbitrage opportunities may exist, since if any agent attempts to exploit this opportunity, he/she will change the resulting prices and taking this into account it may be disadvantageous. However, while they may exist in equilibrium they cannot be exploited, and it is thus not a genuine arbitrage opportunity. Moreover, [29] shows that a necessary condition for this result is the presence of wash-sales, i.e., the simultaneous buying and selling in the same market by a single trader. However, such a trading strategy is eliminated by utility maximizing when (arbitrary small, linear) transaction costs are introduced, see [10]. [10] then proceeds to conclude that the results by [40] are not robust to the introduction of arbitrarily small transaction costs. Our result shows that this relies on the introduction of *linear* transaction costs, while the introduction of non-linear transaction costs not only reestablish, but strengthen the result. [42] argues, like us, that mispricing might persist due to costs of trading, but they assume the existence of noise, and hence irrational, traders. In their model the irrationality creates uncertainty, which again creates costs of trading to risk averse traders, which then eliminates the rational traders from exploiting the arbitrage opportunity. This is in contrast to our model were all traders are rational! Furthermore, they cannot explain *persistent* deviation of prices from the fundamental value, whereas we can. Thus, the explanations of existence of free lunches in equilibrium have been based upon either imperfect competition or irrational behaviour. We show that neither of these assumptions are necessary.

A framework close to ours is [38], which studies the asset pricing in a model with frictions, and, like us, they obtain an opportunity set which is convex. They do, however, not analyze the existence of free lunches, but extend the work of [34] and [41]. They obtain asset prices where atleast one agent is satisfied with consuming her endowment (of income), such a price system is referred to as *viable* and they show that such prices are consistent with an equilibrium and equivalent with the absence of asymptotic free lunches. Furthermore, they characterize the set of viable asset price systems in terms of a linear asset price. They show that the linear functional must lie below the convex pricing rule on the marketed set. While the concept of viable prices in the framework of [34] is characterizing asset prices that can occur in equilibrium, i.e., with no frictions in the markets, our results show that this is not the case with convex transaction costs. Further, our results imply that the claim by [38] that

> "...viability is the minimal requirement for a price system (M, π) to model an economic equilibrium..."

is false. Indeed, we find an economy and an equilibrium, where the viability condition is violated.

Finally, we relate our results to the economics of asymmetric information. The main focal point of this literature is *The Efficient Market Hypothesis* (see e.g. [30] or [25]), which, informally, states that in a competitive market where agents ahve rational expectations, current prices reflect all information available in the economy on the future return of the security. This implies that security prices convey all available information and no trader can gain by exploiting private information. However, as [31] shows when the acquisition of information is costly and traders choose their information level, prices only convey the aggregate information partially, since informed traders have to be compensated for their expenditure on information acquisition. This can only be the case if private information has a value. In this sense, information is a public good provided by informed traders through the price mechanism. Thus, without taking into account the cost of information there is a free lunch to be exploited by the uninformed traders. However, to exploit the free lunch the trader needs to acquire information which more than offsets the gain. Thus, again the free lunch is not genuine. They consider competitive, rational traders as we do, but differ by informational asymmetries.

In section 4.2 we introduce the model and in section 4.3 give our main results, examples illustrating our main result on the existence of arbitrage opportunities. In section 4.4 we show how the model of section 4.2 can be obtained as the trading of assets with intermediation or transaction costs, and thus relates our model with the standard GEI model. Section 4.5 concludes.

4.2 Arbitrage and Equilibrium

Let there be two dates and $S < \infty$ future states, the present $s = 0$ and the future, $s = 1, \dots, S$.

The basic concept of a finance economy is the set of income streams, $M \subset \mathbb{R}^{S+1}$, i.e., the set of income streams the agents can trade among each other.

Definition 7 *A* set of income streams *is a set $M \subset \mathbb{R}^{S+1}_{++}$ such that*

1. *M is closed, convex and non-empty*

2. *if $r \in M$ and $r' \leq r$ then $r' \in M$, and*

3. *$0 \in M$*

Denote by $\mathscr{M}$ the collection of income streams sets. Any element $r = (r^s)_{s=0}^S = (r^0, r^1, r^2, \dots, r^S) \in M \in \mathscr{M}$ is called an *income stream* and r^s is the income available in state s.

We say that an income stream $r \in M \cap \mathbb{R}_+^{S+1}$ is an *arbitrage opportunity* or a *free lunch* if $r \neq 0$. M satisfies the *no-arbitrage principle* when there is no-arbitrage opportunity, i.e., when $M \cap \mathbb{R}_+^{S+1} = \{0\}$. We say that M has *limited free lunches* if there exists $K > 0$ such that $M \cap \{r \geq 0 \mid \|r\| > K\} = \emptyset$.

According to the separation theorem of convex sets, for every income stream set $M \in \mathscr{M}$ the no-arbitrage principle is satisfied if there exists a vector $\lambda_M \in \mathbb{R}^{S+1}$ and a number $\beta_M \in \mathbb{R}$ such that $\lambda_M \cdot r < \beta_M < \lambda_M \cdot p$ for every $r \in M$ and $p \in \{p \in \mathbb{R}_+^{S+1} \mid \sum p = 1\}$.

When M is a cone we obtain the well-known characterization by state prices

Proposition 13 *Assume that M is a convex cone, then M satisfies the no-arbitrage principle if and only if there exist $\lambda \in \mathbb{R}_{++}^{S+1}$ such that $\lambda \cdot M \leq 0$.*

Our concept of an economy basically lies close to [41], but we extend it with some additional objects allowing us to include transaction costs.

Definition 8 *A* finance economy *is a map $M \colon X \to \mathscr{M}$, where X a subset of a vector space, an utility function u_i representing the preferences on $\mathbb{R}^{S+1}$ for each consumer $i \in I$, and a set $Y \subset \mathbb{R}^{I(S+1)}$ referred to as the income production set.*

Denote such an economy by $\mathcal{E} = (M, (u_i)_{i \in I}, Y)$. Y is the set of aggregate income streams which the economy can produce. The usual assumption is that $Y = \{(r_i) \mid \sum r_i = 0\}$, but other cases are also possible, and allow us to consider the case of real transaction costs, i.e., where resources are required to exchange income.

We want to emphasis that introducing utility functions this way implies that the income streams we consider are *net* income streams. However, in order to avoid unnecessary repetation we just call an element r an income stream with the understanding that it is actually a net income stream. Section 4.4 shows how we can derive the utility functions from a more general model.

Definition 9 *A* financial equilibrium *of $\mathcal{E}$ is then an asset price vector $x^\star \in X$ and income transfers $(r_i^\star)_{i \in I}$ such that*

- *$r_i^\star$ maximizes u_i on $M(x^\star)$, and*
- *$(r_i^\star)_{i \in I} \in Y$.*

In this paper, we assume that u_i is a continuous, strictly monotone and quasi-concave function for every agent i. Strict monotonicity implies that agents prefers more income to less income, while quasi-concavity implies risk aversion. Two properties that seem very natural.

4.3 The Results

What is then the relationship between no-arbitrage asset prices and an equilibrium of a finance economy?

When $M(x)$ is a convex, additive cone[2] for all x, and strictly monotone preferences[3] then the no-arbitrage principle is a very natural property that an equilibrium asset price vector must satisfy: if $\bar{r} \in M(x)$ is an arbitrage opportunity then $\gamma\bar{r} \in M(x)$ is also an arbitrage principle for any $\gamma \geq 0$. But then there exists no maximizing consumption bundle: for any income stream $\tilde{r}$, since preferences are strictly monotone $u_i(\tilde{r} + \gamma\bar{r}) > u_i(\tilde{r})$ for any $\gamma > 0$, and $\tilde{r}$ cannot be an income stream chosen in equilibrium since $\tilde{r} + \gamma\bar{r} \in M(x)$.

Thus, we arrive at the result not particularly original:

Lemma 13 *Let $M(x^\star)$ be an additive cone and $(x^\star, (r_i^\star)_{i\in I})$ is a financial equilibrium, then $M(x^\star) \cap \mathbb{R}_+^{S+1} = \emptyset$.*

However, when $M(x)$ is *not* an additive cone, then the close relationship between the no-arbitrage principle and equilibrium does no longer hold! And the fact that $\beta_M = 0$, as in Proposition 13, does not hold any longer. This can also be illustrated with two questions:

- Can there exist equilibria where an arbitrage opportunity is available, but no agent exploit this opportunity? The answer is yes!

- Can an equilibrium exist in which an arbitrage opportunity is exploited by some agent(s)? The answer is, again, yes!

We argue by means of two examples, the first supports the first claim, while the second supports our second claim.

We show that arbitrage opportunities and exploitations of arbitrage opportunities can exist in a model of perfect competition and symmetric information, but with intermediation costs. This also implies that there exists some consumer i with $\lambda^i \cdot r^i > 0$, and the Proposition 13 does no longer hold.

