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PhD thesis

Trine Tornøe Platz

Essays in cooperative game theory

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Contents

Preface

This thesis was written during my years as a ph.d.-student at the Department of Economics, University of Copenhagen. First of all, I owe great thanks to my supervisor, Lars Peter Østerdal, for introducing me to aspects of game theory that sparked my existing interest in the field and for convincing me to apply for a PhD-position. I thank Lars for always being accessible, for co-authorship and for continuous support along the way.

As a PhD-student, I spent a semester visiting the Department of Econometrics & Operations Research at the University of Tilburg, and I would like to thank everyone at the department for their great hospitality. Special thanks should be directed to Professor Herbert Hamers for making time for me in a busy schedule during my stay and subsequent visits and for infectious enthusiasm. Also, I thank both Herbert Hamers and Marieke Quant for co-authorship and inspiring meetings that made my visits to Tilburg extremely educational.

I thank friends, office mates, and colleagues at the Department of Economics for making the time there much more enjoyable, and last but not least, I thank my family and friends for always supporting me. In particular, I thank my wife, Karin, for her love and ongoing support including telling me on a regular basis to get it over with. It is a privilege to have had this opportunity, and I will look back on these years as a time of challenges, fun, frustration and occasional victory.

Trine Tornøe Platz

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Introduction and summary

This thesis consists of three chapters within the field of cooperative game theory. All three chapters are self contained, and while they all deal with aspects of transferable utility games, they are not closely linked. These pages provide an introduction to cooperative games and a summary of each chapter.

Game theory analyzes the strategic interaction between players acting in their own self-interest. In cooperative game theory, groups of individuals are assumed to be able to make enforceable agreements on behavior. The possibility of making binding agreements exists in many economic situations, one example being seller-buyer transactions. By cooperating, an individual may end up being better off than when acting on his own, and the fact that cooperation may be beneficial for the individual is the basis for the most common model in cooperative game theory, the transferable utility model.

A transferable utility game (a TU game) models the situation in which a group of players called the grand coalition jointly earn a surplus or incur a cost by associating a monetary value with every possible (sub)coalition within the coalition. Each monetary value represents the surplus (or cost) that the specific coalition could obtain if it acted on its own without any interaction with the rest of the players. In the transferable utility model, the focus is how to allocate the payoff from the joint actions of the coalition between the players in the coalition. Several solution concepts with different desirable properties have been proposed in the literature. Some of these are one-point solutions, while others describe solution sets. One important solution set is the core. The core of a game consists of all efficient allocations that ensure that no subcoalition has incentives to break away from the grand coalition.

The chapters of this thesis deal with different aspects of TU games. The first chapter considers the incentives of players to manipulate a TU game, by creating binding agreements of so-called partnerships with other players. Chapter two deals with the class of games called compromise stable games, and this chapter provides a characterization of compromise stability in terms of the larginal vectors. The third paper investigates the properties of cooperative cost games arising from Chinese postman problems on graphs, when several depots exist. This is done by characterizing classes of graphs leading to balanced and submodular games.

In the first chapter "Forming and dissolving partnerships in cooperative game situations", we consider the incentives of players in a transferable utility game to manipulate the game by forming binding agreements, called partnerships. When joining a partnership, a player commits to not working with players outside the partnership without the accept of all the members of the partnership. The formation of such partnerships may change the game, implying that players could have incentives to manipulate a game by forming or dissolving partnerships.

For a decision maker deciding on an allocation rule to be implemented, it may be important to know whether using a specific rule could cause agents to manipulate the game by forming or dissolving partnerships. This chapter therefore seeks to explore the existence of allocation rules that are immune to this kind of manipulation. We say that an allocation rule is partnership formation-proof if it is never strictly profitable for any group of players to form a partnership when that particular allocation rule is applied, and partnership dissolution-proof if no group can ever profit from dissolving a partnership. We explore the existence of allocation rules that are partnership formation-proof and/or partnership dissolution-proof, and furthermore, we consider whether some well-known allocation rules are immune to this type of manipulation.

The second chapter, "Characterizing compromise stability of games using larginal vectors" provides a new characterization of the class of compromise stable games. The class of compromise stable games contains several interesting classes of games, such as clan games, big boss games, 1-convex games, and bankruptcy games.

A game is called compromise stable if the core is equal to the core cover. The core cover is a superset of the core and equals the set of efficient solutions in which each player gets at least his minimum right and at most his utopia demand. The core cover is also equal to the convex hull of the larginal vectors, and a game is therefore compromise stable if the core is the convex hull of the larginal vectors. A larginal vector corresponds to an order of the players and describes the efficient payoff vector giving the first players in this order their utopia demand as long as it is still possible to give the remaining players at least their minimum right.

In this chapter, we describe two ways of characterizing sets of larginal vectors that satisfy the condition that if every larginal vector of the set is a core element, then the game is compromise stable. The first characterization of these sets is based on a neighbor argument on orders of the players. The second one uses combinatorial and matching arguments and leads to a complete characterization of these sets. We find characterizing sets of minimum cardinality, a closed formula for the minimum number of orders in these sets, and a partition of the set of all orders in which each element of the partition is a minimum characterizing set.

In the third chapter "On games arising from Chinese postman problems with multiple depots", a special class of Chinese postman games is introduced, and the properties of the games are analyzed. The Chinese postman problem (CPP) models a situation in which a postman must deliver mail to a number of streets using the shortest possible route that both starts and ends at the depot (post office), and the paper studies cooperative cost games arising from CPPs in which multiple depots exist.

A multi-depot CPP (k-CPP) is represented by a graph in which the edges of the graph correspond to the streets to be visited, a fixed set of k vertices are depots, and a weight function is defined on the edges. A solution to the problem is a minimum weight tour visiting every edge in the graph. A multi-depot Chinese postman game (k-CP game) is then the cooperative cost game that arises from associating every edge of the graph with a different player and addressing the problem of allocating between these players the cost of the min. weight tour.

In this chapter, k-CP games are analyzed, and we characterize locally and globally k-CP balanced and submodular graphs. A graph G is called locally (globally) k-CP balanced (respectively submodular), if the k-CP game induced by a k-CP problem on G is balanced (respectively submodular) for some (any) choice of depots and any weight function on G.

Chapter 1

Forming and dissolving partnerships in cooperative game situations

Trine Tornøe Platz and Lars Peter Østerdal

Forming and dissolving partnerships in cooperative game situations*

Trine Tornøe Platz, Lars Peter Østerdal†
Department of Economics, University of Copenhagen

Abstract

A group of players in a cooperative game are *partners* (e.g., as in the form of a union or a joint ownership) if the prospects for cooperation are restricted such that cooperation with players outside the partnership requires the accept of all the partners. The formation of such partnerships through binding aggrements may change the game implying that players could have incentives to manipulate a game by forming or dissolving partnerships. The present paper seeks to explore the existence of allocation rules that are immune to this type of manipulation. An allocation rule that distributes the worth of the grand coalition among players is called *partnership formation-proof* if it ensures that it is never jointly profitable for any group of players to form a partnership and *partnership dissolution-proof* if no group can ever profit from dissolving a partnership. The paper provides results on the existence of such allocation rules for general classes of games as well as more specific results concerning well known allocation rules.

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†Correspondence: Lars Peter Østerdal, Department of Economics, University of Copenhagen, Øster Farimagsgade 5, Building 26, DK-1353 Copenhagen K, Denmark. Email: lars.p.osterdal@econ.ku.dk. Tel: +45 35323561, Fax: +45 35323000.

1 Introduction

A problem common to societies, clubs, joint venture organizations, and other formal social structures is how to allocate the benefit, or cost, of a joint activity among the contributing members. In the language of cooperative game theory, we can describe such a problem in the following stylized way. There is a set of *players*, and each coalition of players has a *worth*. This worth can be thought of as the income or surplus that the coalition can obtain without cooperating with the other players, or it can be thought of as a claim on total income. The problem is then to find an *allocation rule* that specifies how total income (i.e., the worth of the grand coalition) is distributed among the players for any configuration of the coalitional worths.

The present paper is concerned with the players' incentives to create binding agreements – called *partnerships* – in such cooperative game situations. If a group of players create a partnership, they commit to not cooperate with players outside the partnership without the accept of the rest of the group. We may think of such partnerships as if every player in the group is given veto power over activities involving any member of the partnership. The remaining players outside the partnership are also affected; these outside players are deprived of the possibility of collaborating with any strict subset of the players in the partnership.

Examples of partnerships include members of a parliament joined in political parties or particular parties in a coalition government, a couple getting married, countries forming a union (e.g., a trade union or a political union), workers or groups of workers forming a labor union, partners within a firm, or firms establishing joint ownerships over a common pool of assets.

The creation of a partnership may change the game and hence the outcome of any allocation process. Thus the creation or dissolution of a partnership can be seen as a way of changing the cooperative game situation, whether it is due to players seeking to manipulate the game or due to a modelling choice by the analyst. In any case, for someone deciding on an allocation rule to be implemented in a cooperative game situation, knowledge of whether a specific allocation rule gives players incentives to manipulate the game by forming or dissolving partnerships could be highly relevant. Particularly it may be of special interest to consider allocation rules that are "immune" to such manipulation or at least to be aware of if such rules exist at all.

We willl call an allocation rule *partnership formation-proof* if it is never strictly profitable for any coalition of players to form a partnership when applying the allocation rule and *partnership dissolution-proof* if it is never strictly profitable to dissolve a partnership. Thus, implementing a partnership formation-proof allocation rule implies that no manipulation in the form of players forming partnerships will occur while a partnership dissolution-proof allocation rule will be immune to manipulation in the form of players dissolving partnerships. The present paper explores whether allocation rules exist that are in this way immune to manipulation, while still satisfying some desirable properties for allocation rules.

Manipulation of cooperative games has been studied by numerous papers starting with, e.g., Postlewaite and Rosenthal (1974), Charnes and Littlechild (1975), Hart and Kurz (1983), Kalai and Samet (1987), Legros (1987), Lehrer (1988), and Hart and Moore (1990). The present paper is closely related to this literature, however, there are also some important differences. In Postlewaite and Rosenthal (1974), Legros (1987), Lehrer (1988), and more recently Haviv (1995), Derks and Tijs (2000), and Knudsen and Østerdal (2008,) groups of players can amalgamate into a single player.[1] The present paper follows Haller (1994), Carreras (1996) and Segal (2003) and considers environments where the set of players is fixed but the worth of coalitions can be manipulated.

Haller (1994) focuses on *bilateral* agreements (i.e. agreements between two players), and considers so-called *proxy-* and *association-agreements.* In a proxy-agreement one of the players becomes a null player, while the other player's marginal contribution to a coalition is set equal to the two players' joint marginal contribution. Haller (1994, Section 6.4) discusses the similarities and differences between proxy-agreements and amalgamations. If one of the players in an association-agreement enter a coalition, it contributes as if both players entered. Carreras (1996) considers partnerships as defined in the present paper and uses the Shapley value to discuss the effect of partnership formation in (especially) simple games, see also Carreras *et al.* (2005), Carreras *et al.* (2009). Segal (2003) contains a general taxonomy of types of integration.

In contrast to, e.g., the proxy- and association-agreements discussed by Haller (1994), the creation of partnerships does not yield any technical efficiency gains as a partnership does not increase the worth of any coalition as long as the game is monotonic; it only reduces the worth of coalitions containing some but not all members of the partnership. On the other hand, there is a dual effect of creating a partnership, since players outside the partnership cannot obtain the full worth from cooperation with strict subsets of players in the partnership. The purpose of forming a partnership should therefore be to reduce the power of outside players without reducing the power of players in the partnership equally. However, it is generally not clear which of the aforementioned effects dominates.

Section 2 introduces the model and basic definitions. In section 3, some results on the existence of partnership formation- and dissolution-proof allocation rules are given for several

[1] Postlewaite and Rosenthal (1974) and Legros (1987) follow Aumann (1973) and refer to a group of players amalgamating into a single player as a *syndicate.* Charnes and Littlechild (1975) call it a *union.*

general classes of cooperative games. We find that the equal-split rule is the only symmetric allocation rule that is both partnership formation-proof and partnership dissolution-proof. Further, while no symmetric partnership formation-proof allocation rule exists that satisfies the null player condition, we find that on the class of monotonic games[2] there do exist symmetric partnership dissolution-proof allocation rules satisfying the null player condition. In section 4, we restrict attention to convex games and consider some well-known allocation rules with favorable properties on this class of games. We find that while a symmetric probabilistic allocation rule, such as the Shapley value, is partnership dissolution-proof on the class of convex games[3] other well-known core allocation rules are neither partnership formation- nor dissolution-proof.

In section 5, we address the influence of the definition of stability on the results presented in section 3 and 4.

In section 6, we explore the situation in which several disjoint partnerships exist within a population and consider whether more players could be expected to join a partnership or if players have incentives to dissolve existing partnerships and possibly create new ones. A few results on the stability of partnership structures are provided. We comment on the consequence of applying different definitions of stability in this context. Section 7 concludes.

2 Partnerships: model and definitions

A cooperative game with side-payments is a pair (N, v), where $N = \{1, .., n\}$ is a finite set of players with $|N| = n \geq 3$, and v is a mapping from 2^N into $\mathbb{R}_+$, with $v(\varnothing) = 0$.[4] Note that we consider only non-negative coalition worths. Since N is fixed, we refer to a game (N, v) simply as v, when no confusion can arise. Also, for players i and j, we write $v(i)$ instead of $v(\{i\})$, $v(i, j)$ instead of $v(\{i, j\})$ etc. Coalitions of players are subsets S, T, Q... of N. Given a vector $x \in \mathbb{R}^N$, $x(S)$ specifies the aggregate payoff $\sum_{i \in S} x_i$ of coalition $S \subseteq N$. An allocation rule for a family of games $\mathcal{V}$ is a function $\phi : \mathcal{V} \to \mathbb{R}^N$ such that $\sum_{i \in N} \phi_i(v) = v(N)$, i.e., it satisfies efficiency.

The *core* of a game v is the set $C(v) = \{x \in \mathbb{R}^N | x(N) = v(N), x(S) \geq v(S)$ for all $S \subset N\}$. A game v is *balanced* if $C(v) \neq \varnothing$, *monotonic* if $v(S) \leq v(S')$ for all coalitions $S, S' \subseteq N$ with $S \subseteq S'$, *superadditive* if $v(S) + v(S') \leq v(S \cup S')$ for any disjoint coalitions $S, S' \subseteq N$, and *convex* if $v(S) + v(S') \leq v(S \cap S') + v(S \cup S')$ for all coalitions $S, S' \subseteq N$.

[2]Monotonicity implies that no player contributes negatively to a coalition.

[3]Convexity implies that a player's marginal contribution to a coalition (weakly) increases as the coalition grows.

[4]For a general treatment of cooperative games, see, e.g., Owen (1995) or Peleg and Sudhölter (2003).

Convexity implies both superadditivity and balancedness, cf. Shapley (1971). Note also that for convex games non-negativity implies monotonicity.

Players $i, j \in N$ are said to be *symmetric* in v if for all $S \subseteq N \backslash \{i, j\}$ it holds that $v(S \cup \{i\}) = v(S \cup \{j\})$. An allocation rule ϕ is *symmetric* if symmetric players are treated equally, that is, if $\phi_i(v) = \phi_j(v)$ for all symmetric players $i, j \in N$ in v. We say that player i is a *null player* in the game v if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$, and that an allocation rule ϕ *satisfies the null player condition* (or briefly, is *null*) if $\phi_i(v) = 0$ whenever i is a null player.

A coalition $T \subseteq N$ forms a partnership when each player in T commits to not contributing to any coalition S for which $T \nsubseteq S$. More precisely, we follow Kalai and Samet (1987) and Carreras (1996) and say that a coalition $T \subseteq N$ is a partnership in v if

$$v(R \cup S) = v(R) \text{ for all } S \subset T \text{ and all } R \subseteq N \backslash T.$$

This definition of a partnership corresponds to the notion of a p-type coalition introduced in Kalai and Samet (1987) and to what Hart and Moore (1990) call a *joint ownership*, see also Carreras (1996) and Segal (2003, p. 447).[5] As in Carreras (1996), the creation of a partnership changes the game from (N, v) to $\left(N, v^T\right)$ defined by

$$v^T(S) = \begin{cases} v(S), \text{ if } T \subseteq S \\ v(S \backslash T), \text{ otherwise.} \end{cases}$$

Notice that in the *partnership game,* v^T, of v any coalition $S \subset T$ has the same worth as the empty coalition, i.e. $v^T(S) = 0$. Further, all players in T are symmetric in v^T. As mentioned above, we restrict attention to the class of non-negative games. This allows us to preserve the natural interpretation of partnership formation and disregard situations where the formation of a partnership allows the worth of a coalition within the partnership to increase from some negative amount to zero.

Given an allocation rule ϕ, it is not profitable to create any partnership if and only if

$$\sum_{i \in T} \phi_i\left(v^T\right) \leq \sum_{i \in T} \phi_i(v),$$

for all $T \subseteq N$ and all v. Allocation rules that satisfy this condition will be called *partnership*

[5]We follow the line of literature defining a partnership as a property of a coalition with respect to the game. In contrast, Reny *et al.* (2009) consider the "partnership property" which is a property of a collection of coalitions and does not involve the game but plays a role for their study of allocations of a game for which there are no asymmetric dependencies between any two players.

formation-proof. Note that the definition implies that redistribution of the joint profit is possible among members of a partnership. If the reverse inequality always holds we say that the allocation rule is *partnership dissolution-proof.* For convenience we use the abbreviations PFP and PDP respectively throughout this paper.

Lemma 1 below states that the partnership game inherits any properties such as non-negativity, monotonicity, superadditivity, balancedness, and convexity from the original game. The proofs are straightforward for non-negativity, monotonicity, and superadditivity (as pointed out in Carreras (1996)) and are omitted here. The proofs for the latter two properties are given.

Lemma 1 *The following classes of games are closed under partnership formation.*

1. *Non-negative*

2. *Monotonic*

3. *Superadditive*

4. *Balanced*

5. *Convex*

Proof: To show that a balanced game is closed under partnerships, assume that $x \in C(v)$. Note that since $v(i) \geq 0$ for all $i \in N$, we have $C(v) \subseteq \mathbb{R}^N_+$. For coalitions $S \subseteq N$ such that $T \subseteq S$ or $T \cap S = \varnothing$, we have $x(S) \geq v^T(S)$, since $v^T(S) = v(S)$. For coalitions $S \subseteq N$ such that $T \cap S \neq \varnothing$ and $T \not\subseteq S$, we also have $x(S) \geq v^T(S)$, since if not, then $x(S \backslash T) \leq x(S) < v^T(S) = v(S \backslash T)$ by $C(v) \subseteq \mathbb{R}^N_+$, contradicting that $x \in C(v)$. Thus, $x \in C\left(v^T\right)$.

Convex games are closed under partnerships if for any convex game v and any T it holds that v^T is convex, i.e., if for any two coalitions $S, S' \subseteq N$ the inequality $v^T(S) + v^T(S') \leq v^T(S \cup S') + v^T(S \cap S')$ holds. If $T \subseteq S$ and $T \subseteq S'$, the inequality is immediate from the convexity of v. If $T \not\subseteq S$ and $T \not\subseteq S'$, then

$$\begin{aligned} v^T(S) + v^T(S') &= v(S \backslash T) + v(S' \backslash T) \\ &\leq v((S \cup S') \backslash T) + v((S \cap S') \backslash T) \\ &\leq v^T(S \cup S') + v^T(S \cap S'), \end{aligned}$$

where the first inequality follows from convexity of v, and the second follows since $v\left(\left(S\cap S'\right)\backslash T\right) = v^T(S\cap S')$ and $v^T(S\cup S') = v((S\cup S')\backslash T)$ if $T \nsubseteq S\cup S'$ and $v^T(S\cup S') = v(S\cup S') \geq v((S\cup S')\backslash T)$ if $T\subseteq S\cup S'$ (because by monotonicity and convexity v is monotonic).

If $T\subseteq S$ and $T\nsubseteq S'$, then

$$\begin{aligned} v^T\left(S\right)+v^T\left(S'\right) &= v\left(S\right)+v\left(S'\backslash T\right) \\ &\leq v\left(S\cup S'\right)+v\left(\left(S\cap S'\right)\backslash T\right) \\ &= v^T\left(S\cup S'\right)+v^T\left(S\cap S'\right), \end{aligned}$$

where the inequality follows from the convexity of v. Since the remaining case $T\nsubseteq S$ and $T\subseteq S'$ is symmetric, we conclude that v^T is convex. □

3 Partnership formation- and dissolution-proofness

It is easy to construct an allocation rule that is both PFP and PDP. Consider as a trivial example an allocation rule that always allocates the total worth of the grand coalition to the same player, i.e., a dictatorial rule. Since the worth that is allocated to some coalition $S\subseteq N$ will be unchanged in any partnership game the dictatorial rule is both PFP and PDP. There also exists a *symmetric* allocation rule ϕ that is both PFP and PDP: the *equal-split* rule ϕ^{ES} defined by $\phi_i^{ES}(v) = \frac{v(N)}{n}$ for all v and all $i\in N$. It is, in fact, the *only* symmetric rule that is both PFP and PDP.[6]

Proposition 1 *For any class of games that is closed under partnerships, there is one and only one symmetric PFP and PDP allocation rule: the equal split rule.*

Proof: It is clear that the equal split rule is PFP and PDP. We show that it is the *only* rule that satisfies both properties. Suppose that a symmetric rule ϕ is PFP and PDP, and $\phi \neq \phi^{ES}$. Thus, there is a game v such that $\phi(v)\neq\phi^{ES}(v)$. Pick a player $i_{\min}\in N$ for which no other player gets a smaller payoff at the allocation $\phi(v)$, and pick a player $i_{\max}\in N$ for which no other player gets a larger payoff at $\phi(v)$.

