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Noise, Information, and the Favorite-Longshot Bias*

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Abstract

According to the favorite-longshot bias, longshots are overbet relative to favorites. We propose an explanation for this bias (and its reverse) based on an equilibrium model of informed betting in parimutuel markets. The bias arises because bettors take positions without knowing the positions simultaneously taken by other privately informed bettors. The direction and the extent of the bias depend on the amount of private information relative to noise present in the market. With realistic ex-post noise and ex-ante asymmetries, our model replicates the main qualitative features of expected returns observed in horse races.

Keywords: Parimutuel betting, favorite-longshot bias, private information, noise, lotteries.

JEL Classification: D82 (Asymmetric and Private Information), D83 (Search; Learning; Information and Knowledge), D84 (Expectations; Speculations).

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Betting markets provide a natural environment for testing theories of decision making under uncertainty and price formation. The uncertainty about the value of the assets traded in betting markets is resolved unambiguously and the outcome is observed publicly. In addition, it is reasonable in most cases to presume that the realized outcomes are exogenous with respect to market prices. In regular financial markets, instead, the intrinsic value of assets is only observed in the long run and can be affected by market prices.

In horse-racing tracks and lottery games throughout the world, the most common procedure adopted is parimutuel betting.¹ According to the parimutuel system, the holders of winning tickets share the total amount of money bet on all outcomes in proportion to their bets, net of taxes and expenses. Parimutuel payoff *odds* thus depend on the distribution of bets and are determined by the gambling public itself. Because these odds are not skewed by price-setting suppliers (such as bookmakers), parimutuel betting markets are particularly well suited to testing market efficiency.²

The most widely documented empirical regularity observed in horse-betting markets is the *favorite-longshot bias* (hereafter, FLB): horses with “short” odds (i.e., favorites) tend to win more frequently than indicated by their odds, while horses with “long” odds (i.e., longshots) win less frequently.³ Consequently, the expected returns on longshots are much lower than on favorites. This finding is puzzling, because expected returns are not equalized across horses. To further add to the puzzle, note that for parimutuel *lottery* games, such as Lotto, a *reverse* FLB always results: the expected payoff is lower on numbers that attract a higher-than-proportional amount of bets.⁴

The goal of this paper is to formulate a simple theoretical model of parimutuel betting that provides an informational explanation for the occurrence of the FLB and its reverse. In the model, bettors decide simultaneously whether, and on which of several outcomes, to bet. Each bettor’s payoff has two components. First, there is a “common value” compo-

¹Recently, the parimutuel structure was adopted in markets for contingent claims on various economic indices. As Economides and Lange (2005) explain, these markets allow traders to hedge risks related to variables such as U.S. nonfarm payroll employment and European harmonized indices of consumer prices.

²In parimutuel markets, the demand for bets on one outcome generates the supply on all other outcomes. See Levitt (2004) for a discussion of supply-driven distortions in betting markets not using the parimutuel system.

³For surveys of the extensive literature, we refer to Thaler and Ziemba (1988), Hausch and Ziemba (1995), Sauer (1998), and Jullien and Salanié (2002).

⁴See Forrest et al. (2000) for a test of rational expectations in lotteries. The reverse FLB is also present in some betting markets (see e.g. Busche and Hall, 1988, Vaughan Williams and Paton, 1998, and our discussion below).

ment, equal to the expected monetary payoff on the bet based on the final odds.⁵ Second, bettors derive a private utility from gambling (see e.g. Conlisk, 1993). For simplicity, we set this “recreational value” to be the same for all bettors.⁶

Our model allows bettors to have heterogeneous beliefs based on the observation of a private signal. We aim to characterize the effect of private information on market outcomes. While private information clearly is absent in lottery games, there is widespread evidence that it is present in horse betting (see Section 6). In a limit case relevant for lottery games, the signal contains no information.

We model betting as a *simultaneous*-move game. This is a realistic description of lottery games, in which the numbers picked by participants are not made public before the draw. For betting on horse races, the distribution of bets (or, equivalently, the provisional odds given the cumulative bets placed) is displayed over time on the tote board and updated at regular intervals until post time, when the betting is closed. However, a large proportion of bets are placed in the very last seconds before post time (see National Thoroughbred Racing Association, 2004). Thus, we focus on the last-minute simultaneous betting game.⁷

We show that the FLB depends on the amount of information relative to noise that is present in equilibrium. Suppose first that the bettors’ signals are completely uninformative about the outcome, as is the case in Lotto. Market odds then will vary randomly, mostly due to the noise contained in the signal. Since all numbers are equally likely to be drawn, and the jackpot is shared among the lucky few who pick the winning number, the expected payoff is automatically lower for those numbers that attract more than their fair share of bets. Borrowing the terminology used in horse betting, lottery outcomes with short market odds yield lower expected returns than outcomes with long market odds. More generally, when signals contain little information, or there is aggregate uncertainty about the final distribution of bets due to noise, our model predicts a *reverse* FLB—as is observed in parimutuel lotteries.

Now, as bettors have more private information, the realized bets are less affected by noise and have more informational content. If an outcome turns out to attract a larger

⁵By assuming risk neutrality, we depart from a large part of the betting literature since Weitzman (1965) in which bettors are risk loving. Section 6 discusses this and other alternative theories.

⁶In the case of parimutuel derivative markets mentioned in footnote 1, the private value derives from the benefit of hedging against pre-existing risks correlated with the outcome on which betting takes place.

⁷Ottaviani and Sørensen (2006) endogenize the timing of bets in a dynamic model. They show that small privately informed bettors have an incentive to wait till post time, and thus bet simultaneously.

fraction of bets, more bettors must have privately believed that this outcome was likely to result. Conversely, the occurrence of long odds (i.e., few bets) reveals unfavorable evidence. We show that if the population of informed bettors is large and/or the private information is sufficiently precise, then the favorites (or longshots) are more (or less) likely to win than the realized market odds indicate at face value. This results in the FLB, as observed in horse-betting markets. This simple insight is the core of this paper’s contribution.

Theoretical Literature. This is the first paper to propose an informational explanation for the FLB based on equilibrium betting in parimutuel markets. Our explanation is fundamentally different from Ali’s (1977) Theorem 2, that derives the FLB from the heterogeneity of the bettors’ (prior) beliefs, rather than from private information.⁸ While Ali’s explanation hinges on ex-ante asymmetries in the probabilities of the different outcomes, our explanation is valid also when the outcomes are all ex-ante equally likely. To isolate the new informational driver, our baseline model focuses on a fully-symmetric environment.⁹

In a parimutuel market with privately informed bettors, Potters and Wit (1996) derive the FLB as a deviation from a rational expectations equilibrium. In Potters and Wit’s model, bettors are given the opportunity to adjust their bets at the final market odds, but they ignore the information contained in the bets. In the Bayes-Nash equilibrium of our model, bettors instead understand that bets are informative, but they are not able to adjust their bets to incorporate this information because they do not observe the final market odds.

The first paper in the literature to study equilibria in parimutuel markets with asymmetric information is Koessler et al. (forthcoming). They consider bettors with binary signals. Our analysis is greatly simplified by assuming instead that bettors have continuously distributed beliefs, as in other auction-theoretic models of price formation. We are then able to characterize the FLB, endogenize the bettors’ participation decision, and derive a number of empirical predictions.

Shin (1991 and 1992) formulates an information-based explanation of the FLB in the context of *fixed-odds* betting. Even though our informational assumptions are similar to

⁸See also Eisenberg and Gale (1959), Chadha and Quandt (1996), Brown and Lin (2003), and Wolfers and Zitzewitz (2004) for analyses of prediction markets with heterogeneous beliefs, but without private information.

⁹A mechanism similar to Ali’s is also present in the version of our model with ex-ante asymmetries. As explained in footnote 26 that mechanism must be modified for the presence of private information.

Shin's, the logic and the extent of the FLB are different in the two markets. While Shin's explanation for the bias relies on ex-ante asymmetries, our explanation for parimutuel markets does not. As Ottaviani and Sørensen (2005b) show, the parimutuel payoff structure involves a built-in insurance against the winner's curse, which is absent in fixed-odds markets.¹⁰

Plan of Paper. The paper proceeds as follows. Section 1 formulates our general model with an arbitrary number of outcomes. Section 2 then uses a simple example to illustrate the paper's main insight. Section 3 focuses on the case in which the outcomes are equally likely ex ante and the recreational value is so large that no bettor abstains. This symmetric model is ideal for uncovering the role of noise and information in generating the FLB or its reverse. In this setting, we perform comparative statics with respect to the amount of private information, the number of bettors, and the number of outcomes.

Section 4 extends the model to allow for ex-ante asymmetries in the prior probabilities and ex-post noise in the outcome realization. Once we add these two realistic features, the predicted curve of expected returns better matches the empirical pattern documented by Snowberg and Wolfers (2005). Due to the countervailing effects of noise and information in the intermediate range, the expected returns to bets tend to be relatively constant with respect to the market probability, other than for strong favorites and strong longshots.