More specifically, we show that

Proposition 14 *When $I \geq 3$, there exists economies $\mathcal{E}$ and a financial equilibrium $(x^\star, (r_i^\star)_i)$, such that $M(x^\star) \cap \mathbb{R}_+^{S+1} \neq \{0\}$ and $r_i^\star > 0$ for some i.*

Proposition 15 *There exists economies $\mathcal{E}$, $I \geq 2$ and a financial equilibrium $(x^\star, (r_i^\star)_i)$, such that $M(x^\star) \cap \mathbb{R}_+^{S+1} \neq \{0\}$.*

[2] A set $M \subset \mathbb{R}^n$ is an additive cone when $x + \lambda y \in M$ for every $x, y \in M$ and $\lambda \geq 1$.

[3] $f\colon \mathbb{R}^n \to \mathbb{R}$ is strictly monotone if $x > y$ then $f(x) > f(y)$.

The essential feature of this explanation of why arbitrage opportunities might exists and be exploited *in equilibrium* is that intermediation costs are not (piecewise) linear; this implies that, on the margin, small trades have low marginal intermediation costs, while large trades have high marginal intermediation costs. Thus, large trades have no possibility of exploiting any arbitrage opportunity! Agents that do not exploit the arbitrage opportunity does not have the incentive since the arbitrage opportunity does not offer the same risk dispersion as the "normal" income streams.

When $M(x)$ is *not* an additive cone, the no-arbitrage condition, i.e., that $M(x) \cap \mathbb{R}^{S+1}_{+} = \{0\}$, is not a necessary condition for a financial equilibrium to exist. Informally, if the gain in the arbitrage opportunity is sufficiently small, then a no-arbitrage income stream might be preferred by a household: This is illustrated in Figure 4.1. The point is that even when an arbitrage opportunity exist a maximal income stream may exist!

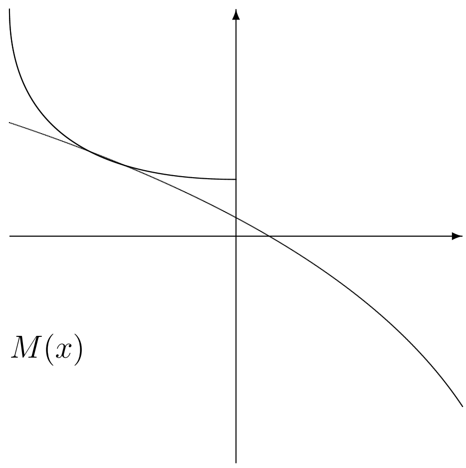

Figure 4.1: Income trade offs

An equilibrium in income streams relative to $M(x)$ could look something like Figure 4.2, where the income production set is of the form $Y = \{(r_i) \mid \sum_i r_i = 0\}$.

The most important thing is that the set $M(x)$ does not recedes in the direcion $\mathbb{R}^{S+1}_{+}$ (see [54, pp. 61]), i.e., for any vector $\xi \in \mathbb{R}^{S+1}_{+}$, $\xi \neq 0$ we must have that $\sup\{\gamma \mid \gamma\xi \in M(x)\} < \infty$. If $M(x)$ has a direction of infinity, then as the arguments go above, the set of consumption bundles in the budget set is not compact, and thus x cannot be an equilibrium asset price vector under any circumstances.

Before we give the second example we make three observations:

Two trivial statements about arbitrage and equilibrium can be stated: *in equilibrium* there cannot exist a consumer which has non-positive (negative) transfers, since $0 \in M(x)$ for any price x. Thus, any arbitrage opportunity must be financed by other households having positive *and* negative transfers. This excludes an arbitrage opportunity to be exploited in equilibrium when there are only two (types of)

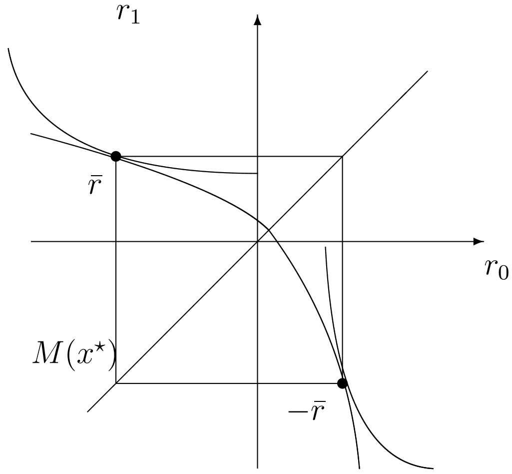

Figure 4.2: Equilibrium in income transfers

agents. Moreover, not all traders can exploit an arbitrage opportunity in equilibrium. Thus, someone has to pay for the exploitation of an arbitrage opportunity. In that sense it is a zero-sum game. Note also, that in any equilibrium it must hold that[4] $r^i \in \partial M$, i.e., the income stream chosen in equilibrium by each agent must be on the boundary of the income stream space. Thus, there cannot exist $r' \in M$ such that $r' > r^i$. Further, this guarantees the exists of state prices $\lambda^i \in \mathbb{R}^{S+1}_{++}$ such that $\lambda^i \cdot r \leq \lambda^i \cdot r^i$ for every $r \in M$.

These results are stated as follows:

Lemma 14 *Let $(x^\star, (r_i^\star)_i)$ be a financial equilibrium, then $r_i^\star \notin -\mathbb{R}^{S+1}_{+} \setminus \{0\}$.*

Lemma 15 *Let $(x^\star, (r_i^\star)_i)$ be a financial equilibrium and $I = 2$, then $r_i^\star \notin \mathbb{R}^{S+1}_{+}$.*

Lemma 16 *Let $(x^\star, (r_i^\star)_i)$ be a financial equilibrium, then for every i there exists $\lambda^i \in \mathbb{R}^{S+1}_{++}$ such that $\lambda^i \cdot (r_i^\star - r) \geq 0$ for every $r \in M(x^\star)$.*

Example 16 *Let there be 3 agents, 2 states and a single consumption good in each state. To see this, let $M \subset \mathbb{R}^2$ be the convex set defined as follows:*

$$M = \text{conv}\{\lambda a + (1,1) \mid a \in \{(2,-5),(-5,2)\} \wedge \lambda \geq 0\}.$$

And consider the income streams $r^1 = (1,1)$, $r^2 = (3,-4)$ and $r^3 = (-4,3)$. Then obviously $r^1 + r^2 + r^3 = (0,0)$. Take further the (indirect) utility functions to be of the form $v_i(r) = \ln(r_0 + b_0^i) + \ln(r_1 + b_1^i)$ for some vector $b^i = (b_0^i, b_1^i) \in \mathbb{R}^2$. Choose

[4] Given a set A, ∂A is the set of boundary points of A, i.e., the set of points $x \in X$ such that any neighbourhood of x intersects with A.

then (b^1, b^2, b^3) *such that*

$$\begin{aligned}\frac{1}{1+b_0^1} - \frac{1}{1+b_1^1} &= 0\\ 2\frac{1}{3+b_0^2} - 5\frac{1}{-4+b_1^2} &= 0\\ -5\frac{1}{-4+b_0^3} + 2\frac{1}{3+b_1^3} &= 0.\end{aligned}$$

This is satisfied when e.g. $b^1 = (1,1)$, $b^2 = (1,6)$, *and* $b^3 = (6,1)$. *Then* (r^1, r^2, r^3) *is an equilibrium of the economy* $(M, (v_i)_{i=1}^3)$ *given above.*

This situation is illustrated in Figure 4.3

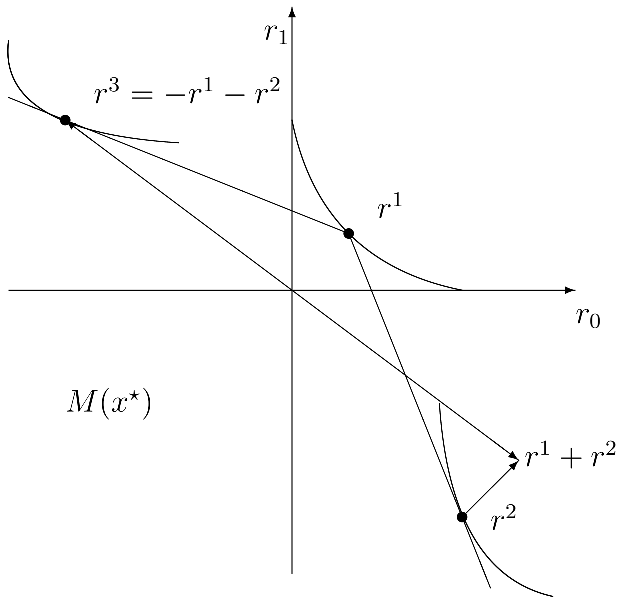

Figure 4.3: Equilibrium with an arbitrage opportunity exploited

In the example M is piecewise linear, however we could easily construct it with a positive curvature, and such that we could construct M with an intermediation cost function which is strictly convex.

4.4 Intermediation Costs in Portfolios and Income Streams

This section contains some miscellaneous of topics related to the results and concepts of sections 4.2 and 4.3.

- How does the concept of intermediation costs of *portfolios* relate to intermediation costs of *income streams*?

