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Topics in Political Economy: Voting, Elections, and Terrorism

by

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Preface

This collection of four papers constitutes my Ph.D. thesis submitted to the Department of Economics, University of Copenhagen in July 2007.

I would like to take this opportunity to thank some of the people who have played a role in the completion of this thesis.

First of all I thank my advisor Mich Tvede for guidance, advice and critique through my years as a Ph.D student. I especially thank Mich for keeping a positive attitude when I (once again) changed the direction of my studies or was a little unenthusiastic about academic life.

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Thomas Jensen
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Introduction and Summary

As the title says, the topics of this thesis are voting, elections and terrorism. While the four papers of the thesis are self-contained and each of them can be read independently of the others, they are certainly related. Most importantly, the papers all consist of formal models within the broadly defined field of political economy. The first two papers both consider electoral competititon, a subject that is obviously at the very core of political economy. In the third paper we study the interaction between terrorists and authorities. The final paper introduces a new decision criterion in collective action situations and studies its implications in some examples. With respect to methodology, the papers are naturally divided into two sub-groups. The first and the last paper both have a behavioral touch, i.e. they both contain assumptions that deviate from the standard rationality paradigm in the field of economics. The models studied in the other two papers are completely standard in that sense. This aspect of the thesis reflects my personal view that both behavioral and standard models can help us understand economic, political, and, more generally, social phenomena.

In the following we will give a summary of each of the four papers. We will not specifically relate them to the existing literature, for that we refer to the papers.

In the first paper, *Projection Effects and Strategic Ambiguity in Electoral Competition*, we suggest a new explanation of ambiguous issue positions in electoral competition. It is based on the psychological concept of cognitive consistency. In this context cognitive consistency implies that voters prefer to believe that they agree with political candidates they like for personal reasons and vice versa. Therefore a voter who likes (dislikes) a candidate will perceive his position as closer to (further from) his own than it really is. This is called projection. Suppose voters' perceptions are not counterfactual, i.e. certain positions are perceived correctly, and that voting is based on perceived issue positions. Then projection gives a candidate who is generally liked (for personal reasons) by the electorate an incentive to be ambiguous. We construct and analyze a formal model to see if this incentive survives in the strategic setting of electoral competition. The model is an extension of the standard Downsian model. We make sure that all results predicting ambiguity are solely driven by projection by assuming that voters dislike ambiguity per se, i.e. if there were no projection effects. Our first results show that the median voter theorem breaks down under quite mild assumptions. Loosely speaking, if a candidate is liked by a few voters and not disliked by too many then he can defeat an opponent positioned at the median by taking an ambiguous position. While this certainly shows that our introduction of projection effects into the standard model of electoral competition does make a difference, it is not enough to predict ambiguity. For that we should have ambiguity in equilibrium. Consider the case where a candidate is liked by a majority and not disliked by any voters while his opponent is not liked by any voters. Ambiguity is then predicted if the advantaged candidate has an ambiguous position that is a winning strategy. We show that this is the case if and only if projection is sufficiently strong. This means that our model does not clearly predict ambiguity if the assimilation is relatively weak (it does not have an equilibrium). While this is a drawback of the model, it is also an interesting observation. It shows that a candidate with a large advantage due to voters personal views of the candidates cannot necessarily make use that advantage because of the strategic nature of electoral competition.

The second paper, *Elections, Private Information, and State-Dependent Candidate Quality*, contributes to the study of how democracy works under the reasonable assumption that politicians are better informed than voters about conditions relevant for policy choice. More specifically, we consider a model of electoral competition where both candidates and voters have private information but candidates' information is more accurate (they are fully informed). There are two states of the world and each voter's preferred policy is state-dependent. The candidates are both purely office-motivated but they are different with respect to state-dependent quality. In one state one candidate has a quality advantage and in the other state it is the other way around. This gives the disadvantaged candidate an incentive not to reveal the true state. We solve the model for Perfect Bayesian Equilibria. These can be classified as either revealing or non-revealing depending on whether the voters can infer the true state from the candidates' policy announcements or not. Our first results show that in revealing equilibria (satisfying a known refinement criterion) there is convergence to the median of the true state and that a revealing equilibrium exists if and only if voters' information is sufficiently accurate. So we conclude that if the electorate is sufficiently well informed then it is at least a possibility that electoral competition works as if they were fully informed. Our results on non-revealing equilibria show that many of these exist, some of them independent of how well informed the voters are. The Intuitive Criterion does not eliminate any of these equilibria. Therefore we impose a monotonicity condition on voters' beliefs. With this condition we get that in any non-revealing equilibrium the candidates diverge and that a non-revealing equilibrium exists if and only if voters are not too well informed. The results on non-revealing equilibria show that we can have policy divergence in equilibrium even though candidates are purely office-motivated. This is because of the state-dependent candidate quality (and the assumptions on information). To our knowledge this is a new possible expanation of policy divergence in electoral competition.

In the third paper of the thesis, *Terrorism, Anti-Terrorism, and the Copycat Effect*, the primary aim is to study the dynamic interaction of terrorists and anti-terrorism authorities when a so called copycat effect exists. We assume that terrorist cells live for one period only and that their only decision is whether to plan a small or a large attack. Planning a large attack involves a higher risk of being rolled up by the authorities before the attack. Our first observation is that an increase in the level of anti-terrorism makes it more likely that a cell will plan a small rather than a large attack. Furthermore, we see that an increase in anti-terrorism can make a terrorist attack more likely. We introduce a copycat effect by assuming that in each period there is a higher probability of a new cell being formed if there was a large attack in the previous period. Solving the problem of optimal anti-terrorism we see that the copycat effect rationalizes an increase in the level of anti-terrorism after a large attack. We use this result to analyze the dynamic pattern of terrorist attacks when a copycat effect exists. Finally, we consider the long run distributions of attacks, damage and anti-terrorism costs. We show that in the long run the copyct effect leads to more anti-terrorism, more small attacks, and a higher sum of damage and costs. On the other hand it leads to less large attacks and less damage.

The final paper, *Group Based Regret and Collective Action*, introduces group based regret minimization as a decision criterion in collective action situations. If regret is group based then an individual feels regret after some outcome if she and other members of the group of people with similar preferences could have improved the outcome by acting differently, i.e. given her (and all other people with similar preferences) a higher payoff. This is different from "rational regret", where rational means that an individual only feels regret if she could personally have changed the outcome. We analyze two specific collective action situations to see how the outcomes change when people are group based regret minimizers rather than payoff maximizers or rational regret minimizers. The examples are voting in large elections and provision of a binary (discrete) public good. The first example is analyzed from a simple decision theoretic perspective while the second example is analyzed by game theory. In both examples we see that there is more contribution when people are group based regret minimizers. Thus the assumptions that regret is group based and that minimization of anticipated regret regret is an important criterion in decision making has the potential to explain over-contribution (relative to the prediction of standard theory) in collective action situations.

Projection Effects and Strategic Ambiguity in Electoral Competition.

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Abstract

Theories from psychology suggest that voters' perceptions of political positions depend on their non-policy related attitudes towards the candidates. A voter who likes (dislikes) a candidate will perceive the candidate's position as closer to (further from) his own than it really is. This is called projection. If voters' perceptions are not counterfactual and voting is based on perceived policy positions then projection gives a generally liked candidate an incentive to be ambiguous. In this paper we construct and analyze a formal model to investigate under which conditions this incentive survives in the strategic setting of electoral competition, even if voters dislike ambiguity per se.

Keywords: Electoral Competition, Ambiguity, Voter Perception, Cognitive Consistency, Projection.

1 Introduction

According to theories from psychology (see e.g. Granberg (1993), Krosnick (2002), and references therein) people prefer to be in a state of cognitive consistency. Therefore a voter prefers to believe that he agrees with political candidates he likes (for non-policy related reasons) and disagrees with candidates he dislikes. One way the voter can achieve this is by distorting his perceptions of the candidates' policy positions. He "pulls" the positions of liked candidates towards his own position and "pushes" the positions of disliked candidates away from it. In general this is called *projection*. More specifically, positive projection ("pulling") is called *assimilation* and negative projection ("pushing") is called *contrast*.

If we assume that voters cannot have counterfactual perceptions then projection of a candidate's policy position can only happen when the candidate is ambiguous. So if projection effects exist and voters vote based on perceived policy positions then a generally liked candidate has an incentive to be ambiguous because of assimilation. This paper investigates under which conditions this incentive survives in the strategic setting of electoral competition.

We formulate and analyze an extension of the standard Downsian model that allows candidates to take ambiguous policy positions and introduces projection effects in voters' perceptions of such positions. Ambiguous positions are modelled by intervals of policies. Each voter has an (exogenous) positive, neutral or negative non-policy related attitude towards each candidate. When we say that a voter likes (dislikes) a candidate it simply means that he has a positive (negative) attitude towards him. Voters' perceptions of announced ambiguous positions are represented by probability distributions. They use perceived expected utility to decide on who to vote for. By assuming that voters are risk averse we get that they dislike ambiguity per se, i.e. if there were no projection effects. Thus any result predicting ambiguity is driven only by projection (assimilation).

Our first results answer the following question: Under which conditions can a candidate defeat the median by being ambiguous, i.e. win the election by taking an ambiguous position when the other candidate's position is fixed at the median? Loosely speaking, we show that if a candidate is liked by some voters and not disliked by too many then he can defeat the median. For example, a candidate who is not disliked by any voters can defeat the median if he is liked by an arbitrarily small group of voters.

Secondly, we consider the question of existence or non-existence of winning strategies (which must be ambiguous) for a candidate with an advantage due to voter attitudes and projection. Consider an advantaged candidate who is not disliked by any voters and liked by a majority and a disadvantaged candidate who is not liked by any voters. We show that the advantaged candidate has winning strategies if the assimilation effect is sufficiently strong. So our model does predict ambiguity in equilibrium when projection is strong. When projection is not sufficiently strong then the advantaged candidate does not have winning strategies and the model does not have an equilibrium (not even in mixed strategies). So the advantaged candidate may not be able to win the election for sure, it depends on the strength of the assimilation effect.

There is a large empirical literature on projection (see Granberg (1993) and Krosnick (2002) for surveys). A lot of studies using cross-sectional data do produce evidence that is consistent with projection effects. Krosnick (2002), however, points out that usually it is also consistent with alternative hypotheses, most notably policy based evaluation and persuation. A fairly recent study using cross-sectional data is Merrill, Grofman and Adams (2001). Their findings are consistent with projection, but they also show that most of their evidence could be explained by policy based evaluation (together with different interpretations of issue scales among voters). Krosnick (2002) reviews a few panel data studies that carefully seek to separate projection from policy based evaluation and persuation. They do not find compelling evidence of projection. He concludes that the existence of projection has not yet been convincingly demonstrated and that further empirical research is needed.

As mentioned above, we make the assumption that projection of political positions is based on exogenous non-policy related attitudes and that the voting decision is policy based (given perceptions of positions). Thus we have a combination of projection and policy based evaluation and it seems hard to cast any verdict on its validity given the existing empirical literature on projection. Furthermore, the type of projection we consider is actually quite modest because we rule out counterfactual perceptions. So unless a candidate is very ambiguous voters' perceptions of his position will not be too inaccurate. Therefore our model can still be relevant even if voters' perceptions are reasonably accurate.

The paper is organized as follows. Section 2 contains a review of related literature. In Section 3 we set up the model and present some examples of our general model of projection of ambiguous policy positions. Section 4 contains our results (all proofs are delegated to the Appendix). In Section 5 we discuss and conlude.

2 Related Literature

A number of theoretical models of ambiguity in electoral competition exist in the literature. Zeckhauser (1969) and Fishburn (1972) both consider lotteries in social choice with a discrete set of alternatives (Zeckhauser only considers sets with three alternatives). They show that a lottery can never be a Condorcet winner.

Shepsle (1972) extends the standard Downsian model by forcing one of the candidates (the challenger) to take a lottery position, i.e. a non-degenerate probability distribution over positions. The voters are expected utility maximizers. The main result is that if a majority of voters are risk loving on an interval containing the median, then the challenger can beat an incumbent at the median by taking a lottery position with mean equal to the median. However, both existence and non-existence of a winning position for the challenger can occur.

Page (1976) is critical of Shepsle's theory of ambiguity. He notes that the prediction of ambiguity is not very strong because the challenger may not have a winning strategy. Also he questions whether (a majority of) voters are really risk loving. Furthermore he argues that lottery positions are not a good way of modelling ambiguous political positions because candidates do not express their positions in ways that can easily be perceived as objective probability distributions. Page also presents his own theory of political ambiguity called emphasis allocation theory. He considers a multidimensional space of policy and valence dimensions. Candidates choose which dimensions (issues) to emphasize and take positions in these dimensions. They are vague/ambiguous on issues they do not put any emphasis on. Voters evaluate a candidate by summing the utilities of the candidate's positions on the issues, weigthed by the candidate's emphasis on each issue. In an example it is shown that this leads to emphasis on consensus issues and ambiguity on issues of conflict, no matter what the risk preferences of the voters are. In a footnote Page mentions that a possible different explanation of ambiguity could be that it allows for projection (p. 748).

McKelvey (1980) generalizes the results of Zeckhauser (1969) and Fishburn (1972) to continuous densities on $\mathbb{R}^n$. Furthermore he looks at the effect of introducing exogenous non-zero levels of ambiguity in a special case of electoral competition. He shows that it does not disrupt existing equilibria.

Glazer (1990) shows by some examples that risk loving voters are not necessary to get ambiguity in equilibrium in a two candidate electoral game. If there is uncertainty about the preferred policy of the candidates and the position of the median voter then ambiguity (not specifying a position) can be the equilibrium outcome of a model with simultaneous announcements. In a model with sequential announcements there can be ambiguity in equilibrium because the first mover does not want to make the second mover more informed about the position of the median voter.

Alesina and Cukierman (1990) considers a two period model. In the first period an incumbent decides on a policy. Before the second period elections are held and in the second period the winner enacts his preferred policy (final period play). Voters are imperfectly informed about the incumbents preferred policy so they vote using an estimate based on the first period policy. Therefore the incumbent can have an incentive to blur his policy preferences by being ambiguous, i.e. by choosing a noisy policy instrument. For a non-empty set of parameter values (including risk averse voters) it is optimal for the incumbent to choose a non-minimal level of ambiguity.

Aragones and Neeman (2000) analyze a two stage electoral game with two candidates. First the candidates simultaneously choose a point in the policy space (an ideology) and the choices become common knowledge. Then they simultaneously choose their level of ambiguity. Candidates have a preference for winning and for being ambiguous because it leaves them with more flexibility in office. Voters dislike ambiguity so candidates face a trade off between a high probability of winning (there is uncertainty about the position of the median voter) and being ambiguous. The main result is that when having flexibility in office is sufficiently important then there is policy divergence with a non-minimal level of ambiguity for both candidates. Otherwise there is policy convergence and no ambiguity.

In Aragones and Postlewaite (2002) a model with only three alternative policies and office motivated politicians is considered. Ambiguous positions are modelled as probability distributions on the set of alternatives. It is known from Fishburn (1972) that if candidates are not restricted in their choice of distributions then any equilibrium will consist of degenerate distributions. But Aragones and Postlewaite restrict the candidates such that each of them must put a minimum of probability mass on one of the alternatives - the alternative that voters think is most likely chosen by the candidate after the election. Under that assumption the result of Fishburn is not valid anymore. The most clean result is found when candidates are uncertain about the level of intensity of the voters preferences (then candidates payoff's are continuous). In that case it holds that (under some further assumptions) there always exists a pure strategy equilibrium and in any such equilibrium candidates are ambiguous (distributions are non-degenerate).

Meirowitz (2005) models a US presidential election with primaries. Each candidate can choose to announce a policy in the primaries or to be ambiguous in the primaries and not announce a policy until the general election. Candidates must stick to their policy announcements and they are imperfectly informed about voter preferences. In equilibrium candidates choose not to announce a policies in the primaries because it enables them to learn more about the electorate before committing to a policy position and because a candidate announcing a position in the primaries will be more vulnerable to an unconstrained opponent in the general election.

Our model is also related to the theoretical literature on valence advantage/candidate quality (see e.g. Ansolabehere and Snyder (2000), Groseclose (2001) and Aragones and Palfrey (2002, 2005)). In these models one candidate has an advantage which makes all voters prefer him over the other candidate if there is policy convergence. In our model a candidate can have an advantage due to voters' attitudes. But he can only make use of that advantage by being ambiguous which makes voters who like him assimilate his position. Voters' attitudes does not directly influence their voting behavior.

3 The Model

Our starting point is a standard one-dimensional spatial model with two candidates. We will extend that model by allowing candidates to take ambiguous policy positions and by introducing projection effects in voters' perceptions of such positions. In the following we describe the model in detail.

3.1 The Candidates

Before the election the two candidates announce policy positions. Each candidate can announce either a certain position or an ambiguous position. A certain position is represented by a point in the policy space $\mathbb{R}$. An ambiguous position is represented by a compact interval of policies. Thus the strategy space for each candidate can be written as

$$S = \{[A - a, A + a] | A \in \mathbb{R}, a \geq 0\}.$$

Announced positions are credible in the sense that the winning candidate must enact a policy in his announced interval. So certain positions are credible in the usual sense.

Each candidate's only objective is to win the election, none of them care about policy. Formally the preference relation of each candidate over the outcome of the election is given by

$$win \succ tie \succ loose.$$

Finally we assume that the candidates are fully informed about the electorate and that this is common knowledge.

3.2 The Electorate

There is a continuum of voters and each of them has a preferred point in the policy space $\mathbb{R}$. The distribution of preferred points is given by a density function v. We assume that v is continuous and that the support of v is an interval (bounded or unbounded). Without loss of generality we assume that the median voter is located at $x = 0$, i.e.

$$\int_{-\infty}^{0} v(x)dx = \int_{0}^{\infty} v(x)dx = \frac{1}{2}.$$

Each voter has a utility function on the policy space. Let the utility function of the median voter be $u_0 : \mathbb{R} \to \mathbb{R}$. Then the utility function u_{x_0} of a voter with preferred point at x_0 is defined by

$$u_{x_0}(x) = u_0(x - x_0) \quad \text{for all} \quad x \in \mathbb{R}.$$

We assume that u_0 is symmetric around 0, continuous on $\mathbb{R}$ and twice continuously differentiable on $\mathbb{R} \setminus \{0\}$ with

$$\begin{array}{rcll} u_0'(x) & \gtrless & 0 & \text{for} \quad x \lessgtr 0, \\ u_0''(x) & < & 0 & \text{for} \quad x \neq 0. \end{array}$$

Thus all voters are strictly risk averse.

We will now model how voters decide on which candidate to vote for. If each candidate announces a certain position then each voter simply votes for the candidate announcing the position with the highest utility. If at least one of the candidates announces an ambiguous position then it is less obvious how the voters should decide on who to vote for. We want them to use expected utility. But that is not straightforward since an ambiguous position is represented by an interval of policies rather than a probability distribution over policies. For a voter to use expected utility to evaluate an ambiguous position he has to somehow associate a probability distribution with the interval representing the position. The distribution represents the voter's *perception* of the ambiguous position. Or, to put it differently, the voter's *belief* about which policy the candidate will enact if elected. How voters perceive ambiguous positions is a crucial element of our model and we will use the rest of this section to describe it.

As mentioned in the introduction, the main idea is that a voter's perception of an ambiguous position depends on whether he has a positive, negative or neutral (non-policy related) attitude towards the candidate announcing it. If the voter likes the candidate, i.e. has a positive attitude towards him, then he will put most of the probability mass on the points of the interval that are closest to his preferred policy (*assimilation*). If the voter dislikes the candidate then he will do the opposite (*contrast*). And if the voter neither likes nor dislikes the candidate then he will spread the probability mass evenly across the interval. We will formalize this below.

For all voters the neutral perception of an ambiguous position is given by the uniform distribution on the interval. So the *perceived expected utility* of the ambiguous position $[A-a, A+a]$ for a neutral voter with preferred point x_0 is

$$\frac{1}{2a} \int_{A-a}^{A+a} u_{x_0}(x) dx.$$

Since voters are strictly risk averse it follows that neutral voters dislike ambiguity.

To model assimilation perceptions first consider a voter with preferred point $x_0 \geq 1$ and a positive attitude towards a candidate announcing $[-1, 1]$. Thus we are modelling assimilation from the right of an ambiguous position centered at the median. The probability distribution that the voter associates with the ambiguous position is given by some cumulative distribution function F_1^1. So the voter's perceived expected utility of the candidate's position is

$$\int_{-\infty}^{\infty} u_{x_0}(x)dF_1^1(x).$$

We assume that F_1^1 satisfies the following two conditions. $U_{[-1,1]}$ denotes the cumulative distribution function of the uniform distribution on $[-1,1]$.

- F_1^1 puts no probability mass outside $[-1,1]$, i.e.

$$F_1^1(x) = 0 \text{ for all } x < -1 \text{ and } F_1^1(1) = 1.$$

- F_1^1 strictly first order stochastically dominates the uniform distribution on $[-1,1]$, i.e.

$$\begin{array}{rcl} F_1^1(x) & \leq & U_{[-1,1]}(x) \text{ for all } x \in \mathbb{R} \\ (\text{with ''} & < & \text{'' for some } x). \end{array}$$

The first condition says that there is no "counterfactual perception". Since the winning candidate must enact a policy in his announced interval the voter does not put any probability mass on policies outside the interval. The second condition is a convenient mathematical way of saying that the voter is assimilating the candidate's position from the right. It means that, for any $x \in [-1,1]$, F_1^1 puts at least as much probability mass to the right of x as the uniform distribution does (and strictly more for some x). Thus we see that, relative to the neutral perception given by the uniform distribution, the voter "pulls" probability mass to the right, i.e. towards his own preferred position.

We model assimilation of $[-1,1]$ from the left by symmetry. Therefore the assimilation perception for a voter with preferred point $x_0 \leq -1$ is given by the distribution function F_1^{-1} defined by

$$F_1^{-1}(x) = 1 - \lim_{y \to (-x)^-} F_1^1(y) \text{ for all } x \in \mathbb{R}.$$

Because with this definition we have that, for any $x \in [-1,1]$, F_1^{-1} puts exactly as much probability mass on $[-1,x]$ as F_1^1 puts on $[-x,1]$.

Then consider assimilation of $[-1,1]$ by some "interior voter", i.e. a voter with preferred point $x_0 \in (-1,1)$. The distribution function representing the perception of such a voter is denoted $F_1^{x_0}$. Again we assume that

$$F_1^{x_0}(x) = 0 \text{ for all } x < -1 \text{ and } F_1^{x_0}(1) = 1$$

to rule out counterfactual perception. Formalizing the pulling of probability mass towards x_0 is a bit more tricky in this case than it was for "exterior voters", but the idea is the same. We want the distribution function to put more probability mass on points close to x_0 than the uniform distribution does. More precisely we assume that, for any $x > 0$, $F_1^{x_0}$ puts at least as much probability mass on the interval $(x_0 - x, x_0 + x)$ as the uniform distribution does (and strictly more for some x). This can be written formally as

$$\begin{aligned}\lim_{y \to x^-} F_1^{x_0}(x_0 + y) - F_1^{x_0}(x_0 - x) \quad &\geq \quad U_{[-1,1]}(x_0 + x) - U_{[-1,1]}(x_0 - x) \text{ for all } x > 0 \\ \text{(with ''} \quad &> \quad \text{'' for some } x).\end{aligned}$$

To have symmetry of perceptions we assume that, for any $x_0 \in (-1, 1)$,

$$F_1^{-x_0}(x) = 1 - \lim_{y \to (-x)^-} F_1^{x_0}(y) \text{ for all } x \in \mathbb{R}.$$

Thus we are done modelling how voters assimilate the ambiguous position $[-1, 1]$. Now we will extend the model to cover assimilation of all ambiguous strategies.