Section 5 investigates two theoretical extensions. First, we allow bettors to abstain. In the limit, as we approach no trade, information dominates noise and the FLB unambiguously results. Second, when individuals are allowed to place bets on more than one outcome, the relative amount of noise is lower and hence the extent of the FLB higher. Section 6 compares our explanation of the empirical evidence with the main alternatives proposed in the literature. Section 7 concludes. Appendix A collects the proofs of all the results. Appendix B contains computations for an analytically tractable example.

¹⁰In parimutuel markets, an increase in the number of informed bettors drives market odds to be more extreme, and thus reduces the FLB. In fixed-odds markets, instead, an increase in the fraction of informed bettors strengthens the FLB, because adverse selection is worsened. See Ottaviani and Sørensen (2005b).

1 Model

Bets can be placed on the realization of a random variable, $k \in \{1, \dots, K\}$, where $K \geq 2$. In horse racing, bets in the win pool depend on the identity, k , of the winning horse.¹¹ With Lotto, k represents the winning combination.

There are N bettors with a common prior belief, $q_k \geq 0$, about the outcome, k .¹² Bettor $i \in \{1, \dots, N\}$ is privately endowed with signal s_i , leading to the private posterior belief p_i . Bettors differ only because of the realization of their private signals, but they are identical ex ante: the joint distribution of posterior beliefs and the state is unchanged if the identities of the players are switched. For any individual, the conditional density of the private belief is denoted by $g(p|k)$.

On the basis of the private posterior belief, each bettor decides the outcome on which to bet a fixed and indivisible amount, normalized to 1, or to abstain from betting.¹³ All bettors are risk neutral and maximize the expected monetary return, plus a fixed recreational utility value received from betting. This recreational value ($u \geq 0$) is foregone when a bettor abstains, and hence generates a demand for betting.¹⁴

The total amount of money bet on all K outcomes is placed in a common pool, from which a fraction, $\tau \in [0, 1)$, is subtracted for taxes and other expenses. The remaining money is returned to those who bet on the winning outcome, k . We assume that there is no payment to the bettors when no bets were placed on the winning outcome.¹⁵ Let b_k denote the total amount bet on k . If k is the outcome, then every unit that is bet on k receives the monetary payoff $(1 - \tau) \left(\sum_{l=1}^K b_l \right) / b_k$. The market probability of outcome k is equal to $b_k / \left(\sum_{l=1}^K b_l \right)$.¹⁶

The strategy of a bettor maps every private posterior belief into one of the $K + 1$

¹¹Exacta bets pick the winner and the runner-up. A race with L horses thus has $K = L(L-1)$ outcomes.

¹²We assume common prior to isolate the effect of asymmetric information. See Ottaviani and Sørensen (2005a) on the interaction of heterogeneous prior beliefs and private information.

¹³See Isaacs (1953) and Ottaviani and Sørensen (2006) for analyses in which bettors can choose how much to bet. See Section 5.2 for a discussion of the effect of allowing bets to be divisible.

¹⁴Without this recreational utility, there is no betting in equilibrium, as predicted by the no-trade theorem (Milgrom and Stokey, 1982). The assumption that bettors are ex-ante identical is not essential for our results, but allows us to obtain a closed-form solution for the equilibrium. See Ottaviani and Sørensen (2006) for a model in which instead some bettors are outsiders (motivated by recreation) while other bettors are privately-informed insiders.

¹⁵Our results continue to hold qualitatively with alternative rules on how the pool is split when no one bets on the winner. For example, the pool could be divided equally among all active bettors.

¹⁶The market odds on outcome k represent the amount of money paid by the system for each dollar placed on that outcome, in addition to the dollar wagered. In order to balance the budget for any outcome

actions: bet on outcome k or abstain. In equilibrium, every bettor correctly conjectures the strategies used by the opponents and then plays the best response to this conjecture. By assumption, the game is always symmetric with respect to the players. Throughout the paper, we focus on symmetric equilibria; that is, on Bayes-Nash equilibria in which all bettors use the same strategy, mapping private posterior beliefs into actions.¹⁷

1.1 Equilibrium Regimes

There are three equilibrium regimes, depending on the level of the recreational value, u :¹⁸

Proposition 1 *Assume that the posterior belief distribution is continuous with full support. There exists a uniquely defined critical value, $u^{*(N)} \in (\tau, 1)$, such that:*

1. *if $u \geq u^{*(N)}$, there exists a symmetric equilibrium in which all bettors bet actively,*
2. *if $u \in (\tau, u^{*(N)})$, in any symmetric equilibrium some (but not all) of the bettors abstain,*
3. *if $u \leq \tau$, in any symmetric equilibrium all bettors abstain.*

When the recreational value is sufficiently high or, equivalently, the takeout rate is sufficiently low, even bettors without strong posterior beliefs will bet on one of the two horses—this is the “no abstention” regime on which we focus for most of the paper. With low recreational value or high takeout rate, bettors without strong private beliefs will prefer not to place any bet—this is the “partial abstention” regime that we analyze in Section 5.1. Additional increases in the takeout will further reduce participation, until the point at which the market completely breaks down, by the logic of the no-trade theorem.¹⁹

realization, the parimutuel odds are $(1 - \tau) \left(\sum_{l=1}^K b_l \right) / b_k - 1$. The results derived in this paper could be rephrased in terms of odds, given the one-to-one correspondence between market odds and probabilities.

¹⁷Koessler et al. (forthcoming) note that there also may be asymmetric equilibria when N is small.

¹⁸Proposition 1 can be extended to the case in which support is not full. Let \bar{p} denote the maximal belief that any bettor attaches to any outcome. Let $\bar{u} = 1 - \bar{p}(1 - \tau)$. Then $u^* > \bar{u} \geq \tau$ and cases 1 and 3 of Proposition 1 hold as stated. The claim in case 2 holds when $u \in (\bar{u}, u^*)$. If $u \in (\tau, \bar{u}]$, our proof implies a weaker version of case 3, that there exists *an* equilibrium in which every bettor abstains.

¹⁹Bettors with beliefs $p = 1$ and $p = 0$ always participate, but these beliefs have probability zero.

2 Main Insight

This section illustrates the main insight of this paper in the context of the simplest possible example. Suppose that there are two horses (1 and 2) and that they are equally likely to win ex ante. There are N bettors, each with imperfect private information. If horse k is going to win, then the independently drawn signal of each bettor is favorable to horse k with probability $2/3$. There is no track take (so $\tau = 0$) and the recreational value is sufficiently high that there is no abstention. Quite intuitively, in equilibrium each bettor wagers on the horse favored by the private signal, as confirmed below by Proposition 2.

It is convenient to see the realized distribution of bets as a sample. Every single bet is like a ball drawn (independently, with replacement) from an urn containing an unknown proportion of balls marked 1 and 2. If horse number 1 is the winner, then the proportion of balls marked with 1 is $2/3$. If instead horse number 2 is the winner, then the proportion of balls marked with 1 is $1/3$. As a sample of N balls, the realized bets contain information on the proportion of balls in the urn and thus on the winning horse.

The thrust of our argument relies on comparing the market probabilities that result from equilibrium betting with the corresponding posterior probabilities that incorporate the revealed information. A horse's market probability is defined as equal to the proportion of money bet on that horse relative to the total pool. For each market probability we can therefore compute the corresponding posterior probability by applying Bayes' rule.

To begin with, consider the case of $N = 2$ bettors. When the bettors wager on different horses, the market probability for horse number 1 is $1/2$, and the posterior probability is

$$\frac{\Pr(\text{bet 1, bet 2} | 1 \text{ wins}) \Pr(1 \text{ wins})}{\Pr(\text{bet 1, bet 2})} = \frac{(4/9)(1/2)}{(4/9)(1/2) + (4/9)(1/2)} = \frac{1}{2},$$

so that the market probability is exactly equal to the posterior probability. If, instead, both bets are placed on horse number 1, it has market probability 1 and posterior probability

$$\frac{\Pr(\text{bet 1, bet 1} | 1 \text{ wins}) \Pr(1 \text{ wins})}{\Pr(\text{bet 1, bet 1})} = \frac{(4/9)(1/2)}{(4/9)(1/2) + (1/9)(1/2)} = \frac{4}{5}.$$

The posterior probability is lower than the market probability—this is the *reverse* of the FLB. Accordingly, the ex-post expected monetary payoff on the favorite horse, 1, is the negative $(4/5)1 - 1 = -1/5$. Intuitively, the favorite has attracted a large proportion of bets and pays off too little when the parimutuel pool is shared among all winning

bettors. Note that the reverse FLB is due to the fact that bettors' signals are a noisy (i.e., imperfectly informative) indication of the winning horse.

Next, suppose that there are $N = 3$ bettors and focus on the case in which two bets are placed on horse number 1 and one on horse number 2. The market probability for horse number 1 is then $2/3$, and this turns out to be exactly equal to the posterior probability computed using Bayes' rule. The reverse FLB induced by the parimutuel payoff structure now is offset perfectly by the fact that the more heavily bet horse is more likely to win.