Let $c\colon \mathbb{R}^K \to \mathbb{R}_+$ be a convex function and denote, for every $x \in \mathbb{R}^K$, the dividends of the assets by $D(x) \in \mathbb{R}^S_K$. Define then the income stream set by

$$M(x) = \{r \in \mathbb{R}^{S+1} \mid \exists z \in \mathbb{R}^K : r \leq (-x \cdot z - c(z), D(x)z)\}.$$

This set is convex: for every $r, r' \in M(q)$ and $\alpha \in [0,1]$ we have that by convexity of $c(\cdot)$

$$\alpha r + (1-\alpha)r' \leq (-x \cdot (\alpha z + (1-\alpha)z') - c(\alpha z + (1-\alpha)z'), D(x)(\alpha z + (1-\alpha)z'))$$

where $(-x \cdot \bar{z} - c(\bar{z}), D(x)\bar{z}) \in M(x)$ with $\bar{z} = \alpha z + (1-\alpha)z'$.

When $c(0) = 0$, we have that $0 \in M(x)$.

If $c(z)$ is the intermediation cost required to obtain a given portfolio, we can define the intermediation costs required to obtain a given income stream $r \in \mathbb{R}^{S+1}$ as the minimum costs of a portfolio that given the gross income stream, i.e.

$$c(r; D) = c_D(r) = \inf\{c(z) \mid r_{\mathbf{1}} \leq D(x)z \wedge r_0 \leq -x \cdot z - c(z)\},$$

whenever $r_{\mathbf{1}} \in \langle D \rangle$[5], while when $r_{\mathbf{1}} \notin \langle D \rangle$ we let $c(r; D) = c(v(r; D); D)$ where $v(r; D) = \arg\min\{\|r - \tilde{r}\| \mid \tilde{r} \in M(x, D)\}$. The reason why we choose $c(r; D)$ in this way is to insure continuity of $c(\cdot; D)$ given D.

Note that for every $r^0 = Dz^0$, when D has a left-inverse D^{-1}, i.e., if it is one-to-one, we have that $z^0 = D^{-1}r^0$ and we obtain $c(r^0, D) = c(D^{-1}r^0) = c(z^0)$.

Note that the set $\{z \in \mathbb{R}^K \mid r_{\mathbf{1}} \leq D(x)z \wedge r_0 \leq -x \cdot z - c(z)\}$ is convex, denote it by $Z(r^0)$. If $c(\cdot)$ is strictly convex there exists a unique z^0 such that $c(r^0, D) = c(z^0)$. But then we have that

$$\begin{aligned} \alpha c(r; D) + (1-\alpha)c(r'; D) &= \alpha c(z) + (1-\alpha)c(z') \geq c(\alpha z + (1-\alpha)z') \\ &\geq c(\alpha r + (1-\alpha)r'; D) \end{aligned}$$

for every $z \in Z(r)$ and $z' \in Z(r')$. Thus, $c(\cdot; D)$ is convex.

Furthermore, obviously it holds that $\lim_{\|r\| \to \infty} c(r; D) = \infty$.

The question is then whether the map[6] $c\colon \mathbb{R}^{S+1} \times \Omega \to \bar{\mathbb{R}}_+$ is[7] not continuous? Let $A_n \to A$ be a convergent sequence of maps $A_n \in \Omega$ with $\langle A_n \rangle = \langle A_m \rangle = E$ for any n, m while $\langle \lim_{n\to\infty} A_n \rangle = \langle A \rangle \neq E$. Let then $r \in E \setminus \langle A \rangle$ then we

[5] If $D\colon \mathbb{R}^J \to \mathbb{R}^S$ is a linear map we denote by $\langle D \rangle$ the image space, i.e., $\langle D \rangle = \{Dz \mid z \in \mathbb{R}^J\}$.

[6] Let Ω be the set of linear one-to-one mappings $A\colon \mathbb{R}^K \to \mathbb{R}^S$ endowed with the usual metric.

[7] We denote by $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ the extended real line endowed with the one-point-compactification topology, i.e., the open sets consists of every open set of $\mathbb{R}_+$ and all sets of the form $\bar{\mathbb{R}}_+ \setminus C$ for some $C \subset \mathbb{R}_+$ compact. E.g., $(a, \infty]$ is an open set.

must have that $\lim_{n\to\infty} c(r, A_n) = \infty$ and also $c(r, \lim_{n\to\infty} A_n) = c(r, A) = \infty$. Thus, $c(\cdot;\cdot)$ is continuous.

But then we obtain an income stream correspondence $(\mathbb{R}, \mathbb{R}^K) \ni (\lambda, x) \mapsto M(\lambda, x)$ by

$$M(\lambda, x) = \{(-\lambda r_0 - c(r; D(x)), r_{\mathbf{1}}) \mid r = (r_0, r_{\mathbf{1}}) \in \mathbb{R} \times \langle D(x) \rangle\}$$

which satisfies the conditions of Definition 7.

- The following example illustrates no-arbitrage asset prices when transaction costs are linear.

Example 17 *Whenever $c(z) = \sum_{k=1}^{K} a_k|z_k| = \sum_{k=1}^{K} a_k(z_k^+ - z_k^-)$, where $z_k^+ = \max\{z_k, 0\}$ and $z_k^- = \max\{-z_k, 0\}$, we have an particular simple characterization of no-arbitrage asset prices, as also shown in [52]. And in such cases, the existence of an arbitrage opportunity is* not *consistent with an equilibrium, because the income space generated is a cone and the result of Proposition 13 applies. In order for x to satisfy the NA condition it must hold that*

$$\tilde{\lambda} \cdot D_k - a_k \leq x_k \leq \tilde{\lambda} \cdot D_k + a_k,$$

for every $k = 1, \ldots, K$ or in vector form

$$\lambda^T D - a \leq x \leq \lambda^T D + a.$$

- The next two examples show how we can obtain income stream sets as in section 4.3 where arbitrage opportunities are present.

Example 18 *As a numerical example of an intermediation cost function which allows arbitrage opportunities is the case where $c(z) = az^2$, where $a > 0$, and let $d_s > 0$ be the dividend of the asset. Then any asset price $x \in]-a, 0[$ allows an arbitrage opportunity with $z \in]0, -\frac{a}{x}[$, but there is an asymptotic absence of a free lunch.*

Example 19 *Consider the asset structure $V = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, the intermediation cost function $c(z) = \alpha z_1^2 + \beta z_2^2$ with $\alpha + \beta < 1$ and the asset pricevector $q = (2, 1)$. Then the portfolio $z = (1, -1)$ generates the income stream $(1 - \alpha - \beta, 0, 2) \in \mathbb{R}_+^3 \setminus 0$. Note also, that $\lim_{\|z\|\to\infty} q \cdot z - \alpha z_1^2 - \beta z_2^2 = -\infty$ and hence there is* limited *arbitrage opportunities.*

- These last examples illustrate an important implication of strictly convex transaction costs, namely that there always exist asset prices which allows limited free lunches. We now show this in more general cases.

  Let c be a strictly convex intermediation cost function, then there exists some $x_0 \in \mathbb{R}^K$ and $z_0 \in \mathbb{R}^K$ such that $-\lambda x_0 \cdot z_0 - c(\lambda z_0) > 0$ for every $0 < \lambda < 1$ but $-\lambda x_0 \cdot z_0 - c(\lambda z_0) \leq 0$ for every $\lambda \geq 1$: Given a $z_0 \neq 0$, the map $x \mapsto f(x; z_0) = -x \cdot z_0 - c(z_0)$ is continuous, $\sup_x f(x; z_0) = \infty$ and $\inf_x f(x; z_0) = -\infty$ and there exists x_0 such that $f(x_0, z_0) = 0$. By strict convexity of c we have that $c(\lambda z + (1-\lambda)z') < \lambda c(z) + (1-\lambda)c(z')$ for every $\lambda \in]0,1[$ and $z \neq z'$, and hence $c(\lambda z) < \lambda c(z)$ for every $\lambda < 1$ while $c(\lambda z) > \lambda c(z)$ for every $\lambda > 1$. By choosing z_0 such that $Dz_0 > 0$ the statement is proved, assuming that such a portfolio exists. Obviously, the set of such x_0 is convex and open. By other words, for every strictly convex intermediation cost function there is a convex, open set of asset prices that allows for free lunches but have the absence of asymptotic free lunches property.

- Finally, we show how to obtain a financial economy as in section 4.2 from a more general model, as e.g., [18]. A typical economy in the Arrow-Debreu sense is characterized by the utility functions v over consumption bundles and the initial endowments of consumption commodities ω. Given a spotmarket price system $(p_s)_{s=0}^S$ and a consumption bundle $x = (x_s)_{s=0}^S$ we obtain a income stream $r = (r_s)_{s=0}^S$ given by

  $$r_s = p_s \cdot (x_s - \omega_s)$$

  for every $s = 0, \ldots, S$. We can thus define the utility of an income stream r, $u(r)$, as the maximal possible utility obtainable given the prices p, i.e.,

  $$u(r) = \max\{v(x) \mid p_s \cdot (x_s - \omega_s) \leq r_s \forall s\}.$$

  If we consider a single-commodity economy then we normalize $p_s = 1$ and we obtain an utility independent of prices. The properties assumed in section 4.2 are easily shown to be satisfied.