First consider assimilation of $[-a, a]$ for some $a > 0$. In this case we use a simple scaling of the distribution functions defining assimilation of $[-1, 1]$. More specifically the assimilation perceptions of voters with $x_0 \geq 0$ are defined as follows (assuming symmetry this is all we need).

- For a voter with $x_0 \geq a$ the distribution function is denoted F_a^a. It is defined by
$$F_a^a(x) = F_1^1(\frac{x}{a}) \text{ for all } x \in \mathbb{R}.$$
- For a voter with $0 \leq x_0 < a$ the distribution function is denoted $F_a^{x_0}$. It is defined by
$$F_a^{x_0}(x) = F_1^{\frac{x_0}{a}}(\frac{x}{a}) \text{ for all } x \in \mathbb{R}.$$

It is easily seen that these distribution functions satisfy conditions that are analogous to the ones we imposed on the $F_1^{x_0}$'s. Counterfactual perception is ruled out because we have, for any $0 \leq x_0 \leq a$,

$$F_a^{x_0}(x) = 0 \text{ for all } x < -a \text{ and } F_a^{x_0}(a) = 1.$$

And the $F_a^{x_0}$'s are assimilation perceptions because

$$\begin{aligned}F_a^a(x) \quad &\leq \quad U_{[-a,a]}(x) \text{ for all } x \in \mathbb{R} \\ \text{(with ''} \quad &< \quad \text{'' for some } x)\end{aligned}$$

and, for any $0 \leq x_0 < a$,

$$\lim_{y \to x^-} F_a^{x_0}(x_0 + y) - F_a^{x_0}(x_0 - x) \geq U_{[-a,a]}(x_0 + x) - U_{[-a,a]}(x_0 - x) \text{ for all } x > 0$$
$$(\text{with }" > " \text{ for some } x).$$

Finally we will define assimilation of intervals of the type $[A - a, A + a]$ where $A \neq 0$. In that case we simply translate the distribution functions defining assimilation of $[-a, a]$ by the constant A. For example, the assimilation perception of $[A - a, A + a]$ by a voter with $x_0 \geq A + a$ is given by the density function $F_{A,a}^{A+a}$ defined by

$$F_{A,a}^{A+a}(x) = F_a^a(x - A).$$

Obviously the translated distribution functions satisfy the "translated" versions of the conditions satisfied by the $F_a^{x_0}$'s.

Contrast perceptions are defined analogously to the way we have just defined assimilation perceptions. So, for example, the distribution function representing the contrast perception of voters to the right of some interval is strictly first order stochastically *dominated by* the uniform distribution on the interval. We use the same notation for contrast perceptions as for assimilation perceptions except that we replace the F's by G's. So the distribution functions representing contrast perception of $[-1, 1]$ by voters to the right of the interval is denoted G_1^1 and so on.

For simplicity we assume that voters with the same preferred policy have the same attitude towards each candidate. Therefore we can define the attitude functions L_i, $i = 1, 2$, by

$$L_i(x) = \left\{ \begin{array}{c} 1 \text{ if voters at } x \text{ have a positive attitude towards Candidate } i \\ 0 \text{ if voters at } x \text{ have a neutral attitude towards Candidate } i \\ -1 \text{ if voters at } x \text{ have a negative attitude towards Candidate } i \end{array} \right\}.$$

We make the technical assumption that, for each $i = 1, 2$, the sets $L_i^{-1}(\{1\})$, $L_i^{-1}(\{0\})$ and $L_i^{-1}(\{-1\})$ are Lebesgue measurable (such that we can integrate over them). Then the fraction of voters that have a positive/neutral/negative towards Candidate i is

$$\int_{L_i^{-1}(\{1\})} v(x)dx \quad / \quad \int_{L_i^{-1}(\{0\})} v(x)dx \quad / \quad \int_{L_i^{-1}(\{-1\})} v(x)dx.$$

Before we move on we will give some examples of our model of assimilation. We will get back to these examples later on.

3.2.1 Example 1

In our first example, the assimilation of $[-1,1]$ by voters to the right is given by the distribution function F_1^1 defined by

$$F_1^1(x) = \left\{ \begin{array}{cl} \frac{1-\delta}{2}x + \frac{1-\delta}{2} & \text{if } x \in [-1,1) \\ 1 & \text{if } x = 1 \end{array} \right\},$$

where $0 < \delta < 1$ is a parameter. This corresponds to a voter to the right believing that with probability δ the policy will be $x = 1$ (the policy in the interval closest to his preferred point) and with probability $1 - \delta$ the policy will be drawn from the uniform distribution on $[-1,1]$.

We want a voter with preferred point $x_0 \in (-1,1)$ to have the same type of perception, i.e. to believe that with probability δ the policy will be $x = x_0$ and with probability $1-\delta$ the policy will be drawn from the uniform distribution. The distribution function corresponding to this perception is

$$F_1^{x_0}(x) = \left\{ \begin{array}{ll} \frac{1-\delta}{2}x - \frac{1-\delta}{2} & \text{if } x \in [-1,x_0) \\ \frac{1-\delta}{2}x + \frac{1+\delta}{2} & \text{if } x \in [x_0,1] \end{array} \right\}.$$

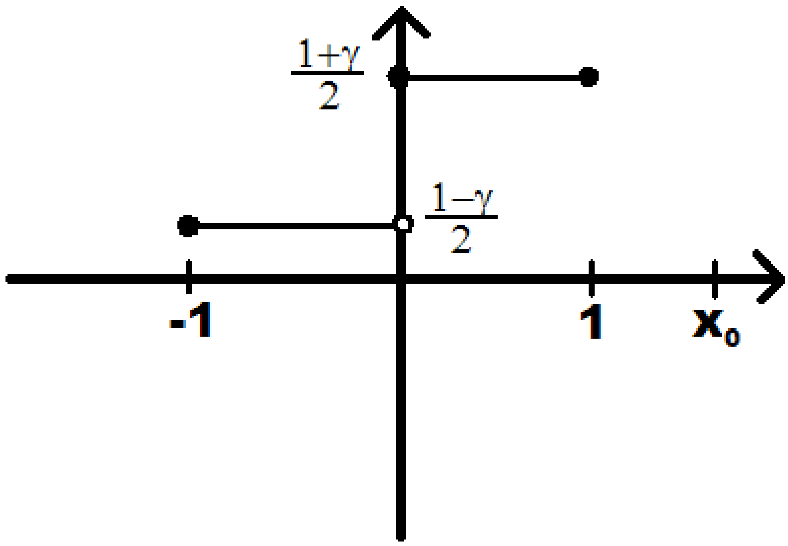
It is easily seen that these distribution functions satisfy the required conditions. The example is extended to cover assimilation of all ambiguous positions as described above (by symmetry, scaling and translation).

3.2.2 Example 2

In our second example, the assimilation of $[-1,1]$ by voters to the right is given by the *density function* f_1^1 defined by

$$f_1^1(x) = \left\{ \begin{array}{ll} \frac{1-\gamma}{2} & \text{if } x \in [-1,0) \\ \frac{1+\gamma}{2} & \text{if } x \in [0,1] \end{array} \right\},$$

where $0 < \gamma \leq 1$ is a parameter.



f_1^1 for some $0 < \gamma < 1$.

We see how a voter to the right of the interval "pulls" probability mass towards his own preferred point. And we see that he does so without putting any probability mass on points outside the interval.

It is easily seen that the distribution function corresponding to the density function above is

$$F_1^1(x) = \left\{ \begin{array}{l} \frac{1-\gamma}{2}x + \frac{1-\gamma}{2} \text{ if } x \in [-1, 0) \\ \frac{1+\gamma}{2}x + \frac{1-\gamma}{2} \text{ if } x \in [0, 1] \end{array} \right\}.$$

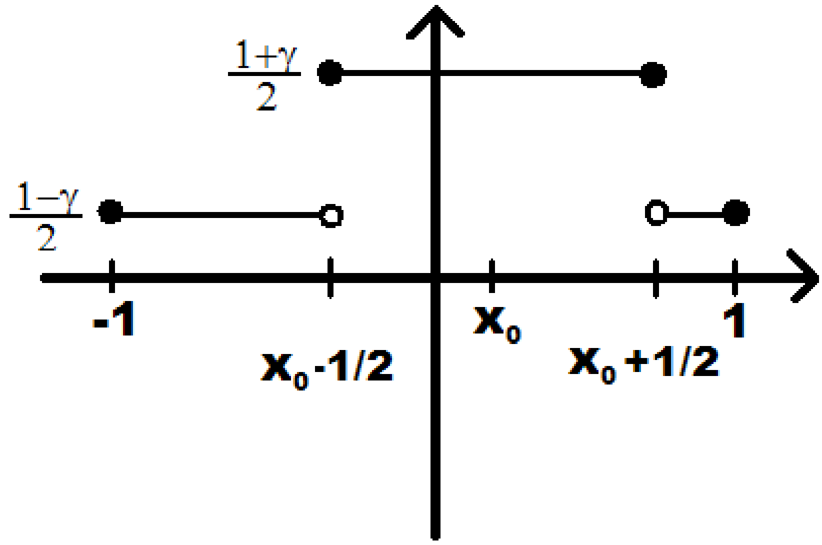
And then it is straightforward to show that the distribution strictly first order stochastically dominates the uniform distribution on $[-1, 1]$.

The assimilation of $[-1, 1]$ by voters with $x_0 \in [0, 1)$ is given by the density functions $f_1^{x_0}$, $x_0 \in [0, 1)$, defined by

$$f_1^{x_0} = f_1^1 \quad \text{if} \quad \frac{1}{2} \leq x_0 < 1$$

and

$$f_1^{x_0}(x) = \left\{ \begin{array}{l} \frac{1-\gamma}{2} \text{ if } |x_0 - x| > \frac{1}{2} \\ \frac{1+\gamma}{2} \text{ if } |x_0 - x| \leq \frac{1}{2} \end{array} \right\} \quad \text{if} \quad 0 \leq x_0 < \frac{1}{2}.$$



$f_1^{x_0}$ for some $x_0 \in (0, \frac{1}{2})$, $0 < \gamma < 1$.

Given our definition of assimilation for "exterior voters" this is a natural way of defining it for "interior voters". In each case a voter puts a constant probability density of $\frac{1+\gamma}{2}$ on the half of the interval that is closest to his preferred point and a constant probability density of $\frac{1-\gamma}{2}$ on the rest.

It is straightforward to check that the distribution functions given by the $f_1^{x_0}$'s satisfy the required conditions. As with our first example it is extended to cover assimilation of all ambiguous positions as described above.

4 Results

In this section we will address the questions in the list below. Our answers will help us understand how and why our introduction of ambiguous positions and projection effects changes the predictions of the standard model.

1. Under which conditions does the median voter theorem break down because candidates can take advantage of the assimilation effect by being ambiguous? More specifically, when can a candidate win the election by taking an ambiguous position when the other candidate's position is fixed at the median?

2. Under which conditions does a candidate with an advantage due to voters' non-policy related attitudes have a winning strategy? I.e. when does such a candidate have an ambiguous position that wins the election for him no matter what position the other candidate takes?

3. What can we say about existence and properties of Nash equilibria?

All proofs are delegated to the Appendix.

4.1 Defeating the Median

Our first result shows that a candidate who is liked by a strict majority of voters can defeat the median, i.e. win the election when the other candidate's position is fixed at the median. Furthermore it shows that he can do so by being ambiguous around the median. Note that no additional assumptions on voter utility functions or voter perceptions are needed. The result holds even if voters are very risk averse and the assimilation effect is very small.

Theorem 4.1 *Suppose Candidate i is liked by a strict majority of voters, i.e.*

$$\int_{L_i^{-1}(\{1\})} v(x)dx > \frac{1}{2}.$$

Then there exists some $a' > 0$ such that, for any $0 < a \leq a'$, Candidate i defeats the median by announcing the ambiguous position $[-a, a]$.

Being liked by a strict majority of voters is a necessary condition for a candidate to be able to defeat the median by an ambiguous position of the type $[-a, a]$, $a > 0$. That follows immediately from the assumption that all voters are strictly risk averse. However, the following result shows that a candidate who is liked by less than a majority may be able to defeat the median by taking an ambiguous position that is not centrered at the median. The sets X_i^+ and X_i^-, $i = 1, 2$, are defined by

$$X_i^+ = \{x \geq 0 | L_i(x) = 0\} \cup \{x | L_i(x) = 1\}$$

and

$$X_i^- = \{x \leq 0 | L_i(x) = 0\} \cup \{x | L_i(x) = 1\}.$$

We let E_1^1 denote the expected value of F_1^1.

Theorem 4.2 *Suppose X_i^+ contains the preferred points of a strict majority of voters, i.e.*

$$\int_{X_i^+} v(x)dx > \frac{1}{2}.$$

Let $A \in (0, E_1^1)$. Then there exists some $a' > 0$ such that, for any $0 < a \leq a'$, Candidate i defeats the median by announcing the ambiguous position

$$[aA - a, aA + a].$$

By symmetry it follows that if a strict majority of voters have preferred points in X_i^- and we let $A \in (-E_1^1, 0)$ then the same conlusion hold. Thus we see that a candidate who is liked by just a few voters may be able to defeat the median. For example that is the case if the candidate is not disliked by any voters.

The last result in this section shows that if neither X_i^+ nor X_i^- contains the preferred points of a strict majority of voters then there exist voter perceptions (given by $F_1^{x_0}$ and $G_1^{x_0}$, $x_0 \in [-1,1]$) such that Candidate i cannot defeat the median. Thus we have a necessary and sufficient condition for Candidate i to be able to defeat the median for any type of voter perceptions.

Theorem 4.3 *Suppose that neither X_i^+ nor X_i^- contains the preferred points of a strict majority of voters, i.e.*

$$\int_{X_i^+} v(x)dx \leq \frac{1}{2} \text{ and } \int_{X_i^-} v(x)dx \leq \frac{1}{2}.$$

Then there exist $F_1^{x_0}$ and $G_1^{x_0}$, $x_0 \in [-1,1]$, (satisfying the assumptions on respectively assimilation and contrast perceptions) such that Candidate i cannot defeat the median.

4.2 Winning Strategies

Our first observation is that Candidate i does not have a winning strategy if the set of voters who like him ($L_i^{-1}(\{1\})$) is not a strict majority. Because in that case Candidate j can always get at least a tie by announcing the midpoint of Candidate i's position. So winning strategies can only exist if at least one candidate is liked by a strict majority of voters. Here we will only consider the case where one candidate (Candidate 1) is not disliked by any voters and liked by a strict majority and the other candidate (Candidate 2) is not liked by any voters. Thus we have

$$L_1(x) \geq 0 \text{ for all } x \in \mathbb{R}, \qquad \int_{L_i^{-1}(\{1\})} v(x)dx > \frac{1}{2},$$

and

$$L_2(x) \leq 0 \text{ for all } x \in \mathbb{R}.$$

Assuming that Candidate 2 is not liked by any voters means that checking if some position of Candidate 1 is a winning strategy becomes a lot simpler. Because in that case we only need to check if Candidate 2 can get at least a tie against it by taking a certain position.

We will make some additional assumptions on voter utility functions and assimilation perceptions. We assume that voter utility functions are of the form

$$u_0(x) = -|x|^\alpha \text{ for some } \alpha > 1.$$

With respect to assimilation perceptions we loosely speaking assume that if two voters have preferred points that are close then their assimilation perceptions are also close. More precisely we assume that for any sequence $(x_n)_{n\in\mathbb{N}}$ in $[-1,1]$ and any continuous function f on $[-1,1]$,

$$x_n \to x_0 \Rightarrow \int_{-1}^{1} f(x)dF_1^{x_n}(x) \to \int_{-1}^{1} f(x)dF_1^{x_0}(x).$$

The following result gives conditions for existence and non-existence of winning strategies for Candidate 1. Remember that E_1^1 denotes the expected value of F_1^1.

Theorem 4.4 *With the additional assumptions from this subsection the following two statements hold.*

1. *Suppose there exists a $\beta > 0$ such that*

$$(-\beta,\beta) \subseteq L_1^{-1}(\{1\})$$

and

$$\int_{-1}^{1} u_{x_0}(x)dF_1^{x_0}(x) > u_{x_0}(E_1^1) \text{ for all } -1 \leq x_0 \leq 0.$$

Then there exists an $a' > 0$ such that, for any $0 < a \leq a'$, $[-a,a]$ is a winning strategy for Candidate 1.

2. *Suppose*

$$u_{x_0}(E_1^1) > \int_{-1}^{1} u_{x_0}(x)dF_1^{x_0}(x) \text{ for all } 0 < x_0 < 1.$$

Then Candidate 1 does not have a winning strategy. More specifically, if Candidate 1 announces the ambiguous position $[A-a, A+a]$ then Candidate 2 can get at least a tie by announcing

$$A - aE_1^1 \text{ if } A \geq 0$$

and

$$A + aE_1^1 \text{ if } A < 0.$$

(And if Candidate 1 announces a certain position then Candidate 2 can get at least a draw by announcing the median.)

The first condition in the first statement says that Candidate 1 is liked by all voters in some neighborhood of the median. The second condition says that if a voter with $x_0 \in [-1,0]$ likes Candidate 1 then he strictly prefers $[-1,1]$ announced by Candidate 1 over the certain position E_1^1. In the second statement the condition says that any voter with $x_0 > 0$ strictly prefers E_1^1 over $[-1, 1]$ announced by Candidate 1.

The following lemma makes it easier to check for non-existence of winning strategies when some additional assumptions on assimilation perceptions are satisfied.

Lemma 4.5 *Suppose that, for all* $0 < x_0 < 1$ *and* $x > 0$*,*

$$\lim_{y \to x^-} F_1^0(y) - F_1^0(-x) \geq \lim_{y \to x^-} F_1^{x_0}(x_0 + y) - F_1^{x_0}(x_0 - x).$$

Furthermore suppose that

$$E_1^{x_0} \leq E_1^1 \text{ for all } 0 < x_0 < 1.$$

Then the condition in part 2. of Theorem 4.4 is satisfied if

$$\int_{-1}^{1} u_0(x) dF_1^0(x) \leq u_0(E_1^1).$$

The new condition on the $F_1^{x_0}$'s says that, for any $x > 0$, F_1^0 puts at least as much probability mass on $(-x, x)$ as $F_1^{x_0}$ puts on $(x_0 - x, x_0 + x)$. It implies that the perceived expected utility of $[-1, 1]$ for a voter at 0 with perception F_1^0 is at least as high as that for a voter at x_0 with perception $F_1^{x_0}$. The second condition says that the mean of the assimilation perception of $[-1, 1]$ for voter a with $x_0 < 1$ is not higher than that for a voter with $x_0 \geq 1$.

We will now use our results to analyze cases where assimilation perceptions are given by the two examples presented earlier.

4.2.1 Example 1

With this type of assimilation we have

$$E_1^1 = (1 - \delta) \int_{-1}^{1} x dx + \delta = \delta.$$

It is straightforward to show that the additional assumptions from this subsection and the assumptions in Lemma 4.5 are satisfied. Thus we see that Candidate 1 does not have a winning strategy if

$$\frac{1 - \delta}{2} \int_{-1}^{1} -|x|^\alpha dx + \delta(-|0|^\alpha) \leq -|\delta|^\alpha.$$

By straightforward calculations this inequality can be reduced to

$$\delta^\alpha + \frac{\delta - 1}{1 + \alpha} \leq 0.$$

For any $\alpha > 1$ the expression on the left hand side is negative for $\delta = 0$, positive for $\delta = 1$ and differentiable (w.r.t. δ) on $[0, 1]$ with positive derivative. Therefore there exists a $\delta^*(\alpha) \in (0, 1)$ such that Candidate 1 does not have a winning strategy if $\delta \leq \delta^*(\alpha)$. We can calculate $\delta^*(2)$ explicitly by solving a second order equation. We get

$$\delta^*(2) = \frac{\sqrt{13} - 1}{6} \approx .43.$$

Our general results does not directly allow us to conclude that Candidate 1 has winning strategies if $\delta > \delta^*(\alpha)$. But we will now show that this is in fact the case (assuming that Candidate 1 is liked by all voters in some neighborhood of the median). For each $-1 \leq x_0 \leq 0$ there exists a unique $C^{x_0} > x_0$ such that

$$\int_{-1}^{1} u_{x_0}(x) dF_1^{x_0}(x) = u_{x_0}(C^{x_0}).$$

(I.e. for a voter at x_0, C^{x_0} is the certainty equivalent of $F_1^{x_0}$ to the right of x_0).

Lemma 4.6 *For each $-1 \leq x_0 \leq 0$ we have $C^{x_0} \leq C^0$.*

It follows from the lemma that

$$\int_{-1}^{1} u_{x_0}(x) dF_1^{x_0}(x) = u_{x_0}(C^{x_0}) \geq u_{x_0}(C^0) \text{ for all } -1 \leq x_0 \leq 0.$$

Since $\delta > \delta^*(\alpha)$ we have $u_0(C^0) \geq u_0(E_1^1)$ and thus $C^0 < E_1^1$. Therefore we have

$$u_{x_0}(C^0) > u_{x_0}(E_1^1) \text{ for all } -1 \leq x_0 \leq 0.$$

And then we can use Theorem 4.4 to conlude that Candidate 1 has winning strategies.

4.2.2 Example 2

With this type of assimilation we have

$$\begin{aligned} E_1^1 &= \int_{-1}^{1} x f_1^1(x) dx \\ &= \frac{1-\gamma}{2} \int_{-1}^{-\frac{1}{2}} x dx + \frac{1+\gamma}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} x dx + \frac{1-\gamma}{2} \int_{\frac{1}{2}}^{1} x dx \\ &= \frac{\gamma}{2}. \end{aligned}$$

Again it is straightforward to show that the additional assumptions from this subsection and the assumptions in Lemma 4.5 are satisfied. Thus we see that Candidate 1 does not have a winning strategy if

$$\frac{1-\gamma}{2}\int_{-1}^{-\frac{1}{2}} -|x|^\alpha dx + \frac{1+\gamma}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} -|x|^\alpha dx + \frac{1-\gamma}{2}\int_{\frac{1}{2}}^{1} -|x|^\alpha dx \leq -|\frac{\gamma}{2}|^\alpha.$$

By straightforward calculations this inequality can be reduced to

$$(\frac{\gamma}{2})^\alpha + \frac{\gamma}{1+\alpha}(1-(\frac{1}{2})^\alpha) - \frac{1}{1+\alpha} \leq 0.$$

For any $\alpha > 1$ the expression on the left hand side is negative for $\gamma = 0$, positive for $\gamma = 1$ and differentiable (w.r.t. γ) on $[0, 1]$ with positive derivative. Therefore there exists a $\gamma^*(\alpha) \in (0, 1)$ such that Candidate 1 does not have a winning strategy if $\gamma \leq \gamma^*(\alpha)$. We can calculate $\gamma^*(2)$ explicitly by solving a second order equation. We get

$$\gamma^*(2) = \frac{\sqrt{57}-3}{6} \approx .76.$$

Our general results does not directly allow us to conclude that Candidate 1 has winning strategies if $\gamma > \gamma^*(\alpha)$. But we will now show that this is in fact the case (assuming that Candidate 1 is liked by all voters in some neighborhood of the median). As in Example 1 it suffices to show that $C^{x_0} \leq C^0$ for all $-1 \leq x_0 \leq 0$. That follows from the two lemmas below.