First, let $T = N\backslash\{i_{\min}\}$ and consider the partnership game v^T based on v. Since ϕ is both PFP and PDP, we have $\sum_{i\in T}\phi_i(v^T) = \sum_{i\in T}\phi_i(v)$, and hence $\phi_{i_{\min}}(v^T) = \phi_{i_{\min}}(v)$. In particular, we have by symmetry that $\phi_{i_{\max}}(v^T) = \frac{\sum_{i\in T}\phi_i(v^T)}{|T|} > \frac{v(N)}{|N|}$. Note that $v^T(S) = v(S)$ for $S = T$, $v^T(S) = v(i_{\min})$ for any coalition $S\ni i_{\min}$, $S\neq N$, and $v^T(S) = 0$ otherwise.

[6] After the initial version of this paper was submitted for publication, it has come to our attention that a similar result has recently and independently been found by Van den Brink (2009).

Second, let $U = N\backslash\{i_{\max}\}$ and consider the partnership game v^{TU} based on v^T. (We write v^{TU} rather than $(v^T)^U$ to avoid cumbersome notation). Since ϕ is both PFP and PDP, we have $\sum_{i\in U}\phi_i(v^{TU}) = \sum_{i\in U}\phi_i(v^T)$, and hence $\phi_{i_{\max}}(v^{TU}) = \phi_{i_{\max}}(v^T) > \phi_{i_{\min}}(v^T)$. In particular,$\phi_{i_{\min}}(v^{TU}) = \frac{\sum_{i\in U}\phi_i(v^{TU})}{|U|} < \frac{v(N)}{|N|}$ by symmetry. Note that $v^{TU}(S) = v(i_{\min})$ if $S = U$, $S \neq N$ and $v^{TU}(S) = 0$ for all $S \neq U, N$.

Third, consider again the coalition $T = N\backslash\{i_{\min}\}$ and the partnership game v^{TUT} based on v^{TU}. Since $v^{TU}(i_{\min}) = 0$ and $v^{TU}(T) = 0$ we have $v^{TUT}(S) = 0$ for all $S \neq N$. In particular, the game is symmetric. But since ϕ is PFP and PDP we have $\sum_{i\in T}\phi(v^{TUT}) = \sum_{i\in T}\phi(v^{TU})$, and therefore $\phi_{i_{\min}}(v^{TUT}) = \phi_{i_{\min}}(v^{TU}) < \frac{v(N)}{|N|}$, contradicting that ϕ is symmetric. □

The equal split rule violates the null player condition. Thus, as a consequence of Proposition 1 we get the following negative result:

Corollary 1 *For no class of games that is closed under partnerships, does there exist a symmetric allocation rule that is both PFP and PDP and satisfies the null player condition.*

It is now natural to ask whether symmetric allocation rules exist that are *either* PFP *or* PDP while satisfying the null player condition. We get the following:

Proposition 2 *i) For the class of non-negative convex games, there exist no symmetric PFP allocation rule satisfying the null player condition. ii) There exist no symmetric PDP allocation rule satisfying the null player condition on the class of non-negative balanced games.*

Proof: For the first part suppose that ϕ is a PFP allocation rule. Let $n \geq 3$ and consider the (convex) game v with $v(N) = v(N \setminus \{1\}) = 1$ and $v(S) = 0$ otherwise. The only allocation consistent with symmetry and the null player condition is $\phi_1(v) = 0, \phi_i(v) = \frac{1}{n-1}$ for all $i \in N \setminus \{1\}$. Now assume that players 1 and 2 form a partnership, $T = \{1, 2\}$. This implies $v^T(N) = 1$ and $v^T(S) = 0$ otherwise. Then by symmetry $\phi_i\left(v^T\right) = \frac{1}{n}$ for all $i \in N$. Since $\phi_1(v) + \phi_2(v) = \frac{1}{n-1} < \frac{2}{n} = \phi_1(v^T) + \phi_2(v^T)$ this contradicts that ϕ is PFP.
For the second part, in order to provoke a counter example suppose that ϕ is PDP and consider the following (balanced) game v, where $n \geq 3$. $v(N) = 1$, $v(1) = 1$, and $v(S) = 0$ otherwise. If $T = \{1, 2\}$ form a partnership, the partnership game v^T is symmetric implying that by symmetry $\phi_i(v^T) = \frac{1}{n}$ for all $i \in N$. For ϕ to be PDP it must therefore hold that $\phi_1(v) + \phi_2(v) \leq \frac{2}{n}$. Next consider instead the formation of a partnership $U = \{2, \ldots, n\}$. Then $v^U(N) = 1$, $v^U(S) = v(S\backslash U) = v(1) = 1$ for all S where $1 \in S$, and $v(S) = 0$ otherwise, implying that all players 2,...,n are null. Thus $\phi_2(v^U) = \cdots = \phi_n\left(v^U\right) = 0$, and

for ϕ to be PDP, it must therefore hold that $\phi_2(v) = \cdots = \phi_n(v) \leq 0$. Since we know that $\sum_{i=1}^{n} \phi_i(v) = 1$, this implies that $\phi_1(v) = 1 - (n-1)\phi_2(v)$. Substituting this into the condition that $\phi_1(v) + \phi_2(v) \leq \frac{2}{n}$ in turn gives $\phi_2(v) \geq \frac{1}{n}$ which contradicts $\phi_2(v) \leq 0$. $\square$

Note that for the first part where the counter example is a non-negative convex game the result holds by implication for the classes of monotonic, superadditive, and balanced games. Note also that the counter example used for the second part of Prop. 2 is not a monotonic game. This is no coincidence as monotonicity does allow for a symmetric PDP rule that satisfies the null player condition:

Proposition 3 *There exist a symmetric PDP allocation rule satisfying the null player condition on the class of monotonic games. Indeed, the equal non-null split rule (which divides $v(N)$ equally between all non-null players in N) satisfies PDP.*

Proof: For a game v, let $D(v) \subseteq N$ denote the set of null players i in N. Let ϕ^* be the rule that gives 0 to the null players and then divides $v(N)$ equally between the remaining players in N; i.e. $\phi_i^*(v) = 0$ if $i \in D(v)$ and $\phi_i^*(v) = \frac{v(N)}{|N \backslash D(v)|}$ otherwise. Clearly, ϕ^* is a symmetric rule satisfying the null player condition. We now show that ϕ^* satisfies the PDP property.

For this, consider a game v and suppose that the players in $T \subseteq N$ form a partnership. We now claim that (i) for all $i \notin T$ we have that $i \in D(v)$ implies $i \in D(v^T)$, and (ii) for all $i \in T$ we have that $i \notin D(v)$ implies $i \notin D(v^T)$.

Ad (i). Let $i \in D(v) \backslash T$. Let $S \subseteq N$ be an arbitrary coalition with $i \notin S$. If $S \cap T = \varnothing$ or $T \subseteq S$, we have $v^T(S) = v(S)$ and $v^T(S \cup \{i\}) = v(S \cup \{i\})$, and thus $v^T(S \cup \{i\}) - v^T(S) = v(S \cup \{i\}) - v(S) = 0$. If $T \nsubseteq S$ and $S \cap T \neq \varnothing$, then $v^T(S) = v(S \backslash T)$ and $v^T(S \cup \{i\}) = v((S \backslash T) \cup \{i\})$, and we have $v^T(S \cup \{i\}) - v^T(S) = v((S \backslash T) \cup \{i\}) - v(S \backslash T) = 0$. Thus, $i \in D(v^T)$.

Ad (ii). Let $i \in T \backslash D(v)$. Since i is not a null player, and the game v is monotonic, there is a coalition $S \subseteq N$ with $i \notin S$ such that $v(S \cup \{i\}) - v(S) > 0$. In particular, we have $v^T((S \cup T) \backslash \{i\}) = v(S \backslash T) \leq v(S)$, and $v^T(S \cup T) = v(S \cup T) \geq v(S \cup \{i\})$ where the inequalities follow by monotonicity of v. We therefore get $v^T(S \cup T) - v^T((S \cup T) \backslash \{i\}) \geq v(S \cup \{i\}) - v(S) > 0$. Thus, $i \notin D(v^T)$.

If $v(N) > 0$, we have $N \backslash D(v) \neq \varnothing$, and since $v^T(N) = v(N)$, we get $N \backslash D(v^T) \neq \varnothing$. In particular, it follows by (i) and (ii) that $\frac{|T \backslash D(v)|}{|N \backslash D(v)|} \leq \frac{|T \backslash D(v^T)|}{|N \backslash D(v^T)|}$. Thus, $\sum_{i \in T} \phi_i^*(v^T) = \frac{|T \backslash D(v^T)|}{|N \backslash D(v^T)|} v^T(N) \geq \frac{|T \backslash D(v)|}{|N \backslash D(v)|} v(N) = \sum_{i \in T} \phi_i^*(v)$ which shows that ϕ^* is PDP. $\square$

Proposition 2 ii) implies that there exists no symmetric PDP allocation rule satisfying the null player condition on the family of non-negative games. In fact, PDP allocation rules cannot be found on this family of games even if the symmetry requirement is dropped.

Proposition 4 *There exist no PDP allocation rules satisfying the null player condition on the class of non-negative games.*

Proof: In order to provoke a counter example, suppose that ϕ is PDP and consider the game v defined by $n \geq 3$, $v(N) = 1$, $v(i) = 1$ for all i, $v(S) = 0$ otherwise. There must exist some two player combination $\{i, j\}$ for which $\phi_i(v) + \phi_j(v) > 0$. Then, if $T = \{i, j\}$ forms a partnership they become null players in v^T with payoff $\phi_i(v^T) = \phi_j(v^T) = 0$ contradicting that ϕ is PDP. □

4 The Shapley value and other core allocation rules

An allocation rule ϕ defined on the family of balanced games is a *core allocation rule* if $\phi(v) \in C(v)$ for all balanced v. Note that all core allocation rules satisfy the null player condition. It turns out that the positive result from Prop. 3 cannot be strengthened to the case of core allocation rules, at least if $n \geq 6$.

Proposition 5 *For $n \geq 6$, there exists no symmetric PDP core allocation rule on the class of monotonic balanced games.*

Proof: Suppose that ϕ is a PDP symmetric core allocation rule. Let $n = 6$ and define a (monotonic balanced) game v as follows: $v(i) = 0$ for all i; $v(1,2) = v(1,3) = v(2,3) = v(1,2,3) = v(4,5,6) = 2$; $v(S) = 2$ if S contains at least two players in $\{1,2,3\}$ but not coalition $\{4,5,6\}$, or if S contains coalition $\{4,5,6\}$ but no more than one of the players in $\{1,2,3\}$; $v(S) = 4$ if S contains $\{4,5,6\}$ and at least one of the coalitions $\{1,2\}$, $\{1,3\}$, $\{2,3\}$; $v(N) = 5$; and $v(S) = 0$ otherwise. Then $C(v)$ only contains one symmetric element and we must have $\phi(v) = \{1, 1, 1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\}$. Let $T = \{1,2,3\}$. Then $v^T(S) = 0$ if $|S| \leq 2$, and $v^T(S) = 2$ if S contains either $\{1,2,3\}$ or $\{4,5,6\}$ but not both of them. By symmetry, we have $\phi_{\{1,2,3\}}(v^T) = \phi_{\{4,5,6\}}(v^T) = \frac{5}{2} < \phi_{\{1,2,3\}}(v) = 3$, contradicting that ϕ is PDP.

Finally, we note that since ϕ is a core allocation rule it satisfies the null player condition. Hence, we can extend the counter example to any $n > 6$ by adding null players. □

As stated in Prop. 5, it is not possible to find symmetric PDP core allocation rules on the family of monotonic balanced games. It is, however, possible to find symmetric PDP core allocation rules on the family of convex games. Segal (2003) gives a condition for a partnership to always be (weakly) (un)profitable when a game is solved by a probabilistic value. It can be inferred from his analysis that partnerships are always (weakly) profitable in convex games. We give a short and illustrative proof for the case of the Shapley value. The Shapley value, which is a symmetric probabilistic value, is defined as

$$\phi_i^{Sh}(N,v) = \sum_{S\subseteq N, S\ni i} p(S)(v(S) - v(S\backslash\{i\})),$$

where $p(S) = \frac{(|S|-1)!(|N|-|S|)!}{|N|!}$, cf. Shapley (1953).

Proposition 6 *The Shapley value (which is indeed a core allocation rule on convex games) is a symmetric PDP core allocation rule on the class of non-negative convex games.*

Proof: For any $n \geq 3$ and $i \notin T$, we have

$$\begin{aligned}
\phi_i^{Sh}\left(v^T\right) &= \sum_{S\subseteq N, i\in S} p\left(S\right)\left[v^T\left(S\right) - v^T\left(S\backslash\left\{i\right\}\right)\right] \\
&= \sum_{\substack{S\subseteq N, i\in S,\\ T\subseteq S}} p\left(S\right)\left[v\left(S\right) - v\left(S\backslash\left\{i\right\}\right)\right] + \sum_{\substack{S\subseteq N, i\in S,\\ T\cap S=\varnothing}} p\left(S\right)\left[v\left(S\right) - v\left(S\backslash\left\{i\right\}\right)\right] + \\
&\quad \sum_{\substack{S\subseteq N, i\in S,\\ T\cap S\neq\varnothing, T\nsubseteq S}} p\left(S\right)\left[v\left(S\backslash T\right) - v\left(S\backslash\left(T\cup\left\{i\right\}\right)\right)\right].
\end{aligned}$$

The values for player $i \notin T$ in the games v and v^T respectively only differ in the last term. That is,

$$\phi_i^{Sh}\left(v^T\right) - \phi_i^{Sh}(v) = \sum_{\substack{S\subseteq N, i\in S,\\ T\cap S\neq\varnothing,\\ T\nsubseteq S}} p\left(S\right)\left[v\left(S\backslash T\right) - v\left(S\backslash\left(T\cup\{i\}\right)\right) - \left(v\left(S\right) - v\left(S\backslash\left\{i\right\}\right)\right)\right].$$

By convexity of v, we have $v\left(S\backslash T\right) - v\left(S\backslash\left(T\cup\{i\}\right)\right) \leq v\left(S\right) - v\left(S\backslash\left\{i\right\}\right)$ implying $\phi_i^{Sh}\left(v^T\right) - \phi_i^{Sh}\left(v\right) \leq 0$ for all $i \notin T$, and thus by efficiency, $\sum_{i\in T}\phi_i^{Sh}\left(v^T\right) \geq \sum_{i\in T}\phi_i^{Sh}\left(v\right)$. Furthermore, in the case of strict convexity we have $\sum_{i\in T}\phi_i^{Sh}\left(v^T\right) > \sum_{i\in T}\phi_i^{Sh}\left(v\right)$. □

As a corollary of Prop. 6 and Lemma 1 it can be noted that when the Shapley value is applied to strictly convex games some set of players can jointly profit from forming a partnership as long as the game is not symmetric in which case further partnership formation has no effect.

The Shapley value was shown to be a PDP core allocation rule on convex games, however, other well-known symmetric core allocation rules do not share this property as shown below.

Fujishige (1980) and Dutta and Ray (1989) and numerous subsequent papers have analyzed the allocation rule that for any convex game selects the unique most egalitarian allocation in the core. This rule will be denoted the Fujishige-Dutta-Ray allocation rule, ϕ^{FDR}, in the following. The algorithm resulting in ϕ^{FDR} in a convex game partitions the set of players N in a game (N, v) into subsets $S_1, S_2, ..., S_m$, where S_1 is the (unique) largest coalition having the highest average worth in (N, v). For any coalition S and any characteristic function v, the average worth of S under v is defined by $e(S, v) = v(S)/|S|$. For $k = 2, ..., m$, S_k is the unique largest coalition with the highest average worth in the game (N_k, v_k) with player set $N_k = N \setminus \{S_1 \cup \cdots \cup S_{k-1}\}$, given that the worth of a coalition S in any game (N_k, v_k) is defined as $v_k(S) = v_{k-1}(S_{k-1} \cup S) - v_{k-1}(S_{k-1})$, where $v_1 = v$, see Dutta and Ray (1989). Then the amount allocated to a player i according to ϕ^{FDR} equals $\phi_i^{FDR} = e(S_k, v_k)$ for all $i \in S_k$. In convex games, ϕ^{FDR} is the unique egalitarian allocation and belongs to the core.

Another well-known allocation rule with favorable properties on the class of convex games is the nucleolus introduced by Schmeidler (1969). The nucleolus is the allocation rule ϕ^{nu} that assigns an allocation $x = \phi^{nu}(v)$ to each game v such that x lexicographically minimizes the vector of excesses $e(S, x) = v(S) - \sum_{i \in S} x_i$. The nucleolus is unique and is in the core whenever the core is non-empty.

While it follows from Prop. 2 i) that none of the above-mentioned core allocation rules are PFP on convex games, it can be shown by way of simple counter examples that neither the Fujishige-Dutta-Ray allocation rule nor the nucleolus is PDP on the class of convex games.

Proposition 7 *Neither the Fujishige-Dutta-Ray allocation rule nor the nucleolus is PDP on the class of non-negative convex games.*

Proof: For $n = 3$, consider a game v defined by $v(N) = 2, v(1) = 1$, $v(1,2) = v(1,3) = 1$ and $v(S) = 0$, otherwise. Then $\phi^{FDR}(v) = \left(1, \frac{1}{2}, \frac{1}{2}\right)$ and $\phi^{nu}(v) = \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}\right)$. If a partnership is formed between players 1 and 2, the game changes to $v^T(N) = 2$, $v^T(1,2) = 1$, and $v(S) = 0$ otherwise. This implies $\phi^{FDR}(v^T) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$, which shows that ϕ^{FDR} cannot be PDP since $\phi_1^{FDR}(v) + \phi_2^{FDR}(v) = \frac{3}{2} > \frac{4}{3} = \phi_1^{FDR}(v^T) + \phi_2^{FDR}(v^T)$. Likewise, we get that $\phi^{nu}(v^T) = \left(\frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right)$ from which we conclude that ϕ^{nu} cannot be PDP since $\phi^{nu}(v) + \phi^{nu}(v) = \frac{5}{3} > \frac{3}{2} = \phi^{nu}(v^T) + \phi^{nu}(v^T)$.

Since the FDR allocation rule as well as the nucleolus satisfy the null player condition, similar counterexamples can be constructed for $n > 3$ by adding null players to the game. □

5 Weak partnership formation- and dissolution-proofness

The definition of PFP and PDP applied in sections 3 and 4 holds an implicit assumption of the ability of members in a partnership to redistribute the joint profit. An obvious alternative to this definition is to consider a deviation to be profitable only if it is individually profitable for every member of a partnership in question. Thus, we define an allocation rule ϕ to be *weakly partnership formation-proof* (WPFP) if and only if for any game v and any nonempty $T \subseteq N$ there exists $i \in T$ such that $\phi_i^T(v) \leq \phi_i(v)$. Likewise, an allocation rule ϕ is said to be *weakly partnership dissolution-proof* (WPDP) if and only if for any game v and any nonempty $T \subseteq N$ there exists $i \in T$ such that $\phi_i^T(v) \geq \phi_i(v)$. In the following, we reconsider some of the previous results in light of these alternative definitions.

We consider again the equal non-null split rule ϕ^* and show that it is both WPFP and WPDP on the class of non-negative convex games. This finding contrasts Prop. 1.

Proposition 8 *i) The equal non-null split rule ϕ^* is both WPDP and WPFP on the class of non-negative convex games, however, ii) There exists no symmetric WPFP rule satisfying the null player condition on the class of monotonic games.*

Proof: For the first part, as in the proof of Prop. 3, let $D(v)$ denote the set of null players in v, and let $\phi_i^*(v) = 0$ if $i \in D(v)$ and $\phi_i^*(v) = \frac{v(N)}{|N \backslash D(v)|}$ otherwise. Note that since v is convex, $i \in N \backslash D(v)$ if and only if $v(N) - v(N \backslash \{i\}) > 0$. Consequently, for any partnership such that $T \subseteq N \backslash D(v)$, it follows that $\left|N \backslash D(v^T)\right| = |N \backslash D(v)|$, and the allocation remains unchanged, i.e., no one can profit from this type of deviation. However, for any partnership T such that $T \nsubseteq N \backslash D(v)$, we get $\left|N \backslash D(v^T)\right| > |N \backslash D(v)|$ and $\phi_i^*(N, v^T) < \phi_i^*(N, v)$ for all $i \in N \backslash D(v)$, implying that forming a partnership can never be strictly profitable for each player in T. Thus, the allocation rule is WPFP. On the other hand, when $T \nsubseteq N \backslash D(v)$ and $\left|N \backslash D(v^T)\right| > |N \backslash D(v)|$ then for those $i \notin N \backslash D(v)$ but in T (and thus in $N \backslash D(v^T)$) we have that $\phi_i^*(v) = 0 < \frac{v(N)}{|N \backslash D(v^T)|} = \phi_i^*(N, v^T)$. Since these players will prefer to keep the partnership intact the allocation rule will also be WPDP.