4.5 Conclusion

We have shown that when intermediation or transaction costs are present and they are convex, then the existence and exploitation of arbitrage opportunities are not inconsistent with a competitive, symmetric information equilibrium.

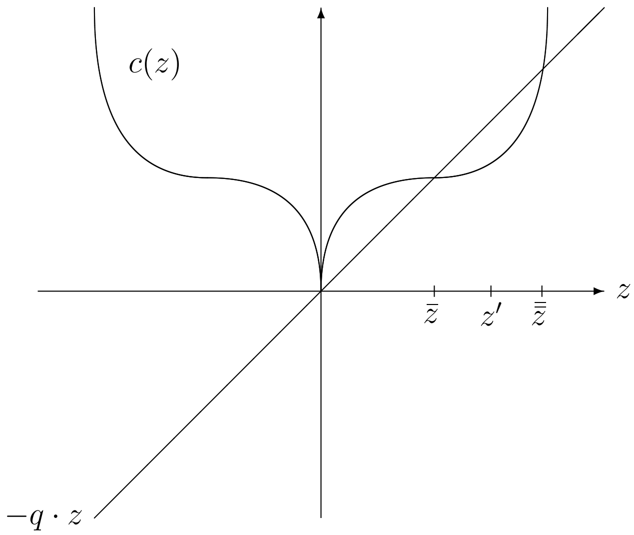

Figure 4.4: Non-convex transaction costs

The question then is of course whether transaction costs have an increasing marginal cost, or equivalently, a diminishing return in the "production of income streams", as implied by the assumption of convex transaction costs.

One argument could be the usual argument that labour has a diminishing return, in that firms hire the most productive units first and as it employs more labour the average product then decreases as the new units employed become less productive. A second argument could be that large quantities of income trade increases the possibility that the market is thin when the income is traded and thus the price is unfavourable to the trader.

Note however that an intermediary will not necessarily be hurt by the existence of arbitrage opportunities since this might increase some of the consumers' income trade. But of course also discourage other consumers' income trade, and the total effect is ambiguous.

As is shown in [42] in the presence of noise traders the prices deviate, and this extra cost increases more than proportionally with the trade by any trader. Thus, any trader is faced with an increasing cost of asset trading in the risky asset.

Note also that "global convexity" is not a necessary condition for this example to hold, but merely that it is convex "asymptotically". An example is in figure 4.4. Here for any portfolio $z' \in [\bar{z}, \bar{\bar{z}}]$ yields an arbitrage opportunity when dividends are positive, since $-q \cdot z' > c(z') > 0$. But asymptotically there is an absence of a free lunch, e.g. when $z > \bar{\bar{z}}$. Note also that the costs are concave for small z's but become convex for z large.

As one presents examples to illustrate some important and interesting results an important question is whether the example is exceptional and are based upon non-generic parameter values. We claim that the examples given in this paper are robust wrt. perturbations in the utility functions, and thus the existence of arbitrage opportunities consistent with equilibrium is a robust property. However, as was noted in section 4.4 the model does not work in cases where costs are concave, i.e., where transaction costs have diminishing marginal costs.

The results in this paper suggest a more general result with separated markets: if the "transportation" costs between two markets are non-linear, arbitrage opportunities, exploited and unexploited, may be consistent with a general equilibrium. The situation is not only restricted to the case of financial markets, but also consumption goods are subject to these results.

We note that according to [45] whenever $M(\cdot)$ is generated according to the form in Section 4.4 there exists an equilibrium. The question is then of course how often a situation like the example in Section 4.2 occurs? As we see, a necessary condition is that consumer preferences are sufficiently different.

Chapter 5

Jumps in Asset Prices: A General Equilibrium Explanation

Tobias Markeprand & Mich Tvede

Abstract. An important question is how incompleteness of the financial markets affects the pricing of securities and the allocation of real commodities in the economy. In this paper we show that incompleteness of financial markets can induce large changes in commodity prices even if the fundamental characteristics do not change significantly. We show that walrasian equilibria, and thus perfect insurance opportunities, would eliminate such changes. Thus, our results provide a test with which incompleteness of markets can be verified. Further, our example exhibit a continuum of equilibria.

JEL classification:. D50, D52, G12, G14.

Keywords:. General Equilibrium, Incomplete Markets, Asset Pricing.

5.1 Introduction

A fundamental question is how shocks transmit into the economy and the resulting allocation and prices. Every day the economy is submitted to shocks influencing the fundamental characteristics of the state of the economy. You slip in the bathroom during your morning shower hurting your back, changing your working capabilities; the frost prevents the train from starting and thereby preventing a load full of passengers to arrive at their planned destination, etc. The question is then whether and how all these shocks affect the overall economy and the allocation of resources in the economy. Obviously, the financial markets are affected by the real economy in that they determine the inflow of funds and they affect the dividend stream of many securities. On the other hand, the financial markets affect the real economy by allowing consumers to reallocate income through time and uncertainty. A fundamental principle in finance is that prices are determined by their fundamental value, i.e., by their future dividend stream. Thus, if the dividends change by a small amount, so will asset prices. Many asset pricing models assume that dividends follow continuous paths and hence prices should move continuous. However, these models have difficulties in generating the observed asset prices, in that, among many things, they fail to account for the "thick tails" in the distribution, i.e., the overrepresentation of large changes in prices, as expressed by [13]

> "In continuous-time setting, jumps in financial prices seem necessary to account for thick tails in asset returns, and the corresponding implied volatility smiles in near-maturity options."

We show how these jumps can be explained by incompleteness of the financial markets, i.e., by the inability of consumers to save and insure perfectly against shocks. [55] argues that asset prices fluctuate too much to be explained by the dividends and hence that the fundamental pricing theory is incorrect. However, we show how asset prices can exhibit excess volatility due to market incompleteness, while still maintaining the market efficiency hypothesis.

Why is it important that prices can jump even if the shocks to the economy are small? This is important since the presence of multiple equilibria is associated with a coordination problem among market participants. Moreover, our result shows that these jumps are connected to *real* jumps and hence have consequences for the welfare of the households. One can relate this to crashes on the stock exchanges experienced in Argentina and Russia. Both examples where financial crashes and a real melt down of the economy happened simultaneously. Furthermore, our example shows that sudden drops in asset prices are *not* necessarily evidence that a bubble has burst, or that a bubble has been present. This makes it even more difficult for the monetary authorities to develop tools to remedy bubbles since prices can change rapidly in a short moment of time, and that they have to be compared with indicators of real activity such that e.g. GDP or others in order to find out whether there is a bubble. However, such indicators are only available *after* the authorities have to make a decision on whether they should intervene or not.

Furthermore, Our result shows the importance of working with a general equilibrium model, taking the interaction between the real economy and financial market into consideration.

In the present paper we show that incompleteness of financial markets can induce discontinuities in commodity prices as the fundamental characteristics changes. We show that Walrasian equilibria, and thus perfect insurance opportunities, would eliminate such changes. Thus, our results provide a test with which incompleteness of markets can be verified. In order to give content to the phrase *"if the fundamental characteristics do not change significantly"*, we consider a non-atomic state space, more specifically, a continuum of states. We assume then that fundamentals, endowments, dividends and densities, are continuous in the states. Furthermore, we show that with differentiable utilities and real assets there is a continuum of equilibria, and hence the equilibrium exhibit real indeterminacy. [47] shows how indeterminacy is a generic phenomenon with real assets and incomplete markets when there is a finite number of securities and infinite states. More specifically, the result is as follows, for any real asset there exists an open set of endowments and utility functions where every element contains an equilibrium set with cardinality equal to the continuum. Our example confirms his result. Our proofs follow along the lines of [47] very tightly, in that the continuity property is proved using Pareto efficiency, while the example also closely relates to his example. However, we focus on the continuity property of the prices. Our result hinges on the fact that the spotmarket equilibrium set can have multiple elements, and thus, that the price expectations must be coordinated given the realization of a future state. Thus, the discontinuity is the result of changing expectations, changes that are not continuous. In the complete market case, these jumps in expectations are forced to be eliminated since contracts can be signed which would leave the parties of an exchange better of and thus prices would change to equilibrate demand and supply.

[6] tells a story to explain the content of the results of analyzing the equilibrium manifold. It is told that passing through irregular economies could, and must eventually, imply a "large" change in commodity prices. This is obviously true in the case of autarky. We formalize and generalize this story, and we show that in order for this story to be true, we need incomplete financial markets, so that traders cannot insure against these changes, since complete markets would prevent such jumps from occurring. Obviously, risk averse traders would like to insure against such risk.

A related problem is the upper hemi continuity of the equilibrium correspondence, i.e., the equilibrium prices and allocations parameterized by the endowment. This property implies that any convergent sequence of endowments must have a converging sequence of equilibrium states. It is, however, easy to see that the equilibrium correspondence is upper hemi continuous, and, thus, our result shows that these questions are independent.