Lemma 4.7 *$C^{x_0} \leq C^0$ for all $-\frac{1}{2} \leq x_0 \leq 0$.*

Lemma 4.8 *$C^{x_0} \leq C^{-\frac{1}{2}}$ for all $-1 \leq x_0 \leq -\frac{1}{2}$.*

4.3 Nash Equilibria

4.3.1 Pure Strategies

Here we will make some observations about the existence and properties of pure strategy Nash equilibria.

First, consider the case where neither candidate can defeat the median. Then $(s_1^*, s_2^*) = (0, 0)$, i.e. convergence to the median, is an equilibrium because neither candidate can win the election by deviating to another position. Furthermore, if neither of the candidates are liked by exactly 50% of the voters then $(0, 0)$ is the unique equilibrium (each candidate can then defeat any position different from the median). Thus the median voter theorem holds in this situation. If a candidate is liked by exactly 50% of the voters (and thus disliked by the other 50%

- otherwise he could defeat the median) then there could exist equilibria where he is ambiguous. But the outcome would always be a tie and he could also get a tie by deviating to the median.

Secondly, consider the case where one candidate (Candidate 1) can defeat the median but the other (Candidate 2) cannot. If Candidate 2 is liked by strictly less than 50% of the voters then Candidate 1 can defeat any position of Candidate 2. Therefore we have that in any equilibrium Candidate 1 must win the election. Thus (s_1^*, s_2^*) is an equilibrium if and only if s_1^* is a winning strategy for Candidate 1 (which must be ambiguous). So we have existence of Nash equilibria if and only if we have existence of winning strategies for Candidate 1. If Candidate 2 is liked by exactly 50% of the voters then the equilibrium outcome is a tie if Candidate 2 has a position that gives him at least a tie against any position of Candidate 1. If Candidate 2 does not have such a position then we again have that in any equilibrium Candidate 1 must win the election. In both situations it follows that in any equilibrium at least one candidate will be ambiguous.

Finally, consider the case where both candidates can defeat the median. Without loss of generality assume that Candidate 1 is liked by at least as many voters as Candidate 2. If Candidate 2 is liked by strictly less than 50% of the voters then we have that (s_1^*, s_2^*) is an equilibrium if and only if s_1^* is a winning strategy for Candidate 1 (Candidate 1 can defeat any position of Candidate 2). If Candidate 2 is liked by at least 50% of the voters then consider the numbers

$$P_i = \int_{\{x|L_i(x)>L_j(x)\}} v(x)dx, \quad i = 1, 2, \quad j \neq i.$$

P_i is the share of voters who has a more positive attitude towards Candidate i than towards Candidate j. If $P_i > P_j$ then Candidate i can defeat any position of Candidate j by imitation. Thus it follows that (s_1^*, s_2^*) is an equilibrium if and only if s_i^* is a winning strategy for Candidate i. If $P_i = P_j$ then each candidate can get a tie against any position of the other candidate (again by imitation). Thus any equilibrium outcome must be a tie and both candidates must be ambiguous in equilibrium.

4.3.2 Mixed Strategies

We have just seen that when at least one candidate can defeat the median, then, except perhaps for some very special cases, pure strategy Nash equilibria only exist if one candidate has a winning strategy. And we know from earlier that winning strategies do not always exist even in the extreme case where one candidate is liked by all voters and the other candidate is disliked by all voters. So it is natural to ask the question if mixed strategy equilibria exist when pure strategy equilibria do not.

Consider again the model with the extra assumptions from the subsection on winning strategies. The question then is if there exists a mixed strategy equilibrium when Candidate 1 does not have a winning strategy. The result below shows that when the condition for non-existence of winning strategies in Theorem 4.4 is satisfied then the answer is no (except perhaps in a knife edge case).

Theorem 4.9 *Suppose all the assumptions from the subsection on winning strategies are satisfied. If*

$$u_{x_0}(E_1^1) > \int_{-1}^{1} u_{x_0}(x) dF_1^{x_0}(x) \text{ for all } 0 \leq x_0 < 1$$

then there does not exist a mixed strategy Nash equilibrium.

With respect to our examples of assimilation, the theorem implies that if $\delta < \delta^*(\alpha)$ (in Example 1) or $\gamma < \gamma^*(\alpha)$ (in Example 2) then a mixed strategy equilibrium does not exist.

5 Discussion

Our goal in this paper has been to theoretically investigate if positive projection (assimilation) of policy positions can explain why some politicians are ambiguous with respect to their issue positions. To do that we have extended the standard Downsian model of electoral competition by allowing candidates to take ambiguous policy positions and by introducing projection effects in voters' perceptions of such positions. By assuming that voters dislike ambiguity per se (if attitudes are neutral then voters dislike ambiguity because of risk aversion) we have made sure that projection is the driving force behind any result predicting ambiguity.

In the standard model the median defeats any other position. Therefore a natural first step was to find out under which conditions a candidate can defeat the median by being ambiguous. We presented necessary and sufficient conditions. We saw that it may suffice for a candidate to be liked by only a small group of voters as long as he is not disliked by too many. For example, if a candidate is not disliked by any voters then he can defeat the median if he is liked by an arbitrarily small group of voters.

While having at least one candidate that is able to defeat the median by being ambiguous is certainly a necessary condition for predicting ambiguity it is far from sufficient. Electoral competition is a strategic situation and there is no reason to assume that one candidate will announce the median if he can do better by taking a different position. Therefore our next step was to look at existence and non-existence of winning strategies (which must be ambiguous). We restricted attention to the case where one candidate is not liked by any voters and made some additional assumptions on voter utility functions and perceptions. We saw that if the assimilation effect is sufficiently strong then a candidate who is not disliked by any voters and liked by a strict majority has winning strategies. But even if he is liked by all voters he may not have winning strategies. So our model does predict that a generally liked candidate will be ambiguous if he is running against a candidate who is generally not liked and the assimilation effect is sufficiently strong. But even a candidate with the largest possible advantage due to voters' non-policy related attitudes may not have a position that wins the election for him (if the assimilation effect is not strong enough). That is a rather striking result.

While the possible non-existence of winning strategies even for a candidate with the largest possible advantage due to voters' attitudes is an interesting feature of the model it is also problematic. Because, as our observations on Nash equilibria revealed, when at least one candidate can defeat the median then non-existence of winning strategies implies non-existence of pure strategy equilibria (except perhaps in some very special cases). So when the median voter theorem breaks down but neither candidate has a winning strategy then our model does not give us clear predictions in terms of pure strategy equilibria. And we have also seen that turning attention to mixed strategy equilibria is not a solution to this problem. So an obvious direction for further research is to come up with a model that give better equilibrium predictions.

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7 Appendix

Proof of Theorem 4.1.

For each $n \in \mathbb{N}$ let

$$X_i^n = \{x | L_i(x) = 1, \frac{1}{n} \leq |x| \leq n\}.$$

Since a strict majority of voters likes Candidate i there exists an $N \in \mathbb{N}$ such that X_i^N contains the preferred points of a strict majority of voters. We will prove that there exists an $a' > 0$ such that, for any $0 < a \leq a'$, all voters with $x_0 \in X_i^N$ strictly prefer $[-a, a]$ announced by Candidate i over the median ($x = 0$).

For each $a > 0$,

$$\max_{-a \leq y \leq a} |u''_{x_0}(y)|$$

is a continuous function of x_0 on (a, ∞). So for $a < \frac{1}{N}$ it follows by compactness that the function is bounded on $[\frac{1}{N}, N]$. Thus we can define

$$C_a = \max_{\frac{1}{N} \leq x_0 \leq N} \max_{-a \leq y \leq a} |u''_{x_0}(y)|.$$

Now let $\frac{1}{N} \leq x_0 \leq N$, $0 < a < \frac{1}{N}$ and $x \in [-a, a]$. Then, by Taylor's theorem, we have

$$u_{x_0}(x) = u_{x_0}(0) + u'_{x_0}(0)x + \frac{u''_{x_0}(\xi)}{2}x^2.$$

for some $\xi \in [-a, a]$ (actually between 0 and x). And thus it follows that

$$u_{x_0}(x) \geq u_{x_0}(0) + u'_{x_0}(0)x - \frac{C_a}{2}x^2.$$

Using this inequality we get (E_1^1 denotes the expected value of F_1^1)

$$\begin{aligned}
\int_{-a}^{a} u_{x_0}(x) dF_a^a(x) &\geq \int_{-a}^{a} (u_{x_0}(0) + u'_{x_0}(0)x - \frac{C_a}{2}x^2) dF_a^a(x) \\
&= u_{x_0}(0) + u'_{x_0}(0) \int_{-a}^{a} x dF_a^a(x) - \frac{C_a}{2} \int_{-a}^{a} x^2 dF_a^a(x) \\
&= u_{x_0}(0) + u'_{x_0}(0) a E_1^1 - \frac{C_a}{2} \int_{-a}^{a} x^2 dF_a^a(x) \\
&\geq u_{x_0}(0) + u'_{x_0}(0) a E_1^1 - \frac{C_a}{2} a^2 \int_{-a}^{a} dF_a^a(x) \\
&= u_{x_0}(0) + u'_{x_0}(0) a E_1^1 - \frac{C_a}{2} a^2 \\
&\geq u_{x_0}(0) + u'_0(-\frac{1}{N}) a E_1^1 - \frac{C_a}{2} a^2.
\end{aligned}$$

Since $u_0'(-\frac{1}{N}) > 0$, $E_1^1 > 0$ and C_a is decreasing with a it follows that, for a sufficiently small,

$$\int_{-a}^{a} u_{x_0}(x) dF_a^a(x) > u_{x_0}(0) \text{ for all } \frac{1}{N} \leq x_0 \leq N$$

So all voters with $\frac{1}{N} \leq x_0 \leq N$ and $L_i(x_0) = 1$ strictly prefer $[-a, a]$ announced by Candidate i over the median for a sufficiently small. By symmetry the same holds for voters with $-N \leq x_0 \leq -\frac{1}{N}$ and $L_i(x_0) = 1$. Thus it holds for all voters with $x_0 \in X_i^N$. □

Proof of Theorem 4.2.

Let $A \in (0, E_1^1)$. Pick an $N \in \mathbb{N}$ such that a majority of voters have preferred points in

$$X_i^{N+} = \{x | x \in X_i^+, \frac{1}{N} \leq |x| \leq N\}.$$

For $a > 0$ with $aA + a < \frac{1}{N}$ we can define

$$C_{A,a} = \max_{\frac{1}{N} \leq |x_0| \leq N} \max_{aA - a \leq y \leq aA + a} |u_{x_0}''(y)|.$$

Then, by Taylor's theorem, we get that for all x_0 with $\frac{1}{N} \leq |x_0| \leq N$ and all $a > 0$ with $aA + a < \frac{1}{N}$,

$$u_{x_0}(x) \geq u_{x_0}(0) + u_{x_0}'(0)x - \frac{C_{A,a}}{2}x^2 \quad \text{for all} \quad x \in [aA - a, aA + a].$$

For all voters with $x_0 \in X_i^+$, $\frac{1}{N} \leq x_0 \leq N$ the perceived expected utility of $[aA - a, aA + a]$ is at least

$$\frac{1}{2a} \int_{aA-a}^{aA+a} u_{x_0}(x) dx.$$

Using the inequality above we get that, for all $a > 0$ with $aA + a < \frac{1}{N}$,

$$\begin{aligned}
\frac{1}{2a} \int_{aA-a}^{aA+a} u_{x_0}(x) dx &\geq \int_{aA-a}^{aA+a} (u_{x_0}(0) + u_{x_0}'(0)x - \frac{C_{A,a}}{2}x^2) dx \\
&= u_{x_0}(0) + u_{x_0}'(0)aA - \frac{C_{A,a}}{2} \int_{aA-a}^{aA+a} x^2 dx \\
&\geq u_{x_0}(0) + u_{x_0}'(0)aA - \frac{C_{A,a}}{2}(aA + a)^2 \\
&\geq u_{x_0}(0) + u_0'(-\frac{1}{N})Aa - \frac{C_{A,a}}{2}(1 + A)^2 a^2.
\end{aligned}$$

The last expression is strictly greater than $u_{x_0}(0)$ for small $a > 0$. So it follows that for a sufficiently small all voters with $x_0 \in X_i^+$, $\frac{1}{N} \le x_0 \le N$ strictly prefer $[aA - a, aA + a]$ over the median.

For all voters with $x_0 \in X_i^+$, $-N \le x_0 \le -\frac{1}{N}$ the perceived expected utility of $[aA - a, aA + a]$ (with $-\frac{1}{N} < aA - a$) is

$$\int_{aA-a}^{aA+a} u_{x_0}(x) dF_{aA,a}^{aA-a}(x).$$

Using the "Taylor inequality" from above we get that, for all $a > 0$ with $aA + a < \frac{1}{N}$,

$$\begin{aligned}
\int_{aA-a}^{aA+a} u_{x_0}(x) dF_{aA,a}^{aA-a}(x) &\ge \int_{aA-a}^{aA+a} (u_{x_0}(0) + u'_{x_0}(0)x - \frac{C_{A,a}}{2}x^2) dF_{aA,a}^{aA-a}(x) \\
&\ge u_{x_0}(0) + u'_{x_0}(0)(aA - aE_1^1) - \frac{C_{A,a}}{2}(aA + a)^2 \\
&\ge u_{x_0}(0) + u'_0(\frac{1}{N})(A - E_1^1)a - \frac{C_{A,a}}{2}(1 + A)^2 a^2.
\end{aligned}$$

The last expression is is strictly greater than $u_{x_0}(0)$ for small $a > 0$. So it follows that for a sufficiently small all voters with $x_0 \in X_i^+$, $-N \le x_0 \le -\frac{1}{N}$ strictly prefer $[aA - a, aA + a]$ over the median.

Thus we have seen that for $a > 0$ sufficiently small, all voters with $x_0 \in X_i^{N+}$ (a majority) prefer $[aA - a, aA + a]$ over the median. □

Proof of Theorem 4.3.

Suppose that assimilation perceptions are as in Example 1 (for some $0 < \delta < 1$). Let contrast perceptions be given by

$$G_1^{x_0} = \left\{ \begin{array}{c} F_1^1 \text{ if } x_0 \in [-1, 0) \\ F_1^{-1} \text{ if } x_0 \in [0, 1] \end{array} \right\}.$$

Suppose Candidate i announces some interval. We will show that at least 50% of the voters strictly prefer the median over the interval.

The interval can be written as

$$[aA - a, aA + a] \text{ for some } A \in \mathbb{R}, a > 0.$$

For $A = 0$ all voters with $L_i(x_0) \le 0$ (at least 50%) strictly prefer the median (by risk aversion).

For $A \ge E_1^1$ it is easily seen that

$$E(F_{aA,a}^{x_0}) \ge aA - aE_1^1 \ge 0 \text{ for all } x_0 \in [aA - a, aA + a].$$

And then it follows by risk aversion that all voters with $x_0 \leq 0$ strictly prefer the median over the interval. Analogously it follows that if $A \leq -E_1^1$ then all voters with $x_0 \geq 0$ prefer strictly the median over the interval.

Thus the only cases left are $A \in (0, E_1^1)$ and $A \in (-E_1^1, 0)$. Suppose $A \in (0, E_1^1)$. Then it is straightforward to check that for each voter with

$$x_0 \in \{x \geq 0 | L_i(x) = -1\} \cup \{x < 0 | L_i(x) \leq 0\}$$

the mean of the voter's perception of the interval is further away from x_0 than the median (0). Thus all these voters strictly prefer the median over the interval (by risk aversion). Since the set above is the complement of X_i^+ these voters constitute at least a weak majority. If $A \in (-E_1^1, 0)$ then it follows analogously that all voters with $x_0 \notin X_i^-$ strictly prefer the median over the interval. Again that is at least a weak majority. □

Proof of Theorem 4.4.

1. First we show that

$$\int_{-1}^{1} u_{x_0}(x) dF_1^{x_0}(x)$$

is a continuous function of x_0 on $[-1, 1]$. Let $x_0 \in [-1, 1]$ and et (x_n) be a sequence in $[-1, 1]$ such that $x_n \to x_0$. Then we have

$$\begin{aligned}
& |\int_{-1}^{1} u_{x_n}(x) dF_1^{x_n}(x) - \int_{-1}^{1} u_{x_0}(x) dF_1^{x_0}(x)| \\
\leq & |\int_{-1}^{1} u_{x_n}(x) dF_1^{x_n}(x) - \int_{-1}^{1} u_{x_0}(x) dF_1^{x_n}(x)| \\
& + |\int_{-1}^{1} u_{x_0}(x) dF_1^{x_n}(x) - \int_{-1}^{1} u_{x_0}(x) dF_1^{x_0}(x)| \\
\leq & \max_{x \in [-1,1]} |u_{x_n}(x) - u_{x_0}(x)| \\
& + |\int_{-1}^{1} u_{x_0}(x) dF_1^{x_n}(x) - \int_{-1}^{1} u_{x_0}(x) dF_1^{x_0}(x)|.
\end{aligned}$$

The first term in the last expression converges to zero because of the continuity of u_0 and the compactness of $[-1, 1]$. The second term converges to zero by the continuity assumption on the $F_1^{x_0}$'s. That proves the continuity of $\int_{-1}^{1} u_{x_0}(x) dF_1^{x_0}(x)$.

Then we can find a $0 < C < E_1^1$ and an $\varepsilon > 0$ such that

$$\int_{-1}^{1} u_{x_0}(x) dF_1^{x_0}(x) > u_{x_0}(C) \text{ for all } x_0 \in [-1, \varepsilon].$$

The following two claims finishes the proof of the first part of the theorem.

Claim A: For sufficiently small $a > 0$, $[-a, a]$ announced by Candidate 1 defeats any $y \in (-aC, aC)$ announced by Candidate 2.

Proof: Choose an $N \in \mathbb{N}$ such that a strict majority of voters have preferred points in the set

$$X_1^N = \{x | L_1(x) = 1, \frac{1}{N} \leq |x| \leq N\}.$$

It suffices to show that for a sufficiently small all voters with $x_0 \in X_1^N$, $x_0 > 0$ will prefer $[-a, a]$ announced by Candidate 1 over aC announced by Candidate 2. Since $0 < C < E_1^1$ that follows by Taylor's theorem as in earlier proofs.

Claim B: For sufficiently small $a > 0$, $[-a, a]$ announced by Candidate 1 defeats any $y \notin (-aC, aC)$ announced by Candidate 2.

Proof: Let $a < \beta$. From the homogeneity (of degree α) of u_0 and what we have shown above it follows that for any $x_0 \in [-a, a\varepsilon]$,

$$\begin{aligned} \int_{-a}^{a} u_{x_0}(x) dF_a^{x_0}(x) &= \int_{-1}^{1} u_{x_0}(ax) dF_1^{x_0}(ax) = a^\alpha \int_{-1}^{1} u_{\frac{x_0}{a}}(x) dF_1^{\frac{x_0}{a}}(x) \\ &> a^\alpha u_{\frac{x_0}{a}}(C) = u_{x_0}(aC). \end{aligned}$$

Thus we see that all voters with $x_0 \in [-a, a\varepsilon]$ strictly prefer $[-a, a]$ announced by Candidate 1 over any certain position $y \geq aC$.

Voters with $x_0 \in (-\beta, -a)$ has the same perception of $[-a, a]$ announced by Candidate 1 as voters with $x_0 = -a$. And they are less (absolute) risk-averse on $[-a, a]$. Therefore these voters also prefer $[-a, a]$ announced by Candidate 1 over any $y \geq aC$.

By Taylor's theorem it is easily seen that voters with $x_0 = -\beta$ and a neutral attitude towards Candidate 1 strictly prefer $[-a, a]$ announced by Candidate 1 over any $y \geq aC$ when a is small enough. The same is true for voters with $x_0 < -\beta$ because they are less risk averse on $[-a, a]$.

Thus all voters with $x_0 \leq a\varepsilon$ strictly prefer $[-a, a]$ announced by Candidate 1 to any $y \geq aC$ for a sufficiently small. By symmetry it follows that, for a sufficiently small, all voters with $x_0 \geq -a\varepsilon$ stricly prefer $[-a, a]$ announced by Candidate 1 to any $y \leq -aC$. That ends the proof of the claim.

2. Suppose Candidate 1 announces $[A - a, A + a]$ for some $A \geq 0$ (if $A < 0$ the proof is analogous). Then, by the homogeneity of u_0 and the condition in the statement, we have that all voters with $x_0 \in (A - a, A)$ strictly prefer $A - aE_1^1$ announced by Candidate 2 over the interval announced by Candidate 1. By risk aversion the same is true for voters with $x_0 \leq A - a$. Thus at least 50% of the voters strictly prefer $A - aE_1^1$ announced by Candidate 2 over the interval announced by Candidate 1. $\square$

Proof of Lemma 4.5.

Suppose

$$\int_{-1}^{1} u_0(x)dF_1^0(x) \le u_0(E_1^1).$$

By the first assumption in the lemma it follows that, for all $0 < x_0 < 1$,

$$\int_{-1}^{1} u_{x_0}(x)dF_1^{x_0}(x) \le \int_{-1}^{1} u_0(x)dF_1^0(x).$$

Therefore we have that, for all $x_0 \in (0, E_1^1]$,

$$\int_{-1}^{1} u_{x_0}(x)dF_1^{x_0}(x) \le \int_{-1}^{1} u_0(x)dF_1^0(x) \le u_0(E_1^1) < u_{x_0}(E_1^1).$$

For $x_0 \in (E_1^1, 1)$ it follows from the second assumption in the lemma that E_1^1 is (weakly) closer to x_0 than $E_1^{x_0}$. So by risk aversion it follows that, for all $x_0 \in (E_1^1, 1)$,

$$\int_{-1}^{1} u_{x_0}(x)dF_1^{x_0}(x) < u_{x_0}(E_1^{x_0}) \le u_{x_0}(E_1^1).$$

Thus the condition in part *2.* of Theorem 4.4 is satisfied. □

Proof of Lemma 4.6.