Now, with $N = 4$ bettors and three out of four bets on horse number 1, that horse's market probability is $3/4$ while Bayes' rule yields a posterior probability of $4/5$. Intuitively, one signal in favor of 2 will cancel one signal in favor of 1. Each of the two additional signals in favor of 1 is twice as likely in outcome 1 as in outcome 2. Hence, outcome 1 is four times as likely as outcome 2, while its market probability indicates literally that it is only three times as likely. Here, information dominates noise, hence the FLB results.

When the number of bettors increases, the sample size becomes large and highly informative. In the limit as $N \rightarrow \infty$, the proportion of balls marked with 1 in the sample converges either to the market probability $2/3$ —in which case the favorite horse, 1, is also the winner with posterior probability 1—or to the market probability $1/3$, in which case the longshot, 1, has a posterior win probability of 0. This is an extreme form of the FLB.

Note that any bias would be eliminated if bettors could instead adjust their positions in response to the final market odds, as in a rational expectations equilibrium. However, the information on the final market odds is typically not available to bettors because a substantial amount of bets are placed at the end of the betting period when the final odds are not yet determined.

The example presented in this section illustrates how the sign and the extent of the FLB depend on the amount of information relative to the noise that is present in equilibrium. While instructive, this example has a number of special features, such as the binary belief distribution, the presence of only two outcomes, the assumption that the horses are ex-ante equally likely to win, the assumption that bettors are forced to make a bet, and the fact that the bet distribution perfectly reveals the outcome of the race when there are infinitely many bettors. By relaxing these special assumptions, in the remainder of the paper we derive richer predictions that shed light on the extant empirical evidence.

3 Symmetric Model

Now we analyze more generally how the occurrence of the FLB depends on the informativeness of the private signals. We assume that the signals are conditionally independent and are identically distributed across bettors. We focus on the symmetric model in which all outcomes are ex-ante equally likely, $q_k = 1/K$, and the posterior beliefs are continuously and symmetrically distributed.

The symmetric model is an important benchmark for two reasons. First, horse races typically are designed to be balanced in order to assure that their outcome is genuinely uncertain. In any given race, horses are sorted into categories depending on their observable characteristics, and known differences are eliminated in part by burdening the advantaged horses with additional weights. While asymmetries are never perfectly eliminated, these procedures are intended to reduce the presence of strong ex-ante favorites or ex-ante longshots. Hence, in our baseline model we assume that the outcomes are ex-ante equiprobable, so as to have only ex-post favorites or longshots. Second, the symmetric model is particularly tractable from the theoretical point of view and it allows us to illustrate that our explanation does not rely on ex-ante asymmetries.

3.1 Equilibrium

In equilibrium, all bettors are optimizing given their opponents' strategies. Because the others' signals and corresponding bets are uncertain, the payoff conditional on winning is also random. Given the opponents' strategies, every bettor can calculate $W(k|l)$, the expected payment from the pool to a bet on k conditional on the realization of outcome l . Since payments are made only to winners, $W(k|l) = 0$ for $k \neq l$. For a bettor who assigns probability p_k to outcome k , the expected payoff from betting on outcome k is $p_k W(k|k) - 1 + u$. To further simplify the analysis, we concentrate throughout the paper, other than in Section 5.1, on the case when u is sufficiently large that there is no abstention.

Proposition 2 *If u is sufficiently large and signals are symmetrically distributed, it is a symmetric equilibrium to bet on the most likely outcome, that is, on the k which maximizes p_k . If bettors have private information, then the probability that any bettor bets on the winner, $\eta(k|k)$, exceeds $\eta(l|k) = (1 - \eta(k|k)) / (K - 1)$, the probability that any bettor bets on any other outcome $l \neq k$.*

3.2 Favorite-Longshot Bias

Now we are ready for the key step in our analysis. Because signals are random, bets on a given outcome will follow a binomial distribution. An observer of the final distribution of bets can update to the *posterior probability* for outcome k , denoted by β_k . When exactly n bets are placed on k , Bayes' rule yields

$$\beta_k = \frac{q_k \Pr(\text{bets}|k \text{ true})}{\Pr(\text{bets})} = \frac{\eta(k|k)^n (1 - \eta(k|k))^{N-n}}{\eta(k|k)^n (1 - \eta(k|k))^{N-n} + (K-1) \eta(k|l)^n (1 - \eta(k|l))^{N-n}}, \quad (1)$$

where $l \neq k$. The law of large numbers guarantees that the empirical frequency of outcome k across many repetitions of the game is approximately equal to β_k . Intuitively, this posterior probability incorporates the information revealed in the betting distribution and adjusts for noise, thus correctly estimating the empirical probability of outcome k .

When n bets are placed on outcome k , the implied *market probability* for outcome k is instead $\pi_k = n/N$, equal to the fraction of money bet on this outcome. The FLB entails a systematic difference between the market probability and the posterior probability: when the market probability is large, it is still smaller than the corresponding posterior probability. That is, a favorite is more likely to win than the market probability suggests.

In our model, the systematic relation between posterior and market probabilities depends on the interplay between the amount of noise and information contained in the bettors' signals. To appreciate the role played by *noise*, note that even with very few bettors, market probabilities can range from zero to infinity. For example, if most bettors happen to draw signals such that they believe k is the most likely outcome, then the market probability for outcome k will be very high. Yet, if the signals contain little information, then the posterior chance is close to the prior chance. In this case, deviations of the market probability from the prior chance are largely due to the randomness contained in the signal, so the reverse FLB is present: the market probabilities are more extreme than the posterior probabilities.

As the number of bettors increases, the realized market bets contain more and more *information*, so the posterior chance is ever more extreme for any given market probability. This implies the FLB, which is our main qualitative finding:

Proposition 3 *Assume that the belief distribution is symmetric and that u is so large that*

no bettor abstains. Let $\pi^* \in (0, 1)$ be defined by

$$\pi^* = \frac{\log\left(\frac{1-\eta(k|l)}{1-\eta(k|k)}\right)}{\log\left(\frac{1-\eta(k|l)}{1-\eta(k|k)}\right) + \log\left(\frac{\eta(k|k)}{\eta(k|l)}\right)} \quad (2)$$

for any pair $l \neq k$. Take as given any market probability $\pi_k \in (0, 1)$ for outcome k . As the number of bettors, N , becomes sufficiently large, a longshot's market probability $\pi_k < \pi^*$ (respectively, a favorite's $\pi_k > \pi^*$) is strictly greater (respectively, smaller) than the associated posterior probability, β_k .

By definition (2), π^* is a neutral realized proportion of bets on outcome k , where the posterior belief is precisely equal to the prior. Next we show that the FLB occurs when bettors are sufficiently well informed:²⁰

Proposition 4 *Assume $K = 2$ and that the belief distribution is symmetric. Take as given any market implied probability $\pi_k \in (0, 1)$ for outcome k . If the bettor's degree of informativeness $\eta(k|k)/\eta(k|l)$ for $l \neq k$ is sufficiently large, then a longshot's market probability $\pi_k < 1/2$ (respectively, a favorite's $\pi_k > 1/2$) is strictly greater (respectively, smaller) than the associated posterior probability β_k .*

3.3 Illustration

We illustrate our general results in a flexible class of examples that is particularly tractable in this context. Suppose that the posterior beliefs have the symmetric Dirichlet density

$$g(p_1, \dots, p_K) = \frac{\Gamma(K\theta)}{(\Gamma(\theta))^K} \prod_{k=1}^K p_k^{\theta-1},$$

with full support on the simplex $\{p \in \mathbb{R}_+^K | p_1 + \dots + p_K = 1\}$, where the parameter $\theta > 0$ measures the amount of private information and $\Gamma(\theta) = \int_0^\infty t^{\theta-1} e^{-t} dt = (\theta-1)\Gamma(\theta-1)$ denotes the Gamma function.²¹

²⁰Note that the FLB inequality (11) in Appendix A is harder to satisfy at more extreme market probabilities. This feature is also apparent from Figures 1 and 2. At the easiest point, $\pi_1 = 1/2$, the condition is $2 < N \log[\eta(k|k)/\eta(k|l)]$.

²¹In a model without private information, Brown and Lin (2003) consider Dirichlet distributed *prior* beliefs. Our setting with Dirichlet distributed *posterior* beliefs is fundamentally different. See footnote 8.

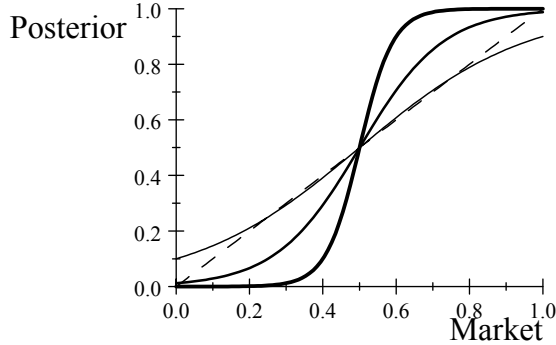


Figure 1: Posterior probability as a function of market probability, for $N = 2, 4, 10$

As demonstrated in Appendix B, in this example with $K \geq 2$ and $\theta = 1$, the equilibrium probability that any bettor bets on the winner is

$$\eta(k|k) = \sum_{j=0}^{K-1} \binom{K-1}{j} \frac{(-1)^j}{(j+1)^2}. \quad (3)$$

For $K = 2$ and any real $\theta > 0$, the equilibrium probability is

$$\eta(k|k) = \frac{1}{2} + \frac{\Gamma(2\theta + 1)}{\Gamma(\theta + 1)\Gamma(\theta)} \frac{4^{-\theta}}{2\theta}. \quad (4)$$

Competition. To illustrate Proposition 3 on how the FLB depends on the number of bettors, N , consider the Dirichlet example with $K = 2$ and $\theta = 1$. Equation (4) yields $\eta(k|k) = 3/4$. Using (1), Figure 1 shows the posterior probability β_k as a function of the market probability $\pi_k = n/N$. The curves correspond to $N = 2, 4$, and 10 , drawn in progressively darker shading. The FLB results if the posterior probability is below the (dashed) diagonal for $\pi < 1/2$ and above the diagonal for $\pi > 1/2$.