Let us briefly take an overview on how volatility in asset prices has been studied in a theoretical framework (We apologise for any unjustified omissions)

In the paper [37] it is shown that in a continuous time model of financial pricing model, prices have continuous sample paths when the information structure is continuous[1] and thus showing that in this model a phenomenon as jumps in asset prices cannot occur. Among those continuous information structures is the Brownian Motions. Basically, it is a continuity of the itô parameters which gives the contiunity. In that sense, our information structure is *dis*continuous since the time of trade is discrete. [12] shows that with CARA utility functions interest rates can fluctuate over time, but not over states since there are idiosyncratic shocks and aggregate certainty. Also, he shows that there is a unique equilibrium. [13] incorporate jumps in the drift and volatility components of the dividend process and they take a statistical approach to the asset pricing. They use a so-called multifractal model which diverges from the normal Gaussian model by allowing jumps in the drift and volatility of the dividend processes, i.e., the itô parameters. However, they do not explain these jumps, while our jumps are perfectly endogenous. [26] shows by means of an example that incompleteness of the markets can induce increased volatility on the prices on durable commodities. He considers a model of asset market which endogenize the default rate and the level of collateral. The use of durable commodities as collateral tends to increase the demand for these commodities and hence increase the price. Again, a general equilibrium model is required for this result, since commodities through collateralization is tied up together. Thus, the price of one commodity affects the other commodities as well. The paper [17] studies volatility of security prices and financial innovation. Their results point in two directions, depending on the nature of risk, more specifically whether there is aggregate risk or not. When there is no aggregate risk, completing the asset market will generically reduce the asset price volatility. While in the case of aggregate risk, reducing the degree of incompleteness per se is not necessarily associated with a volatility reduction. We take the financial structure as exogenous. The paper [36] shows that the equilibrium price has a continuous density. However, we provide a different proof of this result, as we exploit the relationship between Pareto efficiency and Walrasian equilibria

[1] Inforamlly, an information structure is *continuous* if the induced conditional probabilities is continuous, i.e., the map $t \mapsto P(B \mid \mathcal{F}_t)$ is continuous.

given by the first fundamental theorem of welfare economics. Our results moreover show that the continuity property is intimately related to the efficiency property, since any such allocation must be continuous.

Finally, let us just state some remarks on our indeterminacy result. In [19] it is shown that with complete markets, generically, there is a finite set of equilibria, while in [8], [27] it is shown that with *real* assets and a *finite set of states of nature* there is generically a finite set of equilibria. In the other extreme, with *nominal* securities [7] shows that the indeterminacy is large and calculates the dimension of indeterminacy to be of the difference between the number of states and securities. Our example shows that the result of determinacy with real assets and finite states does not extend to the case of a continuum of states. We have not shown that our example is robust, however we claim without any proof that it is actually robust.

The paper is structured as follows: In section 5.2 we introduce notation and the equilibrium concepts, and further state our assumptions on the fundamental characteristics of the economy. Then in section 5.3 we state and prove that with complete markets the prices are continuous in states, and in section 5.4 an example illustrates that asset prices can have jumps when the asset market is incomplete with finitely many securities. Section 5.5 concludes.

5.2 The model

Set-up

There is a finite number $T+1$ of dates with $t \in \{0, \dots, T\}$. There is uncertainty, the set of states at date $t \geq 1$ is $S = [0,1]$ with $s \in S$ and $\pi : S^T \to \mathbb{R}_+$ is the density on the set of states S^T. There is a finite number of goods ℓ at every state with $j \in \{1, \dots, \ell\}$. A collection of maps $p = (p_t)$, where $p_t : S^t \to \mathbb{R}^\ell_{++}$, is a price system for goods. The space S^T is endowed with the σ-algebra induced by the density π.

There is a finite number m of consumers with $i \in \{1, \dots, m\}$. Consumers are described by their identical consumption sets $X = (\mathbb{R}^\ell_{++})^{T+1}$, endowments $\omega_i = (\omega_i^t)_t$, where endowments at date t is described by a map $\omega_i^t : S^t \to X$, and state utility function $u_i : X \to \mathbb{R}$. A collection of maps $x_i = (x_i^t)_t$, where $x_i^t : S^t \to \mathbb{R}^\ell_{++}$, is a consumption bundle. An allocation of goods $x = (x_i)_i$ is a list of individual consumption bundles.

Walras equilibrium

Let $s^t = (s_1, \ldots, s_t)$ denote the history of states to date t, then the problem of consumer i is:

$$\begin{aligned} \max_{x_i} \quad & \int_{S^T} u(x_i^0, \ldots, x_i^T(s^T)) \; \pi(s^T) \; ds^T \\ \text{s.t.} \quad & \int_{S^T} \sum_t p_t(s^t) \cdot x_i^t(s^t) \; ds^T \; \leq \; \int_{S^T} \sum_t p_t(s^t) \cdot \omega_i^t(s^t) \; ds^T \end{aligned}$$

Please note that the problem of consumer i may not have a solution because none of the integrals may be defined. For now it is hoped that informally the problem makes sense.

Definition 1 *A* **Walrasian equilibrium** *is a price system for goods and an allocation of goods* $(\bar{p}, \bar{x})$ *such that:*

- $\bar{x}_i$ *is a solution to the problem of consumer* i *for all* i*, and;*
- *markets clear* $\sum_i \bar{x}_i^t(s^t) = \sum_i \omega_i^t(s^t)$ *for all* t *and* s^t*.*

Financial market equilibrium

There is a finite number n of assets with $k \in \{1, \ldots, n\}$ where the dividend of asset k at date t is described by a map $a_k^t : S^{t+1} \to \mathbb{R}^\ell$. A collection of maps $q = (q_t)$, where $q_t : S^t \to \mathbb{R}^n$, is a price system for assets. A collection of maps $z_i = (z_i^t)_t$, where $z_i^t : S^t \to \mathbb{R}^k$, is portfolio plan. An allocation of assets $z = (z_i)$ is a list of portfolio plans. Portfolios are restricted to be in Z where $Z \subset \mathbb{R}^n$ is convex and closed.

A price system (p, q) is a price system for goods and a price system for assets. An allocation (x, z) is an allocation of goods and an allocation of assets.

Let $a_t(s^t)$ be the $\ell \times n$-matrix of dividends $(a_t^1(s^t) \ldots a_t^n(s^T))$ at date t in state

s^t, then the problem of consumer i is:

$$\max_{(x_i,z_i)} \int_{S^T} u(x_i^0,\ldots,x_i^T(s^T))\ \pi(s^T)\ ds^T$$

$$\text{s.t.} \begin{cases} p_0\cdot x_i^0 + z_i^0\cdot q_0 \ \le\ p_0\cdot \omega_i^0 \\ p_t(s^t)\cdot x_i^t(s^t) + z_i^t(s^t)\cdot q_t(s^t) \\ \qquad \le\ p_t(s^t)\cdot \omega_i^t(s^t) + z_i^{t-1}(s^{t-1})\cdot (q_t(s^t) + p_t(s^t)a^t(s^t)) \\ \qquad \text{for all } t\in\{1,\ldots,T-1\} \\ p_T(s^T)\cdot x_i^T(s^T)\ \le\ p_T(s^T)\cdot\omega_i^T(s^T) + z_i^{t-1}(s^T)\cdot(p_T(s^T)a^T(s^T)) \\ z_i^t(s^t)\ \in\ Z \text{ for all } t\in\{0,\ldots,T-1\} \end{cases}$$

Definition 2 *A* **financial market equilibrium** *is a price system and an allocation* $((\bar p,\bar q),(\bar x,\bar z))$ *such that:*

- $(\bar x_i,\bar z_i)$ *is a solution to the problem of consumer* i *for all* i*, and;*
- *market clears* $\sum_i \bar x_i^t(s^t) = \sum_i \omega_i^t(s^t)$ *and* $\sum_i \bar z_i^t(s^t) = 0$ *for all* t *and* s^t*.*

Assumptions

The consumers are supposed to satisfy the following assumptions:

(A.1) $\omega_i^t \in \mathrm{C}^1([0,1]^t, X)$.

(A.2) $u_i\in \mathrm{C}^2(X,\mathbb{R})$ with $Du_i(x_i)\in\mathbb{R}^{\ell T}_{++}$ for all x_i and $v^T D^2u_i(x_i)v<0$ for all x_i and $v\neq 0$.

(A.3) If $x_i\to\hat x_i$ and $\hat x_i\in\partial X$, then $\|Du_i(x_i)\|\to\infty$.

The economy is supposed to satisfy the following assumptions:

(A.4) $\pi\in \mathrm{C}^1([0,1]^T,\mathbb{R}_{++})$.

(A.5) $a_k^t\in \mathrm{C}^1([0,1]^t,\mathbb{R}^\ell)$ for all k and t.

All the allocations and prices are of course assumed to be measurable wrt. the σ-algebra on S^T induced by π. Moreover, we assume that allocations are uniformly bounded, i.e., they are elements of L^∞. When we consider Walrasian equilibria we assume that prices are elements of L^1, i.e., they are μ-integrable. In the next section, however, we show that more is true, namely that they are continuous.

5.3 Complete financial markets

In the present paper functions that are identical except for a set of measure zero are considered to be identical.