For each $x_0 \in [-1, 0]$ let $C_U^{x_0} > x_0$ be defined by

$$\frac{1}{2}\int_{-1}^{1} u_{x_0}(x)dx = u_{x_0}(C_U^{x_0}).$$

If $C_U^{x_0} \le C_U^0$ then we have

$$\delta u_{x_0}(0) + (1-\delta)u_{x_0}(C_U^0) \le \delta u_{x_0}(x_0) + (1-\delta)u_{x_0}(C_U^{x_0}) = u_{x_0}(C^{x_0}).$$

So the certainty equivalent (the one to the right of the preferred point) of the lottery "0 with probability δ, C_U^0 with probability $1-\delta$" for a voter at x_0 is greater than or equal to C^{x_0}. The certainty equivalent of the same lottery for a voter at 0 is C^0. And since voters at x_0 are less absolute risk averse on $[0, C_U^0]$ than voters at 0 the certainty equivalent for a voter at 0 is greater than that for a voter at x_0. Thus we must have $C^{x_0} \le C^0$. So we see that it suffices to show that $C_U^{x_0} \le C_U^0$ for all $x_0 \in [-1, 0)$.

Let $x_0 \in [-1,0)$. Define $C_{U-}^{x_0}, C_{U+}^{x_0} > x_0$ and $C_{U-}^{0}, C_{U+}^{0} > 0$ by

$$\frac{1}{2+x_0}\int_{-1}^{x_0+1} u_{x_0}(x)dx = u_{x_0}(C_{U-}^{x_0}), \quad \frac{1}{-x_0}\int_{x_0+1}^{1} u_{x_0}(x)dx = u_{x_0}(C_{U+}^{x_0})$$

and

$$\frac{1}{2+x_0}\int_{-1}^{x_0+1} u_0(x)dx = u_0(C_{U-}^{0}), \quad \frac{1}{-x_0}\int_{x_0+1}^{1} u_0(x)dx = u_0(C_{U+}^{0}).$$

It is easily seen that

$$C_{U-}^{x_0} \leq C_{U-}^{0} \text{ and } C_{U+}^{x_0} \leq C_{U+}^{0}.$$

Thus we have that

$$\frac{2+x_0}{2}u_{x_0}(C_{U-}^{0}) + \frac{-x_0}{2}u_{x_0}(C_{U+}^{0}) \leq \frac{2+x_0}{2}u_{x_0}(C_{U-}^{x_0}) + \frac{-x_0}{2}u_{x_0}(C_{U+}^{x_0}) = u_{x_0}(C_U^{x_0}).$$

So the certainty equivalent (the one to the right of the preferred point) of the lottery "C_{U-}^{0} with probability $\frac{2+x_0}{2}$, C_{U+}^{0} with probability $\frac{-x_0}{2}$" for a voter at x_0 is greater than or equal to $C_U^{x_0}$. The certainty equivalent of the same lottery for a voter at 0 is C_U^0. And since voters at x_0 are less absolute risk averse on $[C_{U-}^{0}, C_{U+}^{0}]$ than voters at 0 the certainty equivalent for a voter at 0 is greater than that for a voter at x_0. Thus we must have $C_U^{x_0} \leq C_U^0$. □

Proof of Lemma 4.7.

For each $x_0 \in [-1,0]$ let $C_U^{x_0} > x_0$ be defined an in the proof of lemma *4.6*. Furthermore let $0 < D^0 < C_U^0$ be defined by

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u_0(x)dx = u_0(D^0).$$

From the proof of Lemma 4.6 we know that $C_U^{x_0} \leq C_U^0$ for all $x_0 \in [-1,0]$. Therefore we have, for each $x_0 \in [-\frac{1}{2}, 0]$,

$$\gamma u_{x_0}(D^0) + (1-\gamma)u_{x_0}(C_U^0) \leq \gamma \int_{x_0-\frac{1}{2}}^{x_0+\frac{1}{2}} u_{x_0}(x)dx + (1-\gamma)u_{x_0}(C_U^{x_0}) = u_{x_0}(C^{x_0}).$$

So the certainty equivalent (the one to the right of the preferred point) of the lottery "0 with probability γ, C_U^0 with probability $1-\gamma$" for a voter at x_0 is greater than or equal to C^{x_0}. The certainty of the same lottery for a voter at 0 is C^0. And since voters at x_0 are less risk averse on $[D^0, C_U^0]$ than voters at 0 the certainty equivalent for a voter at 0 is greater than that for a voter at x_0. Thus we must have $C^{x_0} \leq C^0$. $\Box$

Proof of Lemma 4.8.

For each $x_0 \in [-1, -\frac{1}{2}]$ the assimilation perception of $[-1, 1]$ is a convex combination of the uniform distribution on $[-1, 0]$ and the uniform distribution on $[-1, 1]$. Let $C_U^{x_0}$ be defined as in the proof of lemma *4.6*. Furthermore let $D^{x_0} > x_0$ be defined by

$$\int_{-1}^{0} u_{x_0}(x)dx = u_{x_0}(D^{x_0}).$$

Mimicking arguments from the proof of Lemma 4.6 it follows that $C_U^{x_0} \leq C_U^{-\frac{1}{2}}$ and $D^{x_0} \leq D^{-\frac{1}{2}}$ for all $x_0 \in [-1, -\frac{1}{2}]$. Therefore we have that for each $x_0 \in [-1, -\frac{1}{2}]$,

$$\gamma u_{x_0}(D^{-\frac{1}{2}}) + (1-\gamma)u_{x_0}(C_U^{-\frac{1}{2}}) \leq \gamma u_{x_0}(D^{x_0}) + (1-\gamma)u_{x_0}(C_U^{x_0}) = u_{x_0}(C^{x_0}).$$

And since a voter at $x_0 < -\frac{1}{2}$ is less risk averse than a voter at $-\frac{1}{2}$ on $[D^{-\frac{1}{2}}, C_U^{-\frac{1}{2}}]$ it follows that $C^{x_0} \leq C^{-\frac{1}{2}}$for all $x_0 \in [-1, -\frac{1}{2}]$ (same argument as in the two previous proofs). $\Box$

Proof of Theorem 4.9.

Suppose (Δ_1, Δ_2) is a mixed strategy Nash equilibrium. We will show that this leads to a contradiction.

Let $\varepsilon > 0$. Then there exists $a' > 0$ such that Δ_2 puts less than ε probability mass on the certain positions in $(-a', a') \setminus \{0\}$. Therefore, by announcing the ambiguous position $[-a, a]$ for a sufficiently close to zero Candidate 1 can defeat Δ_2 with probability greater than $1 - \varepsilon$ (Candidate 1 defeats all certain positions not in $(-a', a') \setminus \{0\}$). Thus it follows that in the equilibrium (Δ_1, Δ_2), Candidate 1 must win with probability one.

Pick $\bar{A} \in \mathbb{R}$, $\bar{a} > 0$ such that, for any neighborhood B of $(\bar{A}, \bar{a})$, Δ_1 puts positive probability on

$$\{[A - a, A + a] | (A, a) \in B\}.$$

The following claim shows that for some B Candidate 2 has a position that defeats all positions in the set above. Thus it follows that in the equilibrium (Δ_1, Δ_2), Candidate 2 must win with some positive probability. That is a contradiction.

Claim: Suppose Candidate 2 announces

$$\left\{ \begin{array}{l} \bar{A} + \bar{a}E_1^1 \text{ if } \bar{A} \leq 0 \\ \bar{A} - \bar{a}E_1^1 \text{ if } \bar{A} > 0 \end{array} \right\}.$$

Then there exists some neighborhood B of $(\bar{A}, \bar{a})$ such that he defeats all positions of Candidate 1 in the set

$$\{[A - a, A + a] | (A, a) \in B\}.$$

Proof: We will only do the proof for $\bar{A} \leq 0$, the other case is completely analogous. By announcing $\bar{A}+\bar{a}E_1^1$ Candidate 2 defeats $[\bar{A}-\bar{a}, \bar{A}+\bar{a}]$ (by Theorem 4.4, part *2.*). We have to show that he also defeats "nearby" ambiguous positions. Suppose not. Then there exists a sequence (A_n, a_n) that converges to $(\bar{A}, \bar{a})$ and satisfies that, for any n, $[A_n - a_n, A_n + a_n]$ announced by Candidate 1 gets at least a tie against $\bar{A} + \bar{a}E_1^1$ announced by Candidate 2. Therefore, there must exist x_0's such that

$$\int u_{x_0}(x) dF^{x_0}_{A_n,a_n}(x) \nrightarrow \int u_{x_0}(x) dF^{x_0}_{\bar{A},\bar{a}}(x).$$

(If $x_0 < A_n - a_n$ then $F^{x_0}_{A_n,a_n}$ means $F^{A_n-a_n}_{A_n,a_n}$ and so on). By using the definitions of $F^{x_0}_{A_n,a_n}$ and $F^{x_0}_{\bar{A},\bar{a}}$ and the continuity assumption on the $F_1^{x_0}$'s in the section on winning strategies it follows that there is convergence for all x_0's. Thus we have a contradiction. $\square$

Elections, Private Information, and State-Dependent Candidate Quality

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Abstract

In this paper we contribute to the study of how democracy works when politicians are better informed than the electorate about conditions relevant for policy choice. We do so by setting up and analyzing a game theoretic model of electoral competition. An important feature of the model is that candidate quality is state-dependent. Our main insight is that if the electorate is sufficiently well informed then there exists an equilibrium where the candidates' policy positions reveal their information and the policy outcome is the same as it would be if voters were fully informed (the median policy in the true state of the world).

Keywords: Electoral Competition, Uncertainty, Private Information, Candidate Quality, Revealing Equilibria.

1 Introduction

It is a reasonable assumption that politicians are generally better informed than the electorate about conditions relevant for policy choice. They usually have staff to help them receive and process information and sometimes have access to information that is not public, for example information related to national security. Furthermore they have much stronger incentives than voters to be well informed because their carreers depend on how they do as policy makers. In this paper we contribute to the study of how democracy works when politicians are better informed than voters. We do so by setting up and analyzing a game theoretic model of electoral competition.

We consider an election with two candidates and one issue. The candidates are purely office-motivated, i.e. their only objective is to maximize the probability of winning. Before the election the candidates announce credible policy positions. There are two states of the world. Both candidates are informed about the true state when they announce their positions. Voters are only partially informed about the state, they receive a signal that is correlated with the true state. This signal is private information, i.e. it is unknown to the candidates when they announce positions. Each voter has a single peaked policy utility function in each state and the preferred policy is different in the two states.

The voters do not only care about policy, they also care about candidate quality. One candidate has a quality advantage in one state and the other candidate has a quality advantage in the other state. Furthermore there is a stochastic element in voter evaluation of candidates. Suppose for example that the two candidates have announced the same position and that the voters have inferred the true state. Then the candidate with a quality advantage wins with a probability that is greater than one half (because of the quality advantage) but smaller than one (because of the stochastic element of voter evaluation).

A revealing equilibrium is one where at least one of the candidates announces different policies in the two states. Thus voters can infer the true state. Our first main result is that in any such equilibrium (satisfying a known refinement condition) the candidates converge to the median position of the true state. Our second main result is that a revealing equilibrium exists when the electorate is sufficiently well informed about the state of the world. So when voters are sufficiently well informed then it is at least a possibility (there could exist non-revealing equilibria) that electoral competition works as if the voters were fully informed.

Our first result on non-revealing equilibria show that many of these exist. Furthermore, we see that no matter how well informed the electorate is there always exists a non-revealing equilibrium, even with a symmetry restriction. None of these equilibria can be eliminated by the Intuitive Criterion (Cho and Kreps (1987)) which is the most commonly used refinement condition in signalling games. Instead we show that a monotonicity condition on voters' beliefs does eliminate many of the non-revealing equilibria. With that condition a non-revealing equilibrium only exists if the voters are not too well informed. We also see that the candidates diverge by at least the distance between the medians in the two states.

Before we move on we will present a stylized example of a real world situation where our model applies. Suppose a retired general is running against a succesful governor for the US presidency. Both of them primarily care about getting elected, policy preferences are secondary. The main issue is how much of a fixed tax revenue to spend on national security related public goods (e.g. military services, anti-terrorism, a missile defense system). The rest of the budget is spend on other public goods (e.g. health care, education, infrastructure). The candidates know more about the security threat to the country than the voters because they get national security briefings while voters only get information from the media. When the threat is high then each voter wants to spend more on security related public goods than when it is low. Thus the median preferred level of national secuity spending is higher when the threat is high. Furthermore, when the security threat is high then the general has a quality advantage (national security issues are more important) and when the threat is low then the governor has a quality advantage (domestic issues are more important). The possibility of unforeseen events, scandals, campaign mistakes etc. makes voting stochastic.

The paper is organized as follows. In Section 2 we review related literature. Then, in Section 3 and 4, we set up the model and define our notion of equilibrium. Section 5 and 6 contain our results on revealing and non-revealing equilibria. Finally we discuss and conclude in Section 7.

2 Related Literature

The two most immediately related papers are Schultz (1996) and Martinelli (2001). They ask the same general question as we do but they both assume that candidates are policy-motivated. This is fundamentally different from our assumption about completely office-motivated candidates. In Schultz (1996) candidates are fully informed about the state of the economy while voters are uninformed. Thus voters only receive information from the candidates' credible positions. There is revelation in (refined) equilibrium if at least one of the candidates have policy preferences that are sufficiently similar to the preferences of the median voter. In any revealing equilibrium there is convergence to the median policy of the true state of the world.

Martinelli (2001) considers a model where both candidates and voters receive private information about the state of the world but candidates are better informed than voters. The main result is that a revealing equilibrium always exists. This depends crucially on the assumption that voters have private information. The candidates do not converge in revealing equilibria.

Several other papers study models in which politicians are better informed than the electorate. In both Alesina and Cukierman (1990) and Harrington (1993) policy is decided after the election and voters are uncertain about the candidates' policy preferences. Therefore earlier policy decisions by the incumbent reveal information to the voters about what he will do if reelected. That induces the incumbent (who wants to be reelected) to distort his policy choice. In Alesina and Cukierman he does so by choosing a noisy policy instrument, in Harrington it is done by choosing a policy that is more likely to be well received.

Roemer (1994) considers a model where two policy motivated candidates (parties) are better informed about how the economy works than the electorate. Candidates announce both policies and theories of the economy, voters update their beliefs based only on announced theories. In equilibrium there is convergence to the median with respect to policy but divergence with respect to theory.

In Cukierman and Tommasi (1998) the incumbent is better informed than voters about how different policies map into outcomes. Voters update beliefs based on the incumbents (credible) policy announcement and votes for reelection if his announcement is preferred to the expected policy of the challenger. The main insight is that relatively extreme right wing policies are more likely to be implemented by a left wing incumbent (and vice versa) because of credibility issues.

Our model is also related to the literature on candidate quality/valence advantage. Recent contributions to this literature are Ansolabehere and Snyder (2000), Groseclose (2001) and Aragones and Palfrey (2002, 2005). These papers all analyze models of electoral competition where candidates differ in quality such that if they announce sufficiently similar policy positions then each voter votes for the candidate of highest quality. There is no uncertainty about who the high quality candidate is. This is fundamentally different from our paper where no candidate has an a priori quality advantage because quality is state-dependent. We are not aware of other models with uncertainty about candidate quality.

3 The Model

We consider a one issue election. The policy space X is some closed interval (bounded or unbounded) on the real axis. There are two purely office-motivated candidates, i.e. their only objective is to maximize the probability of winning.

The electorate consists of a continuum of voters (indexed by i). The voters have utility functions over the policy space. The utility functions depend on the state of the world ω which can be either L or H. The utility function of voter i is

$$u_i(x|\omega) = -|x - x_i^*(\omega)|, \quad x \in X, \omega \in \{L, H\},$$

where $x_i^*(\omega)$ is the preferred policy of voter i in state ω. The preferred policies of the voters in each state are distributed according to some distribution functions F_L, F_H. In each state there are unique median positions, i.e. unique $x_{m_L}^*, x_{m_H}^* \in X$ such that

$$F_L(x_{m_L}^*) = F_H(x_{m_H}^*) = \frac{1}{2}.$$

We assume that the median is further to the right in state H than in state L, i.e.

$$x_{m_L}^* < x_{m_H}^*.$$

Furthermore, we assume that the ordering of voters by preferred positions is the same in the two states. Formally, for all voters i, j,

$$x_i^*(L) \leq x_j^*(L) \iff x_i^*(H) \leq x_j^*(H)$$

This implies that, for all voters i,

$$x_i^*(L) = x_{m_L}^* \iff x_i^*(H) = x_{m_H}^*$$

Thus a voter with preferred policy equal to the median in one state also has preferred policy equal to the median in the other state. So (with only a slight abuse of language) it makes sense to speak about *the* median voter.

The candidates are fully informed about the state of the world. The voters only receive a signal

$$\omega^V \in \{l, h\}.$$

All voters receive the same signal and the signal is unknown to the candidates when they announce positions. The signal is distributed according to

$$\begin{aligned}
\Pr(l|L) &= \Pr(h|H) = \theta, \\
\Pr(l|H) &= \Pr(h|L) = 1 - \theta,
\end{aligned}$$

where $\theta \in (\frac{1}{2}, 1)$ is a parameter. Each voter has the prior $\Pr(L) = \Pr(H) = \frac{1}{2}$. So if voters update based on their signal then their belief is given by

$$\begin{aligned}
\Pr(L|l) &= \Pr(H|h) = \theta, \\
\Pr(H|l) &= \Pr(L|h) = 1 - \theta.
\end{aligned}$$

Candidate quality is state-dependent. One candidate ("Candidate L") has a quality advantage in the L state while the other candidate ("Candidate H") has a quality advantage in the H state. Furthermore there is a symmetric stochastic element to each voter's candidate preference. These two features are modelled the following way. Suppose Candidate L has announced the policy x^L and that

Candidate H has announced x^H. Then voter i's utility of voting for Candidate L is

$$U_i^L(x^L|\omega) = u_i(x^L|\omega) + \Gamma_L(\omega) + \delta,$$

where, for some parameter $\gamma > 0$,

$$\Gamma_L(\omega) = \left\{ \begin{array}{c} \gamma \text{ if } \omega = L \\ 0 \text{ if } \omega = H \end{array} \right\}$$

and, for some parameter $\sigma > 0$, δ is drawn from a uniform distribution on the interval $[-\frac{1}{2\sigma}, \frac{1}{2\sigma}]$. Note that the realized value of δ is the same for all voters. Voter i's utility of voting for Candidate H is

$$U_i^H(x^H|\omega) = u_i(x^H|\omega) + \Gamma_H(\omega).$$

where

$$\Gamma_H(\omega) = \left\{ \begin{array}{c} 0 \text{ if } \omega = L \\ \gamma \text{ if } \omega = H \end{array} \right\}.$$

Each voter votes for the candidate giving him the highest expected utility based on his belief about the state of the world. So if voter i believes that the probability of state L is μ_L then he votes for Candidate L if

$$\mu_L(u_i(x^L|L)+\gamma+\delta)+(1-\mu_L)(u_i(x^L|H)+\delta) > \mu_L u_i(x^H|L)+(1-\mu_L)(u_i(x^H|H)+\gamma).$$

This is equivalent to

$$\delta > \mu_L(u_i(x^H|L) - u_i(x^L|L) - \gamma) + (1 - \mu_L)(u_i(x^H|H) - u_i(x^L|H) + \gamma).$$

If we plug in the policy utility function of the voter then this inequality becomes

$$\delta > \mu_L(|x^L - x_i^*(L)| - |x^H - x_i^*(L)| - \gamma) + (1 - \mu_L)((|x^L - x_i^*(H)| - |x^H - x_i^*(H)| + \gamma).$$

The timeline of the election game is as follows:

1. The candidates observe the state of the world and then simultaneously announce policy positions.

2. The voters observe the candidates' positions and receive a signal about the state of the world. The value of δ is realized. The voters cast their votes.

3. The winning candidate enacts his announced position (positions are credible).

4 Equilibrium

A strategy profile for the candidates consists of a policy announcement in each state of the world for each candidate and can therefore be written

$$(x^L(L), x^L(H)), (x^H(L), x^H(H)).$$

The belief functions of the voters depend on the two candidates' announcements and the voters' signal. We make the assumption that all voters have the same belief function. The voters' belief about the probability of state L is written

$$\mu_L(x^L, x^H, \omega^V).$$

Each candidate's objective is to maximize the probability of winning in each state given the other candidate's strategy, the belief function of the voters, the distribution of the voters' signal and the distribution of δ. The following lemma shows that the median voter decides the election.

Lemma 4.1 *Suppose that, given the candidates' announcements, the voters' signal, and the realization of δ, the median voter strictly prefers Candidate L (H). Then a strict majority of voters prefers Candidate L (H).*

Proof. Suppose the median voter strictly prefers Candidate L, i.e.

$$\delta > \mu_L(|x^L - x^*_{m_L}| - |x^H - x^*_{m_L}| - \gamma) + (1 - \mu_L)(((|x^L - x^*_{m_H}| - |x^H - x^*_{m_H}| + \gamma).$$

We then have to show that for each voter i in a strict majority,

$$\delta > \mu_L(|x^L - x^*_i(L)| - |x^H - x^*_i(L)| - \gamma) + (1 - \mu_L)(((|x^L - x^*_i(H)| - |x^H - x^*_i(H)| + \gamma).$$

Suppose $x^L \leq x^H$ (the other case is analogous). It then suffices to show that the inequality above holds for all voters i with $x^*_i(L) \leq x^*_{m_L}$ (that is only a weak majority but a simple continuity argument shows that the inequality also holds for voters with a preferred point slightly to the right of the median).

Pick a voter i with $x^*_i(L) \leq x^*_{m_L}$. The inequality is satisfied if

$$|x^L - x^*_i(L)| - |x^H - x^*_i(L)| \leq |x^L - x^*_{m_L}| - |x^H - x^*_{m_L}|$$

and

$$|x^L - x^*_i(H)| - |x^H - x^*_i(H)| \leq |x^L - x^*_{m_H}| - |x^H - x^*_{m_H}|.$$

These inequalities are straightforward to verify.

The proof of the statement when the median voter strictly prefers Candidate H is analogous.$\square$

By the lemma we see that if $(x^H(L), x^H(H))$ is the strategy of Candidate H then the problem of Candidate L in state ω is

$$\max_x \Pr_{\delta,(\omega^V|\omega)}[\delta > \mu_L(x, x^H(\omega), \omega^V)((|x - x^*_{m_L}| - |x^H(\omega) - x^*_{m_L}| - \gamma) +$$
$$(1 - \mu_L(x, x^H(\omega), \omega^V))(|x - x^*_{m_H}| - |x^H(\omega) - x^*_{m_H}| + \gamma)].$$

And if $(x^L(L), x^L(H))$ is the strategy of Candidate L then the problem of Candidate H in state ω is

$$\max_x \Pr_{\delta,(\omega^V|\omega)}[\delta < \mu_L(x^L(\omega), x, \omega^V)((|x^L(\omega) - x^*_{m_L}| - |x - x^*_{m_L}| - \gamma) +$$
$$(1 - \mu_L(x^L(\omega), x, \omega^V))(|x^L(\omega) - x^*_{m_H}| - |x - x^*_{m_H}| + \gamma)].$$

Then we are ready to define our notion of equilibrium. It is that of Perfect Bayesian Equilibrium with the extra condition that all voters have the same belief function.