The more bettors there are, the greater is the extent of the FLB at any fixed market probability, π . This does not take into account the fact that the number of bettors also affects the probability distribution over π . The law of large numbers implies that a greater number of bettors will generate a less random π . Ottaviani and Sørensen (2006) analyze this limit by considering a continuum of informed bettors. In the limit, the implied market probability in outcome k is deterministic and fully reveals the outcome.

Information. To illustrate Proposition 4 on the impact of private information on the FLB, consider again the Dirichlet example with $K = 2$ outcomes, now keeping the num-

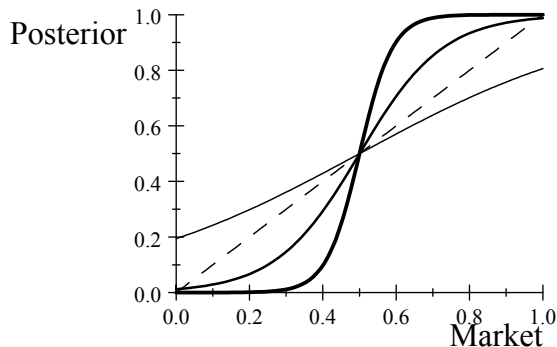


Figure 2: Posterior probability as a function of market probability, for $\theta = 10, 1, 1/10$

ber of bettors fixed at $N = 4$. In this example, bettors have less private information when θ is higher and are completely uninformed in the limit as $\theta \rightarrow \infty$. Figure 2 shows the posterior probability (1) as a function of the market probability, π , for three levels of the Dirichlet parameter $\theta = 10, 1, 1/10$ (corresponding to equilibrium probabilities $\eta(k|k) = .588, .75, .942$), represented in progressively darker shading. Figure 2 illustrates that the FLB will arise for any market probability provided that the signal is sufficiently informative. When the signal is very noisy, there is a reverse FLB. As the signal's informativeness rises above a critical value, the FLB occurs in an ever larger region around $\pi = 1/2$.

Reduced Bias in Exotic Bets. Next consider how the FLB depends on the number of outcomes, K , holding the number of bettors fixed. The analysis of this case is complicated by the fact that the number of outcomes cannot be increased without also changing the bettors' information structures. To derive comparative statics results, we need to make additional assumptions about this change. Focus on the Dirichlet example for the special case with $\theta = 1$, when (p_1, \dots, p_K) is uniformly distributed on the belief simplex. Using (3), we can compute the equilibrium for different values of K :

	$K = 2$	$K = 5$	$K = 10$	$K = 100$	$K = 10,000$
$\eta(k k)$	3/4	0.457	0.293	5.19×10^{-2}	9.79×10^{-4}
$\eta(k l)$	1/4	0.136	0.0786	9.48×10^{-3}	9.99×10^{-5}
$\frac{\eta(k l)}{\eta(k k)}$	3	3.36	3.73	5.47	9.80
$\frac{1-\eta(k l)}{1-\eta(k k)}$	3	1.59	1.30	1.04	1.001
π^*	1/2	0.277	0.161	2.51×10^{-2}	3.86×10^{-4}

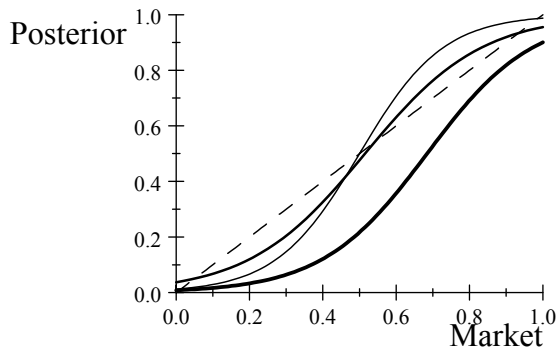


Figure 3: Posterior probability as a function of market probability, for $K = 2, 10, 100$

where π^* from (2) is the neutral market belief when $N \rightarrow \infty$. When $K = 2$, an outcome earns a positive expected monetary return as soon as the market probability is higher than the prior, $1/2$. However, for greater K , the expected return is still negative when the market probability just surpasses the prior probability, $1/K$. For instance, when $K = 100$, the outcome’s market probability must rise from the .01 prior to over .025 before the expected return becomes positive. Intuitively, the fact that an outcome attracts more than a proportional fraction of all the bets does not guarantee that the ex-post chances of that outcome are higher than the prior, because signals are noisy. Despite the fact that all bettors bet on the outcome that achieves the highest posterior, an outcome that attracts slightly more than the fraction $1/K$ of the total bets is unlikely to win. The noise in the bettors’ signals induces this wedge between the prior probability and the market probability above which a horse becomes a good bet.

Figure 3 plots the posterior probability (1) as a function of the market probability, for fixed $N = 4$ and $\theta = 1$. When the number of outcomes is large relative to the number of bettors (as in the thick line for $K = 100$), the noise effect dominates, resulting in a reverse FLB: the posterior probability is higher than the market probability for longshots, but is lower for favorites. This finding is consistent with Asch and Quandt’s (1988) observation of reduced (or reverse) FLB for exotic bets, in which the number of outcomes is high relative to the number of bettors.²² When the number of outcomes is $K = 10$ (medium line), we see the occurrence of a mild FLB; this is further reinforced when $K = 2$ (thin line).

²²For example, when betting on the “pick six” pool, bettors must guess the winners in six consecutive races, usually the second through the seventh. If there are 10 horses in each race, then the pick six pool admits 1,000,000 combinations.

Reverse Bias in Lotto. Unless the game is corrupt, Lotto gamblers have no private information about the outcome drawn, regardless of the number of possible combinations. Lotto corresponds to a degenerate version of our model with completely uninformative signals. In equilibrium, each gambler picks a random combination with probability $1/K$. When N/K tends to infinity, the distribution of bets across all outcomes becomes uniform, according to the law of large numbers. However, the designers of Lotto games typically set K according to the rule of thumb, $K \approx N$ (see e.g. Walker and Young, 2001). Hence, there remains a fair amount of noise in the bet distribution.

As a result of this noise, some combinations receive no bets, others receive one bet, and still others receive two or more bets. The market probability of an outcome is the fraction of bets placed on it. We show here that outcomes with high market probability have low expected return. Note that the jackpot is shared among all those who picked the winning combination and that the posterior probability is equal to the prior probability, $1/K$, regardless of the market probability. The expected return to a bet on an outcome with market probability π is thus $(1 - \tau) / (\pi K) - 1$. Because this return is decreasing in the market probability, the *reverse* FLB always results. This bias is an immediate consequence of how parimutuel payoffs are determined.

Next, we show that when K is large and N is fixed, an increase in K results in a further increase in the reverse FLB. Holding N fixed, as K increases (i.e., as N/K tends to zero), there is probability 1 that all gamblers will bet on different outcomes. A favorite then has market probability $1/N$, while the posterior (as well as prior) probability of that outcome is $1/K$. The reverse FLB, measured by the expected loss to a bet on any favorite, $1 - (1 - \tau)N/K$, is increasing in K .

4 Asymmetric Prior and Common Error

In this section, we extend the model to allow for two ubiquitous elements of horse races: ex-ante asymmetry in the prior belief distribution (Section 4.1) and a common error in the bettors' information (Section 4.2). In the presence of small ex-ante asymmetries and ex-post noise, we find that our model replicates the main qualitative features of expected returns observed in horse races, as documented by Snowberg and Wolfers (2005).

4.1 Ex-ante Asymmetry

Even though horse races are designed to be not too unbalanced, there are in reality known differences in strength among the racing horses. Here we show that the asymmetry in the prior belief induces a contrarian incentive to bet on the ex-ante longshot. By itself, this effect could contribute to the favorite-longshot bias. However, an ex-post favorite now may have been an ex-ante favorite or an ex-ante longshot. We demonstrate that the mixing of these possibilities tends to flatten the expected returns for intermediate market probabilities.