For the second part, let $N = \{1, ..., n\}$ and consider the (monotonic) game v where $v(S) = 0$ if $|S| = 1$ and $v(S) = 1$ otherwise (such that $v(i,j) - v(i) = 1$ for $j \neq i$ and $v(S) - v(S \backslash \{i\}) = 0$ otherwise). Let ϕ be a symmetric rule satisfying the null player condition. Then $\phi_i(v) = \frac{1}{n}$ for all $i \in N$. Now, let $T = \{1, ..., n-1\}$ form a partnership. Then, $v^T(T) = v^T(N) = 1$ and $v^T(S) = 0$ otherwise. Since player n is a null player in v^T, symmetry implies $\phi_i(v^T) = \frac{1}{n-1}$ for all $i \in T$, contradicting WPFP. □

From Prop. 3 we know that the equal non-null split rule (which is a a symmetric rule satisfying the null player condition) is also WPDP on the class of monotonic games. Further, the result from Prop. 5 holds also for the case of WPDP, that is, there exists no symmetric WPDP core allocation rule on the family of monotonic balanced games. To see this, note that in the example given in the proof of Prop. 5, every player in the partnership is strictly better off by dissolving the partnership, implying that no WPDP allocation rule can be found.

Reconsidering the allocation rules from the previous section, we notice first that since the Shapley value is PDP it is also WPDP, but as shown below, it is not WPFP. However, the FDR-allocation rule is WPFP. Recall that on convex games the allocation rule ϕ^{FDR} satisfies the properties that $\phi_i^{FDR} = \phi_j^{FDR}$ for all $i, j \in S_t$ and $t = 1, .., m$ and that $\phi_i^{FDR} > \phi_j^{FDR}$ if $i \in S_k, \;\; j \in S_t$ and $k < t$, cf. Dutta and Ray (1989).

Proposition 9 *i) The FDR-allocation rule (which is indeed a symmetric allocation rule satisfying the null player condition) is WPFP on the class of non-negative convex games, however, ii) Neither the Shapley value nor the nucleolus is WPFP on non-negative convex games.*

Proof: For the first part consider a game (N, v), and let N be partitioned into subsets $S_1, ..., S_m$ according to the description of the FDR-allocation rule in the previous section. Then $i \in S_1$ belongs to the coalition with the highest average worth, and $\phi_i^{FDR}(v^{\mathcal{B}}) = v^{\mathcal{B}}(S_1)/|S_1|$. Since creating a partnership will not strictly increase the payoff of a coalition but may decrease the payoff of certain coalitions, players in S_1 can never strictly profit from joining a partnership. Since no player belonging to S_1 will form a partnership with players outside S_1 and no partnership among players in $N \backslash S_1$ can affect $v^{\mathcal{B}}(S_1)$, each player in of S_1 is secured the payoff $v^{\mathcal{B}}(S_1)/|S_1|$.

The algorithm first allocates the worth to S_1 and then considers the set of remaining players $N \backslash S_1$. Thus, within the player set $N \backslash S_1$ the players in S_2 will be allocated the greatest worth among the remaining coalitions. Given this, the players of S_2 could never (strictly) profit from joining a partnership among the remaining players. This reasoning can be applied to any S_k, $k = 1, .., m,$ in the partitioning of N. Since this holds for any v it can be concluded that the FDR-allocation rule is WPFP.

For the second part, consider the (convex) game v defined by $n = 3, v(N) = 3, v(1) = 1, v(2) = v(3) = 0, v(1,2) = v(1,3) = 2$ and $v(2,3) = 1$. Then $\phi^{Sh}(v) = \phi^{nu}(v) = \left(\frac{5}{3}, \frac{2}{3}, \frac{2}{3}\right)$. If a partnership is formed between players 2 and 3, the game changes to $v(N) = 3, v(1) = 1, v(2) = v(3) = 0, v(1,2) = v(1,3) = v(2,3) = 1$. This implies $\phi^{Sh}(v^T) = \left(\frac{4}{3}, \frac{5}{6}, \frac{5}{6}\right)$, which shows that ϕ^{Sh} cannot be WPFP since $\phi_2^{Sh}(v) = \phi_3^{Sh}(v) = \frac{2}{3} < \frac{5}{6} = \phi_1^{Sh}(v^T) = \phi_2^{Sh}(v^T)$. Likewise, we get that $\phi^{nu}(v^T) = \left(\frac{6}{4}, \frac{3}{4}, \frac{3}{4}\right)$ from which we conclude that ϕ^{nu} cannot be WPFP since $\phi^{nu}(v) = \phi^{nu}(v) = \frac{2}{3} < \frac{3}{4} = \phi^{nu}(v^T) = \phi^{nu}(v^T)$. Again, similar examples can be constructed for $n > 3$, by adding null players to the game. □

Since the FDR rule satisfies symmetry and the null player condition, it shows that Prop. 2 i) does not hold for the case of WPFP.

6 Stability of partnership structures

Until now we have considered the existence of PFP and/or PDP allocation rules on specific classes of games as well as the properties of certain allocation rules. In this context we considered the decision of a group of players to form or dissolve a given partnership. However, given a population where several disjoint partnerships may exist a related problem would be to consider the incentives of any group of players (from the same or from different partnerships) to form a new partnership, possibly breaking up others in the process. In other words we could consider the incentives of any group of players to change the *partnership structure*.

Consider a game v. Define any partition $\mathcal{B} = \{T_1, T_2, ..., T_m\}$ of N as a *partnership structure* with m partnerships and note that we now consider an element of $\mathcal{B}$ with only one member a partnership. The game $v^{\mathcal{B}}$ is then defined by $v^{\mathcal{B}} = (...(v^{T_1})^{T_2})...)^{T_m}$, and the worth of a coalition S is $v^{\mathcal{B}}(S) = v(\bigcup_{T_j \subseteq S} T_j)$. This definition of the coalition worth is also applicable to the case where different partnerships do not necessarily consist of disjoint sets of players. As noted in Carreras (1996), the formation order of the partnerships does not matter. Given a game v and an allocation rule ϕ, we say that a partnership structure $\mathcal{B}$ is stable, if no set of players can profitably leave their respective partnerships and form new (possibly trivial) partnerships. In this context, a change from one partnership structure ($\mathcal{B}$) to another ($\mathcal{B}'$) is considered profitable for a set of players S if the payoff allocated to each player in the set is strictly larger under the new partnership structure, that is, if $\phi_i(v^{\mathcal{B}'}) > \phi_i(v^{\mathcal{B}})$ for all $i \in S$. Alternatively, one could apply a stronger notion of stability by considering a change of partnership structure to be profitable for a set of players S whenever the total worth allocated to S is greater under $\mathcal{B}'$ than $\mathcal{B}$. While the latter definition is in accordance with the analysis of sections 3 and 4, in this section we nevertheless choose to consider its weaker counterpart discussed in the previous section. First, this is the definition applied in other papers where the stability of partnership structures – or alternatively, *coalition structures* - have been analyzed, see e.g., Hart and Kurz (1983, 1984) and Segal (2003).[7] Second,

[7] What we denote "a partnership" is in the terminology of Segal (2003) referred to as a special case of "exclusion" or "exclusive integration".

while the previous analysis concerned only the group decision of whether or not to form a partnership in a given game, the approach taken in the present section also considers an individual's incentives to leave an existing partnership and possibly join another. Concerning such decisions, the strong version of stability may not be a satisfactory concept, as the following example demonstrates.

Consider the game (N, v) defined by $N = \{1, 2, 3\},\ v(N) = 4,\ v(1, 2) = 3$ and $v(S) = 0$ otherwise. Note that for the trivial partnership structure $\mathcal{B} = \{\{1\}, \{2\}, \{3\}\}$ we have that $\left(N, v^{\mathcal{B}}\right) = (N, v)$. In this game the FDR allocation becomes $\phi^{FDR}(v^{\mathcal{B}}) = (\frac{3}{2}, \frac{3}{2}, 1)$. If a partnership $T = \{1, 3\}$ is created, the partnership structure changes to $\mathcal{B}' = \{\{1, 3\}, \{2\}\}$, and the game $\left(N, v^{\mathcal{B}'}\right)$ is defined by $v^{\mathcal{B}'}(N) = 4$ and $v^{\mathcal{B}'}(S) = 0$ otherwise, implying that $\phi^{FDR}(v^{\mathcal{B}'}) = (\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$. Since the change from $\mathcal{B}$ to $\mathcal{B}'$ is jointly profitable for the players in the partnership, the trivial partnership structure cannot be (strongly) stable. However, since $\phi_1^{FDR}(v^{\mathcal{B}'}) = \frac{4}{3} < \frac{3}{2} = \phi_1^{FDR}(v^{\mathcal{B}})$ player 1 as an individual was strictly better off under $\mathcal{B}$ and will therefore have incentives to leave the partnership he just joined. The problem is that when considering the strong version of stability we allow a player to form a partnership even if he is individually worse off assuming that other members of the partnership will be able to compensate him for joining. However, when considering his decision to leave the partnership we only evaluate the payoff allocated to this player individually, according to the allocation rule, and thus fail to take into account the possible compensation offered by other partnership members (i.e., the redistribution of the joint profit). To avoid this problem, we hereafter only consider the weaker notion of stability.

The definition of stability of partnership structures may also depend on what happens when one player (or more) leaves a partnership that is, whether the entire partnership breaks down or a partnership continues to exist among the remaining partners. In Hart and Kurz (1983), this question leads to the definition of two distinct types of stability. For the extent of this paper, we assume that in the case of some member(s) leaving a partnership the remaining members will continue to cooperate. The following results, however, apply to both settings.

Considering the stability of partnership structures, a relevant question is whether a stable partnership structure always exists when employing a specific allocation rule. We find that for the case of the Fujishige-Dutta-Ray allocation rule a stable partnership structure does always exist. In fact, when using the FDR allocation rule the trivial partnership structure is always stable for convex games.

Corollary 2 *For the class of non-negative convex games, the trivial partnership structure consisting of singletons is stable for any game* (N, v) *when the Fujishige-Dutta-Ray allocation* *rule is applied.*

Proof: This follows directly from Prop. 8 i). □

For both the Shapley value and the nucleolus, examples of games where stable partnership structures exist can be found. Consider, e.g., the example from Prop. 9 ii). For this game, it can be shown that the partnership structure $T^{\mathcal{B}} = \{\{1\}, \{2, 3\}\}$ is stable whether the allocation is done by the Shapley value or the nucleolus.

However, for the Shapley value as well as the nucleolus there exist games for any $n \geq 3$ such that no stable partnership structure exist, as shown below.

Proposition 10 *For any $n \geq 3$, there exist non-negative convex games such that no stable partnership structure exists when the Shapley value ϕ^{Sh} is applied.*

Proof: For any given $n \geq 3$, let the convex game v be defined as follows: $v(N) = 2$, $v(N\backslash\{i\}) = 1$ for all $i \neq 1$ and $v(S) = 0$ otherwise. Further, for any partnership structure $\mathcal{B} = \{T_1, T_2, ..., T_m\}$ let $T^{\mathcal{B}}$ denote the set that is the union of player 1 and all players in some T_i that is not a singleton. Then it holds for all $i \in T^{\mathcal{B}}$ that $v^{\mathcal{B}}(N\backslash\{i\}) = 0$.

Recalling the definition of the Shapley value and noting the structure of the game v, it is seen that the allocation to player i will equal

$$\phi_i^{Sh}(v) = \frac{1}{n}(v(N) - v(N\backslash\{i\}) + \sum_{S \ni i, |S|=n-1} \frac{1}{n(n-1)}(v(S) - v(S\backslash\{i\})). \quad (1)$$

This implies that the worth allocated to player i according to the Shapley value and a partnership structure $\mathcal{B}$ will be: $\phi_i^{Sh}(v^{\mathcal{B}}) = \frac{1}{n} * 2 + \frac{1}{n(n-1)}\left(n - |T^{\mathcal{B}}|\right)$ for $i \in T^{\mathcal{B}}$ and $\phi_i^{Sh}(v^{\mathcal{B}}) = \frac{1}{n} + \frac{1}{n(n-1)}\left(n - |T^{\mathcal{B}}| - 1\right)$ for $i \notin T^{\mathcal{B}}$, where the first term in each expression reflects the contribution made to the grand coalition, and the second term reflects the contribution made to coalitions of size $n - 1$.

Now, consider any $i \notin T^{\mathcal{B}}$ and let $\mathcal{B}'$ denote some partnership structure such that $i \in T^{\mathcal{B}'}$ (i.e., where i belongs to a non-trivial partnership). Then a change in the partnership structure from $\mathcal{B}$ to $\mathcal{B}'$ would induce the following change in the Shapley value of player i: $\Delta\phi_i^{Sh}(v^{\mathcal{B}'}, v^{\mathcal{B}}) = \phi_i^{Sh}(v^{\mathcal{B}'}) - \phi_i^{Sh}(v^{\mathcal{B}}) = \frac{1}{n} - ((|T^{\mathcal{B}'}| - |T^{\mathcal{B}}|) - 1)\frac{1}{n(n-1)} > 0$ where $(|T^{\mathcal{B}'}| - |T^{\mathcal{B}}|)$ reflects the change in the number of players belonging to a partnership. The change $\Delta\phi_i^{Sh}(v^{\mathcal{B}'}, v^{\mathcal{B}})$ will always be positive, since $|T^{\mathcal{B}'}| - |T^{\mathcal{B}}| \leq n-1$, and a player currently not in a partnership will therefore always have incentives to join one. Thus, if at least two players (other than player 1) are not in a (non-trivial) partnership, they have incentives to form one.

On the other hand, any player $i \in T^{\mathcal{B}}, T^{\mathcal{B}'}$ is negatively affected when more players join partnerships. To see this note that in this case $\Delta\phi_i^{Sh}(v^{\mathcal{B}'}, v^{\mathcal{B}}) = -(|T^{\mathcal{B}'}| - |T^{\mathcal{B}}|))\frac{1}{n(n-1)}$ is negative whenever $|T^{\mathcal{B}'}| > |T^{\mathcal{B}}|$. A player in $T^{\mathcal{B}}$ will therefore always prefer that fewer players belong to non-trivial partnerships. This can be interpreted as an incentive to exclude other players from partnerships or break up existing partnerships and form new and smaller ones. Therefore, if partnerships with more than two players exists, there will always be incentives to exclude one player. If a partnership with two players (other than player 1) exists, player 1 can benefit from forming a partnership with just one of the two players, and they will both have incentives to break up the partnership and exclude their former partner. However, if player 1 is in a two-player partnership he will have incentives to dissolve it. □

Since the result shows non-existence of stability in the weak sense it also applies to the case where the stronger version of stability is invoked.

Proposition 11 *For any $n \geq 3$, there exist non-negative convex games such that no stable partnership structure exists when the nucleolus ϕ^{nu} is applied.*

Proof: For any given $n \geq 3$, let the convex game v be defined as follows: $v(N) = 2$, $v(N\backslash\{i\}) = \frac{n}{n+2}v(N) = \frac{2n}{n+2}$ for all $i \neq 1$ and $v(S) = 0$ otherwise. Again, let $\mathcal{B} = \{T_1, T_2, ..., T_m\}$ be a partnership structure, and let $T^{\mathcal{B}}$ denote the union of player 1 and the set of all players in some T_i that is not a singleton. Then for any n and any $\mathcal{B}$ the allocation according to the nucleolus will be:

$$\phi^{nu}(v^{\mathcal{B}}) = \frac{2 - \frac{2n}{n+2}}{|T^{\mathcal{B}}|} + \frac{2}{n+2} \quad \text{for all} \quad i \in T^{\mathcal{B}}, \quad \text{and} \tag{2}$$

$$\phi^{nu}(v^{\mathcal{B}}) = \frac{2}{n+2} \qquad\qquad \text{for all} \quad i \notin T^{\mathcal{B}}. \tag{3}$$

First, to show this is true let

$$e_{ij} = \max_{\substack{S \\ i \in S \\ j \notin S}} (v(S) - \sum_{i \in S} x_i)$$

be the maximum excess over coalitions that contain i but not j where x is the vector of allocations. Then since the nucleolus coincides with the prekernel on the domain of convex games it suffices to show that $e_{ij} = e_{ji}$ for all i, j with $i \neq j$, cf. Maschler, Peleg and Shapley (1971).

For any $i \notin T^{\mathcal{B}}$ it holds that $v(i) - x_i = -\frac{2}{n+2}$ while $v(i) - x_i < -\frac{2}{n+2}$ for all $i \in T^{\mathcal{B}}$. Further, for all larger coalitions S where $v(S) = 0$ the excess must be even smaller. Thus, the only coalitions left to consider are those coalitions where $v(S) > 0$. For all coalitions

$S = N \setminus \{i\}$ where $i \notin T^{\mathcal{B}}$ we get:

$$\begin{aligned} v(S) - \sum_{i \in S} x_i &= \frac{2n}{n+2} - \left(2 - \frac{2n}{n+2} + |T^{\mathcal{B}}|\frac{2}{n+2} + (n - |T^{\mathcal{B}}| - 1)\frac{2}{n+2}\right) \\ &= -\frac{2}{n+2}. \end{aligned}$$

Since all players in $T^{\mathcal{B}}$ are symmetric, and all players not in $T^{\mathcal{B}}$ are symmetric, we conclude that $e_{ij} = e_{ji}$ for all $i, j,\ i \neq j$. Specifically, for any $i \notin T^{\mathcal{B}}, j \in N, j \neq i$ we have $e_{ij} = -\frac{2}{n+2}$ and for $i \in T^{\mathcal{B}}, j \notin T^{\mathcal{B}}$ we have $e_{ij} = -\frac{2}{n+2}$ while $e_{ij} = -\phi^{nu}(v^{\mathcal{B}})$ for all $i \in T^{\mathcal{B}}, j \in T^{\mathcal{B}}, j \neq i$. The above allocation therefore equals the nucleolus.

Now, since the worth allocated to a player in $T^{\mathcal{B}}$ always exceeds the worth allocated to players not in $T^{\mathcal{B}}$ (no matter the number of players in $T^{\mathcal{B}}$) any two players not in a partnership will always have incentives to form one. On the other hand, considering two different partnership structures $T^{\mathcal{B}}, T^{\mathcal{B}'}$ where $|T^{\mathcal{B}'}| > |T^{\mathcal{B}}|$ it is seen that for a player $i \in T^{\mathcal{B}}, T^{\mathcal{B}'}$ we have $\phi^{nu}(v^{\mathcal{B}'}) < \phi^{nu}(v^{\mathcal{B}})$. Therefore, if partnerships with more than two players exists, there will always be incentives for the members to exclude one player. Again, if partnerships with two players (other than player 1) exist, player 1 can benefit from forming a partnership with either one of the two, since both have incentives to break up the partnership and exclude their former partner. However, if player 1 is in a two-player partnership he will have incentives to dissolve it. □

7 Concluding remarks

For several classes of games we have considered the existence of partnership formation-proof and partnership dissolution-proof allocation rules. Such allocation rules will be immune to manipulation by players forming or dissolving partnerships. We showed that if allocations rules must satisfy symmetry and the null player condition then for some classes of games neither partnership formation-proof nor partnership dissolution-proof allocation rules exist, while dissolution-proof allocation rules that satisfy these properties do exist for other classes of games.

We considered in particular three well-known allocation rules: the Shapley value, the nucleolus and the Fujishige-Dutta-Ray rule. The first two are classical solution concepts that are widely used and studied in the literature while especially in recent years the egalitarian FDR-rule has attracted considerable attention.

We have focused here on non-negative games that are either convex or satisfy milder regularity conditions such a monotonicity, superadditivity and balancedness. Important cooperative decision problems, such a classes of common pool games, oligopoly games, production games and cost sharing situations, are indeed both non-negative and convex[8]. Thus, many cooperative game situations fall within the classes of games considered here.

When considering the incentives of players to form partnerships, a key distinction is whether partnerships are likely to form when every individual member is better off or simply when members are jointly better off. This leads to our distinction between partnership formation/dissolution-proofness and the weak versions of the concepts. The appropriate concept depends on assumptions of redistribution possibilities between members. These may vary greatly between different game situations and with the nature of the payoff (e.g. money, publicity, seats in a parliament, individual utility). For instance, if partnerships involve private firms establishing joint ownerships or a couple getting married, redistribution between the partners is likely to take place unhindered. In contrast, in game situations where players are, for example, different regions in a country or different departments of a public institution allocations are likely to be determined by a fixed rule, and the possibilities for internal redistributions between players severely limited.