Definition 4.2 (Equilibrium) *An equilibrium consists of candidate strategies*

$$(\hat{x}^L(L), \hat{x}^L(H)), (\hat{x}^H(L), \hat{x}^H(H)),$$

and a voter belief function about the probability of state L

$$\hat{\mu}_L(x^L, x^H, \omega^V)$$

such that

1. *In each state each candidate's announcement maximizes his probability of winning given the other candidates announcement, the belief function of the voter, the distribution of the voter's signal and the distribution of δ;*

2. *The belief function is consistent with Bayes' rule on the equilibrium path. I.e. if $\hat{x}^L(L) \neq \hat{x}^L(H)$ or $\hat{x}^H(L) \neq \hat{x}^H(H)$ then*

$$\begin{aligned} \hat{\mu}_L(\hat{x}^L(L), \hat{x}^H(L), l) &= \hat{\mu}_L(\hat{x}^L(L), \hat{x}^H(L), h) = 1, \\ \hat{\mu}_L(\hat{x}^L(H), \hat{x}^H(H), l) &= \hat{\mu}_L(\hat{x}^L(H), \hat{x}^H(H), h) = 0. \end{aligned}$$

   And if $\hat{x}^L(L) = \hat{x}^L(H)$ and $\hat{x}^H(L) = \hat{x}^H(H)$ then

$$\hat{\mu}_L(\hat{x}^L(L), \hat{x}^H(L), \omega^V) = \hat{\mu}_L(\hat{x}^L(H), \hat{x}^H(H), \omega^V) = \left\{ \begin{array}{cc} \theta & \textit{if } \omega^V = l \\ 1 - \theta & \textit{if } \omega^V = h \end{array} \right\}.$$

An equilibrium where the announcements of the candidates reveal the state to the voters, i.e. at least one of the candidates announces different positions in the two states, is called a *revealing equilibrium.* An equilibrium where each candidate announces the same position in both states is called a *non-revealing equilibrium.*

5 Revealing Equilibria

We will first introduce a refinement condition that puts restrictions on out-of-equilibrium beliefs in revealing equilibria. It has been used by Schultz (1996) in a similar setting. The content of the condition is that if one candidate's strategy reveals the state (i.e. he takes different positions in the two states) and the other candidate deviates to an out-of-equilibrium position then the voters believe the non-deviating candidate.

Definition 5.1 (Refinement Condition (R1)) *Consider a revealing equilibrium where the candidate strategies are $(\hat{x}^L(L), \hat{x}^L(H))$ and $(\hat{x}^H(L), \hat{x}^H(H))$ and the voter belief function is $\hat{\mu}_L$. It satisfies (R1) if the following two conditions are satisfied.*

1. *Suppose $\hat{x}^L(L) \neq \hat{x}^L(H)$. Then*

$$\begin{aligned} \hat{\mu}_L(\hat{x}^L(L), x, l) &= \hat{\mu}_L(\hat{x}^L(L), x, h) = 1 \textit{ and} \\ \hat{\mu}_L(\hat{x}^L(H), x, l) &= \hat{\mu}_L(\hat{x}^L(H), x, h) = 0 \textit{ for all } x \neq \hat{x}^H(L), \hat{x}^H(H). \end{aligned}$$

2. *Suppose $\hat{x}^H(L) \neq \hat{x}^H(H)$. Then*

$$\begin{aligned} \hat{\mu}_L(x, \hat{x}^H(L), l) &= \hat{\mu}_L(x, \hat{x}^H(L), h) = 1 \textit{ and} \\ \hat{\mu}_L(x, \hat{x}^H(H), l) &= \hat{\mu}_L(x, \hat{x}^H(H), h) = 0 \textit{ for all } x \neq \hat{x}^L(L), \hat{x}^L(H). \end{aligned}$$

Let D denote the distance between the median position in the two states, i.e.

$$D = x^*_{m_H} - x^*_{m_L}.$$

For all of our results in this and the following section we will assume that

$$\gamma + D < \frac{1}{2\sigma}.$$

Suppose for example that each candidate announces the median of the state where he has an advantage. Then the assumption implies that, no matter what the belief of the voters is, both candidates have a positive probability of winning the election. The assumption simplifies our analysis considerably because it ensures that in all situations we need to consider the realization of δ matters.

Our first result shows that in any revealing equilibrium satisfying (R1) the candidates converge to the median position of the true state.

Theorem 5.2 *In any revealing equilibrium satisfying (R1) the candidate strategies are*

$$(\hat{x}^L(L), \hat{x}^L(H)) = (\hat{x}^H(L), \hat{x}^H(H)) = (x^*_{m_L}, x^*_{m_H}).$$

Proof. Let $(\hat{x}^L(L), \hat{x}^L(H)), (\hat{x}^H(L), \hat{x}^H(H))$ be the candidate strategies in a revealing equilibrium satisfying (R1). At least one of the candidates must announce different policies in the two states. Suppose that $\hat{x}^L(L) \neq \hat{x}^L(H)$ (the case $\hat{x}^H(L) \neq \hat{x}^H(H)$ is analogous). If $\hat{x}^H(L) \neq x^*_{m_L}$ then Candidate H can win with a higher probability in state L by deviating to a position $x \neq \hat{x}^H(H)$ that is closer to $x^*_{m_L}$ than $\hat{x}^H(L)$ (by (R1) the voter will still be sure that the state is L). Thus we must have $\hat{x}^H(L) = x^*_{m_L}$. Similarly we get $\hat{x}^H(H) = x^*_{m_H}$. And then we can use the same argument for Candidate L to get $\hat{x}^L(L) = x^*_{m_L}$ and $\hat{x}^L(H) = x^*_{m_H}$.$\square$

Our next step is to find the set of parameter values for which a revealing equilibrium satisfying (R1) exists. The following result shows that there is a cut-off value of θ such that a revealing equilibrium satisfying (R1) exists if and only if the voter signal is at least as informative as this cut-off value.

Theorem 5.3 *There exists a revealing equilibrium satisfying (R1) if and only if*

$$\theta \geq \theta^*_R,$$

where

$$\theta^*_R = \frac{1}{2} + \frac{\gamma}{2(\gamma + D)}.$$

Proof. See the Appendix.

Note that θ^*_R is increasing in γ and

$$\lim_{\gamma \to 0} \theta^*_R = \frac{1}{2}.$$

Thus we see that if the difference-in-quality parameter γ increases then the electorate has to be better informed in order to make the candidates reveal their information. The intuition behind this observation is that the higher γ is the more costly (in terms of probability of winning) it is for the disadvantaged candidate to reveal the state relative to not revealing the state. Therefore, when γ increases the new cut-off value of θ must make it more costly for the disadvantaged candidate not to reveal, i.e. it must be higher. We also see that when the difference in candidate quality vanishes then there exists a revealing equilibrium no matter how little information the electorate has.

Also note that θ^*_R is decreasing in D. So when the median positions of the two states are further apart then a revealing equilibrium exist for less informed electorates. The reason is that a higher D makes it more costly not to reveal relative to revealing for the disadvantaged candidate. The good news from this observation is that when the state of the world really matters for policy choice (i.e.

D is high) then it takes less voter information to make the candidates reveal the true state by converging to the median.

We end this section with a remark on the equilibrium voter belief function used in the proof of Theorem 5.3.

Remark 5.4 *In the proof of Theorem 5.3 the equilibrium belief function satisfies*

$$\hat{\mu}_L(x^*_{m_L}, x^*_{m_H}, l) = 1 \quad \textit{and} \quad \hat{\mu}_L(x^*_{m_L}, x^*_{m_H}, h) = 0.$$

So if the disadvantaged candidate deviates to the median of the false state then voters overinfer from their signal. Suppose we require that voters should instead be Bayesians in this case, i.e. that

$$\hat{\mu}_L(x^*_{m_L}, x^*_{m_H}, l) = \theta \quad \textit{and} \quad \hat{\mu}_L(x^*_{m_L}, x^*_{m_H}, h) = 1 - \theta.$$

Then, by mimicking the proof of Theorem 5.3, we get that a revealing equilibrium satisfying (R1) exists if and only if

$$\theta \geq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\gamma}{\gamma + D}}.$$

Compared to θ^*_R the new cut-off value is strictly larger but has qualitatively the same dependence on γ and D.

6 Non-Revealing Equilibria

We will only consider non-revealing equilibria that are symmetric in the following sense.

Definition 6.1 (Symmetry) *Consider a non-revealing equilibrium where Candidate L announces $\hat{x}^L$ and Candidate H announces $\hat{x}^H$. It is* symmetric *if*

$$|\hat{x}^L - x^*_{m_L}| = |\hat{x}^H - x^*_{m_H}| \quad \textit{and} \quad |\hat{x}^L - x^*_{m_H}| = |\hat{x}^H - x^*_{m_L}|.$$

Note that the symmetry condition is equivalent to

$$\frac{\hat{x}^L + \hat{x}^H}{2} = \frac{x^*_{m_L} + x^*_{m_H}}{2}.$$

Also note that when $\hat{x}^L$ is specified, then so is $\hat{x}^H$.

In the following result we find all possible symmetric non-revealing equilibria and the parameter values for which they exist. Remember that we still make the assumption that $\gamma + D < \frac{1}{2\sigma}$.

Theorem 6.2 *Let $\hat{x}^L$, $\hat{x}^H$ be any pair of policy announcements satisfying the symmetry condition. Then the following statements hold.*

1. *Suppose $|\hat{x}^L - x^*_{m_L}| < \gamma$. Then:*

   (a) *If $x^*_{m_L} \leq \hat{x}^L \leq x^*_{m_H}$ then $\hat{x}^L$, $\hat{x}^H$ are equilibrium announcements if and only if*

   $$\theta \leq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\gamma - (\hat{x}^L - x^*_{m_L})}{\gamma + (\hat{x}^H - \hat{x}^L)}};$$

   (b) *If $\hat{x}^L < x^*_{m_L}$ then $\hat{x}^L$, $\hat{x}^H$ are equilibrium announcements if and only if*

   $$\theta \leq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\gamma - (x^*_{m_L} - \hat{x}^L)}{\gamma + D}};$$

   (c) *If $\hat{x}^L > x^*_{m_H}$ then $\hat{x}^L$, $\hat{x}^H$ are equilibrium announcements if and only if*

   $$\theta \leq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\gamma - (\hat{x}^L - x^*_{m_L})}{\gamma - D}}.$$

2. *Suppose $|\hat{x}^L - x^*_{m_L}| \geq \gamma$. Then $\hat{x}^L$, $\hat{x}^H$ are equilibrium announcements if and only if $\hat{x}^L - \hat{x}^H = \gamma$ and $\gamma \leq D$ (this is independent of the value of θ).*

Proof. See the Appendix.

An immediate consequence of the theorem above is that a symmetric non-revealing equilibrium always exists. If $D < \gamma$ then it follows from statement *1.(a)* that there exists an equilibrium with $\hat{x}^L = x^*_{m_H}$ and $\hat{x}^H = x^*_{m_L}$ for $\theta \leq 1$. If $\gamma \leq D$ then it follows from statement *2.* that, independent of the value of θ, there exists an equilibrium with $\hat{x}^L - \hat{x}^H = \gamma$.

The abundance of non-revealing equilibria (even with the symmetry restriction) makes it natural to ask if some of them can be eliminated by a suitable refinement condition. In signalling games the most commonly used refinement condition is the Intuitive Criterion (Cho and Kreps (1987)). For non-revealing equilibria in our model the Intuitive Criterion puts the following restrictions on out-of-equilibrium beliefs. Consider a non-revealing equilibrium $\hat{x}^L$, $\hat{x}^H$, $\hat{\mu}_L$ and a deviation by Candidate L to some x. Suppose we are allowed to change the out-of-equilibrium beliefs and that by doing so we can make the deviation profitable if and only if the state is L (H). Then we must have

$$\begin{aligned}\hat{\mu}_L(x, \hat{x}^H, l) &= \hat{\mu}_L(x, \hat{x}^H, h) = 1 \\ (\hat{\mu}_L(x, \hat{x}^H, l) &= \hat{\mu}_L(x, \hat{x}^H, h) = 0).\end{aligned}$$

Analogous restrictions are put on the belief function in out-of-equilibrium situations where Candidate H deviates.

Unfortunately, as we will now show, the Intuitive Criterion does not eliminate any of the symmetric non-revealing equilibria of our model.

Theorem 6.3 *All symmetric non-revealing equilibria satisfy the Intuitive Criterion.*

Proof. See the Appendix.

One way to eliminate many of the symmetric non-revealing equilibria is to introduce a monotonicity condition on the voter belief function. The content of the condition is that if one candidate moves to a position that is closer to $x^*_{m_L}$ ($x^*_{m_H}$) but not closer to $x^*_{m_H}$ ($x^*_{m_L}$) then μ_L $(1-\mu_L)$ does not decrease.

Definition 6.4 (Monotonicity Condition (M1)) *A voter belief function μ_L satisfies condition (M1) if the following condition holds. Suppose*

$$|x - x^*_{m_L}| \leq |y - x^*_{m_L}| \quad \textit{and} \quad |x - x^*_{m_H}| \geq |y - x^*_{m_H}|.$$

Then, for all $z \in X$, $\omega^V \in \{l, h\}$,

$$\mu_L(x, z, \omega^V) \geq \mu_L(y, z, \omega^V) \quad \textit{and} \quad \mu_L(z, x, \omega^V) \geq \mu_L(z, y, \omega^V).$$

There is no directly state-dependent cost for the candidates that can justify this condition (because they are purely office-motivated). Nevertheless it does seem appealing for voters to think that if one candidate moves closer to e.g. $x^*_{m_L}$ and not closer to $x^*_{m_H}$ then state L is not less likely to be true.

It is worth noting that only considering equilibria that satisfy (M1) (i.e. the voter belief function satisfies (M1)) does not change our result on existence of revealing equilibria. More precisely the conclusion from Theorem 5.3 still holds if we require revealing equilibria to satisfy (R1) *and* (M1). This is easily seen by checking that the equilibrium belief function used in the proof satisfies (M1).

In the following result we find the candidate announcements that are possible in symmetric non-revealing equilibria satisfying (M1).

Theorem 6.5 *The candidate announcements in any symmetric non-revealing equilibrium satisfying (M1) must satisfy*

$$\hat{x}^L \leq x^*_{m_L} \quad \textit{and} \quad x^*_{m_H} \leq \hat{x}^H.$$

Proof. See the Appendix.

It is easily checked that for $\hat{x}^L$, $\hat{x}^H$ satisfying $\hat{x}^L \leq x^*_{m_L}$ and $x^*_{m_H} \leq \hat{x}^H$ the beliefs used in the proof of Theorem 6.2 satisfy (M1). Therefore we can directly use this theorem to find out when the different symmetric non-revealing equilibria satisfying (M1) exist. The following corollary sums up the most important results.

Corollary 6.6 *There exists a symmetric non-revealing equilibrium satisfying (M1) if and only if*

$$\theta \leq \theta^*_N,$$

where

$$\theta^*_N = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\gamma}{\gamma + D}}.$$

Furthermore, for any $\theta \leq \theta^*_N$ there exists a symmetric non-revealing equilibrium satisfying (M1) with

$$\hat{x}^L = x^*_{m_L} \quad \textit{and} \quad \hat{x}^H = x^*_{m_H}.$$

From Theorem 5.3 and Corollary 6.6 it follows that for all parameter values either a revealing equilibrium satisfying (R1) or a symmetric non-revealing equilibrium satisfying (M1) exists. We also see that for some parameter values both types of equilibria exist.

Corollary 6.7 *For all $\gamma, D, \sigma > 0$ with $\gamma + D < \frac{1}{2\sigma}$ we have that*

$$\theta^*_R = \frac{1}{2} + \frac{\gamma}{2(\gamma + D)} < \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\gamma}{\gamma + D}} = \theta^*_N.$$

So for all parameter values there exists either a revealing equilibrium satisfying (R1) or a symmetric non-revealing equilibrium satisfying (M1). Furthermore, both types of equilibria exist for all θ's in an interval of non-zero length.

Finally note that θ^*_N is equal to the cut-off value for existence of revealing equilibria satisfying (R1) *and* the extra condition from Remark 5.4. So with that extra condition on revealing equilibria we still have the existence result from the corollary, but we only have co-existence of the two types of equilibria when $\theta = \theta^*_N$.

7 Discussion

We have analyzed how electoral competition works under the following conditions:

- Candidates are better informed than voters, but voters have some private information;

- Candidates are purely office-motivated;

- Candidate quality is state-dependent.

Our most important insight was that if the electorate is sufficiently well informed then there exists a revealing equilibrium and the policy outcome of such an equilibrium is the median position in the true state of the world. If the electorate is not sufficiently well informed then only non-revealing equilibria exist and in any such equilibrium there is a possibility that the policy outcome is not the median position in the true state. Thus our analysis emphasizes the importance of voters being well informed. It is important to note that voters do not need to be fully informed for electoral competition to function as if they were fully informed. The result that candidates will reveal the true state only if the electorate is sufficiently well informed could be called "The Matthew Principle of Information": *Those who already have good information shall know the truth, but those who do not shall be lied to*[1].

Another interesting feature of our model is that policy divergence is possible in (non-revealing) equilibrium. Thus we see that candidates being better informed than voters and state-dependent candidate quality can lead to policy divergence even when candidates are purely office-motivated. As far as we know this is a new potential explanation of policy divergence in electoral competition (see e.g. section III in the review paper by Osborne (1995) for other explanations).

8 References

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9 Appendix

Proof of Theorem 5.3.

First we show that $\theta \geq \theta_R^* \Rightarrow$ existence.

Consider a belief function $\hat{\mu}_L$ satisfying

$$\begin{aligned}
\hat{\mu}_L(x^*_{m_L}, x, \omega^V) &= 1 \text{ for all } x \neq x^*_{m_H}, \omega^V = l, h; \\
\hat{\mu}_L(x, x^*_{m_H}, \omega^V) &= 0 \text{ for all } x \neq x^*_{m_L}, \omega^V = l, h; \\
\hat{\mu}_L(x^*_{m_L}, x^*_{m_H}, l) &= 1,\ \mu_L(x^*_{m_L}, x^*_{m_H}, h) = 0; \\
\hat{\mu}_L(x^*_{m_H}, x, \omega^V) &= 0 \text{ for all } x \neq x^*_{m_L}, \omega^V = l, h; \\
\hat{\mu}_L(x, x^*_{m_L}, \omega^V) &= 1 \text{ for all } x \neq x^*_{m_H}, \omega^V = l, h; \\
\hat{\mu}_L(x^*_{m_H}, x^*_{m_L}, l) &= 1,\ \mu_L(x^*_{m_H}, x^*_{m_L}, h) = 0.
\end{aligned}$$

We claim that such a belief function together with the candidate strategies

$$(\hat{x}^L(L), \hat{x}^L(H)) = (\hat{x}^H(L), \hat{x}^H(H)) = (x^*_{m_L}, x^*_{m_H})$$

satisfies the equilibrium conditions and (R1). First note that the belief function satisfies Bayes' rule on the equilibrium path and (R1). Thus we just have to check the optimality of each candidate's strategy. First consider the strategies in state L. Using (R1) it follows that none of the candidates can gain by deviating to a $x \neq x^*_{m_H}$. Thus we just have to check that neither candidate can profitably deviate to $x^*_{m_H}$.

In equilibrium Candidate L wins with probability $\frac{1}{2} + \gamma\sigma$ and Candidate H wins with probability $\frac{1}{2} - \gamma\sigma$. If Candidate L deviates to $x^*_{m_H}$ then his probability of winning is

$$\theta(\frac{1}{2\sigma} + (\gamma - D))\sigma + (1-\theta)(\frac{1}{2\sigma} - (\gamma - D))\sigma = \frac{1}{2} + (2\theta - 1)(\gamma - D)\sigma < \frac{1}{2} + \gamma\sigma.$$

So that is never a profitable deviation. If Candidate H deviates to $x^*_{m_H}$ then his probability of winning is

$$\theta(\frac{1}{2\sigma} - (\gamma + D))\sigma + (1-\theta)(\frac{1}{2\sigma} + (\gamma + D))\sigma = \frac{1}{2} - (2\theta - 1)(\gamma + D)\sigma.$$

Thus the deviation is not profitable if

$$\frac{1}{2} - (2\theta - 1)(\gamma + D)\sigma \leq \frac{1}{2} - \gamma\sigma.$$

This inequality is equivalent to

$$\theta \geq \theta^*_R.$$

By symmetry it follows that if no candidate can gain from any deviation in state L, then that is also the case in state H. Thus the equilibrium conditions are satisfied if $\theta \geq \theta^*_R$.

Finally we show that existence $\Rightarrow \theta \geq \theta^*_R$.

Suppose there exists a revealing equilibrium satisfying (R1). We know from Theorem 5.2 that the candidate strategies must be

$$(\hat{x}^L(L), \hat{x}^L(H)) = (\hat{x}^H(L), \hat{x}^H(H)) = (x^*_{m_L}, x^*_{m_H}).$$

Two necessary conditions for equilibrium are that Candidate H cannot gain by deviating to $x^*_{m_H}$ in state L and that Candidate L cannot gain by deviating to $x^*_{m_L}$ in state H. Let $\hat{\mu}_L$ be the equilibrium belief and define

$$\hat{\mu}^l_L = \hat{\mu}_L(x^*_{m_L}, x^*_{m_H}, l) \text{ and } \hat{\mu}^h_L = \hat{\mu}_L(x^*_{m_L}, x^*_{m_H}, h).$$

Then the necessary conditions can be written

$$\theta(\frac{1}{2\sigma} + \hat{\mu}^l_L(\gamma + D) - (1 - \hat{\mu}^l_L)(\gamma + D))\sigma$$
$$+ (1-\theta)(\frac{1}{2\sigma} + \hat{\mu}^h_L(\gamma + D) - (1 - \hat{\mu}^h_L)(\gamma + D))\sigma \leq \frac{1}{2} - \gamma\sigma$$

and

$$\theta(\frac{1}{2\sigma} + (1-\hat{\mu}_L^h)(\gamma + D) - \hat{\mu}_L^h(\gamma + D))\sigma$$
$$+ (1-\theta)(\frac{1}{2\sigma} + (1-\hat{\mu}_L^l)(\gamma + D) - \hat{\mu}_L^l(\gamma + D))\sigma \leq \frac{1}{2} - \gamma\sigma.$$

Thus it suffices to show that if both the two inequalities above are satisfied then we have $\theta \geq \theta_R^*$. By adding the two inequalities and a bit of algebra we get

$$(\hat{\mu}_L^l - \hat{\mu}_L^h)(2\theta - 1)(\gamma + D) \geq \gamma.$$

Thus we see that $\hat{\mu}_L^l > \hat{\mu}_L^h$ and then it follows that

$$(2\theta - 1)(\gamma + D) \geq \frac{\gamma}{(\hat{\mu}_L^l - \hat{\mu}_L^h)} \geq \gamma.$$

Rearranging this inequality we get $\theta \geq \theta_R^*$.□

Proof of Theorem 6.2.