Definition 3 *A measurable function $f : S^T \to \mathbb{R}$ is continuous at $\hat{s}^T$ if and only if there exist a neighborhood A of $\hat{s}^T$ and a function $g : S^T \to \mathbb{R}$, where $g^{-1}(B)$ is open for B open, such that*

$$\int_A 1_{\{s^T | f(s^T) \neq g(s^T)\}} \; \pi(s^T) \; ds^T \;\; = \;\; 0.$$

A function is continuous if and only if it is continuous at all points.

Walrasian equilibrium

At Walrasian equilibria, prices and consumption bundles are differentiable functions of states of nature. The proof consists of two steps: in Lemma 1 it is shown that prices and consumption bundles are continuous functions of states, and; in Theorem 1 it is shown that they are continuous functions of states, then they are differentiable functions of states.

Lemma 1 *Suppose that $(\bar{p}, \bar{x})$ is a Walrasian equilibrium, then it is continuous in s^T.*

In [9] it its shown under which conditions prices are *integrable*, and not only representable as abstract functionals on the commodity space, i.e., $(L^\infty)^\star = ba$ which is the set of bounded additive set functions absolutely continuous with respect to μ. *Proof:* Suppose that $(\bar{p}, \bar{x})$ is a Walrasian equilibrium, then there exists $\lambda_1, \ldots, \lambda_m > 0$ such that $\bar{x}$ is the solution to the following problem

$$\begin{aligned} \max_x \quad & \sum_i \lambda_i \int u_i(x_i^0, \ldots, x_i^T(s^T)) \; \pi(s^T) \; ds^T \\ \text{s.t.} \quad & \sum_i x_i^t(s^t) \;\; = \;\; \sum_i \omega_i^t(s^t) \text{ for all } t \text{ and } s^t \end{aligned} \tag{5.1}$$

The proof that $\bar{x}$ is continuous in s^T is by backward induction on t.

"$t = T$" Suppose that $\hat{c}^{T-1} = (\hat{x}^0, \ldots, \hat{x}^{T-1})$ and $\hat{s}^T$ are fixed and consider the following maximization problem

$$\begin{aligned} \max_{x^T} \quad & \sum_i \lambda_i u_i(\hat{c}_i^{T-1}, x_i^T) \\ \text{s.t.} \quad & \sum_i x_i^T \;\; = \;\; \sum_i \omega_i^T(\hat{s}^T). \end{aligned}$$

Then for every $\hat{c}^{T-1}$ and $\hat{s}^T$ there exists a unique continuous solution to the maximization problem according to assumptions (A.2) and (A.3). Let $f^T : (X^m)^T \times S^T \to X^m$ be the solution, then it is continuous according to Berge's maximum theorem and if $c^{T-1} = (\bar{x}^0(s^0), \ldots, \bar{x}^{T-1}(s^{T-1}))$, then $f^T(c^{T-1}, s^T) = \bar{x}^T(s^T)$. Moreover the function $v_i^T : (X^m)^T \times S^T \to \mathbb{R}$ defined by

$$v_i^T(c^{T-1}, s^{T-1}) \;=\; \int u_i(c_i^{T-1}, f_i^T(c^{T-1}, s^T))\; \pi(s_T|s^{T-1})\; ds_T$$

is strictly concave in $x^0, \ldots, x^{T-1}$.

"$t = T-1$" Suppose that $\hat{c}^{T-2} = (\hat{x}^0, \ldots, \hat{x}^{T-2})$ and $\hat{s}^{T-1}$ are fixed and consider the following maximization problem

$$\begin{aligned} \max_{x^{T-1}} \quad & \sum_i \lambda_i v_i(\hat{c}^{T-2}, x^{T-1}, \hat{s}^{T-1}) \\ \text{s.t.} \quad & \sum_i x_i^{T-1} \;=\; \sum_i \omega_i^{T-1}(\hat{s}^{T-1}). \end{aligned}$$

Then for every $\hat{c}^{T-2}$ and $\hat{s}^{T-1}$ there exists a unique continuous solution to the maximization problem according to assumptions (A.2) and (A.3). Let $f^{T-1} : (X^m)^{T-1} \times S^{T-1} \to X^m$ be the solution, then it is continuous according to Berge's maximum theorem and if $c^{T-2} = (\bar{x}^0(s^0), \ldots, \bar{x}^{T-2}(s^{T-2}))$, then $f^{T-1}(c^{T-2}, s^{T-1}) = \bar{x}^{T-1}(s^{T-1})$. Moreover the function $v_i^{T-1} : (X^m)^{T-1} \times S^{T-1} \to \mathbb{R}$ defined by

$$\begin{aligned} & v_i^{T-1}(c^{T-2}, s^{T-1}) \\ & \quad = \int v_i^T(c^{T-2}, f^{T-1}(c^{T-2}, s^{T-1}), s^{T-1})\; \pi(s_{T-1}|s^{T-2})\; ds_{T-1} \end{aligned}$$

is strictly concave in c^{T-2}.

The steps for $t = T-2, \ldots, 0$ are similar to the step for $t = T-1$. The solution $(\bar{x}^t)_t$, where $\bar{x}_t : S^t \to X^m$, to problem (5.1) is defined as follows

$$\begin{aligned} \bar{x}^0 &= f^0 \\ \bar{x}^1(s^1) &= f^1(\bar{x}^0, s^1) \\ &\vdots \\ \bar{x}^{T-1}(s^{T-1}) &= f^{T-1}(\bar{x}^0, \bar{x}^1(s^1), \ldots, \bar{x}^{T-2}(s^{T-2}), s^{T-1}) \\ \bar{x}^T(s^T) &= f^T(\bar{x}^0, \bar{x}^1(s^1), \ldots, \bar{x}^{T-1}(s^{T-1}), s^T). \end{aligned}$$

The price system $\bar{p}$ is collinear with the gradients of the consumers, so the price system is continuous in s^T too. Indeed there exists $\tau > 0$ such that

$$\bar{p}_t(s^t) \;=\; \tau\lambda_i \int D_{x^t} u_i(\bar{x}_i(s^T))\; \pi(s_{t+1}, \ldots, s_T|s^t)\; d(s_{t+1}, \ldots, s_T).$$

for all i, t and s^t.

□

Remark: In the proof of Lemma 1 it is only used that utility functions are once differentiable and strictly concave, but it is not used that utility functions are twice differentiable with negative definite Hessian matrices.

End of remark

Theorem 1 *Suppose that $(\bar{p}, \bar{x})$ is a Walrasian equilibrium. Then $(\bar{p}, \bar{x})$ is differentiable in s^T.*

Proof: Suppose that $(\bar{p}, \bar{x})$ is a Walrasian equilibrium, then according to Lemma 1 it is continuous in s^T and there exists $\lambda_1, \ldots, \lambda_m > 0$ such that $\bar{x}$ is the solution to the following problem

$$\begin{aligned} \max_x \quad & \sum_i \lambda_i \int u_i(x_i^0, \ldots, x_i^T(s^T))\ \pi(s^T)\ ds^T \\ \text{s.t.} \quad & \sum_i x_i^t(s^t) \ = \ \sum_i \omega_i^t(s^t) \text{ for all } t \text{ and } s^t. \end{aligned}$$

The proof that $\bar{x}$ is differentiable in s^T is by induction on t. At step t it is assumed that x^0 is differentiable in s^0,..., x^{t-1} is differentiable in s^{t-1}.

"$t = 0$" The first-order conditions with respect to x^0 at s^0 are

$$\begin{aligned} \lambda_i \int D_{x^0} u_i(x_i^0, \ldots, x_i^T(s^T))\ \pi(s^T)\ ds^T - \alpha^0 &= 0 \text{ for all } i \\ \sum_i x_i^0 - \sum_i \omega_i^0 &= 0 \end{aligned}$$

The $\ell(m+1) \times \ell(m-1)$-matrix H of derivatives with respect to x^0 and α^0 of the first-order conditions is

$$\begin{pmatrix} D_1^0 & & & -I \\ & \ddots & & \vdots \\ & & D_m^0 & -I \\ I & \cdots & I & \end{pmatrix}$$

where D_i^0 is a $\ell \times \ell$-matrix defined by

$$D_i^0 \ = \ \lambda_i \int D^2_{x^0 x^0} u_i(x_i^0, \ldots, x_i^T(s^T))\ \pi(s^T)\ ds^T$$

and I is a the $\ell \times \ell$-identity matrix. The matrix of derivatives with respect to x^0 and α^0 of the first-order conditions has full rank. Therefore according to the Implicit Function Theorem x^0 is a differentiable function of s^0, because x^1 is a continuous function of s^1,..., x^T is a continuous function of s^T.

"$t = T$" The first-order conditions with respect to x^T at s^T are

$$\begin{aligned}\lambda_i D_{x^T} u_i(x_i^0, \ldots, x_i^T(s^T)) - \alpha^T &= 0 \text{ for all } i \\ \sum_i x_i^T(s^T) - \sum_i \omega_i^T(s^T) &= 0\end{aligned}$$

The $\ell(m+1) \times \ell(m-1)$-matrix H of derivatives with respect to x^1 and α^1 of the first-order conditions is

$$\begin{pmatrix} D_1^T & & & -I \\ & \ddots & & \vdots \\ & & D_m^T & -I \\ I & \cdots & I & \end{pmatrix}$$

where D_i^T is a $\ell \times \ell$-matrix defined by

$$D_i^T \;=\; \lambda_i D^2_{x^T x^T} u_i(x_i^0, \ldots, x_i^T(s^T)).$$

The matrix of derivatives with respect to x^1 and α^1 of the first-order conditions has full rank. Therefore according to the Implicit Function Theorem x^T is a differentiable function of s^T, because x^0 is a differentiable function of s^0,..., x^{T-1} is a differentiable function of s^{T-1}.