To keep the analysis tractable and to obtain a simple characterization of equilibrium behavior, we further assume that there are $K = 2$ outcomes, with $q_1 = q = 1 - q_2 \geq 1/2$. Assume that the bettors' signals are identically and independently distributed conditional on outcome k , with posterior beliefs, $p = \Pr(k = 1|s)$, distributed according to the continuous cumulative distribution G with density g on $[0, 1]$.²³ Bayesian updating implies that $g(p|1) = pg(p)/q$ and $g(p|2) = (1 - p)g(p)/(1 - q)$. In particular, $g(p|1)/g(p|2) = p/(1 - p)$ is monotone, thus $G(p|1)$ is higher than $G(p|2)$ in the likelihood ratio dominance order, and hence in the first-order stochastic dominance order.

Consider the problem of a bettor with posterior belief $p = \Pr(k = 1|s)$. The payoff from abstention is 0. The expected payoff of a bet on outcome 1 is $pW(1|1) - 1 + u$, while a bet on outcome 2 yields $(1 - p)W(2|2) - 1 + u$. Since $W(1|1), W(2|2) > 0$, the best response of each individual bettor has a cutoff characterization. There exist threshold posterior beliefs— $\hat{p}_2, \hat{p}_1 \in [0, 1]$ with $\hat{p}_2 \leq \hat{p}_1$ —such that if $p < \hat{p}_2$ it is optimal to bet on 2; if $p \in [\hat{p}_2, \hat{p}_1)$, it is optimal to abstain; and if $p \geq \hat{p}_1$ it is optimal to bet on 1.

Suppose that all opponents adopt the cutoff strategy defined by the thresholds \hat{p}_2 and \hat{p}_1 . Because the others' signals and corresponding bets are uncertain, the conditional winning payoff is also random. For instance, $G(\hat{p}_2|1)$ is the probability that any opponent bets on outcome 2 conditional on the true outcome being 1.

Lemma 1 *When all opponents use thresholds $\hat{p}_2 \leq \hat{p}_1$, the expected return on a successful*

²³In the presence of discontinuities in the posterior belief distribution, the only symmetric equilibria might involve mixed strategies. Our results can be extended to allow for these discontinuities.

bet is

$$W(1|1) = \begin{cases} (1 - \tau) \frac{1 - G(\hat{p}_1|1) + G(\hat{p}_2|1)[1 - G(\hat{p}_1|1)^{N-1}]}{1 - G(\hat{p}_1|1)} & \text{if } G(\hat{p}_1|1) < 1, \\ (1 - \tau) [1 + (N - 1) G(\hat{p}_2|1)] & \text{if } G(\hat{p}_1|1) = 1, \end{cases} \quad (5)$$

$$W(2|2) = \begin{cases} (1 - \tau) \frac{G(\hat{p}_2|2) + [1 - G(\hat{p}_1|2)][1 - [1 - G(\hat{p}_2|2)]^{N-1}]}{G(\hat{p}_2|2)} & \text{if } G(\hat{p}_2|2) > 0, \\ (1 - \tau) [1 + (N - 1) [1 - G(\hat{p}_1|2)]] & \text{if } G(\hat{p}_2|2) = 0. \end{cases} \quad (6)$$

Focusing on the no-abstention regime, the symmetric equilibrium strategy is characterized by the single threshold, $\hat{p} \equiv \hat{p}_1 = \hat{p}_2 \in [0, 1]$, such that for $p < \hat{p}$ it is optimal to bet on 2, and for $p \geq \hat{p}$ it is optimal to bet on 1.²⁴

Proposition 5 *If u is sufficiently large, then there is a unique symmetric equilibrium in which all bettors with $p \geq \hat{p}^{(N)}$ bet on 1, and all bettors with $p < \hat{p}^{(N)}$ bet on 2, where $\hat{p}^{(N)} \in (0, 1)$ is the unique solution to*

$$\frac{\hat{p}}{1 - \hat{p}} = \frac{1 - G(\hat{p}|1)}{G(\hat{p}|2)} \frac{1 - [1 - G(\hat{p}|2)]^N}{1 - G(\hat{p}|1)^N}. \quad (7)$$

As the number of bettors, N , tends to infinity, the threshold belief, $\hat{p}^{(N)}$, tends to the unique solution to

$$\frac{p}{1 - p} = \frac{1 - G(p|1)}{G(p|2)}. \quad (8)$$

To interpret the equilibrium condition (8), consider a bettor with posterior p . When winning, the bettor shares the pool with all those who bet on the same outcome. In the limit case with a large number of bettors, the law of large numbers implies that there is no uncertainty in the conditional distribution of the opponents. Since all bettors use the same cutoff strategy, \hat{p} , the fraction of bettors who pick 1 when $k = 1$ is $1 - G(\hat{p}|1)$. The expected payoff from a bet on outcome 1 is then $p(1 - \tau) / (1 - G(\hat{p}|1))$. Similarly, the expected payoff from 2 is $(1 - p)(1 - \tau) / G(\hat{p}|2)$. The payoff of an indifferent bettor satisfies equation (8).

In the special case with symmetric prior and belief distributions, we have $1 - G(p|1) = G(1 - p|2)$ for every p , so that $\hat{p} = 1/2$ solves (8). Once we introduce ex-ante asymmetries instead, it is not an equilibrium to bet on the ex-post most likely outcome.²⁵

²⁴Note that the game with asymmetric prior is still symmetric with respect to the players.

²⁵Note the contrast with the positive results on truth-telling obtained by Cabrales et al. (2003) for mutual insurance schemes that use proportional payment/reimbursement rules with a parimutuel structure.

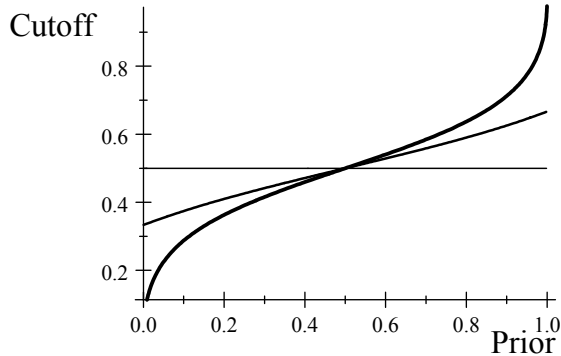


Figure 4: Equilibrium cutoff belief as a function of prior belief, for $N = 1, 2, 100$

Proposition 6 *Holding fixed the signal structure, the greater is the prior chance, q , for outcome 1, the greater is the limit threshold belief, \hat{p} .*

If the belief distribution is symmetric when $q = 1/2$, then Proposition 6 implies that $\hat{p} > 1/2$ when $q > 1/2$. In this case, bettors with beliefs in the interval $(1/2, \hat{p})$ will bet on the ex-ante longshot, 2, while privately believing that the ex-ante favorite, 1, is more likely to win. These bettors shun away from the outcome they privately believe to be more likely, because they expect that outcome to attract relatively more competing bets. Intuitively, since the opponents are more likely to bet on the ex-ante favorite outcome, the parimutuel payoffs are automatically adjusted adversely against that outcome. This contrarian incentive to bet on the ex-ante longshot, 2, persists in equilibrium.²⁶

Example. To illustrate the contrarian incentive for general N , consider the *linear signal example* with conditional densities $f(s|k=1) = 2s$ and $f(s|2) = 2(1-s)$ for $s \in [0, 1]$.²⁷ The cutoff private belief is obtained by substituting the conditional belief distributions $G(p|1) = [p(1-q)]^2 / [p(1-q) + (1-p)q]^2$ and $G(p|2) = 1 - [(1-p)q]^2 / [p(1-q) + (1-p)q]^2$ into the equilibrium condition (7). Figure 4 plots the equilibrium cutoff belief,

²⁶This effect is similar to the one identified in Ali's (1977) Theorem 2 in an environment in which bettors have heterogeneous (prior) beliefs. When instead the heterogeneous beliefs are correlated with the outcome realization, as in our model, the ex-ante longshot is more subject to ex-post unfavorable adjustments in the odds, because long odds are more elastic than short odds. Hence, it can be shown that the ex-ante favorite attracts more bets in our equilibrium than it would if the bettors did not take into account the correlation between odds and outcome realizations.

²⁷When $q = 1/2$, this is the symmetric Dirichlet example with $K = 2$ and $\theta = 1$, i.e., $p = s$ is uniformly distributed on $[0, 1]$.

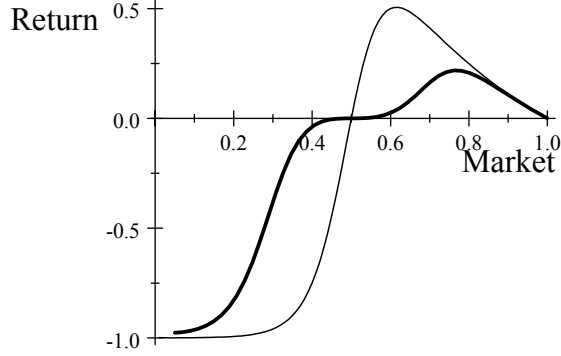


Figure 5: Ex-post expected returns as a function of market probability, for $q = 1/2, .22$

$\hat{p}^{(N)}(q)$, as a function of the prior belief, $q \in [0, 1]$, for $N = 1, 2, 100$, in progressively darker shading. A single bettor ($N = 1$) has $\hat{p}^{(1)}(q) = 1/2$. With $N > 1$ bettors instead, the cutoff belief satisfies $\hat{p}^{(N)}(q) \geq 1/2$ for $q \geq 1/2$. When $q > 1/2$, bettors with beliefs within the interval $[1/2, \hat{p}^{(N)}(q)]$ bet on the ex-ante longshot 2, in accordance with Proposition 6. Since $|\hat{p}^{(N)}(q) - 1/2|$ increases in N , competition exacerbates contrarian betting behavior.