1.(a). First we show that $\theta \geq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\gamma - (\hat{x}^L - x_{m_L}^*)}{\gamma + (\hat{x}^H - \hat{x}^L)}} \Rightarrow$ existence.

Consider a belief function $\hat{\mu}_L$ satisfying

$$\begin{aligned}
\hat{\mu}_L(\hat{x}^L, \hat{x}^H, l) &= \theta \text{ and } \hat{\mu}_L(\hat{x}^L, \hat{x}^H, h) = 1 - \theta; \\
\hat{\mu}_L(\hat{x}^L, x, \omega^V) &= 1 \text{ for all } x \neq \hat{x}^H, \omega^V = l, h; \\
\hat{\mu}_L(x, \hat{x}^H, \omega^V) &= 0 \text{ for all } x \neq \hat{x}^L, \omega^V = l, h.
\end{aligned}$$

We will show that this belief function supports $\hat{x}^L$, $\hat{x}^H$ as an equilibrium. By symmetry it suffices to show that Candidate H does not have a profitable deviation. It is easily seen that this is the case if deviating to $x_{m_L}^*$ in state L is not profitable. In state L Candidate H's equilibrium probability of winning is

$$\frac{1}{2} - (2\theta - 1)^2(\gamma + (\hat{x}^H - \hat{x}^L))\sigma.$$

By deviating to $x_{m_L}^*$ in state L Candidate H wins with probability

$$\frac{1}{2} - (\gamma - (\hat{x}^L - x_{m_L}^*))\sigma.$$

Thus the deviation is not profitable if

$$\frac{1}{2} - (2\theta - 1)^2(\gamma + (\hat{x}^H - \hat{x}^L))\sigma \geq \frac{1}{2} - (\gamma - (\hat{x}^L - x_{m_L}^*))\sigma.$$

This inequality is satisfied if

$$\theta \geq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\gamma - (\hat{x}^L - x^*_{m_L})}{\gamma + (\hat{x}^H - \hat{x}^L)}}.$$

Then we show that existence $\Rightarrow \theta \geq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\gamma - (\hat{x}^L - x^*_{m_L})}{\gamma + (\hat{x}^H - \hat{x}^L)}}$.

Let $\hat{\mu}_L$ be the equilibrium belief function. Define

$$\hat{\mu}^l_L = \hat{\mu}_L(\hat{x}^L, x^*_{m_L}, l) \quad \text{and} \quad \hat{\mu}^h_L = \hat{\mu}_L(\hat{x}^L, x^*_{m_L}, h).$$

If Candidate H deviates to $x^*_{m_L}$ in state L he wins with probability

$$\frac{1}{2} + \theta(1 - 2\hat{\mu}^l_L)(\gamma - (\hat{x}^L - x^*_{m_L}))\sigma + (1-\theta)(1 - 2\hat{\mu}^h_L)(\gamma - (\hat{x}^L - x^*_{m_L}))\sigma.$$

No candidate can profitably deviate so we must have

$$\frac{1}{2} + \theta(1 - 2\hat{\mu}^l_L)(\gamma - (\hat{x}^L - x^*_{m_L}))\sigma + (1-\theta)(1 - 2\hat{\mu}^h_L)(\gamma - (\hat{x}^L - x^*_{m_L}))\sigma \\ \leq \frac{1}{2} - (2\theta - 1)^2(\gamma + (\hat{x}^H - \hat{x}^L))\sigma.$$

Since the left hand side is decreasing in $\hat{\mu}^l_L$and $\hat{\mu}^h_L$ the inequality still holds if we replace these numbers by 1's, i.e.

$$\frac{1}{2} + (\gamma - (\hat{x}^L - x^*_{m_L}))\sigma \leq \frac{1}{2} - (2\theta - 1)^2(\gamma + (\hat{x}^H - \hat{x}^L))\sigma.$$

From this inequality we easily get

$$\theta \geq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\gamma - (\hat{x}^L - x^*_{m_L})}{\gamma + (\hat{x}^H - \hat{x}^L)}}.$$

1.(b). The proof is analogous to the proof of *1.(a)*.

1.(c). The proof is analogous to the proof of *1.(a)*.

2. First we show that if $\hat{x}^L - \hat{x}^H = \gamma$ and $\gamma \leq D$ then $\hat{x}^L$, $\hat{x}^H$ are equilibrium announcements. Consider a voter belief function $\hat{\mu}_L$ satisfying

$$\begin{aligned}
\hat{\mu}_L(\hat{x}^L, \hat{x}^H, l) &= \theta \text{ and } \hat{\mu}_L(\hat{x}^L, \hat{x}^H, h) = 1 - \theta; \\
\hat{\mu}_L(\hat{x}^L, x, \omega^V) &= 0 \text{ for all } x < \hat{x}^H, \omega^V = l, h; \\
\hat{\mu}_L(\hat{x}^L, x, \omega^V) &= 1 \text{ for all } x > \hat{x}^H, \omega^V = l, h; \\
\hat{\mu}_L(x, \hat{x}^H, \omega^V) &= 0 \text{ for all } x < \hat{x}^L, \omega^V = l, h; \\
\hat{\mu}_L(x, \hat{x}^H, \omega^V) &= 1 \text{ for all } x > \hat{x}^L, \omega^V = l, h.
\end{aligned}$$

Obviously Bayes' rule is satisfied on the equilibrium path. Thus we just have to show that no candidate can profitably deviate. In equilibrium each candidate wins with probability $\frac{1}{2}$ in each state. It is easily seen that if a candidate deviates in some state then he wins with a probability that is strictly smaller that $\frac{1}{2}$. Therefore we have an equilibrium.

Finally we show that if $\hat{x}^L - \hat{x}^H \neq \gamma$ or $\gamma > D$ then $\hat{x}^L$, $\hat{x}^H$ are not equilibrium announcements.

First suppose that $\gamma > D$. If $\hat{x}^L$, $\hat{x}^H$ are equilibrium announcements then in state L Candidate H wins with a probability strictly less than $\frac{1}{2}$ (remember that $|\hat{x}^L - x^*_{m_L}| \geq \gamma$). But no matter what voters out-of-equilibrium belief are Candidate H can win with a probability of at least $\frac{1}{2}$ by deviating to $x^*_{m_L}$. Thus $\hat{x}^L$, $\hat{x}^H$ are not equilibrium announcements.

Then suppose that $\hat{x}^L - \hat{x}^H \neq \gamma$ and $\gamma \leq D$, but for now disregard the special case $\hat{x}^L - \hat{x}^H > \gamma$ and $\gamma = D$ which is handled later. If $\hat{x}^L$, $\hat{x}^H$ are equilibrium announcements then we have that in each state one of the candidates wins with probability strictly greater than $\frac{1}{2}$. Consider the state where Candidate L wins with probability greater than $\frac{1}{2}$. It is straightforward to check that if Candidate H deviates to the position x given by

$$\begin{aligned} \hat{x}^L - x = \gamma \quad &\text{if} \quad \hat{x}^L > x^*_{m_L}, \\ x = x^*_{m_L} \quad &\text{if} \quad \hat{x}^L < x^*_{m_L} \end{aligned}$$

then he wins with a probability of at least $\frac{1}{2}$ no matter what the voters' out-of-equilibrium beliefs are. Thus $\hat{x}^L$, $\hat{x}^H$ are not equilibrium announcements.

Finally consider the special case $\hat{x}^L - \hat{x}^H > \gamma$ and $\gamma = D$. Each candidate wins with probability $\frac{1}{2}$ in each state. But, in each state, by deviating to $x^*_{m_L}$ Candidate H can win with a probability stricly greater than $\frac{1}{2}$ no matter what the voters' out-of-equilibrium beliefs are. Thus $\hat{x}^L$, $\hat{x}^H$ are not equilibrium announcements. $\square$

Proof of Theorem 6.3.

Let $\hat{x}^L$, $\hat{x}^H$, $\hat{\mu}_L$ be a symmetric non-revealing equilibrium. Consider a deviation by Candidate L to a position x. If we want to change the out-of-equilibrium beliefs such that the probability of winning for Candidate L after the deviation is maximal then we should choose μ'_L such that

$$\mu'_L(x, \hat{x}^H, l) = \mu'_L(x, \hat{x}^H, h) = 1$$

or

$$\mu'_L(x, \hat{x}^H, l) = \mu'_L(x, \hat{x}^H, h) = 0.$$

Which of these equations that should be satisfied depends on $\hat{x}^H$ and x but not on the state. Hence we see that Candidate L's maximal probability of winning after the deviation is independent of the state. In equilibrium Candidate L wins with a probability $p \geq \frac{1}{2}$ if the state is L and $1-p$ if the state is H. So if x can (by changes in the belief function) be made a profitable deviation for Candidate L in state L then the same is true in state H. Similarly we get that if x can be made a profitable deviation for Candidate H in state H then the same is true in state L. Therefore the Intuitive Criterion does not eliminate the equilibrium if

$$\hat{\mu}_L(x, \hat{x}^H, l) = \hat{\mu}_L(x, \hat{x}^H, h) = 0 \quad \text{for all} \quad x \neq \hat{x}^L$$

and

$$\hat{\mu}_L(\hat{x}^L, x, l) = \hat{\mu}_L(\hat{x}^L, x, h) = 1 \quad \text{for all} \quad x \neq \hat{x}^H.$$

All equilibrium announcements in part *1.* of Theorem 6.2 are supported by such beliefs (see the proof). That is not true for the equilibrium announcements in part *2.* of the theorem. But for those equilibrium announcements each candidate wins with probability $\frac{1}{2}$ in both states. Therefore the Intuitive Criterion does not put any restrictions at all on out-of-equilibrium beliefs. □

Proof of Theorem 6.5.

Suppose $\hat{x}^L$, $\hat{x}^H$, $\hat{\mu}_L$ is a symmetric non-revealing equilibrium satisfying (M1) with $\hat{x}^L > x^*_{m_L}$. We split the proof into two cases, $\hat{x}^L \leq x^*_{m_H}$ and $\hat{x}^L > x^*_{m_H}$.

If $\hat{x}^L \leq x^*_{m_H}$ then in state L Candidate L wins with probability

$$\frac{1}{2} + (2\theta - 1)^2(\gamma + (\hat{x}^H - \hat{x}^L))\sigma.$$

If Candidate L deviates to $x^*_{m_L}$ then, by using that $\hat{\mu}_L$ satisfies (M1), we get that he wins with a probability of at least

$$\frac{1}{2} + (2\theta - 1)^2(\gamma + (\hat{x}^H - x^*_{m_L}))\sigma.$$

Since $\hat{x}^L > x^*_{m_L}$ we have $\hat{x}^H - x^*_{m_L} > \hat{x}^H - \hat{x}^L$ and thus the deviation is profitable. That is a contradiction.

If $\hat{x}^L > x^*_{m_H}$ then in state L Candidate L wins with probability

$$\frac{1}{2} + (2\theta - 1)^2(\gamma - D)\sigma.$$

Suppose Candidate L deviates to the position x satisfying $x^*_{m_H} - x = \hat{x}^L - x^*_{m_H}$. Then Candidate L has moved closer to $x^*_{m_L}$ and not closer to $x^*_{m_H}$. And then it follows by (M1) that his probability of winning has increased. That is a contradiction. □

Terrorism, Anti-Terrorism, and the Copycat Effect

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Abstract

In this paper we formulate and analyze a simple dynamic model of the interaction between terrorists and authorities. Our primary aim is to analyze how the introduction of a so called copycat effect influences behavior and outcomes. We first show that our simple model of terrorist cells implies that an increase in anti-terrorism makes it more likely that cells will plan small rather than large attacks. Furthermore, we see that an increase in anti-terrorism can make a terrorist attack more likely. Analyzing the problem of optimal anti-terrorism we see that the introduction of a copycat effect rationalizes an increase in the level of anti-terrorism after a large attack. Using this result we show how the copycat effect changes the dynamic pattern of terrorism attacks and what the long run consequences are.

Keywords: Terrorist Cells, Optimal Anti-Terrorism, Copycat Effect, Dynamic Pattern of Terrorism.

1 Introduction

In this paper we formulate and analyze a simple dynamic model of the interaction between terrorists and authorities. The primary aim of the paper is to analyze how the introduction of a so called copycat effect influences behavior and outcomes. We say that a copycat effect exists if terrorist cells are more likely to be formed after a period with a high level of terrorism (for example a large attack) than after a period with low terrorist activity. There are several good reasons to expect that a copycat effect exists. Media attention to terrorism is higher when there has been a lot of terrorist activity in the recent past. Therefore the possibility of becoming a terrorist is more salient for potential terrorists. Furthermore, the increased media attention means that even relatively minor terrorist acts get a lot of publicity. Therefore it becomes more attractive to form a terrorist cell and thus it is likely that more cells are formed.

In our model a terrorist cell lives for one period only and its sole decision is whether to plan a small or a large attack. Planning a large attack is more risky because it requires more planning and therefore involves a higher risk of being rolled up by the authorities. The difference in risk between the two types of attacks is increasing in the authorities' spending on anti-terrorism. Therefore it follows that if the authorities increase the level of anti-terrorism then a cell is more likely to plan a small attack. The effect of an increase in anti-terrorism on the probability that a cell will be succesful in making an attack (small or large) is ambiguous. It can be the case that increased spending on anti-terrorism makes a terrorist attack more likely.

In each period of time a terrorist cell is formed with some probability. The aim of the authorities is to minimize the sum of (discounted) expected damage from terrorism and anti-terrorism costs over all periods by choosing the level of anti-terrorism in each period. The horizon is infinite. We solve for optimal anti-terrorism in two cases. First we consider a benchmark case where the probability of a cell being formed is the same in all periods. Then we move on to a case where a copycat effect is in play. More specifically we assume that the probability of a cell being formed is higher if there was a large attack in the previous period. We show that the authorities choose a higher level of anti-terrorism after a large attack. Using that result we see that if a cell is formed then the probability of a small attack is highest and the probability of a large attack and the expected damage is lowest after a large attack. Finally, we compare long run distributions for the benchmark case and the copycat case. In the long run the copycat effect implies more anti-terrorism, more small attacks and a higher per period sum of terrorism damage and anti-terrorism costs. On the other hand it implies less large attacks and less damage from terrorism.

A substantial number of papers have studied economic and game theoretic models of the interaction between terrorists and authorities. For a review see for example Sandler and Enders (2004). Among the specific problems that have been studied are terrorists choice of targets (see e.g. Sandler and Lapan (1988)), hostage taking (Lapan and Sandler (1988)), substitutions by terrorists after policy changes (Enders and Sandler (1993)), the choice between proactive and defensive counterterrorism measures (Rosendorff and Sandler (2004)), and the effect of concessions to terrorists (Bueno de Mesquita (2005)). We are not aware of papers studying the implications of copycat effects. Furthermore, our model is distinct from most of the literature because it is dynamic (with infinite horizon), although it should be noted that the dynamic structure is very simple because terrorist cells live for one period only. Another dynamic model is Faria (2003).

The paper is organized as follows. In Section 2 we set up the model. Then we consider the behavior of the terrorist cells in Section 3 and the problem of optimal anti-terrorism in the two cases in Section 4. Finally, in Section 5, we discuss our results and some ideas for further research.

2 The Model

In each period of time $t = 0, 1, 2, ...$ a terrorist cell is formed with probability $\lambda_t \in (0, 1]$. A cell lives for one period only and its only decision is whether to plan a small or a large attack (it can only make one attack). If the cell formed in period t succeeds in making a small attack then the damage is $D > 0$. If the cell succeeds in making a large attack then the damage is

$$D(1 + \varepsilon_t),$$

where ε_t is drawn from a probability distribution on $[0, \infty)$ with cumulative distribution function F. The realization of ε_t is known to the cell when it makes its decision. We assume that $F(0) = 0$ and that $F(\varepsilon) < 1$ for all ε. Furthermore we assume that F is differentiable on $[0, \infty)$ such that it has the density function $f = F'$.

The authorities choose a level $a \in [0, \infty)$ of anti-terrorism in each period. The level of anti-terrorism in some period t, a_t, decides how likely it is that a cell formed in period t is rolled up before it attacks *if the cell plans a large attack.* If the cell plans a small attack then the probability that it is rolled up is zero. While this is hardly realistic it is a simple way of modelling that a cell preparing a small attack is less likely to be rolled up because a small attack requires less planning. Formally, if the cell decides to plan a large attack then the probability that is rolled up before the attack is $p(a_t)$, where $p : [0, \infty) \to [0, 1]$ is a differentiable and strictly increasing function satisfying $p(0) = 0$. The level of anti-terrorism is known to the cell when it makes its decision. We assume that the cell maximizes expected damage. Therefore the cell plans a small attack if

$$D > (1 - p(a_t))D(1 + \varepsilon_t)$$

and a large attack if we have the opposite inequality. If the cell is indifferent then we assume that it plans a small attack.

The aim of the authorities is to minimize the sum of discounted expected damages and anti-terrorism costs by choosing the level of anti-terrorism in each period. The discounting rate of the authorities is $\delta \in (0, 1)$. The cost of anti-terrorism is given by a differentiable and strictly increasing function $c : [0, \infty) \to [0, \infty)$ with $c(0) = 0$.

The timing of events and decisions in period t is as described in the list below. It is important to note that when the authorities decide on the level of anti-terrorism they know λ_t and F but they do not know whether a cell will be formed and what the realized value of ε_t will be in that case. r_t denotes the damage from terrorism in period t.

Timing of events and decisions in period t:

1. The authorities decide on a_t and pays the cost $c(a_t)$;
2. A new cell is formed with probability λ_t;
3. If a cell was formed then the value of ε_t is realized and the cell decides on what kind of attack to plan;
4. If a cell was formed and planned a small attack then it launches its attack ($\Rightarrow r_t = D$);
5. If a cell was formed and planned a large attack then it is rolled up with probability $p(a_t)$ ($\Rightarrow r_t = 0$);
6. If a cell was formed, planned a large attack and was not rolled up then it launches its attack ($\Rightarrow r_t = D(1 + \varepsilon_t)$).

In the following we first take a closer look at the behavior of the terrorist cells and its consequences. Then we move on to the problem of optimal anti-terrorism. We will focus on two cases. First, we consider a benchmark case where in each period the probability of a new cell being formed does not depend on actions or events in previous periods. Secondly, we consider a case where a copycat effect is in play, i.e. it is more likely that a cell is formed in the current period if there was a large terrorist attack in the previous period than if there was not. Finally, we compare the copycat case to the benchmark case.

3 The Behavior of the Terrorist Cells

Suppose that in some period t the authorities choose the level of anti-terrorism a. Furthermore suppose that a cell is formed. As noted above the cell plans a small attack if (and only if)

$$D \geq (1 - p(a))D(1 + \varepsilon_t).$$

Thus the probability that the cell launches a small attack is

$$\Pr(D \geq (1 - p(a))D(1 + \varepsilon_t)) = \Pr(\varepsilon_t \leq \varepsilon^*) = F(\varepsilon^*),$$

where

$$\varepsilon^* = \frac{1}{(1 - p(a))} - 1 = \frac{p(a)}{(1 - p(a))}.$$

Note that

$$\frac{\partial \varepsilon^*}{\partial a} = \frac{p'(a)}{(1 - p(a))^2} > 0$$

which implies

$$\frac{\partial F(\varepsilon^*)}{\partial a} = \frac{\partial \varepsilon^*}{\partial a} f(\varepsilon^*) > 0.$$

Thus we see that an increase in the level of anti-terrorism makes it more likely that the cell will make a small attack. The probability of the cell succesfully launching a large attack is

$$(1 - p(a))(1 - F(\varepsilon^*)).$$

Since $p' > 0$ and $\frac{\partial F(\varepsilon^*)}{\partial a} > 0$ this expression is decreasing in a, so an increase in a makes a large attack less likely. Adding the two probabilities above we get the probability that the cell launches some kind of attack (i.e. the probability that it is not rolled up):

$$P(a) = F(\varepsilon^*) + (1 - p(a))(1 - F(\varepsilon^*)) = 1 - p(a)(1 - F(\varepsilon^*)).$$

We see that

$$\frac{\partial P}{\partial a} = -p'(a)(1 - F(\varepsilon^*)) + p(a)\frac{\partial \varepsilon^*}{\partial a} f(\varepsilon^*)$$

The first term arises from a's effect on the probability of the cell being rolled up. This term is obviously negative. The second term arises from a's effect on the cells decision about what kind of attack to plan. An increase in a makes it more likely that the cell will plan a small attack which decreases the probability that it is rolled up. Thus this term is positive. Generally we cannot say which of the two effects that dominates, it depends on the functions p and F and the value of a. Below we show by an example that an increase in the level of anti-terrorism can make a terrorist attack more likely. That is an interesting observation.

Consider the following simple example:

$$\begin{aligned} p(a) &= \frac{a}{1+a}, \\ F(\varepsilon) &= 1 - \exp(-\varepsilon). \end{aligned}$$

Then we have

$$\varepsilon^* = \frac{a}{(1+a)(1-\frac{a}{1+a})} = a$$

and

$$f(\varepsilon) = \exp(-\varepsilon).$$

Thus the probability of the cell making an attack is

$$P(a) = 1 - \frac{a}{1+a}\exp(-a)$$

and hence we have

$$\frac{\partial P}{\partial a} = \frac{\exp(-a)}{(1+a)^2}(a(1+a)-1).$$

Loosely speaking, for small levels of a an increase in the level of anti-terrorism makes a terrorist attack less likely and for large levels of a we have the opposite effect. More precisely, $\frac{\partial P}{\partial a}$ is negative for a's below the positive root of $a(1+a)-1$ and positive for a's above this root.

4 Optimal Anti-Terrorism

In this section we consider the problem of the authorities. First, we consider our benchmark case where, for each t, λ_t (the probability of a cell being born in period t) does not depend on what has happened in earlier periods. In that case the authorities' problem is just a sequence of independent static problems which are easy to solve. Secondly, we introduce a simple type of copycat effect which makes the authorities problem truly dynamic. We solve this problem by dynamic programming. Finally, we compare the two cases.

4.1 The Benchmark Case

In this case the level of anti-terrorism chosen in some period t does not influence the problem of the authorities in future periods. Thus in each period the authorities simply choose the level of anti-terrorism that minimizes the sum of the expected damage from the cell possibly formed in that period and the cost of anti-terrorism.