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[8]See, for example, Shapley and Shubik (1969), Champsaur (1975), Curiel *et al.* 1989, Granot and Hojati (1990), Meinhardt (1999), Zhao (1999), Driessen and Meinhardt (2005).

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Chapter 2

Characterizing compromise stability of games using larginal vectors

Trine Tornøe Platz, Herbert Hamers and Marieke Quant

Characterizing compromise stability of games using larginal vectors

Trine Tornøe Platz[1] Herbert Hamers[2] Marieke Quant[2]

Abstract

The core cover of a TU-game is a superset of the core and equals the convex hull of the larginal vectors. A larginal vector corresponds to an ordering of the players and describes the efficient payoff vector giving the first players in the order their utopia demand as long as it is still possible to assign the remaining players at least their minimum right. A game is called compromise stable if the core is equal to the core cover, i.e., the core is equal to the convex hull of the larginal vectors. In this paper we describe two ways of characterizing sets of larginal vectors that satisfy the condition that if every larginal vector of the set is a core element, then the game is compromise stable. The first characterization of these sets is based on a neighbor argument on orders of the players. The second one uses combinatorial and matching arguments and leads to a complete characterization of these sets. We find characterizing sets of minimum cardinality, a closed formula for the minimum number of orders in these sets, and a partition of the set of all orders in which each element of the partition is a minimum characterizing set.

Keywords: Core, core cover, larginal vectors, matchings.

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[1]Department of Economics, University of Copenhagen, Øster Farimagsgade 5, building 26, 1353 Copenhagen K, Denmark

[2]Department of Econometrics & OR and CentER, Tilburg University, P.O. Box 90153, 5000 LE, Tilburg, The Netherlands.

1 Introduction

The core, introduced by Gillies (1953), is a well-established solution concept for TU-games and equals the set of efficient allocations that satisfy the property that no sub-coalition has an incentive to leave the grand coalition and act on their own. The Weber set (Weber (1988)) and the core cover (Tijs and Lipperts (1982)) are well-known supersets of the core.

The Weber set of a TU-game is the convex hull of the marginal vectors. A marginal vector corresponds to an order of the players and is the efficient allocation vector that assigns to every player his marginal contribution to the coalition consisting of players preceding him in the order. Shapley (1971) and Ichiisi (1981) showed that a TU-game is convex if and only if the core is equal to the Weber set, i.e., if the core is the convex hull of the marginal vectors.

A set of marginal vectors characterizes convexity if it satisfies the condition that the game is convex whenever all marginal vectors of this set are core elements. Rafels and Ybern (1995) showed that the sets consisting of either all even or all odd marginal vectors are sets that characterize convexity. Van Velzen et al. (2002) improved this result and found such characterizing sets with a smaller cardinality by using a neighbor argument showing that if two consecutive neighbors of a marginal vector are in the core, so is the marginal vector itself. Using combinatorial arguments, Van Velzen et al. (2004) derive the minimum cardinality of sets that characterize convexity.

Quant et al. (2005) showed that the core cover equals the convex hull of the larginal vectors. A larginal vector corresponds to an ordering of the players and equals the efficient payoff vector giving the first players in the order their utopia demand as long as it is still possible to assign the remaining players at least their minimum right. A TU-game is compromise stable if and only if the core cover equals the core, i.e., if the core is the convex hull of the larginal vectors. The interest in compromise stable games is two-fold. In many TU-games the nucleolus (Schmeidler (1969)) is hard or even impossible to compute, but for the class of compromise stable games, Quant et al. (2005) provide a closed formula for the nucleolus. Moreover, the class of compromise stable games contains several interesting classes of games such as clan games (Potters et al. (1989)), big boss games (Muto et al. (1989)), 1-convex games (Driessen(1988)) and bankruptcy games (Curiel et al. (1988)). In fact, the class of bankruptcy games is the intersection between the classes of convex and compromise stable games. This means that any game that is both convex and compromise stable is strategically equivalent to a bankruptcy game.

This paper is in the spirit of Van Velzen et al. (2002, 2004). We study sets of larginal vectors that characterize compromise stability. A set of larginal vectors characterizes compromise stability if it satisfies the condition that a game is compromise stable whenever all larginal vectors of this set are core elements. To do so, we follow two different approaches.

First, we use the same neighbor argument as Van Velzen et al. (2002) to provide an upper bound on the cardinality of characterizing sets. Second, by using combinatorial arguments and results on matching in bipartite graphs, we are able to identify the minimum cardinality of characterizing sets and construct a procedure for finding such sets. Furthermore, we show that the set of all orders can be partitioned into disjoint characterizing sets of minimum cardinality.

While the results in this paper are similar to those of van Velzen et al. (2002,2004), the difference in the structure of the larginal and the marginal vectors proves to significantly change the reasoning in the proofs. In the first part of the paper, the proofs differ due to the differences between marginal and larginal vectors while the results turn out to be the same. In the second part of the paper, the different structure of the vectors leads to a new approach based on the combination of a combinatorial and graph theoretical argument. It leads also to different results on the lower bound on the minimum cardinality of characterizing sets, and we find the minimum cardinality of characterizing sets to be lower for compromise stability than for convexity.

The paper is organized as follows: Section 2 presents some notation. Section 3 contains the main body of the paper where we start by using the neighbor argument to compute an upper bound on the cardinality of characterizing sets in section 3.1, while the minimum cardinality of characterizing sets is derived in section 3.2. Section 4 describes the partition of the orders into disjoint characterizing sets of minimum cardinality.

2 Preliminaries

A *transferable utility game* (TU-game) is a pair (N, v), where $N = \{1, \ldots, n\}$, the grand coalition, is a finite set of players and $v : 2^N \to \mathbb{R}$ is a function that assigns to every coalition $S \subseteq N$ a *worth* $v(S)$, with $v(\emptyset) = 0$. We often refer to a game as v rather than (N, v) when no confusion can arise. The set of transferable utility games with player set N is denoted by TU^N.

For $k \in \{1, \ldots, n\}$, $\mathcal{S}_k$ denotes the set of all coalitions with cardinality k, i.e., $S_k = \{S \in 2^N | \; |S| = k\}$.

Let N be a finite set of players. An order is a bijective function $\sigma: \{1, ..., |N|\} \rightarrow N$. The set of all orders is denoted $\Pi(N)$, and $\sigma(h)$ denotes the player at position h in the order σ. An order σ_h denotes the *h'th neighbor* of σ which is obtained by switching players at positions h and $(h+1)$ in σ. Thus, $\sigma_h = (\sigma(1) \ldots \sigma(h-1)\, \sigma(h+1)\, \sigma(h)\, \sigma(h+2) \ldots \sigma(n))$. As an example let $N = \{1, 2, 3, 4\}$. If $\sigma = (1234)$ we get $\sigma_1 = (2134)$, $\sigma_2 = (1324)$, and $\sigma_3 = (1243)$.

Let the identity order e be the order such that $e(i) = i$ for all $i \in N$. Then an *even* order is an order that can be obtained from e by switching positions of neighboring players an even number of times. An order that is not even is called *odd*. The neighbor of an odd order is even and vice versa.

Let $\sigma \in \Pi(N)$ be an order and let $k \in \{1, \ldots, n\}$. Then the *k-head* of σ refers to the first k positions of σ and the *k-tail* of σ refer to the last k positions of σ. Further, denote the set of players belonging to the k-head of σ by $H_k^\sigma = \{\sigma(1), \ldots, \sigma(k)\}$, hence $H_l^\sigma \subseteq H_k^\sigma$ if $l \leq k$. Likewise, $T_k^\sigma = \{\sigma(n-k+1), \ldots, \sigma(n)\}$ denotes the set of players belonging to the k-tail of σ, and $T_l^\sigma \subseteq T_k^\sigma$ if $l \leq k$.

The *core* of a game v is defined by

$$C(v) = \{x \in \mathbb{R}^N | \sum_{i=1}^{n} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N\}$$

and describes the set of efficient allocation vectors such that no coalition has an incentive to split off from the grand coalition. The core of a game may be empty.

The *utopia demand* of player $i \in N$ is given by

$$M_i(v) = v(N) - v(N \backslash \{i\})$$

and describes the maximum amount player i can achieve from cooperation, since the coalition consisting of the rest of the players will never settle for less than $v(N \backslash \{i\})$.

Player i can gather a coalition by promising the rest of the players in the coalition their utopia demand. The maximum amount i can obtain in this way from some coalition is denoted the *minimum right* of player i:

$$m_i(v) = \max_{S \subseteq N : i \in S} \{v(S) - \sum_{j \in S \setminus \{i\}} M_j(v)\}.$$

The *core cover* of a game v equals

$$CC(v) = \{x \in \mathbb{R}^N | \sum_{i=1}^{n} x_i = v(N), m(v) \leq x \leq M(v)\}$$

and thus gives the set of all efficient allocation vectors such that players receive at least their minimum right but no more than their utopia demand. Observe that the core is always a subset of the core cover (cf. Tijs and Lipperts (1982)). A game $v \in TU^N$ is said to be compromise admissible if $m(v) \leq M(v)$ and $\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)$, that is, if the core cover is non-empty. The class of compromise admissible games with player set N is denoted CA^N.

The core cover equals the convex hull of the *larginal* vectors of a game $v \in CA^N$. Let $v \in CA^N$ and $\sigma \in \Pi(N)$. The larginal vector $l^\sigma(v)$ is defined by

$$l^\sigma_{\sigma(k)}(v) = \begin{cases} M_{\sigma(k)}(v) \text{ if } \sum_{j=1}^{k} M_{\sigma(j)}(v) + \sum_{j=k+1}^{n} m_{\sigma(j)}(v) \leq v(N), \\ m_{\sigma(k)}(v) \text{ if } \sum_{j=1}^{k-1} M_{\sigma(j)}(v) + \sum_{j=k}^{n} m_{\sigma(j)}(v) \geq v(N), \\ v(N) - \sum_{j=1}^{k-1} M_{\sigma(j)}(v) - \sum_{j=k+1}^{n} m_{\sigma(j)}(v) \text{ otherwise.} \end{cases}$$

for each $k \in \{1, ..., n\}$. For each order $\sigma \in \Pi(N)$ the larginal vector $l^\sigma(v)$ is the efficient payoff vector that assigns the utopia demand to the first players in σ as long as it is still possible to give the remaining players their minimum rights. The first player that does not receive his utopia demand is called the *pivot* player. A larginal $l^\sigma(v)$ is called even (odd) if the corresponding order σ is even (odd). Furthermore, $l^{\sigma_h}(v)$ is a said to be a neighbor of $l^\sigma(v)$, whenever σ_h is a neighbor of σ.

The following theorem is a straightforward consequence of the results of Quant et al. (2005).

Theorem 2.1. *Let $v \in CA^N$. Then the following statements are equivalent.*

(i) *v is compromise stable,*

(ii) *$C(v) = CC(v)$,*

(iii) *$C(v) = conv\{l^\sigma(v) | \sigma \in \Pi(N)\}$,*

(iv) $v(S) \leq \max\left\{\sum_{i\in S} m_i(v), v(N) - \sum_{i\in N\setminus S} M_i(v)\right\}$ *for all* $S \subseteq N$.

3 Characterizing compromise stable games using larginal vectors

In this section, we describe specific sets of larginals satisfying the condition that a game is compromise stable whenever all larginals of the set are core elements. While this is known to be true for the full set of larginals (see Theorem 2.1) we can use the specific structure of the larginal vectors to identify much smaller sets of larginals that characterize compromise stability. We will describe such specific sets of larginals using two different approaches. First, we use a neighbor argument to show that smaller sets characterizing compromise stability can be found. Next, we combine combinatorial arguments and graph theoretical results, in particular a matching argument, to construct characterizing sets of minimum cardinality.

3.1 The neighbor argument

As a first approach, we consider the neighbor argument of Van Velzen et al. (2002) and show that this argument can also be applied in the context of compromise stable games despite the differences between the marginal and larginal vectors. Thus, we show that if two consecutive neighbors of a larginal are in the core, then the larginal itself belongs to the core. This result in turn implies that if all even larginals or all odd larginals are in the core, then the game is compromise stable. While the results in the two cases are parallel, the argumentation in the proofs depends on the structure of the marginal and larginal vectors and therefore differs.

Lemma 3.1. Let $v \in CA^N$, $n \geq 3$, and $\sigma \in \Pi(N)$. Suppose there is an $h \in \{1, ..., n-2\}$ such that $l^{\sigma_h}(v), l^{\sigma_{h+1}}(v) \in C(v)$. Then $l^{\sigma}(v) \in C(v)$.

Proof. Since $l^{\sigma}(v)$ is by definition efficient, we only have to show that $\sum_{i\in S} l_i^{\sigma}(v) \geq v(S)$ for all $S \subset N$. We distinguish between three cases depending on the position of the pivot player.

Case 1. The pivot player in $l^{\sigma}(v)$ is at position $j \in \{1, \ldots, h\}$. Therefore, $l_i^{\sigma}(v) =$

$l_i^{\sigma_{h+1}}(v)$ for all $i \in N$, and

$$\sum_{i\in S} l_i^{\sigma}(v) = \sum_{i\in S} l_i^{\sigma_{h+1}}(v) \geq v(S)$$

for all $S \subset N$, where the inequality holds since $l^{\sigma_{h+1}}(v) \in C(v)$.

Case 2. The pivot player in $l^{\sigma}(v)$ is at position $j \in \{h+2, \ldots, n\}$. It follows that $l_i^{\sigma}(v) = l_i^{\sigma_h}(v)$ for all $i \in N$, and therefore

$$\sum_{i\in S} l_i^{\sigma}(v) = \sum_{i\in S} l_i^{\sigma_h}(v) \geq v(S)$$

for all $S \subset N$, where the inequality holds since $l^{\sigma_h}(v) \in C(v)$.

Case 3. The pivot player in $l^{\sigma}(v)$ is at position $h+1$. Here, we distinguish between two cases depending on whether the pivot player $\sigma(h+1)$ belongs to S or not.

Case 3a. $\sigma(h+1) \in S$. Since $\sigma(h+1)$ is the pivot player it follows from the definition of the larginal vector that

$$l_{\sigma(h+1)}^{\sigma}(v) + l_{\sigma(h+2)}^{\sigma}(v) = l_{\sigma(h+1)}^{\sigma_{h+1}}(v) + l_{\sigma(h+2)}^{\sigma_{h+1}}(v)$$

and that $l_i^{\sigma}(v) = l_i^{\sigma_{h+1}}(v)$ for all $i \in N \setminus \{\sigma(h+1), \sigma(h+2)\}$. Furthermore, $l_{\sigma(h+1)}^{\sigma}(v) \geq l_{\sigma(h+1)}^{\sigma_{h+1}}(v)$ implying that

$$\sum_{i\in S} l_i^{\sigma}(v) \geq \sum_{i\in S} l_i^{\sigma_{h+1}}(v) \geq v(S),$$

where the first inequality is an equality if $\sigma(h+2) \in S$.

Case 3b. $\sigma(h+1) \notin S$. Then since $l_{\sigma(h)}^{\sigma}(v) \geq l_{\sigma(h)}^{\sigma_h}(v)$ and $l_i^{\sigma}(v) = l_i^{\sigma_h}(v)$ for every $i \in N \setminus \{\sigma(h), \sigma(h+1)\}$ we have

$$\sum_{i\in S} l_i^{\sigma}(v) \geq \sum_{i\in S} l_i^{\sigma_h}(v) \geq v(S).$$

Combining all three cases yields $l^{\sigma}(v) \in C(v)$. □

The above lemma states that if two consecutive neighbors of a larginal vector are in the core so is the larginal vector itself. This implies in particular that if all odd or all even larginals of a game belong to the core, then all larginals belong to the core, so the following theorem is a straightforward consequence of Lemma 3.1.

Theorem 3.1. *Let $v \in TU^N$, $n \geq 3$. Then the following statements are equivalent:*

1. *v is compromise stable,*
2. *$l^\sigma(v) \in C(v)$ for all odd $\sigma \in \Pi(N)$,*
3. *$l^\sigma(v) \in C(v)$ for all even $\sigma \in \Pi(N)$.*

This result is analogous to that of Rafels and Ybern (1995) for convex games. In fact, we can use Lemma 3.1 to find smaller sets that imply compromise stability whenever all the larginals of the set are in the core.

If $A \subseteq \Pi(N)$ is a set of orders such that the corresponding larginals belong to the core, then this set can be expanded by applying Lemma 3.1 and adding to the set any order that has two consecutive neighbors in A. Consequently, all larginals corresponding to the orders of this new set are core elements. By repeatedly applying this neighbor argument, we ultimately arrive at a set that can not be expanded further. If this resulting set is equal to $\Pi(N)$ then the original set A is said to be *neighbor-complete* or *n-complete*. This indicates that if in some game all larginals corresponding to the orders of an n-complete set are core elements, then all larginals of the game are core elements, and hence, the game is compromise stable.

Example 3.1. Let $N = \{1, 2, 3, 4\}$ and consider the set of orders $A = \{(1234), (1342), (1423), (2314), (2431), (2413), (3124), (3241), (3412), (4132), (4213), (4231)\}$ consisting of both odd and even orders. Then A is an n-complete set. Indeed, let $A' = A \cup \{(1243), (1324), (1432), (2134), (2341), (3143), (3214), (3421), (4123), (4312)\}$ where A' arises from A by adding to the set all orders with two consecutive neighbors in A. Observe that $A' \neq \Pi(N)$. Next, $A'' = \Pi(N)$ where A'' is obtained from A' by adding to the set all orders with two consecutive neighbors in A'. Consider the set $B = \Pi(N) \backslash \{(2143), (2413)\}$ and note that B is not an n-complete set. Since (2143) is the 2nd neighbor of (2413), and vice versa, none of the two can have two consecutive neighbors in B. Therefore, B cannot be expanded to $\Pi(N)$ by applying the neighbor argument, and it is not an n-complete set. $\triangle$

In general, the sets of all odd or all even orders, as well as any set containing all odd (or all even) orders, are examples of n-complete sets. Van Velzen et al. (2002) derive upper and lower bounds for the minimum cardinality of n-complete sets. These results are based on the properties of orders and the neighbor argument alone, and they can therefore be directly applied to our setting. This is true since Lemma 3.1 of this paper is identical to Lemma 1 (pp.325) of Van Velzen et al. (2002), except that the first is stated with respect to larginal vectors and the latter with respect to marginal vectors.

For a game with n players let $Q_n = \min\{|A| : A \subseteq \Pi(N) \text{ is } n\text{-complete}\}$ denote the minimum cardinality of a n-complete set. Van Velzen et al. (2002) show that the lower bound for Q_n is: $n!\frac{1}{2^{\frac{n-2}{2}}}$ if n is even and $n!\frac{1}{2^{\frac{n-1}{2}}}$ if n is odd. Table 1 summarizes the results for $n = 3, ..., 9$. Two numbers in brackets represent a lower and upper bound respectively. Due to Lemma 3.1 the results from the table below carry over to the present setting.

n	3	4	5	6	7	8	9
$n!$	6	24	120	720	5040	40320	362880
Q_n	3	12	30	180	[630,1260]	5040	[22680,45360]

Table 1: Minimum cardinality of n-complete sets

In the following section, we no longer make use of the neighbor argument but take a different approach that enables us to find characterizing sets of lower cardinality.

3.2 The cardinality of minimum characterizing sets

In this subsection, we present a necessary and sufficient condition for a set of orders to characterize compromise stability. Based on this condition, we develop a procedure for constructing characterizing sets of minimum cardinality.

We start by introducing the notion of *compromise-complete* (or *c-complete*) sets. A set $A \subseteq \Pi(N)$ is called c-complete if every game $(N, v) \in CA^N$ for which $l^\sigma(v) \in C(v)$ for every $\sigma \in A$ is compromise stable. In the preceding section several examples of c-complete sets were encountered. The full set of orders $\Pi(N)$ is a c-complete set, and Theorem 3.1 shows that the sets of all odd and all even orders are c-complete sets. In fact, all n-complete sets are also c-complete. The rest of this section explores the structure and minimum cardinality of c-complete sets.

First, we introduce a necessary and sufficient condition for a set $A \subseteq \Pi(N)$ to be c-complete. Let $P(N\backslash S, S)$ denote the set of orders that begins with the players of $N\backslash S$ and ends with the players of S, i.e., $\sigma \in P(N\backslash S, S)$ if $\sigma(i) \in S$ for all $i \in \{n - |S| + 1, \ldots, n\}$.

Lemma 3.2. A set $A \subseteq \Pi(N)$ is c-complete if and only if

$$A \cap P(N \backslash S, S) \neq \emptyset \text{ for all } S \subset N \text{ with } 1 < |S| < n-1. \tag{3.1}$$

Proof. First we prove the 'if' part. Let $A \subset N$ be such that (3.1) holds and let $(N, v) \in CA^N$. Assume that $l^\sigma(v) \in C(v)$ for all $\sigma \in A$. We will show that A is c-complete by showing that (N, v) is compromise stable. To do so, it is sufficient to show that the inequality in Theorem 2.1 (iv) is satisfied for all S. Observe that if $S = N, S = N\backslash\{i\}$ or $S = \{i\}$ with $i \in N$, then this inequality is satisfied.