Expected Returns. To compute the expected returns, empiricists typically group the observations according to the market probability. As Snowberg and Wolfers (2005) document, expected returns in horse races are an increasing function of the market probability, with a flat segment for intermediate market probabilities. Now we show how this flat segment results in our model when the prior distribution is asymmetric. Using the prior belief distribution and the equilibrium conditional probabilities $\eta(m|l)$, we can compute the expected return associated with a realized market probability:

Lemma 2 *With N active bettors, the posterior expected return corresponding to the market probability π is*

$$\bar{R}_\pi = \frac{1 - \tau}{\pi} \frac{\sum_{k=1}^K q_k \eta(k|k)^{N\pi} (1 - \eta(k|k))^{N(1-\pi)}}{\sum_{m=1}^K \sum_{l=1}^K q_l \eta(m|l)^{N\pi} (1 - \eta(m|l))^{N(1-\pi)}} - 1. \quad (9)$$

To illustrate the effect of ex-ante asymmetries on expected returns, consider the linear signal example with $K = 2$ outcomes, $\tau = 0$, and $N = 10$. Figure 5 displays the posterior expected return (9) as a function of the market probability, for $q_1 = 1/2$ (thin curve) and $q_1 = 0.22$ (thick curve). Note that the return tends to be close to zero for odds in the

intermediate region and that the FLB is pronounced only for strong favorites and strong longshots, as empirically observed.

Intuitively, intermediate market probabilities can arise in two ways in an asymmetric race. Either the ex-ante favorite has received relatively few bets, and therefore yields a negative return by the informational FLB, or the ex-ante longshot has received relatively many bets, and thus yields a positive return. These two effects counteract each other and tend to flatten the expected return curve in the intermediate range. We expect this flattening to be even more prevalent when there are more than two outcomes.

4.2 Ex-post Noise

So far we have maintained the hypothesis that the private signals are independent, conditional on the true outcome, k . However, there is often an unpredictable component in the outcome of the race. Equivalently, there is residual uncertainty about the outcome that could not be resolved even if infinitely many private signals were observed. For expositional simplicity, we revert here to the ex-ante symmetric case, and define a new random variable, $x \in \{1, \dots, K\}$, to capture the information state. Let $\Pr(x = k|k) = \sigma > 1/K$ denote the chance that the information state is identical to the true outcome. Knowing information state x , the posterior chance that the outcome is x is then σ . Individual signals are informative about x , but carry no further information about the outcome.

By the symmetry of the setup, it continues to be a symmetric equilibrium that every bettor bets on the most likely outcome according to the private posterior belief. Denote by $\eta(l|x)$ the chance of an individual betting on l in information state x . Generalizing (1), an outcome k with market probability π now has posterior probability

$$\hat{\beta} = \frac{\sigma \eta(k|k)^{N\pi} (1 - \eta(k|k))^{N(1-\pi)} + (1 - \sigma) \eta(k|x)^{N\pi} (1 - \eta(k|x))^{N(1-\pi)}}{\eta(k|k)^{N\pi} (1 - \eta(k|k))^{N(1-\pi)} + (K + 1 - 2\sigma) \eta(k|x)^{N\pi} (1 - \eta(k|x))^{N(1-\pi)}},$$

where $x \neq k$. In this symmetric setting, the posterior expected return is $\hat{\beta}(1 - \tau)/\pi - 1$. By simple algebra, it can be verified that the expected return corresponding to market probability π is an increasing function of σ when $\pi > \pi^*$. Hence, an increase in ex-post noise (i.e., a reduction of σ) flattens the expected return curve for favorites, thus aligning our predictions to Snowberg and Wolfers' (2005) empirical findings.

5 Participation and Selection

Now we extend the model to allow the players to abstain from betting (Section 5.1) or to bet on more than one outcome (Section 5.2). We find that a lower recreational value from betting leads to reduced participation by poorly informed bettors and exacerbates the FLB. Similarly, the divisibility of bets results in a reduction in the relative amount of noise and therefore in an increase in the FLB.

5.1 Abstention

Turn to the second case of Proposition 1, in which bettors with intermediate private beliefs abstain. As the recreational value, u , of betting decreases or, equivalently, the level of the track take increases, participation decreases because their expected loss from betting is not compensated by the recreational value.

Proposition 7 *When $u \in (\tau, u^{*(N)})$, there exists a unique symmetric equilibrium with thresholds $0 < \hat{p}_2 < \hat{p}_1 < 1$. When u increases or τ decreases, participation by bettors increases: \hat{p}_2 rises and \hat{p}_1 falls. In the limit as $u \rightarrow u^{*(N)}$, we have $\hat{p}_1 - \hat{p}_2 \rightarrow 0$. In the limit as $u \rightarrow \tau$, we have $\hat{p}_2 \rightarrow 0$ and $\hat{p}_1 \rightarrow 1$.*

When u decreases, more bettors abstain; hence, the overall amount of information that is present in the market is also reduced. However, the selection effect is such that the least informed of the individuals drop out first. Thus, the realized bets will contain *relatively* more information and less noise. According to the logic of Proposition 4, more informative bets contribute to the FLB.

Proposition 8 *Assume that $K = 2$ and that the belief distribution is symmetric and unbounded. Take as given any bet realization with total amounts $b_1, b_2 > 0$ placed on the two outcomes. If $u \in (\tau, u^{*(N)})$ is sufficiently small, a longshot's market probability $\pi_1 = b_1 / (b_1 + b_2) < 1/2$ (respectively a favorite's $\pi_1 > 1/2$) is strictly greater (respectively smaller) than the associated posterior probability β_1 .*

Note that if there is a reverse FLB at u^* , it will persist as u falls slightly below u^* because the equilibrium changes continuously. But, according to Proposition 8, the reverse bias is overturned as u falls further towards τ , at which point active betting ceases. Note

that while the overall amount of information that is present in the market is very small, when u approaches τ , abstention increases the amount of information *relative* to noise that is contained in the equilibrium bets. Thus, abstention strengthens the FLB.

Example. Consider the symmetric Dirichlet example with $K = 2$ and $\theta = 1$. For this symmetric example, we have $W(1|1) = W(2|2)$ and the equilibrium satisfies $\hat{p}_1 = 1 - \hat{p}_2$. The equilibrium condition (14) from the Appendix becomes

$$\frac{1-u}{1-\tau} = \hat{p}_1 \frac{1 - G(\hat{p}_1|1) + G(\hat{p}_2|1) \left[1 - G(\hat{p}_1|1)^{N-1}\right]}{1 - G(\hat{p}_1|1)} = \hat{p}_1 \frac{2 - (1 - \hat{p}_1) \hat{p}_1^{2(N-1)}}{1 + \hat{p}_1}.$$

The critical $u^{*(N)}$ corresponds to the cutoff belief $\hat{p}_1 = 1/2$ at which $(1 - u^{*(N)}) / (1 - \tau) = (2 - 2^{1-2N}) / 3$. Note that as $N \rightarrow \infty$, we obtain the limit equation $(1 - u) / (1 - \tau) = 2\hat{p}_1 / (1 + \hat{p}_1)$, solved by $\hat{p}_1 = (1 - u) / (1 + u - 2\tau)$. For any N , the solution for \hat{p}_1 rises from $1/2$ to 1 as u decreases from $u^{*(N)}$ to τ . Moreover, abstention becomes more attractive as the number of bettors increases.

5.2 Divisible Bets

We illustrate the role of our indivisibility assumption through two simple examples. Consider first the case without private information, with K equally likely outcomes. If bets are perfectly divisible, then in the symmetric equilibrium each bettor places $1/K$ on each outcome. As a result, the market probability is always equal to the empirical probability. The reverse FLB then disappears completely! In reality, however, the divisibility constraint binds because the minimum bet is bounded by the price of an individual Lotto ticket. Since it is impossible to bet infinitesimal amounts and N/K is small, the substantial amount of noise that is present in Lotto drives the reverse FLB.

Second, the incentive to bet on more than one outcome exists also when bettors are privately informed. However, this incentive is reduced because it entails acting against one's private information. To illustrate this point, we return to a generalization of the example with binary signals presented in Section 2:

Proposition 9 *Assume that there are $K = 2$ equally likely outcomes, $N = 2$ bettors, and sufficiently high recreational utility. If each bettor observes a binary symmetric signal with precision $\pi = \Pr(s = k) > 1/2$ and is allowed to split the bet between the two outcomes,*

then in the symmetric equilibrium they place the fraction $y^* > \pi$ on the outcome that they believe to be more likely conditional on the signal observed, and the FLB always results.