Consider the authorities problem in period t. For simplicity we suppress subscript t's such that we write λ, a and ε instead of λ_t, a_t and ε_t. The expected damage from a cell formed in period t is

$$\Delta(a) = F(\varepsilon^*)D + (1 - F(\varepsilon^*))(1 - p(a))D(1 + E[\varepsilon|\varepsilon > \varepsilon^*]).$$

Note that

$$E[\varepsilon|\varepsilon > \varepsilon^*] = \frac{\int_{\varepsilon^*}^{\infty} \epsilon f(\varepsilon) d\epsilon}{1 - F(\varepsilon^*)}.$$

We can write the problem of the authorities as

$$\min_{a \in [0,\infty)} \pi(a),$$

where

$$\pi(a) = \lambda\Delta(a) + c(a).$$

Since p, c and F are differentiable so is π. The first order condition for an interior solution is

$$\pi'(a) = 0,$$

which can also be written

$$c'(a) = -\lambda\Delta'(a).$$

This is a simple "marginal cost equals marginal benefit" equation. The left hand side is of course the marginal cost of anti-terrorism. The right hand side is minus the marginal effect of anti-terrorism on the expected damage from terrorism. By differentiating Δ and collecting terms (see the Appendix for details) we see that the condition can be written as

$$c'(a) = \lambda p'(a)(1 - F(\varepsilon^*))D(1 + E[\varepsilon|\varepsilon > \varepsilon^*]).$$

Under some additional assumptions on the functions p and c the solution to the authorities problem is unique, interior and the only solution to the first order condition.

Theorem 4.1 *Suppose p and c are twice differentiable and that we have the following conditions:*

1. $p'(0) > 0$, $c'(0) = 0$ *and* $\lim_{a\to\infty} c'(a) = \infty$;
2. $p'' \leq 0$ *and* $c'' > 0$.

Then there is a unique solution to the authorities problem and it is interior and the only solution to the first order condition.

Proof. See the Appendix.

If the conditions in the theorem are satisfied then we let $\bar{a}$ denote the unique solution to the authorities problem. By using the Implicit Function Theorem on the first order condition it is easily seen that $\bar{a}$ is a differentiable function of λ and that

$$\frac{\partial \bar{a}}{\partial \lambda} > 0$$

(simply note that $\frac{\partial \pi'}{\partial \lambda} < 0$ and, by the proof of Theorem 4.1, $\pi''(a) > 0$). So if the probability of a cell being formed is increased then the authorities will choose a higher level of anti-terrorism. This immediately implies that the probability of a small attack,

$$\lambda F(\varepsilon^*(\bar{a})),$$

is increasing in λ (remember that ε^* increases with the level of anti-terrorism). We cannot generally say in which direction the probability of a large attack,

$$\lambda(1 - F(\varepsilon^*(\bar{a})))(1 - p(\bar{a})),$$

changes when λ increases. The same is true for the sum of the two probabilities and for the expected damage, $\lambda\Delta(\bar{a})$. Note, however, that conditional on a cell being formed we have that the probability of a large attack and the expected damage from terrorism are decreasing in λ. Conditioning on a cell being formed does not change the result that the probability of a small attack is increasing in λ or the fact that we cannot determine in which direction the probability of some kind of attack changes when λ increases.

As an example of functions satisfying the conditions in Theorem 4.1 we can take the example of p from the previous section and $c(a) = a^2$. If we again let $F(\varepsilon) = 1 - \exp(-\varepsilon)$ then the first order condition for optimal anti-terrorism becomes

$$2a - \lambda \frac{\exp(-a)}{(1+a)^2}(2+a)D = 0.$$

4.2 Introducing a Copycat Effect

Now we introduce a simple type of copycat effect. More specifically we assume that λ_t is higher if there was a large terrorist attack in period $t-1$ than if there was not. To model the copycat effect define the variable x at time t as

$$x_t = \left\{ \begin{array}{l} s \text{ if } r_{t-1} \leq D \\ l \text{ if } r_{t-1} > D \end{array} \right\}.$$

So $x_t = s$ (for small) if the damage from terrrorism was at most D in period $t-1$. If the damage was higher than D in period $t-1$ then $x_t = l$ (for large). We then let λ_t depend on x_t and assume that

$$\lambda_t(s) < \lambda_t(l),$$

which reflects the copycat effect. We furthermore assume that $\lambda_t(s)$ and $\lambda_t(l)$ does not depend on t. Thus we can write

$$\lambda_t(s) = \lambda^s < \lambda^l = \lambda_t(l) \text{ for all } t.$$

Having modelled the copycat effect it is obvious that the level of anti-terrorism chosen in period t influences the probability that a cell is born in period $t+1$ because it influences r_t and thus x_{t+1}. Therefore the authorities must solve a truly dynamic problem in order to find their optimal level of anti-terrorism in each period. We solve the problem by dynamic programming.

The Bellman equation for the dynamic programming problem can be written as

$$V(x) = \inf_{a \in [0,\infty)} [\pi(a,x) + \delta(P(s,x,a)V(s) + P(l,x,a)V(l))],$$

where

$$\pi(a,x) = \lambda^x \Delta(a) + c(a)$$

and

$$P(x',x,a) = \Pr(x_{t+1} = x' | x_t = x, a_t = a) \text{ for all } x, x' \in \{s,l\}.$$

Note that the Bellman equation is a little non-standard because the transition probabilities depend on a. Writing the transition probabilities in detail we get

$$P(l,x,a) = \lambda^x(1-p(a))(1-F(\varepsilon^*))$$

and

$$\begin{aligned} P(s,x,a) &= 1 - P(l,x,a) \\ &= 1 - \lambda^x(1-p(a))(1-F(\varepsilon^*)) \end{aligned}$$

for all $x \in \{s,l\}$, $a \in [0,\infty)$. By plugging in the transition probabilities the Bellman equation becomes

$$V(x) = \inf_{a \in [0,\infty)} [\pi(a,x) + \delta\lambda^x(1-p(a))(1-F(\varepsilon^*))(V(l) - V(s)) + \delta V(s)].$$

Lemma 4.2 *There exists a unique solution $\bar{V}$ to the Bellman equation above. It satisfies $\bar{V}(s) < \bar{V}(l)$.*

Proof. See the Appendix.

Now, for each $x \in \{s, l\}$, consider the problem

$$\min_{a \in [0,\infty)} [\pi(a, x) + \delta(P(s, x, a)\bar{V}(s) + P(l, x, a)\bar{V}(l))].$$

Pairs of solutions to these two minimization problems are solutions to the dynamic programming problem of the authorities. In the theorem below we present an existence and uniqueness result. For simplicity we define

$$g(a, x) = \pi(a, x) + \delta(P(s, x, a)\bar{V}(s) + P(l, x, a)\bar{V}(l)).$$

Theorem 4.3 *Suppose the assumptions from Theorem 4.1 are satisfied, that F is twice differentiable, and that*

$$\frac{\partial^2}{\partial a^2} F(\varepsilon^*) = \frac{\partial}{\partial a}(\frac{\partial \varepsilon^*}{\partial a} f(\varepsilon^*)) \leq 0.$$

Then, for each $x \in \{s, l\}$, there is a unique solution to the problem considered above and it is interior and the only solution to the first order condition

$$\frac{\partial}{\partial a} g(a, x) = 0$$

Furthermore, letting $\bar{a}_+(x)$ denote the optimal level of terrorism in state x we have

$$\bar{a}_+(s) < \bar{a}_+(l).$$

Proof. See the Appendix.

The result that authorities choose a higher level of anti-terrorism when $x = l$ than when $x = s$ is perhaps not surprising but it does have some interesting implications. Consider the probability of a small attack as a function of x. This probability is

$$\lambda^x F(\varepsilon^*(\bar{a}_+(x)))$$

and thus it is highest when $x = l$. So if there was a large attack in the previous period then there is a higher probability of a small attack than if there was not. This is also true if we instead consider the probability of a small attack conditional on a cell being born, which is of course equal to $F(\varepsilon^*(\bar{a}_+(x)))$. The probability of a large attack as a function of x is

$$\lambda^x(1 - p(\bar{a}_+(x)))(1 - F(\varepsilon^*(\bar{a}_+(x)))).$$

We cannot generally say whether this function is highest when $x = s$ or when $x = l$. However, note that the probability of a large attack conditional on a cell being formed is highest when there was not a large attack in the previous period. With respect to the expected damage from terrorism, $\lambda^x \Delta(\bar{a}_+(x))$, we again cannot say whether it is highest when $x = s$ or when $x = l$. But conditional on a cell being born it is highest when there was not a large attack in the previous period. Finally, note that the per period sum of damage from terrorism and anti-terrorism costs,

$$\pi(a, x) = \lambda^x \Delta(\bar{a}_+(x)) + c(\bar{a}_+(x)),$$

is highest when $x = l$. This follows from the observation that $\pi(a, s)$ is increasing for $a \geq \bar{a}_+(s)$, which follows easily from Theorem 4.4 in the following section.

Now we will consider the problem of finding the long run distribution of x when the authorities behave optimally. Define

$$Q(x', x) = P(x', x, \bar{a}_+(x)) \text{ for all } x', x \in \{s, l\}$$

and note that

$$Q(x', x) \in (0, 1) \text{ for all } x', x \in \{s, l\}.$$

These transition probabilities defines a map Q^* on the set of probability distributions on $\{s, l\}$ into itself by

$$(Q^*\beta)(x') = Q(x', s)\beta(s) + Q(x', l)\beta(l), \quad x' \in \{s, l\}.$$

Define $\bar{\beta}$ by

$$\bar{\beta}(s) = \frac{Q(s, l)}{1 - Q(s, s) + Q(s, l)}.$$

It is easily seen that $\bar{\beta}$ is the unique fixed point for Q^* and that $(Q^*)^t\beta \to \bar{\beta}$ for any β. Hence $\bar{\beta}$ is the unique stationary distribution of x and for any distribution of x_0 the distribution of x_t converges to $\bar{\beta}$. Therefore we conclude that the long run distribution of x is given by $\bar{\beta}$. Of course $\bar{\beta}$ then determines the long run distribution of the level of anti-terrorism and all functions thereof.

Finally we return briefly to the example of p, F and c considered earlier. Since F is twice differentiable and

$$\frac{\partial^2}{\partial a^2} F(\varepsilon^*) = \frac{\partial^2}{\partial a^2}(1 - \exp(-a)) = -\exp(-a) < 0$$

we have that the conditions in Theorem 4.3 are satisfied. For each x the first order condition for optimal anti-terrorism is

$$2a - \lambda^x \frac{\exp(-a)}{(1+a)^2}(2+a)D - \delta\lambda^x [\frac{\exp(-a)}{(1+a)^2}(2+a)](\bar{V}(l) - \bar{V}(s)) = 0,$$

which can be simplified to

$$2a - \lambda^x \frac{\exp(-a)}{(1+a)^2}(2+a)(D + \delta(\bar{V}(l) - \bar{V}(s))) = 0.$$

By comparing with the first order condition from the benchmark case we see that in each state the authorities behave as if they were in a case where there is no copycat effect and D is replaced by $D + \delta(\bar{V}(l) - \bar{V}(s))$.

4.3 Comparing the Two Cases

Consider the authorities' problem with and without the copycat effect in some period t. Suppose that $\lambda = \lambda^{x_t}$, i.e. that the probability of a new cell being formed is the same whether or not there is a copycat effect. Then the following result shows that the authorities will choose a strictly higher level of anti-terrorism if the copycat effect is present. Note that we still assume that the assumptions in Theorem 4.3 (which include the assumptions in Theorem 4.1) are satisfied.

Theorem 4.4 *If* $\lambda = \lambda^x$ *then*

$$\bar{a} < \bar{a}_+(x).$$

Proof. By the first order conditions for the two cases we have that

$$\lambda \Delta'(\bar{a}) + c'(\bar{a}) = 0$$

and

$$\lambda^x \Delta'(\bar{a}_+(x)) + c'(\bar{a}_+(x)) > 0.$$

Since $\lambda = \lambda^x$ and $\lambda\Delta(a) + c(a)$ is convex it follows that $\bar{a} < \bar{a}_+(x)$. □

The intuition behind this result is the following. With the copycat effect the authorities do not only consider the sum of expected damage and anti-terrorism costs in the present period, they also take into account that raising the anti-terrorism level makes it less likely that a cell will be formed in the following period. Thus the marginal benefit from anti-terrorism is higher with the copycat effect and therefore a higher level is chosen.

From the result it follows easily that if we are in a period with $\lambda = \lambda^{x_t}$ then the probability of a small attack is higher with the copycat effect than without it. On the other hand the probability of a large attack and the expected damage from terrorism is lower with the copycat effect (note, however, that the sum of damages and costs is higher with the copycat effect).

Ultimately we want to compare long run distributions for the two cases. The problem with this is how to choose the parameters λ, λ^s and λ^l in order to get meaningful comparisons. We do it the following way. Fix λ^s and λ^l and suppose that in each period the probability of a cell being formed in the benchmark case is

$$\lambda = \left\{ \begin{array}{c} \lambda^s \text{ with probability } \bar{\beta}(s) \\ \lambda^l \text{ with probability } \bar{\beta}(l) \end{array} \right\},$$

where $\bar{\beta}$ is the long run distribution of x from the copycat case. The realization of λ is known to the authorities when they choose the level of anti-terrorism. We define λ this way to ensure that the long run distributions of the probability of a cell being formed are the same in the benchmark and the copycat case. Thus any difference between the two cases does not arise because of differences in these long run distributions.

We say that a variable (e.g. the level of anti-terrorism or the expected damage from terrorism) is higher in the long run with (without) the copycat effect if the long run distribution of the variable with (without) the effect strictly first order stochastically dominates the long run distribution without (with) the effect. Note that this implies that the long run average of the variable is higher with (without) the effect. With this definition we have the following results.

Theorem 4.5 *Assuming λ is distributed as described above the following statements hold.*

1. *The level of anti-terrorism is higher in the long run* with *the copycat effect.*
2. *The probability of a small attack ($r = D$) is higher in the long run* with *the copycat effect.*
3. *The probability of a large attack ($r > D$) is higher in the long run* without *the copycat effect*
4. *The expected damage from terrorism is higher in the long run* without *the copycat effect.*
5. *The sum of expected damage and anti-terrorism costs is higher in the long run* with *the copycat effect.*

Proof. When λ is distributed as described above then any variable v depending on the level of anti-terrorism is higher in the long run with (without) the copycat effect if and only if

$$\begin{array}{rcl} v(\bar{a}_+(x)) & \geq & v(\bar{a}(\lambda^x)) \text{ for each } x \in \{s, l\} \\ (& \leq &) \end{array}$$

with strict inequality for at least one x. Using that observation all conclusions follow easily from Theorem 4.4. □

It is worth noting that we cannot generally say whether the long run probability of some kind of attack ($r > 0$) is highest with or without the copycat effect.

5 Discussion

Using our simple model of terrorist cells we saw that an increase in the level of anti-terrorism makes it more likely that a cell will make a small attack and less likely that it will make a large attack. The probability that a cell makes some kind of attack (which is equal to the probability that it is not rolled up by the authorities) can change in either direction. This is an interesting observation - spending more on anti-terrorism may increase the probability of a terrorist attack. Suppose that there has just been a terrorist attack and that the authorities increase the level of anti-terrorism only to try to calm down the public. This effort can have the effect that another terrorist attack becomes more likely! Note, however, that an increase in the level of anti-terrorism always decreases the expected damage made by a terrorist cell.

By analyzing the problem of optimal anti-terrorism we saw that the existence of a copycat effect offers a rational choice explanation of why authorities increase the level of anti-terrorism after a large attack. Therefore, when a copycat effect exists a terrorist cell formed after a large attack is more likely to make a small attack and less likely to make a large attack. This implies that after a large attack there is a larger probability of a small attack. But, because of the increased likelihood of a cell being formed, it does not necessarily imply that there is a smaller probability of a large attack. By the same argument we have that while the expected damage made by a terrorist cell is smaller after a large attack, the a priori expected damage from terrorism may be higher.

In our comparison of the copycat case and the benchmark case we saw that the long run distribution of several variables differs systematically in the two cases. With the copycat effect there is more anti-terrorism, more small attacks and a higher sum of damages and costs while there is less large attacks and less damage. Note that the benchmark case is the better one for the authorities because the sum of damages and costs are lower.

The way we define the copycat effect is evidently stylized. Instead of assuming that λ_t is a piecewise constant function of r_{t-1} with a jump at D it would be more desirable to assume only that it is some increasing function of r_{t-1}. That would, however, also make the model more technically challenging to analyze. Our intuition tells us that a model with a more realistic assumption on λ_t's dependence on r_{t-1} would give results that are qualitatively similar to ours. Still, it would be nice to see the analysis of such a model carried out.

The copycat effect is introduced exogenously into the model. As we have mentioned earlier there are good reasons for assuming that a copycat effect exists. Nevertheless, it would be desirable to have a model where the copycat effect follows endogenously from the dynamic interaction between terrorists and authorities (and perhaps the public and the media). This is an interesting direction for further research.

A different way of rationalizing that authorities increase the anti-terrorism level after a large attack is to assume that such an attack reveals information that the authorities use to update beliefs. For example, it could be information about the number of existing cells, the probability that a cell is formed during some period of time, or the striking capabilities of existing or new cells. Modelling this is another possible direction for further research.

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7 Appendix

Proof of $\Delta'(a) = -p'(a)(1-F(\varepsilon^*))D(1+E[\varepsilon|\varepsilon > \varepsilon^*])$.

First write $\Delta(a)$ as

$$\Delta(a) = F(\varepsilon^*)D + (1-F(\varepsilon^*))(1-p(a))D + (1-p(a))D\int_{\varepsilon^*}^{\infty} \epsilon f(\varepsilon)d\epsilon.$$

By differentiation we get

$$\Delta'(a) = \frac{\partial \varepsilon^*}{\partial a} f(\varepsilon^*)D - \frac{\partial \varepsilon^*}{\partial a} f(\varepsilon^*)(1-p(a))D - p'(a)(1-F(\varepsilon^*))D \\ - p'(a)D\int_{\varepsilon^*}^{\infty} \epsilon f(\varepsilon)d\epsilon - (1-p(a))D\frac{\partial \varepsilon^*}{\partial a}\varepsilon^* f(\varepsilon^*).$$

By collecting terms we then get

$$\Delta'(a) = -p'(a)(1-F(\varepsilon^*))D(1+E[\varepsilon|\varepsilon > \varepsilon^*]) \\ + \frac{\partial \varepsilon^*}{\partial a} f(\varepsilon^*)D(1-(1-p(a))(1+\varepsilon^*)).$$

Since $\varepsilon^* = \frac{1}{(1-p(a))} - 1$ it follows that the last term is equal to zero and thus we are done. □

Proof of Theorem 4.1.

First note that it suffices to show that

$$\pi'(0) < 0, \quad \lim_{a\to\infty} \pi'(a) = \infty \quad \text{and} \quad \pi'' > 0.$$

We know that

$$\pi'(a) = c'(a) - \lambda p'(a)(1-F(\varepsilon^*))D(1+E[\varepsilon|\varepsilon > \varepsilon^*]).$$

Thus we have

$$\begin{aligned} \pi'(0) &= c'(0) - \lambda p'(0)(1-F(0))D(1+E[\varepsilon|\varepsilon > 0]) \\ &= c'(0) - \lambda p'(0)D(1+E[\varepsilon]). \end{aligned}$$

And then it follows from the two first assumptions in 1. that $\pi'(0) < 0$. Now rewrite π' as

$$\pi'(a) = c'(a) - \lambda p'(a)(1-F(\varepsilon^*))D - \lambda p'(a)D\int_{\varepsilon^*}^{\infty} \epsilon f(\varepsilon)d\epsilon.$$

By that expression and the assumption that $p'' \le 0$ we see that

$$\begin{aligned}\pi'(a) &\ge c'(a) - \lambda p'(a) D(1 + E[\varepsilon]) \\ &\ge c'(a) - \lambda p'(0) D(1 + E[\varepsilon])\end{aligned}$$

Using that inequality it follows from the last assumption in 1. that $\lim_{a\to\infty} \pi'(a) = \infty$. By differentiating π' we get

$$\pi''(a) = c''(a) - \lambda p''(a)(1 - F(\varepsilon^*))D(1 + E[\varepsilon|\varepsilon > \varepsilon^*]) \\ + \lambda p'(a) \frac{\partial \varepsilon^*}{\partial a} f(\varepsilon^*) D(1 + \varepsilon^*).$$

By our assumptions the first term is strictly positive and each of the last two terms are non-negative. Thus we have $\pi'' > 0$. $\square$

Proof of Lemma 4.2.

Define the map T from the set of real functions on $\{s, l\}$ (which can be identified with $\mathbb{R}^2$) into itself by

$$(Tf)(x) = \inf_{a \in [0,\infty)} [\pi(a, x) + \delta(P(s, x, a)f(s) + P(l, x, a)f(l))].$$

It is easily checked that T satisfies Blackwells sufficient conditions for a contraction (see e.g. Stokey and Lucas (1989), Theorem 3.3, p. 54). And then it follows by Banach's Fixed Point Theorem / The Contraction Mapping Theorem (see e.g. Stokey and Lucas (1989), Theorem 3.2, p. 50) that there exists a unique $\bar{V}$ such that

$$T\bar{V} = \bar{V}.$$

Furthermore we have $T^n f \to \bar{V}$ for all f.