The fact that $\bar{p}$ is differentiable in s^T follows from the proof that $\bar{p}$ is continuous in s^T in the proof of Lemma 1 and that $\bar{x}$ is differentiable in s^T.

□

Financial market equilibrium

At financial market equilibria, where the allocation is Pareto optimal, prices including asset prices, consumption bundles and portfolios are differentiable functions of states.

Corollary 1 *Suppose that $(\bar{p}, \bar{x})$ is a Walrasian equilibrium and $a = (a_k)_k$, where $a_k = (a_k^t)_t$ and $a_k^t : S^t \to \mathbb{R}^\ell$, is an asset structure such that $((\bar{p}, \bar{q}), (\bar{x}, \bar{z}))$ is a financial market equilibrium. Then $\bar{q}$ is differentiable in s^T.*

Proof: The proof that $\bar{q}$ is differentiable in s^T is by backward induction on t.

"$t = T-1$" The asset price of asset k at date $T-1$ in state s^{T-1} is

$$\bar{q}_k^{T-1}(s^{T-1}) \;=\; \int \bar{p}_T(s^{T-1}, s_T) \cdot a_k^T(s^{T-1}, s_T)\, ds_T$$

where $\bar{p}_T$ is continuous in s^T according to Lemma 1 and a_k^T is continuous in s^T according to assumption (A.5). Therefore $\bar{q}_k^{T-1}$ is continuous in s^{T-1}.

"$t = 0$" Trivial because $\bar{q}_k^0$ is a number rather than a function. However the asset price of asset k at date 0 is

$$\bar{q}_k^0 = \int (\bar{p}_1(s_1) \cdot a_k^1(s_1) + q_k^1(s_1)) \, ds_1$$

where $\bar{p}_1$ is continuous in s^1 according to Lemma 1 and a_k^1 is continuous in s^1 according to assumption (A.5). Therefore $\bar{q}_k^0$ is continuous.

$\square$

Remark: In the proof of Corollary 1 it is only used that $(\bar{p}, \bar{x})$ is continuous and that a is continuous, but it is not used that a is differentiable.

End of remark

5.4 Incomplete financial markets

Financial market equilibrium

At financial market equilibria there may be jumps in prices including asset prices, consumption bundles and portfolios. The proof is based on an example.

Theorem 2 *There exists an economy such that if $((\bar{p}, \bar{q}), (\bar{x}, \bar{z}))$ is a financial market equilibrium, then $\bar{q}$ is not continuous in s^T.*

Proof: Consider an economy with three dates $T = 2$, one good per state $\ell = 1$, two consumers $m = 2$, one asset $n = 1$ and $Z = \mathbb{R}$. We let $S = [0, 1]$ be the set of states. The dividend of the asset is supposed to be one unit of the good at the last date. Endowments and asset dividends are supposed to independent of the state at the first and last date. For the density $\pi \colon S \to \mathbb{R}_{++}$ suppose that $\pi(s_1) = 1$ for all $s_1 \in S$. The information algebra is such that the information set at $t = 0$ is $F_0 = \{\emptyset, S\}$ and finally, the σ-algebra generated by $F_1 = F_2 = \{\{s\} \mid s \in S\}$.

Endowments at the first date are supposed to be identical $\omega_2^0 = \omega_1^0$ and endowments at the last two dates are supposed to be reverse $\omega_2^1(s_1) = \omega_1^2(1 - s_1)$ and $\omega_2^2(s_1) = \omega_1^1(1 - s_1)$. Similarly, utility functions are supposed to be identical for the first date and reverse for the last two dates $u_2(x^0, x^1, x^2) = u_1(x^0, x^2, x^1)$.

For c_i^0 let $f(\cdot; c_i^0) : \mathbb{R}_{++}^2 \times \mathbb{R}_{++} \to \mathbb{R}_{++}^2$ denote the demand function for the consumer with endowments $e_i(s_1) = (\omega_i^1(s_1), \omega_i^2(s_1))$ and utility function $v_i(\cdot; c_i^0) : \mathbb{R}_{++}^2 \to \mathbb{R}$ defined by $v_i(x_i^1, x_i^2; c_i^0) = u_i(c_i^0, x_i^1, x_i^2)$. Then $(p, s_1) \in \mathbb{R}^2 \times \mathbb{R}_{++}^2 \times S$ is an equilibrium for the Edgeworth box economy $\mathcal{E}(s_1; (c_i^0)_i) = (e_i(s_1), v_i(\cdot, ; c_i^0))_i$ if and only if

$$f_1(p, p \cdot e_1(s_1); c_1^0) + f_2(p, p \cdot e_2(s_1); c_2^0) = e_1(s_1) + e_2(s_1)$$

Clearly (p_1, p_2, s_1) is an equilibrium for $\mathcal{E}(s_1; (c_i^0)_i)$ if and only if $(p_2, p_1, 1 - s_1)$ is an equilibrium for $\mathcal{E}(s_1; (d_i^0)_i)$, where $d_1^0 = c_2^0$ and $d_2^0 = c_1^0$.

Suppose that equilibrium prices are normalized such that the sum of the prices is equal to one and let $E \subset \mathbb{R}^2_{++} \times S$ be the equilibrium set for the collection of Edgeworth economies $(\mathcal{E}(s_1; (c_i^0)_i)_{s_1}$, where $c_i^0 = \omega_i^0(s_0)$, so

$$E \;=\; \{\, (p, s_1) \,|\, (p, s_1; (c_i^0)_i) \text{ is an equilibrium for } \mathcal{E}(s_1; (c_i^0))\,\}.$$

Suppose that E is S-shaped as shown in Figure 5.1 and let $r \colon S \to \mathbb{R}^2_{++}$ be a selection from $\mathcal{E}$ such that $r_1(s_1)$ is the lowest equilibrium price for $s_1 < 1/2$, $r_1(s_1) = (1/2, 1/2)$ for $s_1 = 1/2$ and $r_1(s_1)$ is the highest equilibrium price for $s_1 > 1/2$. In

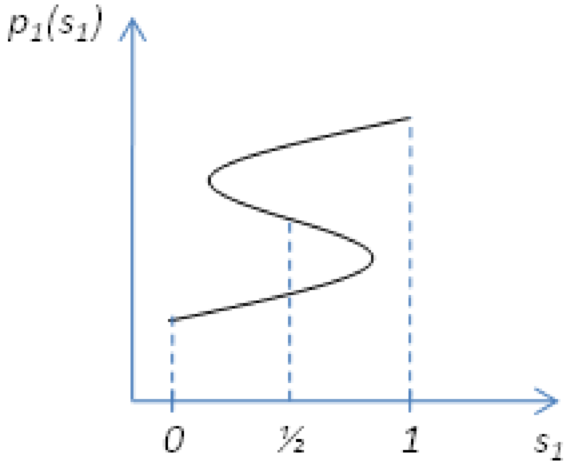

Figure 5.1: The equilibrium set E.

order to construct a financial market equilibrium: let the allocation x be defined by $x_i^0 = \omega_i^0$, $x_i^j(s_1) = f_i^j(r(s_1), e_i(s_1); \omega_i^0)$ for $j \in \{1, 2\}$; let the portfolio plan z be defined by $z_i^0 = 0$ and $z_i^1(s_1) = (r_1(s_1)/r_2(s_1))(\omega_i^1(s_1) - f_i^1(r(s_1), e_i(s_1); \omega_i^0)) = f_i^2(r(s_1), e_i(s_1); \omega_i^0) - \omega_i^2(s_1)$; let the price system p be defined by $p^2(s^2) = p^1(s^1) = p^0(s_0) = 1$, and; let the price system for assets q be defined by $q^1(s^1) = r_2(s^1)/r_1(s^1)$ and $q^0 > 0$ such that

$$\int \left(-q^0 \frac{\partial u_i(x_i(s_1))}{\partial x_i^0} + q^1(s_1) \frac{\partial u_i(x_i(s_1))}{\partial x_i^1} \right) \, ds_1 \;=\; 0.$$

Then $((p, q), (x, z))$ is a financial market equilibrium and the asset price at date 1 is discontinuous at $s_1 = 1/2$.

Finally, We note that the portfolio z_1^0 of consumer 1 at date 0 is bounded from below by

$$\underline{z} = -\min_{s_1} \frac{\omega_1^1(s_1) + q^1(s_1)\omega_1^2(s_1)}{q^1(s_1)}$$

and from above by

$$\overline{z} = \min_{s_1} \frac{\omega_2^1(s_1) + q^1(s_1)\omega_2^2(s_1)}{q^1(s_1)},$$

i.e., $\underline{z} \leq z_1^0(s_1) \leq \overline{z}$ for every $s_1 \in S$. When $\|(\omega_1^1(s_1), \omega_1^2(s_1))\|$ is bounded from above by $M > 0$ for $s_1 \in [0,1]$ and the marginal rates of substitution at the Pareto optimal allocations in the economies $\mathcal{E}(s_1)$ for $s_1 \in [0,1]$ are bounded away from zero and infinity, then for M sufficiently small, the set of equilibria for the collection of economies $\{\mathcal{E}(s_1)\}$ is S-shaped for all feasible portfolios so there is a discontinuity in prices.