If players can bet on more than one outcome, then the amount of noise relative to information present in equilibrium decreases and the FLB increases. More generally, we expect that the incentive to bet on multiple outcomes is particularly important in exotic bets, where the ratio of outcomes to bettors is high. For example, someone who received a tip on the outcome of two races fears the noise on the outcomes of the other four races, making a “pick six” gamble (see footnote 22) akin to a lottery. Consistent with popular advice, the best strategy is to bet on many combinations involving different outcomes in those other races.

6 Comparison with Other Explanations

In this section, we compare the performance of our theory with the main alternative explanations for the FLB in parimutuel betting markets. The most notable alternative theories that have been proposed in the literature are the following: (1) Griffith (1949) suggested that the FLB is due to a tendency of individual bettors to *overestimate* low probability events. (2) Isaacs (1953) noted that an informed *monopolist bettor* who can place multiple bets does not set the expected return on his marginal bet at zero, because this destroys the return on inframarginal bets. (3) Weitzman (1965) hypothesized that individual bettors are *risk loving*, and thus willing to accept a lower expected payoff when they bet on riskier longshots.²⁸ (4) Ali (1977) showed that if bettors have *heterogeneous* (prior) *beliefs*, then the market probability of the favorite is lower than the median bettor’s belief. The FLB then results if the belief of the median bettor is correct. (5) Hurley and McDonough (1995) and Terrell and Farmer (1996) showed that the FLB can result because the amount of *arbitrage* is *limited* by the track take.

While we believe that all of these theories can contribute to the explanation of the empirical facts, the information-based theory developed in this paper has a number of merits:

²⁸See also Rosett (1965), Quandt (1976), and Ali (1977) on the risk-loving explanation. According to Golec and Tamarkin (1998), the bias is compatible with preferences for skewness rather than risk. Jullien and Salanié (2000) use data from fixed-odds markets to argue in favor of non-expected utility models.

- Our theory builds on the realistic assumption that the differences in beliefs among bettors are generated by *private information* (see e.g. Crafts, 1985). Only by modeling the informational determinants of beliefs explicitly, can we address the natural question of information aggregation.²⁹
- Our theory offers a parsimonious explanation for the FLB *and its reverse*. Our theory also predicts that the FLB is lower, or reversed, when the number of bettors is low (relative to the number of outcomes), as Sobel and Raines (2003) and Gramm and Owens (2005) verify empirically.³⁰
- Our theory is compatible with the *reduced* level of FLB that Asch and Quandt (1988) document in “exotic” bets, such as exactas (see footnote 11).³¹ Asch and Quandt (1988) conclude in favor of private information because the payoffs on winners tend to be more depressed in the exacta than in the win pool.³²
- Our theory is compatible with the fact that *late bets* tend to contain more information about the horses’ finishing order than earlier bets, as Asch et al. (1982) observe. Ottaviani and Sørensen (2006) demonstrate that late informed betting will result in equilibrium when (many small) bettors are allowed to optimally time their bets.
- Our explanation can account for the pattern of *expected returns* that Snowberg and Wolfers (2005) document. Asymmetries in the prior probabilities of the different outcomes and common errors in the bettors’ beliefs tend to flatten the expected returns for intermediate market probabilities.³³
- Theories based on private information can explain the occurrence of the FLB in

²⁹See Ottaviani and Sørensen (2005a) on the interplay of heterogeneous priors and private information in prediction markets.

³⁰The preponderance of noise might account for some of Metrick’s (1996) findings in basketball betting.

³¹Risk loving does not seem compatible with arbitrage across exacta and win pools. As Snowberg and Wolfers (2005) stress, probability weighting would need to be combined with additional misperceptions to be compatible with arbitrage across betting pools.

³²Asch and Quandt (1988) observe that the market probabilities recovered from the win pool overestimate the market probabilities on the exacta pool, by a much larger margin for winning than for losing horses.

³³The observed flat segment is theoretically compatible with risk loving or probability weighting, but would imply a very specific preference pattern. For example, if bettors had mean-variance preferences, as posited by Quandt (1986), then the expected return would be strictly increasing in the market probability. Limited arbitrage à la Hurley and McDonough (1995) can explain the flat segment for horses with probability above a certain threshold, but not the decreasing segment for strong favorites.

both *fixed-odds* and parimutuel markets, as well as the lower level of bias observed in parimutuel markets, as Bruce and Johnson (2001) among others document (see footnote 10 above).

7 Conclusion

This paper builds a simple model of parimutuel betting with private information. The sign and the extent of the FLB depend on the amount of information relative to noise that is present in the market. When there is little private information, posterior odds are close to prior odds, even when the market odds are extreme. In this case, deviations of market odds from prior odds are mostly due to the noise contained in the signal. Systematically, the market odds are more extreme than the posterior odds, and the reverse FLB results.

As the number of bettors increases, the realized market odds contain more information and less noise. For any fixed market odds, the posterior odds are then more extreme, increasing the extent of the FLB. Note that the FLB always arises with a large number of bettors, provided that they have *some* private information. This is confirmed by Ottaviani and Sørensen (2006) in a model with a continuum of privately informed bettors. In that setting there is no noise, so that the FLB always results.

Our theory sheds new light on the available evidence. A formal test of the theory could exploit the variation across betting environments. The amount of private information tends to vary consistently depending on the prominence of the underlying event. Similarly, the amount of noise present depends on the number of outcomes, as well as on the observability of past bets. For example, there is a sizeable amount of noise in lotteries and exotic bets, because the number of outcomes is high relative to the number of tickets sold and the opponents' bets are not observed.

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Appendix A: Proofs

Proof of Proposition 1. First, suppose that $u > \tau$, but no one is betting. Individuals with posterior p_k sufficiently close to 1 gain from deviating to betting on outcome k , since the expected utility from so doing is arbitrarily close to $(1 - \tau) - 1 + u > 0$.

In any symmetric equilibrium where everyone is betting, we can find a belief \hat{p} at which the interim expected utility is at the lowest level. By upper hemi-continuity of the equilibrium correspondence and compactness of the strategy space, there exists an equilibrium where this lowest interim utility U is, in turn, minimal. Holding fixed these equilibrium strategies, the critical value $u^{*(N)}$ is defined as the solution to $U = 0$.

Finally, suppose that $u \leq \tau$. Any bet is reduced to $1 - \tau$ before being placed in the pool. By the logic of the no-trade theorem, it is impossible that all active bettors expect a return in excess of $1 - \tau$. If there is a positive chance of betting, some active bettors expect a return less than $1 - \tau \leq 1 - u$ and are better off abstaining. The contradiction implies that there can be no active bettors at all.

Proof of Proposition 2. By symmetry of the betting strategy and the belief distribution, $\eta(1|1) = \dots = \eta(K|K)$ and hence $W(1|1) = \dots = W(K|K) > 0$. The expected utility from a bet on k is $p_k W(k|k) + u - 1$, thus the best response is to bet on the outcome with the greatest p_k , as claimed. Again, by symmetry, $\eta(k|l)$ assumes the same value for every pair $k \neq l$. Since probabilities sum to one, the proof is complete once we show that $\eta(1|1) > 1/K$. The prior chance of outcome 1 is $1/K$, so Bayes' rule implies that the density of beliefs satisfies $g(p|1) = K p_1 g(p)$. Where p_1 is greatest, almost surely $p_1 > 1/K$. Finally, symmetry implies that there is ex-ante chance $1/K$ that p_1 is greatest. Hence,

$$\eta(1|1) = \int_{p|p_1 \text{ greatest}} g(p|1) dp = \int_{p|p_1 \text{ greatest}} K p_1 g(p) dp > \int_{p|p_1 \text{ greatest}} g(p) dp = \frac{1}{K}.$$

Proof of Proposition 3. Let $\pi_k < \pi^*$ be given. The desired inequality is

$$\frac{1 - \pi_k}{\pi_k} < (K - 1) \left(\frac{\eta(k|l)}{\eta(k|k)} \right)^n \left(\frac{1 - \eta(k|l)}{1 - \eta(k|k)} \right)^{N-n} \quad (10)$$

where $l \neq k$. Take the natural logarithm, use $n/N = \pi_k$, and re-arrange (10) to arrive at

$$\frac{1}{N} \log \left(\frac{1 - \pi_k}{\pi_k (K - 1)} \right) < \pi_k \log \left(\frac{\eta(k|l)}{\eta(k|k)} \right) + (1 - \pi_k) \log \left(\frac{1 - \eta(k|l)}{1 - \eta(k|k)} \right).$$

The left-hand side tends to 0 as $N \rightarrow \infty$. The right-hand side is positive, precisely since $\pi_k < \pi^*$.

Proof of Proposition 4. With two outcomes, $\eta(k|k) = 1 - \eta(k|l)$. From (2) we obtain $\pi^* = 1/2$. Let $\pi_k < 1/2$ be given. Inequality (10) reduces to

$$\frac{1 - \pi_k}{\pi_k} < \left(\frac{\eta(k|k)}{\eta(k|l)} \right)^{N-2n}.$$

Take the natural logarithm, use $n/N = \pi_k$, and re-arrange to obtain the inequality

$$\frac{1}{1 - 2\pi_k} \log \left(\frac{1 - \pi_k}{\pi_k} \right) < N \log \left(\frac{\eta(k|k)}{\eta(k|l)} \right). \quad (11)$$

Since $\pi_k < 1/2$ and $\eta(k|k) > \eta(k|l)$, all terms are positive. The right-hand side of (11) tends to ∞ when $\eta(k|k)/\eta(k|l)$ tends to ∞ . The inequality is reversed if $\pi_k > 1/2$.