Let $S \in 2^N$ with $1 < |S| < n - 1$. Take $\sigma \in A \cap P(N\backslash S, S)$. Considering the corresponding larginal, $l^\sigma(v)$, we distinguish between two cases.

Case 1. The pivot player of $l^\sigma(v)$ is in $N\backslash S$. This implies $\sum_{i\in S} l_i^\sigma(v) = \sum_{i\in S} m_i(v)$, and thus,

$$v(S) \leq \sum_{i\in S} l_i^\sigma(v) = \sum_{i\in S} m_i(v), \tag{3.2}$$

where the inequality holds since $l^\sigma(v) \in C(v)$.

Case 2. The pivot player of $l^\sigma(v)$ is in S. This implies $l_i^\sigma(v) = M_i(v)$ for all $i \in N\backslash S$. Therefore,

$$v(S) \leq \sum_{i\in S} l_i^\sigma(v) = v(N) - \sum_{i\in N\backslash S} l_i^\sigma(v) = v(N) - \sum_{i\in N\backslash S} M_i(v), \tag{3.3}$$

where the inequality follows since $l_i^\sigma(v) \in C(v)$.

Combining (3.2) and (3.3) yields that

$$v(S) \leq \max\left\{\sum_{i\in S} m_i(v), v(N) - \sum_{i\in N\backslash S} M_i(v)\right\}.$$

We conclude that v is compromise stable, and therefore, A is c-complete.

Second, we prove the 'only if' part. We show that A is not c-complete if (3.1) is not satisfied by providing a game such that all larginals corresponding to orders in A are core elements while the game is not compromise stable.

Assume that A does not fulfill (3.1). Then there exists a coalition $S^* \subset N, 1 <$

$|S^*| < n-1$ such that $A \cap P(N\backslash S^*, S^*) = \emptyset$. Define the game (N, v) by:

$$v(T) = \begin{cases} 1 & \text{if } T = S^*, \\ 0 & \text{if } |T| \leq |S^*|, T \neq S^*, \\ |T| - |S^*| & \text{if } |T| > |S^*|. \end{cases} \tag{3.4}$$

Note, that the utopia demand and minimum right will be $M_i(v) = 1$ and $m_i(v) = 0$ respectively for all $i \in N$. Then, for each $\sigma \in \Pi(N)$ the larginal $l^\sigma(v)$ becomes

$$l^\sigma_{\sigma(h)}(v) = \begin{cases} 1 & \text{for all } h \in \{1, \ldots, n - |S^*|\} \\ 0 & \text{for all } h \in \{n - |S^*| + 1, \ldots, n\}. \end{cases} \tag{3.5}$$

First, we show that $l^\sigma(v) \in C(v)$ for all $\sigma \in A$.

Let $\sigma \in A$. We have to show that

$$\sum_{i \in T} l^\sigma_i(v) \geq v(T) \tag{3.6}$$

for all $T \in 2^N \backslash \{\emptyset, N\}$. Let $T \in 2^N \backslash \{\emptyset, N\}$.

If $|T| \leq |S^*|, T \neq S^*$, then $v(T) = 0$, and since $l^\sigma_i(v) \geq 0$ for all $i \in N$, (3.6) follows immediately.

If $T = S^*$, then $v(T) = 1$, and since $\sigma \notin P(N\backslash S^*, S^*)$ at least one player in T is at position h, with $h \in \{1, \ldots, n - |S^*|\}$, and again (3.6) holds.

If $|T| > |S^*|$, then $v(T) = |T| - |S^*|$, and by (3.5), (3.6) holds. Hence, $l^\sigma(v) \in C(v)$.

Second, we show that v is not compromise stable. Let $\sigma \in P(N\backslash S^*, S^*)$. Then

$$\sum_{i \in S^*} l^\sigma_i(v) = 0 < v(S^*) = 1,$$

hence $l^\sigma(v) \notin C(v)$ and v is not compromise stable. Thus A is not c-complete. □

According to Lemma 3.2, a set A of orders can only be c-complete if for each $S \subset N$ with $1 < |S| < n-1$ there exists an order $\sigma \in A$ such that $T^\sigma_{|S|} = S$. As an illustration, we consider the following example.

Example 3.2. Let $N = \{1, 2, 3, 4\}$ and let $A, B \subset \Pi(N)$. An example of a c-complete set is $A = \{1234, 1324, 1423, 2314, 2413, 3412\}$ since every coalition of size 2 is contained in the 2-tail of some order in A. Thus, A is c-complete, according to (3.1). However, much larger sets of orders may not be c-complete. Consider for example the set $B =$

$\Pi(N) \setminus \{3412, 3421, 4312, 4321\}$. This set contains 20 orders, but it cannot be a c-complete set, since the coalition $\{1, 2\}$ is not contained in the 2-tail of any order in B. $\triangle$

The above example illustrates that even large sets of orders with corresponding larginals in the core may not be c-complete. An upper bound on the cardinality of sets that are not c-complete is given in the proposition below.

Proposition 3.1. Let $A \subseteq \Pi(N)$ be a set of orders with $|A| > n! - \lceil \frac{n}{2} \rceil!(n - \lceil \frac{n}{2} \rceil)!$. Then A is c-complete.

Proof. For any set of players $S \in \mathcal{S}_k$ we have $|P(N \backslash S, S)| = (n-k)!k!$, i.e., there exist $(n-k)!k!$ different orders $\sigma \in \Pi(N)$ such that $S = T_k^\sigma$. Since $|P(N \backslash S, S)| \geq \lceil \frac{n}{2} \rceil!(n - \lceil \frac{n}{2} \rceil)!$ for any $S \subset N$, it holds that $A \cap P(N \backslash S, S) \neq \emptyset$ for all $S \subseteq N$. Thus, A is c-complete. $\square$

Proposition 3.1 gives an upper bound for the cardinality of a c-complete set. Next, we focus on the minimum cardinality of a c-complete set of orders which is defined as follows:

$$L_n = \min\{|A| : A \subseteq \Pi(N) \text{ is c-complete}\}.$$

From Lemma 3.2 follows immediately the lower bound for the minimum cardinality:

$$L_n \geq \binom{n}{\lceil \frac{n}{2} \rceil} \tag{3.7}$$

In order to determine the minimum cardinality L_n, we identify the smallest possible set of orders A, such that $A \cap P(N \backslash S, S) \neq \emptyset$ for all $S \subset N$. A c-complete set can easily be created by choosing an order from $P(N \backslash S, S)$ for each S, resulting in a cardinality of $\sum_{k=2}^{n-2} \binom{n}{k}$. However, instead of picking for each S a new order where S is contained in the tail, we can make use of the fact that the k-tail of any order $\sigma \in A$ also contains sets of smaller sizes. The final part of this section is devoted to finding the minimum cardinality of c-complete sets and a procedure for creating c-complete sets of minimum cardinality. Before we proceed, some intermediate observations are stated. The proofs are trivial and omitted.

The first lemma states that given an order $\sigma \in \Pi(N)$ rearranging entries within the k-tail of σ will not affect the set of players in the k-tail.

Lemma 3.3. Let $k \in \{1, \dots, n-1\}$ and let $\sigma, \tau \in \Pi(N)$ be orders such that $\sigma(i) = \tau(i)$ for all $i \in \{1, \dots, n-k\}$. Then $T_k^\sigma = T_k^\tau$.

The second lemma states the observation that whenever every subset of size k is contained in the k-tail of some order in a set $A \subseteq \Pi(N)$ then every subset of size $n-k$ will be contained in the $(n-k)$-head of some order in A and vice versa.

Lemma 3.4. Let $A \subseteq \Pi(N)$ be a set of orders and let $k \in \{1, \ldots, n-1\}$. For each $S \in \mathcal{S}_k$ there exists a $\sigma \in A$ such that $T_k^\sigma = S$ if and only if for each $S' \in \mathcal{S}_{n-k}$ there exists a $\tau \in A$ such that $H_{n-k}^\tau = S'$.

Observe that if A satisfies condition (3.1), this is equivalent to saying that for each $k \in \{2, \ldots, n-2\}$ and for each $S \in \mathcal{S}_k$ there exists a $\sigma \in A$ such that $T_k^\sigma = S$. Then, according to Lemma 3.4 there exists also for each $k \in \{2, \ldots, n-2\}$ and for each $S \in \mathcal{S}_k$ a $\sigma \in A$ such that $H_k^\sigma = S$.

The following example illustrates lemma 3.4.

Example 3.3. Let $N = \{1, 2, 3, 4, 5\}$. Then $|\mathcal{S}_3| = 10$. For each $S \in \mathcal{S}_3$ choose a σ such that $T_3^\sigma = S$. Ten such orders are listed below. A vertical line separates the 2-heads from the 3-tails.

$$\begin{array}{ccccc} 45|123 & 35|124 & 34|125 & 25|134 & 24|135 \\ 23|145 & 15|234 & 14|235 & 13|245 & 12|345 \end{array}$$

Observe that for each $S \in \mathcal{S}_3$ there exists a σ such that $T_3^\sigma = S$ and for each $S \in \mathcal{S}_2$ there exists a σ such that $H_2^\sigma = S$. △

Next, we describe a procedure for generating c-complete sets. The key element of this procedure is an iterative step starting from a set of orders where every subset of size k is contained in a k-tail. By rearranging the entries within the k-tails, a new set of orders is created where, besides every subset of size k being contained in a k-tail, also every subset of size $k-1$ is contained in the $(k-1)$-tail of some order in this set.

We determine the right positions of the players in the tails by describing the problem of rearranging tails as a matching problem. We construct suitable bipartite graphs representing the problem and change the orders according to maximum matchings in these specific graphs. Before we formally introduce the procedure, we provide some notation and results from graph theory.

A *graph* G is a pair (V, E) where V is a non-empty, finite set of *vertices* and E is a set of pairs of vertices called *edges*. For an edge $\{a, b\}$ the vertices a, b are said to be the *endpoints* of the edge. The edge $\{a, b\}$ is said to be *incident* to each of the vertices a and b, and the two vertices are said to be *adjacent*. The *degree* of a vertex w is equal to the number of edges incident to w. The *maximum degree*, $\Delta(G)$, of a graph equals the largest degree of a vertex in the graph.

A *bipartite* graph $G(V_1, V_2, E)$ is a graph with vertex set $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and a set of edges E, where every edge in E has one endpoint in V_1 and one in V_2. A bipartite graph is said to be *k-regular* if the degree of all vertices in V equals k, and it is said to be *(k,l)-semiregular* if the degree of all vertices in V_1 and V_2 is k and l respectively.

A subset M of E is called a *matching* in G if no edges in M are incident to the same vertex. A vertex w is said to be covered under M if there is an edge in M that is incident to w.

Consider a set of orders, A, where for any $S \in \mathcal{S}_k$ there exists an order σ such that $T_k^\sigma = S$. We now show that it is always possible to modify the k-tails of the orders in A to obtain a new set A' where, not only does there exist for every $S \in \mathcal{S}_k$ an order $\sigma \in A'$ such that $T_k^\sigma = S$, there also exists for every $S \in \mathcal{S}_{k-1}$ an order $\sigma' \in A'$ such that $T_{k-1}^{\sigma'} = S$. To do so, we first create a graph in which each coalition of size k and $k-1$ is represented by a unique node.

For some $n \geq 3$ and $k \leq \lceil \frac{n}{2} \rceil$, let $G(V_1, V_2, E)$ be a graph with vertex set $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$. Let $|V_1| = \binom{n}{k}$ and define the map $S^k : V_1 \to \mathcal{S}_k$ that assigns to each vertex of V_1 a different coalition of size k. Let $|V_2| = \binom{n}{k-1}$, and define the map $S^{k-1} : V_2 \to \mathcal{S}_{k-1}$ that assigns to each vertex of V_2 a different coalition of size $k-1$. Let $w_1 \in V_1$ and $w_2 \in V_2$ be adjacent if and only if $S^{k-1}(w_2) \subset S^k(w_1)$. Then two nodes within V_1 (or V_2) will never be adjacent and $G(V_1, V_2, E)$ is thus a bipartite graph. Further, every vertex in V_1 has degree k, while every vertex in V_2 has degree $n-k+1$. Hence, G is a $(k, n-k+1)$-semiregular graph. Observe that $n-k+1 \geq k$.

Example 3.4. Let $N = \{1,2,3,4,5\}$. If $k = 3$, we have $|V_1| = |V_2| = 10$, and each vertex in V has degree 3, see Figure 1. If we consider instead the graph for $k = 2$, we see that $|V_1| = 10 > 5 = |V_2|$, and every vertex in V_1 has degree 2 while every vertex in V_2 has degree 4, see Figure 2. $\triangle$

In the graph $G(V_1, V_2, E)$, a matching covering every node in V_2 can be used as a basis for rearranging the k-tails of a set of orders that contains all coalitions of $\mathcal{S}_k$ such that every subset of size $k-1$ is contained in one of the $(k-1)$-tails of the set. The existence of such a matching is straightforward (cf. Berge (1973)).

Next, it is shown how such a matching can be used to determine a proper way of arranging the players within the tails.

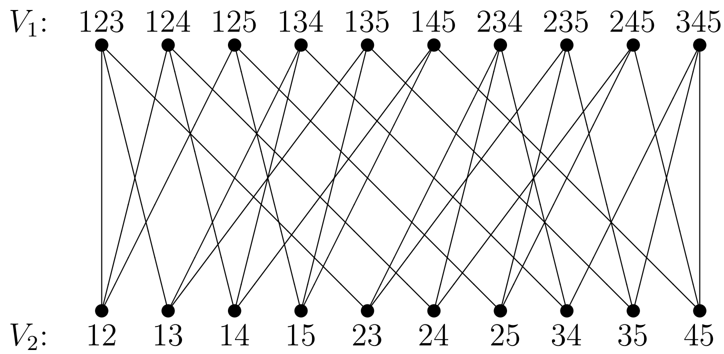

Figure 1: $G(V_1, V_2, E)$ for $n = 5$, $k = 3$

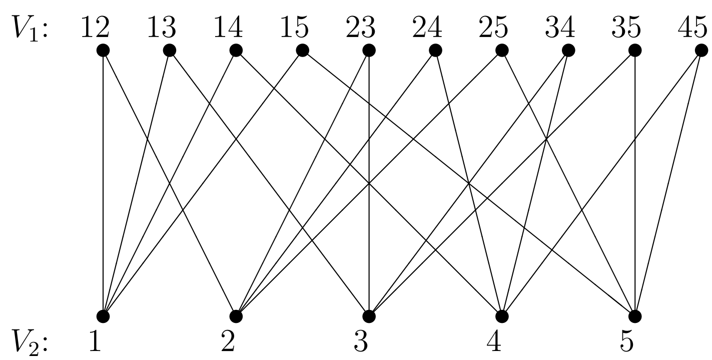

Figure 2: $G(V_1, V_2, E)$ for $n = 5$, $k = 2$

Lemma 3.5. Let $k \in \{1, \ldots, \lceil \frac{n}{2} \rceil\}$ and let A_1, with $|A_1| = \binom{n}{k}$, be a set of orders such that for any $S \in \mathcal{S}_k$ there exists a $\sigma \in A_1$ with $T_k^\sigma = S$. Then there exists a set of orders A_2 that satisfies the following three properties:

(i) for every σ in A_1 there exists a $\tau \in A_2$ such that $\tau(h) = \sigma(h)$ for all $h \in \{1, \ldots, n-k\}$,

(ii) for all $S \in \mathcal{S}_{k-1}$ there exists a $\tau \in A_2$ such that $T_{k-1}^\tau = S$,

(iii) $|A_1| = |A_2|$.

Proof. Consider the graph $G(V_1, V_2, E)$. Let M be a matching covering V_2 in $G(V_1, V_2, E)$. Let $\{w_1, w_2\}$ be an edge in M such that $w_1 \in V_1$ and $w_2 \in V_1$. Take $\sigma \in A_1$ such that $T_k^\sigma = S^k(w_1)$. Consider $S^{k-1}(w_2)$. If $T_{k-1}^\sigma = S^{k-1}(w_2)$, then define $\tau = \sigma$. Otherwise, there exists a j such that $\sigma(j) = S^k(w_1) \backslash S^{k-1}(w_2)$. Now define τ as follows:

$$\begin{aligned} \tau(h) &= \sigma(h) && \text{if } h \in \{1, \dots, n-k\} \cup \{j+1, \dots, n\}, \\ \tau(h) &= \sigma(j) && \text{if } h = n-k+1, \\ \tau(h) &= \sigma(h-1) && \text{if } h \in \{n-k+2, \dots, j\}. \end{aligned}$$

Let A_2 be the set that arises by changing each $\sigma \in A_1$ as above. Then A_2 satisfies properties (i), (ii) and (iii). □

Note that Lemma 3.5 can also be stated in term of heads instead of tails and similarly proven. The following example illustrates Lemma 3.5.

Example 3.5. Let $N = \{1, 2, 3, 4, 5\}$ and consider the set of orders A_1 in which the 3-tails contain every subset of size 3:

45|123	35|124	34|125	25|134	24|135
23|145	15|234	14|235	13|245	12|345

We want to ensure that every 2-player coalition is contained in at least one 2-tail of the ten orders. This is currently not the case since none of the sets $\{1, 2\}, \{1, 3\}, \{1, 4\}$ or $\{1, 5\}$ are contained in any 2-tail. To solve this, consider the graph in Figure 1 and identify a matching covering V_2. The matching M illustrated in Figure 3 is such a matching. Next, we will transform the set A_1 into the new set of orders A_2 in

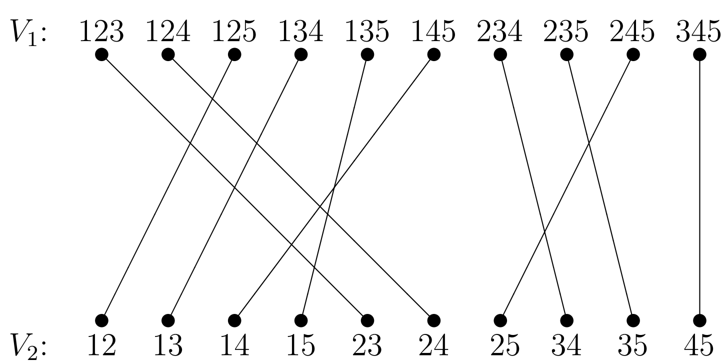

Figure 3: A maximum matching M in $G(V_1, V_2, E)$ for $n = 5$, $k = 3$

which the 2-tails and 3-tails contain every subset of size 2 and 3, respectively. For example, the order $\sigma = (45123) \in A_1$ defines an edge in Figure 3 that connects the sets $S^3(w_1) = \{1, 2, 3\}$ and $S^2(w_2) = \{2, 3\}$. Since $T_2^\sigma = \{2, 3\} = S^2(w_2)$, we define τ

equal to σ. On the other hand $\sigma = (24135) \in A_1$ defines an edge that connects the sets $S^3(w_1) = \{1,3,5\}$ and $S^2(w_2) = \{1,5\}$. Since $T_2^\sigma = \{3,5\} \neq S^2(w_2)$, we define τ by $\tau(1) = \sigma(1), \tau(2) = \sigma(2), \tau(3) = \sigma(4), \tau(4) = \sigma(3)$ and $\tau(5) = \sigma(5)$. By applying this procedure to each $\sigma \in A_1$, we obtain the set A_2:

$$\begin{array}{ccccc} 45|123 & 35|124 & 34|512 & 25|413 & 24|315 \\ 23|514 & 15|234 & 14|235 & 13|425 & 12|345 \end{array}$$

Notice that A_2 is c-complete, according to Lemma 3.2, since $n = 5$ and the tails of the set of orders contain every coalition of size 2 and 3. △

Note that for $k < \lceil \frac{n}{2} \rceil$, a (maximum) matching covering V_2 will leave some nodes in V_1 unmatched since $\binom{n}{k} > \binom{n}{k-1}$ for $k < \lceil \frac{n}{2} \rceil$.

We are now ready to state the main results.

Theorem 3.2. *For any $n \geq 3$ there exists a c-complete set of cardinality $\binom{n}{\lceil \frac{n}{2} \rceil}$.*

Proof. Let $n \geq 3$ and $k = \lceil \frac{n}{2} \rceil$. We show that a c-complete set of cardinality $\binom{n}{\lceil \frac{n}{2} \rceil}$ can be constructed for any $n \geq 3$.

Let A_1, with $|A_1| = \binom{n}{\lceil \frac{n}{2} \rceil}$, be a set of orders that satisfies (3.1) for all $S \in \mathcal{S}_{\lceil \frac{n}{2} \rceil}$. Then there exists for every $S \in \mathcal{S}_{\lceil \frac{n}{2} \rceil}$ a $\sigma \in A_1$ such that $T^\sigma_{\lceil \frac{n}{2} \rceil} = S$.