$\square$

Remark: The proof of Theorem 2 reveals that any measurable selection $r\colon S \to \mathbb{R}^2_{++}$ such that $r_1(s_1) = 1 - r_1(1 - s_1)$ and $r_2(s_1) = 1 - r_2(1 - s_1)$ is part of a financial market equilibrium. Therefore as shown in [47] there is a continuum of financial market equilibria.

End of remark

On the example in the proof of Theorem 2

Let us try, informally, to argue that the example in the proof of Theorem 2 is robust. In order to consider pertubations of fundamentals suppose that the set of fundamentals is endowed with the Whitney topology, endowments and dividends with the C^1-topology and utility functions with the C^2-topology.

The S-shape of the equilibrium set E is robust to perturbations in fundamentals and small changes in portfolios. Therefore every selection from the equilibrium set is discontinuous. Hence assets prices are discontinuous. The robustness of the example in the proof of Theorem 2 shows that the symmetry in the example is not essential, but merely convenient.

5.5 Final remarks

In the present paper we have shown, admittedly by use of an example, that jumps in asset prices are possible in case of incomplete financial markets. Moreover, we have shown that jumps are impossible in case of complete financial markets. Our results show that with incomplete financial markets on the one hand consumers are not able to insure themselves against all uncertainty and on the other hand induce price effects such as jumps in asset prices.

From a finance perspective it would be interesting to calibrate a parametric model such as an optimal growth model or an overlapping generations model to see whether jumps in asset prices are compatible with data.

From a general equilibrium perspective a partial answer to the question of the appropriate commodity for economies with infinite dimensional commodity spaces. Indeed we have shown that for Walrasian economies restricting attention to continuous maps on the underlying state space as in [16] is no real restriction.

One might argue that such jumps as are observed in real life are abrupt changes of prices over time, and they are not easily studied in a model with discrete time. However, one might conjecture that our model could be transformed into a model of continuous time, by utilizing the fact that we have a continuum of states. One could argue as follows[2]: interpret the states as the time between the two periods and let there be no uncertainty. Let there be a single asset. If there is no boundary on short sale, the asset market is complete and our first result applies. However, introducing short-sale boundaries income transfers becomes limited and the S-shaped curve in the example re-emerges. Then as time passes by the asset price must jump.

[2] This is an interpretation attributed to Yves Balasko, in a conversation with one of the authors.

Appendix A

Existence of finite horizon equilibrium

We need to extend the result of [45] to the case of multiple periods: Again, let Y be finite and let $T < \infty$ be the number of periods, then the state space is $\Omega = Y^T$ and the probability of a state $s = (s_1, \ldots, s_T)$ is given by $P(s) = \pi(s_1, s_2) \cdot \cdots \cdot \pi(s_{T-1}, s_T)$. Let $(\mathbb{D}_t)_{t=1}^T$ be a finite sequence of partitions of Ω, wlog. $\{\Omega\} = \mathbb{D}_1 \subset \mathbb{D}_2 \subset \cdots \subset \mathbb{D}_{T-1} \subset \mathbb{D}_T = \bigcup_{\omega \in \Omega} \{\omega\}$.

A process ($\mathbb{R}^m$-valued) is a finite sequence $(x_t)_{t=1}^T$ such that $x_t \colon \Omega \to \mathbb{R}^m$ is $\mathbb{D}_t$-measurable, i.e., for every $U \subset \mathbb{R}^m$ open and every $t = 1, \ldots, T$, $x_t^{-1}(U)$ is measurable wrt. $\sigma(\mathbb{D}_t)$[1]. This implies that $x_t(\omega) = x_t(\omega')$ whenever there exists $Q \in \mathbb{D}_t$ such that $\omega, \omega' \in Q$. This follows since any $\sigma(\mathbb{D}_t)$-measurable set E is the union of atoms from $\mathbb{D}_t$. Assume that there exists $Q \in \mathbb{D}_t$, $\omega, \omega' \in Q$ and $x_t(\omega) \neq x_t(\omega')$. Then take open disjoint sets $U, V \subset \mathbb{R}^m$, $x_t(\omega) \in U$ and $x_t(\omega') \in V$. But then $x_t^{-1}(U \setminus V) = x_t^{-1}(U) \setminus x_t^{-1}(V)$ which is not a union of atoms of $\mathbb{D}_t$ since $\omega \in x_t^{-1}(U) \setminus x_t^{-1}(V)$ but $\omega' \notin x_t^{-1}(U) \setminus x_t^{-1}(V)$.

Denote by L the set of consumption processes ($\mathbb{R}^l$-valued) and Θ the set of portfolio processes ($\mathbb{R}^k$-valued). An allocation is a process $(x, \theta) \in (L \times \Theta)^n$ such that

$$\begin{aligned} \sum_i x_{it}(\omega) - e_{it}(\omega) &= 0 \\ \sum_i \theta_{it}(\omega) &= 0 \end{aligned}$$

P-a.e., i.e., for a subset $E \subset \Omega$ with $P(E) = 1$. A commodity pricevector is a process $p \in L_+$ and an asset pricevector is a process $q \in \Theta$. We say that $(x, \theta) \in L \times \Theta$ is

[1]Where $\sigma(\mathbb{D}_t)$ is the σ-algebra generated by $\mathbb{D}_t$, i.e., the smallest σ-algebra containing $\mathbb{D}_t$. Since $\mathbb{D}_t$ is a partition this makes $\sigma(\mathbb{D}_t)$ particular simple, since if $A, B \in \mathbb{D}_t$ then either $A \cap B = \emptyset$ or $A = B$, and thus any set $E \in \sigma(\mathbb{D}_t)$ is of the form $E = \bigcup_{D \in \mathscr{D}} D$ where $\mathscr{D} \subset \mathbb{D}_t$.

feasible if

$$p_0 \cdot (x_0 - e_{i0}) + q_0 \cdot \theta_0 \leq w_{i0} - c(\theta_0)$$

and for every $t = 1, \ldots, T$

$$p_t(\omega) \cdot (x_t(\omega) - e_t(\omega)) + q_t(\omega) \cdot \theta_t(\omega) \leq D(\omega) \cdot \theta_{t-1}(\omega) + w_t(\omega) - c(\theta_t(\omega))$$

P-a.e. Denote by $B(p, q, w)$ the set of feasible policies given the price processes $(p_t, q_t)_{t=1}^T$ and transfer process $(w_t)_{t=1}^T$.

A T-horizon equilibrium is then a finite sequence $(s_t)_{t=0}^T$ with S-values such that for every i the strategy $(x_{it}, \theta_{it})_{t=0}^T$ is optimal.

Proposition 20 *For any T there exists a T-horizon equilibrium.*

The difficult part of the result is to show that the budget constraint is lower hemicontinuous. Note first that since $p_t \cdot e_t > 0$ and $w_t \geq 0$ the Slater condition is satisfied; i.e., that there exists a strategy (x_t, θ_t) such that the inequalities are all satisfied with strict inequalities. Let $(p^n, q^n, w^n) \to (p, q, w)$ and consider $(x, \theta) \in B(p, q, w)$.

The desired sequence $(x^n, \theta^n) \in B(p^n, q^n, w^n)$ is obtained using the method of forward induction: first solve $t = 0$ and obtain (x_0^n, θ_0^n), then given this solution let

$$\eta_1^n = \frac{p_1^n \cdot e_1 + D^n(q^n) \cdot \theta_0^n + w_1^n}{p_1^n \cdot x_1 + q_1^n \cdot \theta_1 + c(\theta_1)},$$

when n is large enough (since this guarantee that the fraction is well-defined). Obviously, $0 \leq \eta^n \leq 1$ and $\lim \eta_1^n = 1$ and thus, the sequence $(x_1^n, \theta_1^n) = (\eta_1^n x_1, \eta_1^n \theta_1)$ satisfies the required properties. Assume that $(\eta_\tau)_{\tau=0}^{t-1}$ is constructed such that $(x_\tau^n, \theta_\tau^n) = \eta_\tau^n (x_\tau, \theta_\tau)$ for any $\tau \leq t - 1$ and $n \in \mathbb{Z}_0$. Then we define η_t^n using the formula

$$\eta_t^n = \frac{p_t^n \cdot e(y_t) + D_t^n \cdot \theta_{t-1}^n + w_t^n}{p_t^n \cdot x_t + q_t^n \cdot \theta_t + c(\theta_t)}.$$

Using this algorithm, we obtain the desired sequence $(x^n, \theta^n) = (x_t^n, \theta_t^n)_{t=0}^T$.

We note that we have showed the existence of a spotless equilibrium, since the endogenous variables only depend on the fundamental states given by Ω.

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