To show $\bar{V}(s) < \bar{V}(l)$ it suffices to show that, for any f,

$$f(s) \le f(l) \Rightarrow (Tf)(s) < (Tf)(l).$$

Because then we can pick such an f to get

$$\bar{V}(s) = \lim_n (T^n f)(s) \le \lim_n (T^n f)(l) = \bar{V}(l)$$

and thus

$$\bar{V}(s) = (T\bar{V})(s) < (T\bar{V})(l) = \bar{V}(l).$$

Suppose $f(s) \leq f(l)$. Then we have

$$\begin{aligned}(Tf)(l) = \inf_{a\in[0,\infty)} & [\pi(a,s) + \delta(P(s,s,a)f(s) + P(l,s,a)f(l)) \\ & + (\lambda^l - \lambda^s)\Delta(a) + \delta(\lambda^l - \lambda^s)(1-p(a))(1-F(\varepsilon^*))(f(l) - f(s))].\end{aligned}$$

From this equation we see that

$$\begin{aligned}(Tf)(l) &\geq \inf_{a\in[0,\infty)}[\pi(a,s) + \delta(P(s,s,a)f(s) + P(l,s,a)f(l))] + (\lambda^l - \lambda^s)\inf_{a\in[0,\infty)}[\Delta(a)] \\ &= (Tf)(s) + (\lambda^l - \lambda^s)\inf_{a\in[0,\infty)}[\Delta(a)]\end{aligned}$$

and since $\inf_{a\in[0,\infty)}[\Delta(a)] = D$ it then follows that

$$(Tf)(l) > (Tf)(s).$$

□

Proof of Theorem 4.3.
First note that to prove the first statement of the theorem it suffices to show that, for each $x \in \{s,l\}$,

$$\frac{\partial}{\partial a}g(a,x)|_{a=0} < 0, \quad \lim_{a\to\infty}\frac{\partial}{\partial a}g(a,x) = \infty \quad \text{and} \quad \frac{\partial^2}{\partial a^2}g(a,x) > 0.$$

By differentiation we get (after collecting some terms)

$$\begin{aligned}\frac{\partial}{\partial a}g(a,x) = & \frac{\partial}{\partial a}\pi(a,x) \\ & + \delta\lambda^x[-p'(a)(1-F(\varepsilon^*)) - (1-p(a))\frac{\partial\varepsilon^*}{\partial a}f(\varepsilon^*)](\bar{V}(l) - \bar{V}(s)).\end{aligned}$$

By plugging in $a = 0$ and using Theorem 4.1 we get

$$\frac{\partial}{\partial a}g(a,x)|_{a=0} < \frac{\partial}{\partial a}\pi(a,x)|_{a=0} < 0.$$

Since $p'' \leq 0$ and $\frac{\partial}{\partial a}(\frac{\partial\varepsilon^*}{\partial a}f(\varepsilon^*)) \leq 0$ it follows that the term

$$p'(a)(1-F(\varepsilon^*)) + (1-p(a))\frac{\partial\varepsilon^*}{\partial a}f(\varepsilon^*)$$

is bounded. By Theorem 4.1 we have $\lim_{a\to\infty} \frac{\partial}{\partial a}\pi(a,x) = \infty$ and thus we can conclude that

$$\lim_{a\to\infty} \frac{\partial}{\partial a} g(a,x) = \infty.$$

By differentiation of $\frac{\partial}{\partial a} g(a,x)$ we get

$$\frac{\partial^2}{\partial^2 a} g(a,x) = \frac{\partial^2}{\partial^2 a}\pi(a,x)$$
$$+\delta\lambda^x[-p''(a)(1-F(\varepsilon^*))+2p'(a)\frac{\partial\varepsilon^*}{\partial a}f(\varepsilon^*)-(1-p(a))\frac{\partial}{\partial a}(\frac{\partial\varepsilon^*}{\partial a}f(\varepsilon^*))](\bar{V}(l)-\bar{V}(s)).$$

By our assumptions the term in the square brackets is non-negative and by Theorem 4.1 we have $\frac{\partial^2}{\partial^2 a}\pi(a,x) > 0$. Thus we see that

$$\frac{\partial^2}{\partial a^2} g(a,x) > 0.$$

To prove the last statement of the theorem note that

$$\frac{\partial}{\partial a} g(a,l) = \frac{\partial}{\partial a} g(a,s)$$
$$+ (\lambda^l - \lambda^s)[\Delta'(a) - \delta(\bar{V}(l) - \bar{V}(s))(p'(a)(1 - F(\varepsilon^*)) + (1 - p(a))\frac{\partial\varepsilon^*}{\partial a} f(\varepsilon^*))].$$

Since the term in the square brackets is negative we have

$$\frac{\partial}{\partial a} g(a,l) < \frac{\partial}{\partial a} g(a,s).$$

Therefore

$$\frac{\partial}{\partial a} g(a,l)|_{a=\bar{a}_+(s)} < \frac{\partial}{\partial a} g(a,s)|_{a=\bar{a}_+(s)} = 0.$$

And then it easily follows by the convexity of $g(a,l)$ that

$$\bar{a}_+(s) < \bar{a}_+(l).$$

□

Group Based Regret and Collective Action

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Abstract

We introduce the notion of group based regret in collective action situations. When regret is group based an individual feels regret if she and members of the group of people with similar preferences could have improved the outcome by acting differently. This is different from "rational regret", where rational means that regret is only felt if the individual herself could have improved the outcome. We show that in two specific examples of collective action situations individuals contribute more if they are group based regret minimizers rather than payoff maximizers or rational regret minimizers.

Keywords: Regret, Collective Action, Voting, Binary Public Goods.

1 Introduction

The outcome of the 2000 US presidential election was decided when Bush was certified as the winner in Florida. His official margin of victory was 537 votes. Consider a potential voter in Florida who preferred Gore over Bush but for some reason did not vote. What was her state of mind when it became clear that Bush had won the election? Did she regret not voting? If regret is "rational" in the sense that an individual only feels regret if she couldpersonally have improved the outcome by choosing differently then the answer is no. Even though Bush's margin of victory in Florida was remarkably slim, one extra vote for Gore would not have changed anything. However, it is not given that the feeling of regret is so closely linked to the observation of whether she as an individual could have changed the outcome. In this paper we hypothesize that an individual feels regret after a realized outcome of a collective action situation if she *and members of the group of people with similar preferences* could have changed the outcome. If that is true then the voter would have felt regret because she and just 537 other abstaining Gore supporters in Florida could have given Gore the victory by turning out to vote. In a CBS News/New York Times national post-election poll (CBS and NYT (2000)) 55% of non-voters answer that given the closeness of the election they wish they had voted. The poll was done well before the official result from Florida was known and the reliability of such a poll is obviously questionable in many ways. Nevertheless it does suggest that regret was widespread among non-voters after the 2000 election and thus that regret is not always rational in the sense described above.

In individual decision making people often take into account the regret they will feel after an outcome is realized (see e.g. Zeelenberg (1999)). Therefore it is reasonable to believe that this is also the case when individuals decide what to do in collective action situations. However, if regret is rational then it is unlikely that it can lead to costly participation in collective action situations where the action of one individual is very unlikely to make a difference, e.g. large elections. But if an individual can feel regret even though she could not personally have changed the outcome then it is entirely possible that anticipating this regret can lead to costly participation. Consider the example from above. Suppose the voter knows that if she does not vote and it turns out that abstaining Gore supporters in Florida could have tipped result of the election (not a very unlikely event) then she will feel regret. In that case she may prefer to avoid the risk of feeling this regret by turning out to vote even though it is costly.

In this paper we analyze some examples of collective action situations under the assumption that people are regret minimizers. As mentioned above we will assume that the regret of an individual is based on what she and members of the group of people with similar preferences could have achieved if they had acted differently.

We refer to this as *group based regret*. While this type of regret obviously differs from rational regret (as defined above) it is still completely determined by what did happen, what realistically could have happened and the individual's payoffs in those situations. Also, it is not asymmetric in the sense that individuals can only regret non-participation in collective action. Suppose an individual has contributed to a binary public good and it turns out that the good is provided and would have been so even without her contribution. Then she regrets her contribution just as she would have if regret was rational.

The results from our examples show that we get more participation when people minimize group based regret than when they minimize rational regret or are standard payoff maximizers. Even though our examples are very simple they do indicate that when we observe over-participation in collective action (relative to the prediction of individual payoff maximization), group based regret is a possible explanation.

The role of regret in the decision to vote or not has been studied by Ferejohn and Fiorina (1974) and Li and Majumdar (2007). Ferejohn and Fiorina uses the minimax regret criterion (due to Savage) to analyze the voting decision. That means that each voter makes the choice that minimizes the maximal regret that she can come to feel when an outcome is realized. With reasonable parameter values a voter will feel more regret if she did not vote and would have been pivotal than if she voted and was not pivotal. When that is the case the voter will choose to vote for her preferred candidate. Invoking the minimax regret criterion has been criticized (see e.g. Beck (1975) and Mayer and Good (1975)) because it is only reasonable when the voting decision is taken without any knowledge about the probability of being pivotal, i.e. under uncertainty rather than under risk. While voters may have quite limited knowledge about the probability of being pivotal it is reasonable to believe that they do know it is very small in large elections.

Li and Majumdar studies a game theoretic model where potential voters vote if their anticipated regret of abstaining is higher than the voting cost. Since their model assumes a continuum of voters their notion of regret is clearly not rational (the decision of a single voter does not have any impact on the outcome). Also note that regret is asymmetric in the sense that only abstaining voters feel regret and that abstainers feel regret even if their preferred candidate wins. Thus their notion of regret clearly differs from our notion of group based regret. It is assumed that regret is decreasing in the margin of victory and that abstainers feel more regret if their preferred candidate looses than if he wins. The authors prove the existence of a unique equilibrium with positive turnout and find comparative statics results that fit well with stylized facts. For example, turnout increases with the importance of the election.

We are not aware of other papers that directly explores the link between regret and collective action. In the field of individual decision theory under risk models based on anticipated regret/rejoice has been studied. The first of these were Bell (1982) and Loomes and Sugden (1982).

The paper is organized as follows. In Section 2 we consider the collective action situation of voting in large elections. We take a simple decision theoretic approach. Then, in Section 3, we look at the provision of a binary (discrete) public good. We study this collective action situation from a game theoretic perspective. Finally we discuss and conclude in Section 4.

2 Voting in Large Elections

Consider an election with two candidates (1 and 2) and a large number of voters. We will analyze the decision whether to vote or not for an individual who prefers Candidate 1 over Candidate 2. Let N_1 denote the number of people preferring Candidate 1 (excluding the individual under consideration) and let N_2 denote the number of people supporting Candidate 2. Furthermore let V_i, $i = 1, 2$, denote the number of people (other than the individual under consideration) who turns out to vote for candidate i. When our individual makes her decision by maximizing expected payoff or minimizing rational regret there are four scenarios that she has to take into consideration:

- $S_1 : V_1 > V_2$;
- $S_2 : V_1 = V_2$;
- $S_3 : V_1 = V_2 - 1$;
- $S_4 : V_1 < V_2 - 1$.

In the scenarios two and three the individual is pivotal, in scenarios one and four she is not. Let her payoff be one if Candidate 1 wins and zero if Candidate 2 wins. A tie gives her a payoff of one half. The voting cost is $c > 0$. The payoff and rational regret for voting and abstaining in each of the four scenarios are given in the following table.

	S_1	S_2	S_3	S_4
Payoff, voting	$1 - c$	$1 - c$	$\frac{1}{2} - c$	$-c$
Payoff, abstaining	1	$\frac{1}{2}$	0	0
Rat. regret, voting	c	$\max\{c - \frac{1}{2}, 0\}$	$\max\{c - \frac{1}{2}, 0\}$	c
Rat. regret, abstaining	0	$\max\{\frac{1}{2} - c, 0\}$	$\max\{\frac{1}{2} - c, 0\}$	0

Suppose that $c < \frac{1}{2}$ (otherwise voting would be dominated by abstaining). Let p_i, $i = 1, 2, 3, 4$, denote the individuals belief about the probability of scenario i. If our individual is an expected payoff maximizer then she votes if

$$p_1 + p_2 + \frac{p_3}{2} - c > p_1 + \frac{p_2}{2},$$

i.e. if

$$p_2 + p_3 > 2c.$$

If she is a rational regret minimizer then she votes if

$$(p_1 + p_4)c < (p_2 + p_3)(\frac{1}{2} - c),$$

i.e. if

$$p_2 + p_3 > 2c.$$

Thus we see that if the individual is an expected payoff maximizer or a rational regret minimizer then she votes if (and only if) the probability of being pivotal is greater than two times the cost of voting. In large elections that means that the individual will vote only if her cost of voting is essentially zero.

Now suppose that our individual is a group based regret minimizer. More precisely this means the following:

1. Her regret after some outcome is realized is given by the maximal payoff she could have gotten if she and other members of the group of Candidate 1 supporters had acted differently minus her realized payoff;

2. She takes the action that minimizes her expected regret.

In this case some of the scenarios considered above has to be split into sub-scenarios. Scenario three has to be split into two sub-scenarios and scenario four has to be split into three sub-scenarios:

- $S_{3A} : V_1 = V_2 - 1$ and $N_1 = V_1$;
- $S_{3B} : V_1 = V_2 - 1$ and $N_1 > V_1$;
- $S_{4A} : V_1 < V_2 - 1$ and $N_1 < V_2 - 1$;
- $S_{4B} : V_1 < V_2 - 1$ and $N_1 = V_2 - 1$;
- $S_{4C} : V_1 < V_2 - 1$ and $N_1 > V_2 - 1$.

In S_{3A} and S_{3B} our individual can make the election a tie if she votes. But they are different because in S_{3B} there are abstainers who prefer Candidate 1 while that is not the case in S_{3A}. So if our individual abstains in scenario S_{3B} then she and the other abstainers preferring Candidate 1 could have given him the victory by voting. That is not the case in scenario S_{3A} simply because there are no other abstainers in the group of Candidate 1 supporters. In a similar way the sub-scenarios of S_4 differs in the number of Candidate 1 supporters that abstain.

The group based regret for voting and abstaining in each of the seven scenarios are given in the following table. Note that we assume $c < \frac{1}{2}$.

	S_1	S_2	S_{3A}	S_{3B}	S_{4A}	S_{4B}	S_{4C}
Group b. regret, voting	c	0	0	0	c	0	0
Group b. regret, abstaining	0	$\frac{1}{2} - c$	$\frac{1}{2} - c$	$1 - c$	0	$\frac{1}{2} - c$	$1 - c$

We see that a group based regret minimizer votes if

$$(p_1 + p_{4A})c < (p_2 + p_{3A} + p_{4B})(\frac{1}{2} - c) + (p_{3B} + p_{4C})(1 - c),$$

i.e. if

$$p_2 + p_{3A} + p_{4B} + 2p_{3B} + 2p_{4C} > 2c.$$

In a large election it is reasonable to believe that the probabilities p_2, p_{3A}, p_{3B} and p_{4B} are essentially zero. In that case the voting condition becomes

$$p_{4C} > c.$$

p_{4C} is the probability that, excluding the considered individual, Candidate 2 gets at least two more votes than Candidate 1 but if all supporters of Candidate 1 had voted then he would have gotten at least a tie. So we see that a group based regret minimizer votes if this probability is greater than the voting cost. There is no reason to expect that p_{4C} is close to zero in large elections. Therefore we conclude that group based regret can lead individuals with non-negligible voting costs to vote in large elections.

3 Provision of a Binary Public Good

In this section we consider a game where a number of individuals simultaneously decide whether to contribute a fixed amount to a public good or not. If a sufficient number (which is common knowledge) of people contribute then the public good is provided, if not then all contributions are wasted. This type of game has been analyzed extensively in the literature (see Rapoport (1999) for a review). Here we will show how the prediction of behavior and outcomes is changed if individuals are group based regret minimizers instead of expected payoff maximizers or rational regret minimizers.

Suppose that there are three individuals and that at least two of them have to contribute for the good to be provided. The contribution cost is $c > 0$ and the value of the public good to each individual is $r > c$. The payoff structure is then summarized by the following pair of tri-matrices where individual 1 chooses a row, individual 2 chooses a column and individual 3 chooses a matrix. "C" means that the individual contributes while "NC" means that she does not.

	C	NC
C	$r-c, r-c, r-c$	$r-c, r, r-c$
NC	$r, r-c, r-c$	$0, 0, -c$

C

	C	NC
C	$r-c, r-c, r$	$-c, 0, 0$
NC	$0, -c, 0$	$0, 0, 0$

NC

It is easily seen that when individuals are payoff maximizers then there are four pure strategy Nash equilibria. Either exactly 2 individuals contribute or nobody contributes. We will also consider symmetric mixed Nash equilibria. Suppose all individuals contribute with probability $p \in (0,1)$. That is a mixed Nash equilibrium if (and only if) each individual is indifferent between contributing and not contributing, i.e. if

$$(1-(1-p)^2)(r-c) + (1-p)^2(-c) = p^2 r.$$

Simplifying this equation we get

$$2p(1-p) = \frac{c}{r}.$$

If $\frac{c}{r} < \frac{1}{2}$ this equation has two solutions, if $\frac{c}{r} = \frac{1}{2}$ it has one solution and if $\frac{c}{r} > \frac{1}{2}$ it has no solutions. So there exists a symmetric mixed Nash equilibrium (other than the pure strategy equilibrium where nobody contributes) if and only if $\frac{c}{r} \leq \frac{1}{2}$.

When individuals are rational regret minimizers the game can be summarized by the following pair of rational regret tri-matrices.

	C	NC
C	c, c, c	$0, 0, 0$
NC	$0, 0, 0$	$r-c, r-c, c$

C

	C	NC
C	$0, 0, 0$	$c, r-c, r-c$
NC	$r-c, c, r-c$	$0, 0, 0$

NC

It is easily seen that we get the same pure strategy Nash equilibria as above. Symmetric mixed Nash equilibria are given by the equation

$$(p^2 + (1-p)^2)c = 2p(1-p)(r-c),$$

which can be simplified to

$$2p(1-p) = \frac{c}{r}.$$

So with respect to symmetric mixed Nash equilibria we also get the same results as when the individuals were assumed to be expected payoff maximizers.

Now assume that the individuals are group based regret minimizers. All three individuals belong to the same group. The group based regret for each individual in each possible outcome is summarized by the following pair of tri-matrices.

	C	NC
C	c, c, c	$0, 0, 0$
NC	$0, 0, 0$	$r-c, r-c, c$

C

	C	NC
C	$0, 0, 0$	$c, r-c, r-c$
NC	$r-c, c, r-c$	$r-c, r-c, r-c$

NC

The only change from the rational regret case is in the situation where nobody contributes. In the rational regret case neither of the individuals could have personally improved the situation and therefore none of them feel regret. When regret is group based each individual feels regret in this case because if the individual herself and at least one of the other two had contributed then she (and the co-contributor(s)) would have received a payoff of $r-c$ instead of a payoff of zero.

Suppose $r-c > c$. It is easily seen that in this case there are three pure strategy Nash equilibria, namely the ones where exactly two individuals contribute. The strategy profile (NC, NC, NC) is no longer an equilibrium. The group based regret has destroyed the no-contribution equilibrium and thus the public good is provided in all of the pure strategy equilibria. If instead $r-c \leq c$ then we still have the no-contribution equilibrium.

Next consider symmetric mixed strategy equilibria that are not pure strategy equilibria (i.e. we disregard the pure no-contribution equilibrium if it exists). Contribution with probability $p \in (0,1)$ is an equilibrium if (and only if)

$$(p^2 + (1-p)^2)c = (1-p^2)(r-c).$$

This equation can be simplified to

$$(1 + (1-p)^2)c = (1-p^2)r$$

or

$$\frac{1-p^2}{p^2-2p+2} = \frac{c}{r}.$$

By a little more algebra we get

$$(\frac{c}{r}+1)p^2 - (\frac{2c}{r})p + (\frac{2c}{r}-1) = 0.$$

The solutions of this quadratic equation are

$$\frac{\frac{c}{r} \pm \sqrt{-(\frac{c}{r})^2 - \frac{c}{r} + 1}}{\frac{c}{r} + 1}$$

It is straightforward to see that if the solutions are real then the largest one is in $(0, 1)$. So there exists a symmetric mixed strategy equilibrium if (and only if)

$$-(\frac{c}{r})^2 - \frac{c}{r} + 1 \geq 0,$$

i.e. if

$$\frac{c}{r} \leq \frac{-1 + \sqrt{5}}{2} \approx .62.$$

Furthermore, we see that there are two symmetric mixed strategy equilibria if (and only if)

$$0 < -(\frac{c}{r})^2 - \frac{c}{r} + 1 < (\frac{c}{r})^2,$$

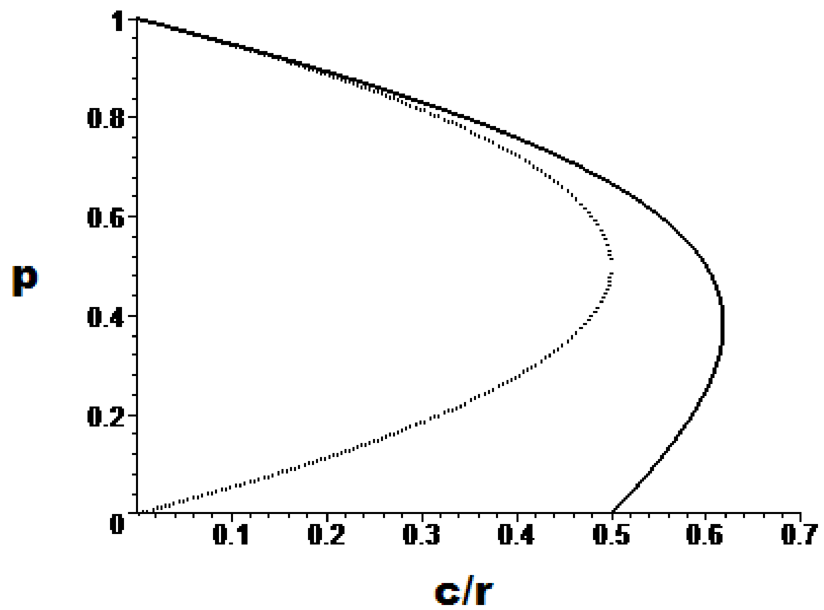
which is equivalent to

$$\frac{1}{2} < \frac{c}{r} < \frac{-1 + \sqrt{5}}{2}.$$

Thus there exists only one equilibrium if (and only if)

$$\frac{c}{r} \leq \frac{1}{2} \quad \text{or} \quad \frac{c}{r} = \frac{-1 + \sqrt{5}}{2}.$$

Below we show the symmetric mixed equilibria in the group based regret case (solid) and in the payoff/rational regret case (dots) as functions of $\frac{c}{r}$.



We see that when we have existence of a symmetric mixed equilibrium in both cases ($\frac{c}{r} \leq \frac{1}{2}$) then the probability of contribution is higher in the unique equilibrium in the group based regret case than in any of the equilibria in the payoff/rational regret case. This statement is formally proved in the Appendix.

4 Discussion

In this paper we have introduced the notion of group based regret in collective action situations. We have seen that in two specific collective action situations there is more contribution if individuals are group based regret minimizers than if they are payoff maximizers (or rational regret minimizers). Our definition of group based regret and the assumption that individuals sole aim of decision making is to minimize regret are obviously stylized. It is reasonable to believe that the feeling of regret is worse when it would have taken the contribution of only a few individuals to change the outcome than when it would have taken the contribution of many. For example, after the official result of the 2000 US presidential election was known, an abstaining Gore supporter from Florida was probably more regretful than one from Ohio (a state that could also have tipped the election), where Bush won with more than 175,000 votes. Furthermore, it is surely more realistic to assume that minimization of regret is one aim among others that are taken into account when an individual makes a decision. Despite all this, our examples do suggest that when we observe more contribution than predicted by individual payoff maximization then part of the explanation can be that regret is group based and that minimizing anticipated regret plays a non-negligible role in decision making.

5 References

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6 Appendix

Proof of statement from Section 3.

It suffices to show that for all $\frac{c}{r} \leq \frac{1}{2}$ we have

$$\frac{\frac{c}{r} + \sqrt{-(\frac{c}{r})^2 - \frac{c}{r} + 1}}{\frac{c}{r} + 1} > \frac{1 + \sqrt{1 - 2\frac{c}{r}}}{2}$$

(note that the right hand side is the largest solution to $2p(1-p) = \frac{c}{r}$). For simplicity define $k = \frac{c}{r}$. By a little algebra we see that the inequality above is equivalent to

$$\frac{\sqrt{-k^2 - k + 1} - \sqrt{-2k + 1}}{k(1-k)} > \frac{1 - \sqrt{-k^2 - k + 1}}{k(1+k)}.$$

The left hand side is the slope of the straight line through the points

$$(-2k + 1, \sqrt{-2k + 1}) \quad \text{and} \quad (-k^2 - k + 1, \sqrt{-k^2 - k + 1}).$$

The right hand side is the slope of the straight line through the points

$$(-k^2 - k + 1, \sqrt{-k^2 - k + 1}) \quad \text{and} \quad (1, 1).$$

Since the square root function is strictly concave it follows that the inequality holds. □