From Lemma 3.5 we can construct a set of orders $A_2 \subset \Pi(N), |A_2| = |A_1|$, such that there exists for any $S \in \mathcal{S}_{\lceil \frac{n}{2} \rceil - 1}$ a $\tau \in A_2$ with $T^\tau_{\lceil \frac{n}{2} \rceil - 1} = S$. Since Lemma 3.5 holds for any k, we may iteratively apply this argument. This will, after j iterations, result in a set of orders A_{1+j} such that $|A_{1+j}| = |A_1|$ and such that for any $k \in \{\lceil \frac{n}{2} \rceil - j, \ldots, \lceil \frac{n}{2} \rceil\}$ and any $S \in \mathcal{S}_k$ there exists a $\tau \in A_{1+j}$ with $T_k^\tau = S$. Thus, after $\lceil \frac{n}{2} \rceil - 2$ iterations we have constructed the set $A_{\lceil \frac{n}{2} \rceil - 1}$ where for each $S \in \mathcal{S}_k$ and any $k \in \{2, \ldots, \lceil \frac{n}{2} \rceil\}$ there exists a $\tau \in A_{\lceil \frac{n}{2} \rceil - 1}$ with $T_k^\tau = S$.

Since Lemma 3.5 can also be stated in terms of heads, we may apply a similar argument to $A_{\lceil \frac{n}{2} \rceil - 1}$ to iteratively construct new sets by changing the heads of orders. After $\lfloor \frac{n}{2} \rfloor - 2$ iterations this will result in a set A^* such that for each $S \in \mathcal{S}_k$ with $k \in \{2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ there exists a $\tau \in A^*$ with $H_k^\tau = S$.

Then, by Lemma 3.4 there exists for any $S \in \mathcal{S}_k$ with $k \in \{n - \lfloor \frac{n}{2} \rfloor, \ldots, n-2\}$ a $\tau \in A^*$ such that $T_k^\tau = S$. Since there already exists for any $S \in \mathcal{S}_k$ with $k \in \{2, \ldots, \lceil \frac{n}{2} \rceil\}$ a $\tau \in A^*$ such that $T_k^\tau = S$, A^* is c-complete by Lemma 3.2. □

Thus, we have showed that there exists a c-complete set of cardinality $\binom{n}{\lceil \frac{n}{2} \rceil}$ for any $n \geq 3$. The following corollary is a straightforward consequence of Theorem 3.2 and (3.7).

Corollary 3.1. For $n \geq 3$, the minimum cardinality of a c-complete set is $L_n = \binom{n}{\lceil \frac{n}{2} \rceil}$.

From Van Velzen et al. (2004), the minimum cardinality of sets characterising convexity is $M_n = \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}$ for odd n and $M_n = \frac{n!}{2(\frac{n-2}{2})!(\frac{n-2}{2})!}$ for even n. A comparison of the minimum cardinality of complete sets for the cases of compromise stability and convexity is given in Table 2, for $n = 3, \ldots, 9$. It shows that $L_n < M_n$ for any $n > 3$. In fact, the relative size of the minimum cardinalities can be calculated as $L_n/M_n = \frac{8}{n^2-1}$ for odd n and $L_n/M_n = \frac{8}{n^2}$ for even n .

n	3	4	5	6	7	8	9
$n!$	6	24	120	720	5040	40320	362880
M_n	3	12	30	90	210	560	1260
L_n	3	6	10	20	35	70	126

Table 2: Summary of results

4 Partitioning the set of orders

In the previous section, we showed that a c-complete set of cardinality $\binom{n}{\lceil \frac{n}{2} \rceil}$ can be found for any $n \geq 3$. It is easily observable that for e.g. $n = 3$ it is possible to partition the full set of orders, $\Pi(N)$, into disjoint c-complete sets of minimum cardinality, namely the two sets consisting of the odd and even orders, respectively. In this section we show that a partition of $\Pi(N)$ such that each element of the partition is a c-complete set of cardinality $\binom{n}{\lceil \frac{n}{2} \rceil}$ can be found for any $n \geq 3$. Note that such a partition will result in $n!/\binom{n}{\lceil \frac{n}{2} \rceil} = (n - \lceil \frac{n}{2} \rceil)!\lceil \frac{n}{2} \rceil!$ minimum c-complete sets.

We use the procedure from Theorem 3.2 along with properties of the bipartite graphs from the previous section to create the partition.

Let $n \geq 3$ and $k \leq \lceil \frac{n}{2} \rceil$. Let $G(V_1, V_2, V_3)$ be the $(k, n-k+1)$-regular bipartite graph, as defined in the previous section, in which V_1 and V_2 represent all coalitions of size k and $k-1$ respectively. The Integer Flow Theorem immediately implies that there exists k disjoint maximum matchings of G, say $E_1, E_2, \ldots, E_k$. Hence, each matching covers V_2 completely. Now, define the following k edge-disjoint bipartite graphs $G_i = (V_1, V_2, E_i \cup E_i^*)$, $i \in \{1, \ldots, k\}$, where $E_1^*, E_2^*, \ldots, E_k^*$ is a partition of the edges not included in any matching, i.e. $\cup_{j=1}^{k} E_j^* = E \setminus \cup_{j=1}^{k} E_j$, and $|E_j^*| = \binom{n}{k}\left(\frac{n-2k+1}{n-k+1}\right)$, for all $j \in \{1, 2, \ldots, k\}$, and the degree of each vertex of V_1 in G is equal to one, We illustrate this partition in the following example.

Example 4.1. Let $N = \{1, 2, 3, 4, 5\}$, and let $k = 2$. Then the corresponding $(2, 4)$-regular bipartite graph $G = (V_1, V_2, E)$ is displayed in Figure 4. Figure 5 represents the edge-disjoint partition G_1, G_2 in which the bold edges correspond to the maximum matchings. △

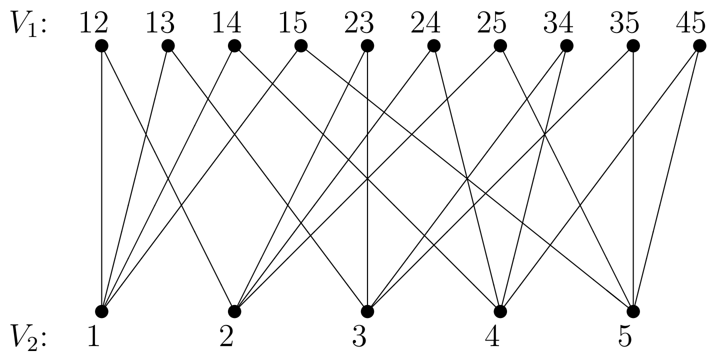

Figure 4: $G(V_1, V_2, E)$ for $n = 5$, $k = 2$

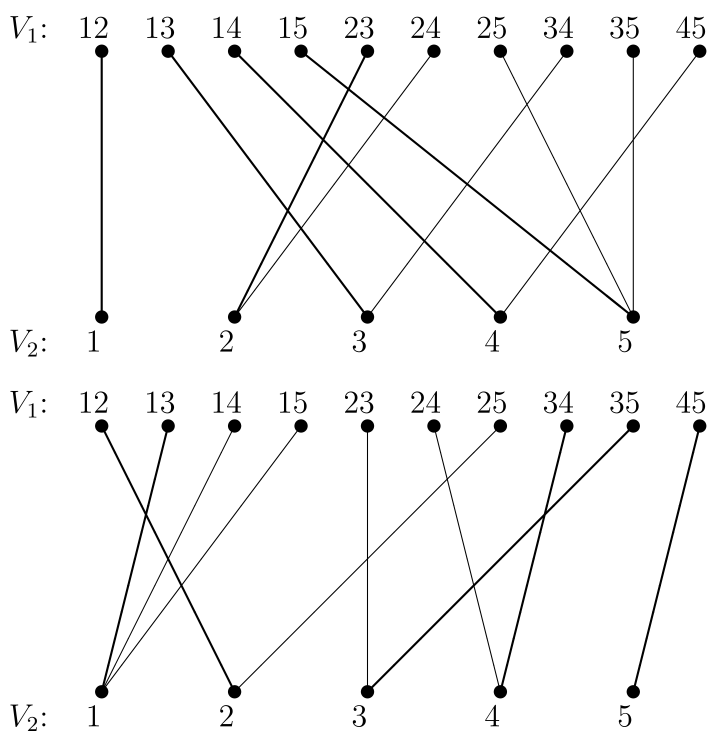

Figure 5: An edge-partitioning G_1, G_2 of G

Since the above described partition is based on disjoint matchings, the following lemma is a straightforward generalization of Lemma 3.5. The proof is similar to the proof of Lemma 3.5., and therefore, it is omitted.

Lemma 4.1. Let $n \geq 3$ and $k \in \{1, \ldots, \lceil \frac{n}{2} \rceil\}$, and let A be a set of orders such that for any $S \in \mathcal{S}_k$ there exists a $\sigma \in A$ with $T_k^\sigma = S$. Then we can construct k disjoint sets $A_1, \ldots, A_k \subseteq N$, such that for each $j \in \{1, \ldots, k\}$ it holds that:

(i) for each σ in A, there exists a $\tau \in A_j$ such that $\tau(h) = \sigma(h)$ for all $h \in \{1, \ldots, n-k\}$,

(ii) for every $S \in \mathcal{S}_{k-1}$, there exists a $\tau \in A_j$ such that $T_{k-1}^\tau = S$,

(iii) $|A_j| = |A|$.

Note, that also this lemma can be stated in terms of heads as well as tails.

The following theorem is a straightforward result of Lemma 4.1 and Theorem 3.2.

Theorem 4.1. *For any $n \geq 3$ the set of orders $\Pi(N)$ can be partitioned into $\lceil \frac{n}{2} \rceil!(n - \lceil \frac{n}{2} \rceil)!$ disjoint c-complete sets of cardinality $\binom{n}{\lceil \frac{n}{2} \rceil}$.*

In the final example we provide a partition of the orders if $N = \{1, 2, 3, 4, 5\}$.

Example 4.2. Table 3 provides a partition of the set of orders of $\Pi(N)$ with $N = \{1, 2, 3, 4, 5\}$ where every row corresponds to a c-complete set. Observe that it is not a unique partition, because the collection of disjoint maximum matchings is not unique.

54|132	35|142	43|521	25|413	42|315	32|514	15|243	14|235	13|425	21|354
54|123	35|124	43|512	25|431	42|351	32| 541	15|234	14|253	13|452	21|345
54|321	35|214	43|125	25|143	42|513	32|415	15|342	14|532	13|254	21|435
54|312	35|241	43|125	25|134	42|531	32|451	15|324	14|523	13|245	21|453
54|213	35|421	43|215	25|314	42|135	32|154	15|432	14|325	13|542	21|543
54|231	35|412	43|251	25|341	42|153	32|145	15|423	14|352	13|524	21|534
45|132	53|142	34|521	52|413	24|315	23|514	51|243	41|235	31|425	12|354
45|123	53|124	34|512	52|431	24|351	23| 541	51|234	41|253	31|452	12|345
45|321	53|214	34|125	52|143	24|513	23|415	51|342	41|532	31|254	12|435
45|312	53|241	34|125	52|134	24|531	23|451	51|324	41|523	31|245	12|453
45|213	53|421	34|215	52|314	24|135	23|154	51|432	41|325	31|542	12|543
45|231	53|412	34|251	52|341	24|153	23|145	51|423	41|352	31|524	12|534

Table 3: Twelve disjoint c-complete sets of minimum cardinality for $n = 5$

△

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Chapter 3

On games arising from multi-depot Chinese postman problems

Trine Tornøe Platz, Herbert Hamers

On games arising from multi-depot Chinese postman problems

Trine Tornøe Platz[1] Herbert Hamers[2]

Abstract

A multi-depot Chinese postman problem is represented by a connected graph G, a set of k depots that is a subset of the vertices of G, and a non-negative weight function on the edges of G. A solution to the problem is a minimum weight tour of the graph consisting of a collection of subtours, such that the subtours originate from different depots, and each subtour starts and ends at the same depot. A cooperative Chinese postman game is induced from a multi-depot Chinese postman problem by associating every edge of the graph with a different player and addressing the problem of allocating between these players the cost of the minimum weight tour. This paper considers multi-depot Chinese postman (k-CP) games and characterizes locally and globally k-CP balanced and submodular graphs. A graph G is called locally (globally) k-CP balanced (respectively submodular), if the k-CP game induced by a k-CP problem on G is balanced (respectively submodular), for some (any) choice of depots and every non-negative weight function.

Keywords: Chinese postman problem, cooperative game, submodularity, balancedness.

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[1]Department of Economics, University of Copenhagen, Øster Farimagsgade 5, building 26, 1353 Copenhagen K, Denmark

[2]Department of Econometrics & OR and CentER, Tilburg University, P.O. Box 90153, 5000 LE, Tilburg, The Netherlands.

1 Introduction

A Chinese postman problem (CPP) models the situation in which a postman must deliver mail to a given set of streets using the shortest possible route, under the constraint that he must start and end at the post office, see e.g. Edmonds and Johnson (1973). In this paper, we consider Chinese postman problems in which multiple depots exists. Applications of the (multi-depot) CPP are many and include road sweeping, snow plowing, and garbage collection.

A multi-depot Chinese postman problem (k-CPP) is represented by a graph in which the edges of the graph correspond to the streets to be visited, a fixed set of k vertices serves as depots, and a non-negative weight function is defined on the edges. A solution to the problem is a minimum weight tour consisting of a collection of sub-tours such that every edge of the graph is visited, and such that the subtours originate at different depots. Furthermore, each subtour must start at a depot, visit a set of edges, and return to same depot.[1] The 'standard' Chinese postman problem arises as a special case of the k-CPP, when $k = 1$.

If each edge of the graph is associated with a different player, a related problem is how to allocate between these players the incurred cost of the minimum weight tour. This cost allocation problem can be addressed in the form of a cooperative (cost) game, and we refer to cooperative games induced by k-CP problems as k-CP games. The topic of this paper is the analysis of k-CP games.

A k-CP game is an example of a so-called operations research game (OR-game). OR-games are cooperative games addressing (cost) allocation problems that arise from optimization problems in the operations research literature, see Borm et al. (2001) for a survey. The OR-games considered in this paper arise from an underlying network problem. Other examples of OR-games arising from network problems include the Chinese postman (CP) games (Hamers et al. (1999), Hamers (1997), Granot et al. (1999)), traveling salesman games (Potters et al. (1992), Herer and Penn (1995)), highway games (Çiftçi et al. (2010)), and minimum coloring games (Deng et al. (1999)).

In the present paper, we explore the (total) balancedness and submodularity of k-CP games. In a (totally) balanced game, the core (of every subgame) is non-empty, and submodular games have several desirable properties. For example, submodular games are (totally) balanced, and the Shapley value of a submodular game is the barycenter

[1] The k-CPP can be seen as a special case of the multi-depot capacitated arc routing problem (MD-CARP) described in, e.g., Wøhlk (2008), and Kansou and Yassine (2010).

of the core, Shapley (1971). Furthermore, some solution concepts coincide for this class of games. The kernel is equal to the nucleolus, and the bargaining set coincides with the core, Maschler et al. (1972).

A connected graph or a strongly connected digraph G is said to be globally k-CP submodular (balanced) (totally balanced) if for all $Q \in V(G)$ with $|Q| = k$, the induced k-CP game is submodular (balanced) (totally balanced) for every non-negative weight function.[2] G is locally k-CP submodular (balanced) (totally balanced) if for some $Q \in V(G)$ with $|Q| = k$, the induced game is submodular (balanced) (totally balanced) for all non-negative weight functions.

We characterize classes of globally and locally k-CP balanced and k-CP submodular graphs and digraphs. We show that for $k = 1$ and $k \geq |V(G) - 1|$, a graph G is globally k-CP submodular if and only if G is weakly cyclic, where a graph is called weakly cyclic if every edge is part of at most one cycle. Furthermore, we find that no connected graph is globally k-CP submodular, for $1 < k < |V(G) - 1$. On the other hand, G is locally k-CP submodular for $k \in \{1, \ldots, |V(G)|\}$ if and only if G is weakly cyclic, and the depots can be located in a specific pattern.

The corresponding characterizations for the standard version of the CP game can be found in the literature. Hamers et al. (1999) introduced CP games and showed that weakly Eulerian graphs are CP-balanced, while Hamers (1997) showed that weakly cyclic graphs are CP-submodular.[3] Full characterizations of the classes of CP-balanced and CP-submodular graphs were given in Granot et al. (1999). In Granot et al. (2004), the distinction between global and local requirements was made, and the authors characterized the classes of locally CP-submodular graphs and digraphs. Similar characterizations of classes of graphs exists in the literature for other types of OR-games, see e.g. Herer and Penn (1995) and Granot et al. (2000) for the case of traveling salesman games.

The paper is organized as follows. In section 2, some terms and notions from game theory and graph theory are introduced. In section 3, the model is presented, and in section 4, we analyze k-CP games and characterize the classes of k-CP balanced and submodular graphs. Section 4.1 considers undirected graphs, while the case of directed

[2] This notation follows Granot et al. (1999), where a graph is said to be CP balanced (submodular) (totally balanced) if every single-depot CP-problem on this graph gives rise to a balanced (submodular) (totally balanced) CP game.

[3] A graph is said to be weakly Eulerian if the components that remain after removing all bridges in the graph are either Eulerian or singletons, where a component is Eulerian if every vertex is of even degree.

graphs is analyzed in 4.2. Finally, section 5 presents some results on k-CP totally balanced games.

2 Preliminaries

Before we present the model, we first recall some definitions and terms from cooperative games and graph theory, respectively.

A cooperative (cost) game is a pair (N, c) (often referred to simply as c when no confusion arises) in which $N = \{1, \dots, n\}$ is a finite set of players denoted the grand coalition, and $c : 2^N \to \mathbb{R}$ is a function that assigns to every *coalition* $S \subseteq N$ a *cost* $c(S)$, with $c(\emptyset) = 0$. $x \in \mathbb{R}^N$ is an allocation of the cost of the grand coalition, $c(N)$, between the players. The *core* of a game c is defined by

$$Core(c) = \{x \in \mathbb{R}^N | \sum_{i=1}^{n} x_i = c(N), \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N\}.$$

Thus, the core is the set of efficient allocations in which no coalition has an incentive to split from the grand coalition. The core of a game may be empty. A game in which the core is non-empty is said to be *balanced*. Let c^S denote the restriction of c to the subsets of players in S. Then, if the subgame (S, c^S) is balanced for every $S \subseteq N$, the game, (N, c), is said to be *totally balanced*. A game is submodular if it holds for all $j \in N$ and all $S \subset T \subseteq N \setminus \{j\}$ that:

$$c(T \cup \{j\}) - c(T) \leq c(S \cup \{j\}) - c(S). \tag{2.1}$$

A submodular game is totally balanced, Shapley (1971).

An undirected (directed) *graph* G is a pair $(V(G), E(G))$ where $V(G)$ is a non-empty, finite set of *vertices*, and $E(G)$ is a set of pairs of vertices called *edges (arcs)*. Throughout the paper, we let $m = |V(G)|$ denote the cardinality of $V(G)$. An edge $\{a, b\}$ joins the vertices a, b in an undirected graph. An arc (a, b) that joins the vertices a and b in a directed graph is directed from a to b and can only be traversed in this direction. The vertices of an edge (arc) $\{a, b\}$ are the *endpoints* of the edge. An edge (arc) is said to be *incident* to each of its endpoints, and the two endpoints are said to be *adjacent*.

A *(directed) walk* is a sequence of vertices and edges (arcs) $v_0, e_1, v_1, \dots, v_{m-1}, e_m, v_m$, in which $m \geq 0$, $v_0, \ldots, v_m \in V(G)$, and $e_1, \ldots, e_m \in E(G)$ such that $e_j = \{v_{j-1}, v_j\}$ for all $j \in \{1, \ldots m\}$. If $v_0 = v_m$, the walk is said to be closed. A *(directed) path* is a (directed) walk in which no edge (arc) or vertex is visited more than once. A (directed) circuit is a closed (directed) path. In a path $v_0, e_1, v_1, \ldots, v_{m-1}, e_m, v_m$, the vertices v_0 and v_m are called the *endpoints* of the path. If there exists an undirected (directed) path between any to vertices in a graph, then the graph is said to be *connected* (*strongly connected*). In a connected graph G, an edge $b \in E(G)$ is called a *bridge*, if removing b results in a graph that is not connected. The set of all bridges in G is denoted $B(G)$. A graph G is said to be *Eulerian*, if the degree of every edge in $E(G)$ is even. We say that a connected graph G is *weakly Eulerian*, if removing all bridges in G results in a disconnected graph in which every connected component is Eulerian. Furthermore, we say that a graph G is *weakly cyclic*, if every edge in G belongs to a most one cycle. Note that a weakly cyclic graph is weakly Eulerian.

3 k-CP games

Let $G = (V(G), E(G))$ be a connected undirected (or strongly connected directed) graph in which $V(G)$ denotes the set of vertices, and $E(G)$ denotes the set of edges (arcs). Furthermore, let $Q \subseteq V(G)$ be a fixed subset of the vertices, which will be referred to as depots. Let Q be of cardinality k with $k \in \{1, \ldots, m\}$, and let $S \subseteq E(G)$. Then an S-tour wrt. Q is a collection of closed walks such that every player in S is visited, and such that each closed walk originates from a different depot in Q where it both starts and ends. Formally, let $Q = \{v_0^1, \ldots, v_0^k\}$. Then for an S-tour $d(S)$ in G, there exists a $p \in \{1, \ldots, k\}$ such that $d(S) = \{(v_0^1, e_1^1, v_1^1, ..., e_{m^1}^1, v_0^1) ..., (v_0^p, e_1^p, v_1^p, ..., e_{m^p}^p, v_0^p)\}$. The set of S-tours associated with $S \subseteq E(G)$ is denoted by $D(S)$.

Let $t : E(G) \to [0 : \infty)$ be a non-negative weight function defined on the edges (arcs) of G. Then for every $S \subseteq N$ we can assign a cost to the S-tours in the graph. The cost of the S-tour $d(S) = \{(v_0^1, e_1^1, v_1^1, ..., e_{m^1}^1, v_0^1) ..., (v_0^p, e_1^p, v_1^p, ..., e_{m^p}^p, v_0^p)\}$ is equal to

$$C(d(S)) = \sum_{l=1}^{p} \sum_{j=1}^{m^l} t(e_j^l).$$

Let $\Gamma = (E(G), (G, Q), t)$ be a multi-depot CP problem in which $E(G)$ is the set of players (edges, arcs), Q is a set of depots with cardinality k, and $t : E(G) \to [0 : \infty)$ is the non-negative weight function. The induced k-CP game (N, c) is then defined by

$$c(S) = \min_{d(S) \in D(S)} C(d(S)).$$

That is, for any $S \subseteq N$, $c(S)$ equals the cost of a minimum weight S-tour in G. To emphasize the set of depots in the underlying k-CP problem, we refer to the k-CP game induced by $\Gamma = (E(G), (G, Q), t)$ as (N, c_Q) whenever relevant. Note that for $k = 1$, the k-CP situation and its induced game will coincide with the class introduced in Hamers (1999).[4]

An example of a k-CP game is given below. It shows that not all graphs are globally k-CP balanced.

Example 3.1. Consider Figure 1 below. Assume that $Q = \{v_0, v_2\}$, and let $t(e) =$

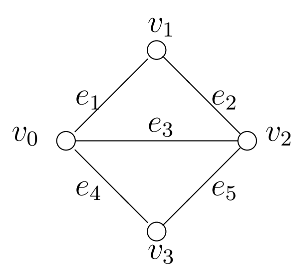

Figure 1: Non k-CP-balanced graph

1 for all edges in the graph. Then we see that $c(e_1, e_2, e_3) = c(e_3, e_4, e_5) = 3$, and $c(e_1, e_2, e_4, e_5) = 4$. However, since $c(N) = 6$, the worth of the grand coalition can not be allocated between the players such that $x(N) = c(N)$ without violating $x(S) \leq c(S)$ for some $S \subset N$. The game is therefore not balanced, and the graph is not k-CP balanced for $k = 2$. $\triangle$

The above example showed that even when $k > 1$, some graphs are not globally k-CP balanced. From Granot et al. (1999), all weakly cyclic graphs are globally k-CP submodular for $k = 1$, but for the case of $k > 1$, a weakly cyclic graph is not necessarily globally k-CP submodular, as the following example shows. This example also illustrates how different choices of Q with the same cardinality may lead to different games.

[4]In Hamers (1999), the games are called delivery games, while the term Chinese postman (CP) games is used in subsequent publications on this topic.

Example 3.2. Consider the undirected graph in Figure 2, and let $t(e) = 1$ for all edges in the graph. Let $Q = \{v_0, v_1\}$ and $Q' = \{v_0, v_3\}$. The induced games are shown in Table 1. The second row corresponds to the induced game when depots are located at $\{v_0, v_1\}$, while the third row illustrates the costs of the coalitions in the induced game, when depots are located at $\{v_0, v_3\}$. For example, in case the depots are at $\{v_0, v_3\}$, the min. S-tour of $S = \{e_1, e_3\}$ is equal to $\{(v_0, e_1, v_1, e_1, v_0), (v_3, e_3, v_3)\}$ with an associated cost of 4. For ease of exposition a coalition $\{e_i, e_j\}$ is in the table written as ij.

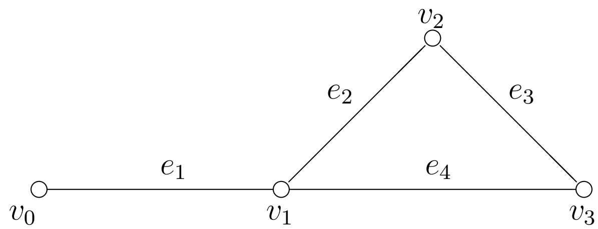

Figure 2

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$c_{\{v_0,v_1\}}(S)$	2	2	3	2	4	5	4	3	3	3	5	5	5	3	5
$c_{\{v_0,v_3\}}(S)$	2	3	2	2	4	4	4	3	3	3	5	5	5	3	5

Table 1: Two different 2-CP games arising from the same graph

From Table 1, it is evident that different choices of depots leads to different games. Furthermore, while the two games are both balanced, only the first is submodular, since $c_{\{v_0,v_3\}}(e_1, e_2, e_3) - c_{\{v_0,v_3\}}(e_2, e_3) = 2 > 1 = c_{\{v_0,v_3\}}(e_1, e_2) - c_{\{v_0,v_3\}}(e_2)$. $\triangle$

While we have just shown that not all k-CP games are balanced or submodular, it is straightforward to verify that every k-CP game (N, c) satisfies the following: $c(S) \leq c(T)$ for all $S \subset T \subseteq N$ (monotonicity), and $c(S \cup T) \leq c(S) + c(T)$ for all S, T with $S \cap T = \emptyset$ (subadditivity). Furthermore, the worth of the grand coalition is independent of Q and k. That is, if we let $Q, Q' \subset V(G)$, $Q' \neq Q$ and consider the games (N, c_Q) and $(N, c_{Q'})$, then

$$c_Q(N) = c_{Q'}(N), \text{ for all } Q, Q' \subset V(G).$$

It follows that a 2-player k-CP game is submodular, and it can be verified that all graphs with $|V(G)| \leq 3$ are k-CP submodular. In the following, we therefore restrict attention to graphs $G = (V(G), E(G))$ with $|V(G)| \geq 4$.

Below we provide a lemma stating how the presence of certain structures in a k-CP problem precludes submodularity of the induced game. First some notation is, however, needed.

Let $\{a, b\}$ be an edge in a graph G. Then $\{a, b\}$ is said to be *subdivided*, if we replace it by a path of length two, that is, if we construct a new vertex c and replace the edge $\{a, b\}$, by the edges $\{a, c\}$, $\{c, b\}$. A subdivision of a graph G is then any graph that can be obtained by recursively subdividing edges in G. Furthermore, let P_4 denote a path with four vertices in which the endpoints of the path are (the only) depots, and let S_4 denote a star graph with four vertices in which exactly one edge is not incident to a depot. P_4 and S_4 are illustrated in Figure 3. Let $G = (E(G), V(G))$ be an undirected

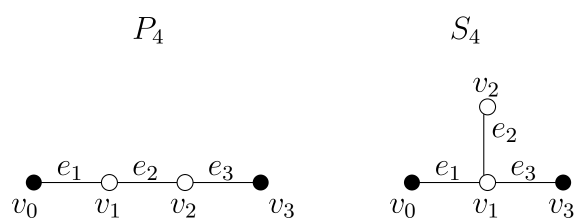

Figure 3: Path graph and star graph

connected graph, and let $Q \subseteq V(G)$. Then we say that G is $(P_4, S_4)-$*free wrt.* Q, if no subdivision of P_4 or S_4 exists in G when depots are located at the vertices in Q. We state the following lemma.

Lemma 3.1. Let $G = (E(G), V(G))$ be a connected undirected graph. Let $\Gamma = (E(G), (G, Q), t)$, and let (N, c) be the induced k-CP game. If G is not $(P_4, S_4)-$free wrt. Q, then (N, c) is not submodular for all non-negative weight functions.

Proof. Let $Q \subseteq V(G)$ be such that G is not (P_4, S_4)-free wrt. Q. Then there exists a subgame (U, c^U) such that this subgame is defined on a subdivision of P_4 or S_4. We show that (U, c^U) is not submodular. Consider the graphs in Figure 3, and let $t(e) = 1$ for all edges in the graphs. Let $S = \{e_2\}$ and $T = \{e_1, e_2\}$. Then, in both cases,

$$c(T \cup \{e_3\}) - c(T) = 6 - 4 = 2 > 0 = 4 - 4 = c(S \cup \{e_3\}) - c(S),$$

and the game is not submodular. Since we can choose a weight function on U so as to mimic the situation in Figure 3, (U, c^U) is not submodular for all non-negative weight functions, and neither is (N, c). $\square$

We are now ready to proceed to characterizing classes of graphs leading to k-CP games with desirable properties.

4 k-CP graphs

In this section, we characterize k-CP submodular and k-CP-balanced graphs and digraphs, and in addition, we consider both global and local requirements, for each of the different properties.

4.1 Undirected k-CP graphs

We analyze first the case of undirected graphs and characterize connected k-CP balanced graphs. We find that the class of connected k-CP balanced graphs coincide with the class of connected CP-balanced graphs characterized in Granot et al. (1999).

Theorem 4.1. *Let $G = (V(G), E(G))$ be a connected graph. Then the following statements are equivalent:*

(i) G is weakly Eulerian,

(ii) G is globally k-CP balanced for all $k \in \{1, \dots, m\}$,

(iii) G is locally k-CP balanced for all $k \in \{1, \dots, m\}$.

Proof. $(i) \to (ii)$: Let $k \in \{1, \dots, m\}$, and let $Q \subseteq V(G)$ have cardinality k. Let $\Gamma = (E(G), (G, Q), t)$ be a k-CP situation for which (N, c) is the induced k-CP game. We have to show that (N, c) is balanced.

Define an allocation $x \in \mathbb{R}^N$ as:

$$x(e) = \begin{cases} 2t(e) & \text{if } e \in B(G), \\ t(e) & \text{otherwise.} \end{cases}$$

Since G is weakly Eulerian, $c(N) = \sum_{e \in E(G)} t(e) + \sum_{e \in B(G)} t(e)$, and consequently, x is efficient.[5] Furthermore, $c(S) \geq \sum_{e \in S} t(e) + \sum_{e \in B(G) \cap S} t(e) = x(S)$, where the

[5] In a weakly Eulerian graph there exists a min N-tour that visits every bridge in the graph twice and all other edges exactly once.

inequality holds since every player in S must be visited at least once, and every player on a bridge must be visited twice, for any location of the depots. Hence, (N, c) is balanced for every Q of cardinality k, and G is globally k-CP balanced.

(ii) $\rightarrow$ *(iii)*. Follows readily, since each globally k-CP balanced graph is locally k-CP balanced.

(iii) $\rightarrow$ *(i)*: Let $\Gamma = (E(G), (G, Q), t)$ and assume that G is not weakly Eulerian, but (N, c) is balanced. Then by Granot et al. (1999) the cardinality of Q is at least two. Consider a $v \in Q$, the corresponding $\Gamma_1 = (E(G), (G, v), t)$, and the induced CP-game (N, c_v). Then since $c_v(N) = c_Q(N)$ and $c_v(S) \geq c_Q(S)$, we may infer that $Core(c_Q) \subseteq Core(c_v)$. From Granot et al. (1999), there exists a t such that $Core(c_v) = \emptyset$, and therefore $Core(c_Q) = \emptyset$. Thus, G is not locally k-CP balanced. □

We proceed to consider k-CP submodular graphs. We show that for $k \in \{2, \ldots, m-2\}$, no undirected connected graph is globally k-CP submodular, and we characterize globally k-CP submodular graphs, for $k \in \{1, m-1, m\}$.

Theorem 4.2. *Let $G = (V(G), E(G))$ be a connected graph. If $k \in \{2, \ldots, m-2\}$, then G is not globally k-CP submodular.*

Proof. Since $|V(G)| \geq 4$, there exists in G a connected subgraph with four vertices. The possible non-isomorphic structures of this subgraph are displayed in Figure 4. Let

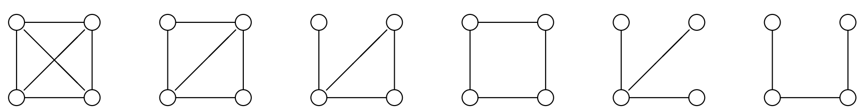

Figure 4: Non-isomorphic connected graphs with 4 vertices

$\Gamma = (E(G), (G, Q), t)$, and let (N, c) be the induced k-CP game. Then there exists a 3-player subgame (U, c^U) that is defined on a path with three edges or a star graph with three edges. Since $k \leq m-2$, we can in either case assign the k depots to the vertices of G, such that G is not (P_4, S_4)-free wrt. Q. It, therefore, follows from Lemma 3.1 that G is not globally k-CP submodular. □

For $k \in \{1, m-1, m\}$ we state the following result:

Theorem 4.3. *Let $G = (V(G), E(G))$ be a connected graph and let $k \in \{1, m-1, m\}$. Then G is globally k-CP submodular if and only if G is weakly cyclic.*

Proof. From Granot et al. (1999), it is sufficient to consider the case of $k \geq m-1$. Let $\Gamma = (E(G), (G, Q), t)$, let (N, c) be the induced game, and let $k \in \{m-1, m\}$. For the 'if part', we have to show that (2.1) holds for all $e \in N$ and all $S \subset T \subseteq N \setminus \{e\}$. Let G be weakly cyclic. Then G consists of a set of edge-disjoint circuits and bridges. Let $C(G)$ and $B(G)$ denote the set of circuits and bridges, respectively, and observe that for any $S \subseteq N$, we have:

$$c(S) = \sum_{C \in C(G): C \cap S \neq \emptyset} \min\{\sum_{a \in C} t(a), \sum_{a \in S \cap C} 2t(a)\} + \sum_{b \in B(G): b \in S} 2t(b), \tag{4.1}$$

and recall that we need to show

$$c(S \cup \{e\}) - c(S) \geq c(T \cup \{e\}) - c(T) \text{ for all } S \subset T \subseteq N. \tag{4.2}$$

We distinguish between two cases:

Case 1. $e \in B(G)$. From (4.1) it readily follows that $c(S \cup \{e\}) - c(S) = c(T \cup \{e\}) - c(T) = 2t(e)$ for all $e \in N$ and all $S \subset T \subseteq N \setminus \{e\}$, so (4.2) is satisfied.

Case 2. $e \notin B(G)$. Then there exists a $C \in C(G)$ such that $e \in C$. Let $A = \sum_{a \in C} t(a)$, $B = \sum_{a \in S \cap C} 2t(a)$, and $D = \sum_{a \in T \cap C} 2t(a)$. Then (4.1) implies that

$$\begin{aligned} c(S \cup \{e\}) - c(S) &= \min\{A, B + 2t(e)\} - \min\{A, B\}, \text{ and} \\ c(T \cup \{e\}) - c(T) &= \min\{A, D + 2t(e)\} - \min\{A, D\}. \end{aligned}$$

Observe that $B \leq D$. Now,

$$\begin{aligned} &\text{if } A \leq B \text{ then } c(S \cup \{e\}) - c(S) = c(T \cup \{e\}) - c(T) = 0, \\ &\text{if } A > B \text{ and } A > D + 2t(e), \text{ then } c(S \cup \{e\}) - c(S) = c(T \cup \{e\}) - c(T) = 2t(e), \\ &\text{if } A > B \text{ and } A \leq D + 2t(e), \text{ then } c(T \cup \{e\}) - c(T) = A - \min\{A, D\} \\ &\qquad = \max\{0, A - D\} \leq \min\{A - B, 2t(e)\} = c(S \cup \{e\}) - c(S), \end{aligned}$$

where the last inequality follows from $B \leq D$ and $A \leq D + 2t(e)$. Thus, (4.2) is satisfied for all $e \in N$ and all $S \subset T \subseteq N \setminus \{e\}$.

Turning to the 'only if' part, we note that if G is not weakly cyclic, there exists a subgraph G_1 in G that is on the form of the graph in Figure 5. Consider Figure 5 and let E_1, E_2 and E_3 denote the set of edges in each of the three paths between w_1

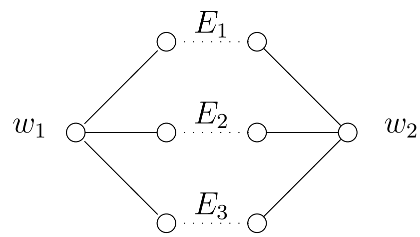

Figure 5

and w_2, respectively. Let the weight function on the edges of G be such that $t(e) = 1$ for every $e \in E_1 \cup E_2 \cup E_3$, while $t(e)$ is arbitrarily large for all $e \in E(G) \setminus E(G_1)$. Since $k \geq m - 1$, either w_1 or w_2 is in Q, and then since a (sub)tour in the graph must start and end at the same depot, an allocation $x \in \mathbb{R}^N$ in the core of the (sub)game $(E(G_1), c^{E(G_1)})$ must fulfill the following:

$$x(E_1 \cup E_2) \leq c(E_1, E_2) = |E_1| + |E_2|,$$
$$x(E_1 \cup E_3) \leq c(E_1, E_3) = |E_1| + |E_3|,$$
$$x(E_2 \cup E_3) \leq c(E_2, E_3) = |E_2| + |E_3|.$$

Adding the inequalities leads to

$$x(N) \leq |E_1| + |E_2| + |E_3| < c(N),$$

and $(E(G_1), c^{E(G_1)})$ is not balanced. Since G_1 is not globally k-CP balanced, we know that G (and any other supergraph of G_1) is not globally k-CP totally balanced, and therefore, G is not globally k-CP submodular. □

The theorems above stated that only weakly cyclic graphs are globally k-CP submodular, and that this is only the case if there is just one depot or every edge is incident to a depot. In the following subsection, we relax the strict global requirement and consider undirected locally k-CP submodular graphs. The following theorem follows readily from Lemma 3.1

Theorem 4.4. *Let $G = (V(G), E(G))$ be a connected graph. If there exists no $Q \subseteq V(G)$ with cardinality k such that G is (P_4, S_4)-free wrt. Q, then G is not locally k-CP submodular.*

Being (P_4, S_4)-free is, however, not sufficient for connected graphs in general to be k-CP submodular, as the following example shows.

Example 4.1. Consider Figure 5 and assume that $Q = V(G)$ implying that every vertex in the graph is associated with a depot, and that the graph is (P_4, S_4)-free. Let $S = \{e_4, e_5\}$ and $T = \{e_1, e_2, e_4, e_5\}$. Then $c(T \cup \{e_3\}) - c(T) = 2 > 1 = c(S \cup \{e_3\}) - c(S)$, and the game is not submodular. △

This leads us to the following result:

Theorem 4.5. *Let G be a connected graph. If there exists a $Q \subseteq V(G)$ with cardinality k such that G is $(P_4, S_4)-$free wrt. Q, then G is locally k-CP submodular if and only if G is weakly cyclic.*

Proof. For the if ' part, let G be a weakly cyclic graph, let $\Gamma = (E(G), (G, Q), t)$, and let (N, c) be the induced k-CP game. We have to show that $c(S \cup \{e\}) - c(S) \geq c(T \cup \{e\}) - c(T)$ for all $e \in N$ and all $S \subset T \subseteq N \setminus \{e\}$.

First, let G_1 be the subgraph induced by all paths between the vertices of Q, and observe that G_1 is also weakly cyclic. Then every edge in $E(G_1)$ is incident to at least one vertex of Q, since G is (P_4)-free. Furthermore, since G is (S_4)-free, the edges in $E(G) \setminus E(G_1)$ can be partitioned into r sets $\{A_1, \ldots, A_r\}$ such that for each A_i with $i \in \{1, \ldots, r\}$, there exists a unique $a_i \in Q$ such that $G \setminus \{a_i\}$ is disconnected, and a path from a_i to any edge $e \in A_i$ contains no other depot than a_i. We let c_{a_i} denote the one-depot game in which the single depot is located at a_i, and consequently, we have:

$$c(S) = c(S \cap E(G_1)) + \sum_{i=1}^{r} c_{a_i}(S \cap A_i).$$

Observe that G_1 is weakly cyclic, and that every edge in G_1 is incident to at least one vertex in Q. Then for every coalition $U \subseteq E(G_1)$, the cost $c(U)$ can be expressed as in (4.1), and we can follow the proof of Theorem 4.3 to prove that (2.1) holds for every $e \in E(G_1)$. Furthermore, it follows from Granot et al. (1999) that (N, c_{a_i}) is submodular for all $i \in \{1, \ldots, r\}$, so (2.1) holds for every $e \in A_i$, $i \in \{1, \ldots, r\}$. We conclude that (N, c) is submodular.

For the 'only if' part, note that if G is not weakly cyclic, there exists a subgraph G^* with a structure on the form of Figure 5. We need to show that for every k, there exists a t and a location of the k depots in G for which (N, c) is not totally balanced and hence not submodular. For $k = 1$ the result follows from Granot and Hamers (2004). We consider $k \geq 2$ and distinguish between three cases:

Case 1. $Q \cap V(G^*) = \emptyset$: Since G is connected, there exists for every vertex $v_0 \in Q$ a vertex $v \in V(G^*)$ such that the path from v_0 to v contains no other vertices in $V(G^*)$. Choose a pair v_0, v such that the path P_1 from v_0 to v contains no vertices of Q. If $v \neq w_1, w_2$, let P_2 be the path from v to w_1 that visits only the edges of E_i, for some $i \in \{1, 2, 3\}$. Let $E(P_i)$ denote the set of edges in the path P_i, and let $t(E_i \setminus E(P_2))$ denote the sum of the weights of the edges in $E_i \setminus E(P_2)$. We choose a weight function t such that $t(e) = 0$ for all $e \in E(P_1) \cup E(P_2)$, $t(E_i \setminus E(P_2)) = 1$ for all $i \in \{1, 2, 3\}$, and $t(e) = 100$ for all other edges in G. Now, the cost of a min weight tour that visits every edge in G^* is equal to 4, while the cost of a min weight tour visiting all edges of $E_i \cup E_j$, $i \neq j$, $i, j \in \{1, 2, 3\}$ is equal to 2.

Case 2. $|Q \cap V(G^*)| = 1$. Consider $v_0 \in Q \cap V(G^*)$ and note that for this choice of v_0, P_1 consists only of v_0. If $v_0 \neq w_1, w_2$, let P_2 denote the path from v_0 to w_1 that visits only the edges of E_i for some $i \in \{1, 2, 3\}$. As before, we let $t(e) = 0$ for all $e \in E(P_1) \cup E(P_2)$, $t(E_i \setminus E(P_2)) = 1$ for all $i \in \{1, 2, 3\}$, and $t(e) = 100$ for all other edges in G. Then $c(E(G^*))$ is again equal to 4, as well as $c(E_i \cup E_j) = 2$, for all $i \neq j$, $i, j \in \{1, 2, 3\}$.

Case 3. $|Q \cap V(G^*)| \geq 2$. Now, at least two vertices of Q are in G^*. If these depots belong to (at least) two different paths from w_1 to w_2, we choose a weight function $t(E_i) = 1$ for $i \in \{1, 2, 3\}$, and $t(e) = 100$ otherwise. If, on the other hand, the vertices in $Q \cap V(G^*)$ are all on the same path between w_1 and w_2, we choose one of these depots, and construct a P_2 as before. Again, we let $t(e) = 0$ for all $e \in E(P_2)$, $t(E_i \setminus E(P_2)) = 1$ for all $i \in \{1, 2, 3\}$, and $t(e) = 100$ for all other edges in G. It can easily be verified that in both subcases we get $c(E(G^*)) = 4$ and $c(E_i \cup E_j) = 2$ for all $i \neq j$, $i, j \in \{1, 2, 3\}$.

In each of the three cases above, we reach the conclusion that $c(E(G^*)) = 4$, while $c(E_i \cup E_j) = 2$, for all $i \neq j$, $i, j \in \{1, 2, 3\}$. This implies that the subgame $(E(G^*), c^{E(G^*)})$ is not balanced. To see this note that an allocation $x \in \mathcal{R}^N$ in the core must fulfill the following:

$$
\begin{aligned}
x(E_1 \cup E_2) &\leq c(E_1 \cup E_2) = 2, \\
x(E_1 \cup E_3) &\leq c(E_1 \cup E_3) = 2, \\
x(E_2 \cup E_3) &\leq c(E_2 \cup E_3) = 2, \\
x(E(G^*)) &= c(E(G^*) = 4.
\end{aligned} \tag{4.3}
$$

If we add the inequalities, we get $2x(E(G^*)) = 6$ which implies that $x(E(G^*)) = 3 <$

$4 = c(E(G^*)$, and this contradicts the assumption that $x(E(G^*)) = c(E(G^*)$. Since the subgame is not balanced, (N, c) is not totally balanced and therefore not submodular. This completes the proof. □

Theorem 4.5 shows that the class of locally k-CP submodular graphs does not coincide with the class of globally k-CP submodular graphs. This differs from the result in Granot et al. (1999) that the classes of globally and locally CP-submodular graphs coincide. We now turn to the case of directed graphs.

4.2 Directed k-CP graphs

In this section, we consider strongly connected directed graphs, and as in the previous section, we characterize k-CP balanced and k-CP submodular graphs. The results turn out to be similar in structure to the undirected case.

First, we extend to the case of multiple depots in the underlying graph the result from Granot et al. (1999) that every strongly connected digraph is CP-balanced. Thus, also in the case of directed graphs do the classes of globally k-CP balanced and globally CP-balanced graphs coincide. Furthermore, the classes of globally k-CP balanced and locally k-CP balanced graphs coincide as well. We state the following equivalence theorem.

Theorem 4.6. *Let $G = (V(G), E(G))$ be a directed graph. Then the following statements are equivalent:*

(i) G is strongly connected,

(ii) G is globally k-CPP balanced,

(iii) G is locally k-CPP balanced.

Proof. $(i) \to (ii)$: Let $\Gamma = (E(G), (G, Q), t)$ where $Q \subset V(G)$ is of cardinality k, and let $(E(G), c)$ be the induced game. Consider the game $(E(G), c^*)$ for which the linear programming problem in (4.4) represents a linear production game formulation of $(E(G), c^*)$. Let x_{ij} denote the flow in arc (v_i, v_j) and t_{ij} the cost of the arc and consider the LP-problem below.

$$
\begin{aligned}
& c^*(S) = \min \sum_{i,j \in E(G)} t_{ij} x_{ij} \\
\text{subject to} & \\
& \sum_{j \in E(G)} x_{ji} - \sum_{j \in E(G)} x_{ij} = 0 \text{ for all } i \in E(G) \\
& x_{ij} \geq 1 \text{ for all arcs } (v_i, v_j) \in S \\
& x_{ij} \geq 0 \text{ for all arcs } (v_i, v_j) \notin S
\end{aligned}
\tag{4.4}
$$

If $S = E(G)$, the solution to the problem is a minimum cost circulation in which the flow in every arc is at least 1. According to Orloff (1974), this is equivalent to an optimal Chinese postman tour of $E(G)$ with cost function t. Thus, $c(E(G)) = c^*(E(G))$. For $S \subset E(G)$, the solution to the linear programming problem will be a minimum cost circulation that may consist of several disconnected min cost subtours. In the k-CP problem, each subtour must visit a depot, and consequently, $c(S) \geq c^*(S)$. Owen (1975) has shown that $(E(G), c^*)$ is a totally balanced game, and since $c(E(G)) = c^*(E(G))$, and $c(S) \geq c^*(S)$ for each $S \subset E(G)$, this implies that $(E(G), c)$ is balanced. Hence, G is globally k-CP balanced.

$(ii) \rightarrow (iii)$: Follows readily, since every globally k-CP balanced digraph is locally k-CP balanced. $(iii) \rightarrow (i)$: Follows per definition, since the k-CP problem is only defined for strongly connected digraphs. □

Next, we turn to the submodularity of games arising from directed graphs. We note that whenever a digraph, G, is a directed circuit, submodularity of the induced game (N, c) is trivial, since $c(S) = \sum_{e \in N} t(e)$ for all $S \subseteq N$ and all $Q \subseteq V(G)$. A characterization of globally k-CP submodular digraphs is given below. First, however, we state a few definitions.

A directed weakly cyclic graph is a 1-sum of directed circuits, where a 1-sum of two graphs H and G is the graph that arises from coalescing one vertex in H with a vertex in G. Furthermore, we say that a directed circuit C is *internal* if it shares vertices with at least two other circuits, and we let $|C|$ denote the number of edges in C. Furthermore, let $\mathcal{C}(G)$ denote the set of internal circuits in G. We are now ready to characterize globally k-CP submodular digraphs.

Theorem 4.7. *Let $G = (V(G), E(G))$ be a strongly connected digraph, and let $k >$*

$m - \min\{|C||C \in \mathcal{C}(G)\}$. *Then G is globally k-CPP submodular if and only if G is weakly cyclic.*

Proof. For the 'if' part, let $\Gamma = (E(G), (G, Q), t)$, and let (N, c) be the induced game. We have to prove that (2.1) holds. Recall that G is weakly cyclic, and every internal circuit contains at least one depot, while the non-internal circuits may contain no depots. For a non-internal circuit C let C_n denote the neighboring circuit sharing a vertex with C. Finally, let C^* denote the circuit containing e, and let $t(C^*)$ denote the sum of the weights on the edges in C^*. We distinguish between three cases:
Case 1. $T \cap E(C^*) \neq \emptyset$: Then $c(T \cup \{e\}) - c(T) = 0$, and (2.1) clearly holds.
Case 2. $T \cap E(C^*) = \emptyset$ and $Q \cap V(C^*) \neq \emptyset$: Then $c(T \cup \{e\}) - c(T) = c(S \cup \{e\}) - c(S) = t(C^*)$, and (2.1) holds.
Case 3. $T \cap E(C^*) = \emptyset$ and $Q \cap V(C^*) = \emptyset$: then $c(T \cup \{e\}) - c(T) \leq c(S \cup \{e\} - c(S) \leq t(C^*) + t(C_n^*)$. Again (2.1) holds.

For the 'only if' part: If G is not weakly cyclic, it is not a 1-sum of circuits, and there exists a subgraph G_1 in G, such that at least one arc in G_1 belongs to more than one cycle, as in Figure 6. Consider Figure 6, and let the set of arcs contained in the three

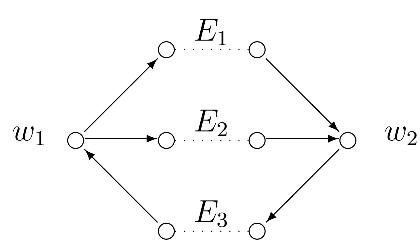

Figure 6: Non-weakly cyclic directed graph

directed paths between w_1 and w_2 be given by E_1, E_2 and E_3 respectively. Let $t(e) = 1$ for all $e \in E_1 \cup E_2 \cup E_3$, and let $t(e)$ be arbitrarily large for all $e \in E(G) \setminus E(G_1)$. If we assume, for example, that $w_1 \in Q$, then

$$\begin{aligned} c(E_1 \cup E_2 \cup E_3) + c(E_3) &= (|E_1| + |E_2| + 2|E_3|) + (|E_3| + \min\{|E_1|, |E_2|\}) \\ &> (|E_1| + |E_3|) + (|E_2| + |E_3|) \\ &= c(E_1 \cup E_3) + c(E_2 \cup E_3), \end{aligned}$$

and therefore, $c(E_1 \cup E_2 \cup E_3) - c(E_1 \cup E_3) > c(E_2 \cup E_3) - c(E_3)$, which shows that the game is not submodular. Hence, G is not globally k-CP submodular. □

The theorem above showed that even if every vertex is associated with a depot, only weakly cyclic digraphs are globally k-CP submodular. If, on the other hand, there are too few depots in the multi-depot CP problem, a connected digraph is not globally k-CP submodular.

Theorem 4.8. *Let G be a strongly connected directed graph, and let $1 < k \leq m - \min\{|C||C \in \mathcal{C}(G)\}$. Then G is not globally k-CP submodular.*

Proof. From Theorem 4.7 it follows that a digraph that is not weakly cyclic is not k-CP submodular for any $k \in \{1, \ldots, m\}$. To prove the present theorem, we therefore only need to show that a directed weakly cyclic graph is not globally k-CP submodular, for $1 < k < m - \min\{|C||C \in \mathcal{C}(G)\}$. Let $\Gamma = (E(G), (G, Q), t)$, let (N, c) be the induced game, and let $1 < k < m - \min\{|C||C \in \mathcal{C}(G)\}$. Then we can choose a $Q \subset V(G)$ such that at least one internal circuit does not contain a depot but is situated 'between' circuits containing depots, e.g., as in Figure 7 below. To see that the induced game

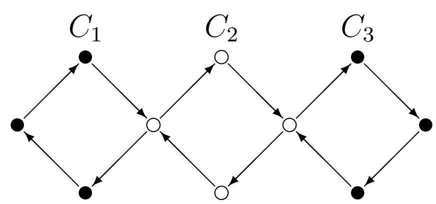

Figure 7: Internal circuit containing no depots in a weakly cyclic digraph

is not submodular for every non-negative weight function, consider the three circuits C_1, C_2, C_3 in Figure 7, and let $S = \{E(C_2)\}$ and $T = \{E(C_1), E(C2)\}$. If we let $t(e) = 1$ for all edges in the graph, we see that $c(S \cup E(C_3)) = c(S) = 8$ whereas $c(T \cup E(C_3)) = 12 > 8 = c(T)$. Thus, the induced game is not submodular for all non-negative weight functions, and G is not globally k-CP submodular. □

Before we turn to consider locally k-CP submodular graphs, we describe a simple procedure for placing depots in a graph G in a convenient way, when G is directed weakly cyclic.

Recall that G is a 1-sum of directed circuits. Choose a circuit C in the graph and place a depot at some vertex in C. Continue by placing depots at the remaining vertices in C. For $k > |C|$, we make use of a neighboring circuit (i.e, a circuit sharing a vertex with C) and place depots at vertices in this circuit. We continue by repeatedly placing depots at all vertices in one circuit and then moving to a neighboring circuit. By placing depots in this way, we ensure that the subgraph consisting of all circuits containing depots is strongly connected.

The final result of this section show that all directed weakly cyclic graphs are locally k-CP submodular.

Theorem 4.9. *Let G be a strongly connected digraph. If G is weakly cyclic, then G is locally k-CP-submodular.*

Proof. Let G be weakly cyclic, let $\Gamma = (E(G), (G, Q), t)$, and let (N, c) be the induced game. We need to show that (2.1) holds. For every $k \in \{1, \dots, m\}$, there exists a $Q \subseteq V(G)$ such that the depots are located according to the procedure above. Let G_1 denote the subgraph consisting of all circuits in G that contain depots according to Q, and note that G_1 is strongly connected. Let C^* denote the circuit containing e. We consider two cases:

Case 1. $e \in E(G_1)$: If $c(T \cup \{e\}) - c(T) = 0$, then (2.1) holds, and otherwise, $c(T \cup \{e\}) - c(T) = c(S \cup \{e\}) - c(S) = t(C^*)$.

Case 2. If $e \in E(G) \setminus E(G_1)$, there exists a vertex $v \in Q$ such that every $S \cup \{e\}$-tour must pass v in order to visit e. Thus, for any $S \subseteq N \setminus \{e\}$ there is a min. weight $S \cup \{e\}$-tour in which e is serviced by v, along with all other edges in C^*. Then for every $e \in E(G) \setminus E(G_1)$ there exists a submodular one-depot CP game (N, c_v) such that $c_Q(S \cup \{e\}) - c_Q(S) = c_v(S \cup \{e\}) - c_v(S)$ for all $S \subset T \subseteq N \setminus \{e\}$, and (2.1) is fulfilled. □

From Theorem 4.7 and Theorem 4.9, it follows that the set of locally k-CP submodular digraphs is a superset of the set of globally k-CP submodular digraphs.

5 k-CP totally balanced graphs

In this section we state some insights on k-CP totally balanced graphs. First, note that in contrast to the CP case with $k = 1$, the classes of undirected globally k-CP submodular and globally k-CP totally balanced graphs do not coincide, for $k > 1$. We show that there exists globally k-CP totally balanced graphs that are not globally k-CP submodular. Consider for example a line graph with three edges. From Lemma 3.1 this graph is not globally k-CP submodular, but it is globally k-CP totally balanced, as shown below.

Proposition 5.1. The class of connected globally k-CP totally balanced graphs is a superset of the class of connected globally k-CP submodular graphs.

Proof. Let G be a path with three edges. Let $\Gamma = (E(G), (G, Q), t)$ and let (N, c) be the induced game. Let $k \in \{3, 4\}$ and note that for all $Q \subset V(G)$ with $|Q| = k$, every edge in the graph is incident to a depot. We need to show that the core of the subgame (S, c^S) is non-empty, for all $S \subseteq N$. For every $e \in S$, let x be defined as $x(e) = 2t(e)$. Clearly, x is efficient, and $x(U) = c^S(U)$, for all $U \subseteq S$. Thus, G is globally k-CP totally balanced for $k \in \{3, 4\}$. □

In addition, we find that there exists locally k-CP totally balanced graph that are not locally k-CP submodular.

Proposition 5.2. The class of connected locally k-CP totally balanced graphs is a superset of the class of connected locally k-CP submodular graphs.

Proof. Let G be the circuit with 5 edges illustrated in Figure 8. Let $\Gamma = (E(G), (G, Q), t)$ and let (N, c) be the induced game. G is not locally k-CP submodular for $k = 2$, since it is not (P_4)-free wrt. Q, for any choice of $Q \subset V(G)$. Assume that $Q = \{v_1, v_3\}$.

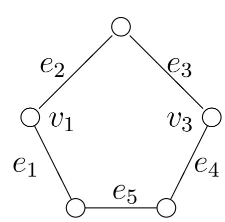

Figure 8

We show for every $S \subseteq N$ that the subgame (S, c^S) is balanced, for every non-negative weight function. We consider two separate cases:

Case 1. $e_5 \notin S$, or at least one of the pairs $\{e_1, e_5\}$ or $\{e_4, e_5\}$ is contained in S: It follows that $c(S) = \min\{\sum_{e \in S} 2t(e), \sum_{e \in N} t(e)\}$. We define x as:

$$x(e) = 2t(e) - \max\left\{0, \left(\sum_{e \in S} 2t(e) - \sum_{e \in N} t(e)\right) \frac{1}{|S|}\right\}$$

for every $e \in S$. Note that x is efficient.

Now, for a subset $U \subset S$, either

$$c(U) \geq \sum_{e \in U} 2t(e) \geq x(U), \text{ or } c(U) = \sum_{e \in N} t(e) = c(S) = x(S),$$

implying that $x(U) \leq c(U)$ for every $U \subseteq S$. Thus x is a core element in (S, c^S).

Case 2. S is equal to one of the subsets, $\{e_5\}, \{e_2, e_5\}, \{e_3, e_5\}$ or $\{e_2, e_3, e_5\}$: We get $c(S) = \min\{c(e_5) + \sum_{e \in S \setminus \{e_5\}} 2t(e), \sum_{e \in N} t(e)\}$, and we define an efficient allocation x' according to:

$$x'(e) = \begin{cases} c(e) - \alpha & \text{if } e = e_5, \\ 2t(e) - \alpha & \text{if } e \in \{e_2, e_3\}, \end{cases}$$

where $\alpha = \max\{0, \frac{1}{|S|}\big(c(5) + \sum_{e \in S \setminus \{e_5\}} 2t(e) - \sum_{e \in N} t(e)\}$. For every $U \subset S$ that does not contain e_5, we have

$$c(U) = \min\{\sum_{e \in U} 2t(e), \sum_{e \in N} t(e)\},$$

which readily shows that $c(U) \geq x'(U)$. For every $U \subseteq S$ that contains e_5,

$$c(U) = \min\{c(e_5) + \sum_{e \in U \setminus \{e_5\}} 2t(e), \sum_{e \in N} t(e)\},$$

and again $c(U) \geq x'(U)$.

When $Q = \{v_1, v_3\}$, the induced subgame is, therefore, balanced, for all non-negative weight functions, and the graph is locally k-CP totally balanced, for $k = 2$. □

The above proposition contrasts the result from Granot et al. (1999) that the classes of locally CP submodular and locally CP-totally balanced graphs coincide.

For directed graphs we find similar results, and we show that the set of globally (respectively locally) k-CP submodular digraphs does not coincide with the set of globally (respectively locally) k-CP totally balanced digraphs.

Proposition 5.3. The class of strongly connected globally k-CP totally balanced digraphs is a superset of the class of strongly connected globally k-CP submodular digraphs.

Proof. Consider Figure 9 and note that from Theorem 4.7 and 4.8, the graph G in Figure 9 is not globally submodular for any $k \in \{1, \ldots, 4\}$. However, for $k \in \{2, 3, 4\}$, both cycles in G contains a depot for every $Q \subseteq V(G)$ with $|Q| = k$, and it follows from the proof of Theorem 4.6 that G is globally k-CP totally balanced, for $k \in \{2, 3, 4\}$. □

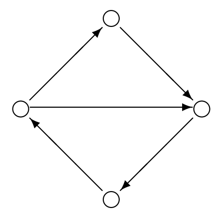

Figure 9

Consequently, if Q contains sufficiently many depots, then every strongly connected digraph is globally k-CP totally balanced. In addition to this, we show that the set of globally k-CP totally balanced digraphs is contained in the set of locally k-CP totally balanced digraphs.

Proposition 5.4. The class of locally k-CP totally balanced digraphs is a superset of the class of globally k-CP totally balanced digraphs.

Proof. Consider again Figure 9 and note that for every $k \in \{1, 2, 3, 4\}$, it is possible to choose a $Q \subseteq V(G)$ with $|Q| = k$ such that both cycles in G contain a depot. Again, it follows from the proof of Theorem 4.6 that the induced game is totally balanced for all non-negative weight functions, and therefore, G is locally k-CP totally balanced for $k \in \{1, 2, 3, 4\}$. □

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