Proof of Lemma 1. We derive $W(1|1)$. Let $\eta_2 = G(\hat{p}_2|1)$, $\eta_1 = 1 - G(\hat{p}_1|1)$ and $\eta_0 = G(\hat{p}_1|1) - G(\hat{p}_2|1)$ denote the equilibrium chances for the respective actions given outcome 1. By definition,

$$W(1|1) = (1 - \tau) \sum_{\hat{b}_1=0}^{N-1} \sum_{\hat{b}_2=0}^{N-1-\hat{b}_1} \frac{\hat{b}_2 + \hat{b}_1 + 1}{\hat{b}_1 + 1} p(\hat{b}_2, \hat{b}_1|1),$$

where $p(\hat{b}_2, \hat{b}_1|1)$ is the chance that \hat{b}_2 opponents bet on 2 and \hat{b}_1 opponents bet on 1.

If $\eta_1 = 0$, the distribution degenerates to be binomial with chance η_2 for bets on 2. Then, $W(1|1) = (1 - \tau) ((N-1)\eta_2 + 1)$ since $(N-1)\eta_2$ is the expected number of bets on 2.

For the remainder, assume $\eta_1 > 0$. Let $n = \hat{b}_2 + \hat{b}_1$ denote the realized number of bets by the opponents. Conditional on n , \hat{b}_1 is binomially distributed with (updated) chance $\eta_1/(\eta_2 + \eta_1)$. Given n , the expected value, $W(1|1)$, of betting on outcome 1 is

$$\begin{aligned} & (1 - \tau) \sum_{\hat{b}_1=0}^{n-1} \frac{n}{\hat{b}_1 + 1} \binom{n-1}{\hat{b}_1} \left(\frac{\eta_1}{\eta_2 + \eta_1} \right)^{\hat{b}_1} \left(\frac{\eta_2}{\eta_2 + \eta_1} \right)^{n-1-\hat{b}_1} \\ &= \frac{(1 - \tau)(\eta_2 + \eta_1)}{\eta_1} \sum_{\hat{b}_1=1}^n \binom{n}{\hat{b}_1} \left(\frac{\eta_1}{\eta_2 + \eta_1} \right)^{\hat{b}_1} \left(\frac{\eta_2}{\eta_2 + \eta_1} \right)^{N-\hat{b}_1} = (1 - \tau) \frac{1 - \left(\frac{\eta_2}{\eta_2 + \eta_1} \right)^{n+1}}{\left(\frac{\eta_1}{\eta_2 + \eta_1} \right)}. \end{aligned}$$

Since n is binomially distributed with parameter $\eta_2 + \eta_1$, $W(1|1)/(1 - \tau)$ is

$$\begin{aligned} & \sum_{n=0}^{N-1} \frac{1 - \left(\frac{\eta_2}{\eta_2 + \eta_1} \right)^{n+1}}{\left(\frac{\eta_1}{\eta_2 + \eta_1} \right)} \binom{N-1}{n} (\eta_2 + \eta_1)^n \eta_0^{N-1-n} \\ &= \frac{\eta_2 + \eta_1}{\eta_1} - (1 - \tau) \frac{\eta_2}{\eta_1} \sum_{n=0}^{N-1} \binom{N-1}{n} \eta_2^n \eta_0^{N-1-n} = \frac{\eta_2 + \eta_1 - \eta_2(\eta_2 + \eta_0)^{N-1}}{\eta_1}, \end{aligned}$$

as desired.

Proof of Proposition 5. Consider the optimal response of a bettor to all others using threshold \hat{p} . From Lemma 1, $W(1|1), W(2|2) \in [1 - \tau, (1 - \tau)(N - 1)]$. Let $\check{p} \in (0, 1)$ be the unique solution to $\check{p}W(1|1) = (1 - \check{p})W(2|2)$. The bettor with belief \check{p} is indifferent between betting on either of the two outcomes. Symmetric equilibrium requires $\check{p} = \hat{p}$.

If $G(\hat{p}) = 1$, Lemma 1 reduces the indifference condition to $\check{p}(N - 1) = (1 - \check{p})$ or $\check{p} = 1/N$. Thus, if $G(1/N) = 1$, then $\hat{p}^{(N)} = 1/N$ defines the equilibrium with all bets on 2. Likewise, if $G(1 - 1/N) = 0$, then $\hat{p}^{(N)} = 1 - 1/N$ defines the equilibrium with all bets on 1. If $G(\hat{p}) \in (0, 1)$, then (5) and (6) reduce to $W(1|1) = (1 - \tau) \left[\frac{1 - G(\hat{p}|1)^N}{1 - G(\hat{p}|1)} \right]$ and $W(2|2) = (1 - \tau) \left[\frac{1 - (1 - G(\hat{p}|2))^N}{G(\hat{p}|2)} \right]$. Then the equilibrium condition $\hat{p}W(1|1) = (1 - \hat{p})W(2|2)$ gives the desired (7). We now show that there exists a unique solution $\hat{p}^{(N)}$ to (7). The left-hand side of (7) is strictly increasing, ranging from zero to infinity as \hat{p} ranges over $(0, 1)$. The positive right-hand side of (7) is weakly decreasing in \hat{p} since $W(1|1) = (1 - \tau) \sum_{k=0}^{N-1} [G(\hat{p}|1)]^k / N$ is increasing in \hat{p} and similarly $W(2|2)$ is decreasing in \hat{p} . Since $1/(N - 1) \leq W(1|1)/W(2|2) \leq N - 1$, the solution is in the range $[1/N, 1 - 1/N]$. We conclude that this cutoff $\hat{p}^{(N)}$ defines the unique symmetric equilibrium when $G(1 - 1/N) > 0$ and $G(1/N) < 1$.

Finally, let \hat{p} solve the limit equation (8) and fix any $\varepsilon > 0$. Notice that for N sufficiently large, $G(1 - 1/N) > 0$ and $G(1/N) < 1$, so that $\hat{p}^{(N)}$ solves (7). By monotonicity, at $\hat{p} + \varepsilon$ the left-hand side of (8) exceeds the right-hand side. By pointwise convergence of the right-hand side of (7) to that of (8), for sufficiently large N , the left-hand side of (7) exceeds the right-hand side at $\hat{p} + \varepsilon$. Likewise, for sufficiently large N , the right-hand side of (7) exceeds the left-hand side at $\hat{p} - \varepsilon$. It follows that $\hat{p}^{(N)} \in (\hat{p} - \varepsilon, \hat{p} + \varepsilon)$.

Proof of Proposition 6. For any signal s , the greater is the prior q , the greater is the corresponding posterior belief $p = qf(s|1) / [qf(s|1) + (1 - q)f(s|2)]$. Hence, when q rises, the conditional distribution functions $G(p|1)$ and $G(p|2)$ both fall at any p . At any p , the right-hand side of (8) rises. Since the left-hand side is increasing and the right-hand side is decreasing, it follows that the solution \hat{p} to (8) also rises.

Proof of Lemma 2. Conditional on outcome $l \in \{1, \dots, K\}$, the chance that $b_k = n$ is

$$\Pr(b_k = n|l) = \binom{N}{n} \eta(k|l)^n (1 - \eta(k|l))^{N-n}.$$

Conditional on $b_k = n$, the chance that k wins is then

$$\Pr(k|b_k = n) = \frac{q_k \eta(k|k)^n (1 - \eta(k|k))^{N-n}}{\sum_{l=1}^K q_l \eta(k|l)^n (1 - \eta(k|l))^{N-n}}. \quad (12)$$

Given that some outcome has market probability $\pi = n/N$, the chance that k is this outcome is

$$\Pr(b_k = n|b = n) = \frac{\Pr(b_k = n)}{\Pr(b = n)} = \frac{\sum_{j=1}^K q_j \eta(k|j)^n (1 - \eta(k|j))^{N-n}}{\sum_{m=1}^K \sum_{l=1}^K q_l \eta(m|l)^n (1 - \eta(m|l))^{N-n}}. \quad (13)$$

Substituting (12) and (13) into

$$\Pr(k|\pi = n/N) = \sum_{k=1}^K \Pr(k|b_k = \pi N) \Pr(b_k = \pi N|b = \pi N),$$

we conclude that (9) is the expected return to a bet with market probability π .

Proof of Proposition 7. We are considering the case $0 < \hat{p}_2 < \hat{p}_1 < 1$, so optimality of the threshold rule implies

$$\hat{p}_1 W(1|1) = 1 - u, \quad (14)$$

$$(1 - \hat{p}_2) W(2|2) = 1 - u. \quad (15)$$

Using the notation from the proof of Lemma 1, rewrite (14) as

$$\eta_2 = \frac{\frac{\eta_1}{\hat{p}_1} \frac{1-u}{1-\tau} - \eta_1}{1 - (1 - \eta_1)^{N-1}} \equiv H(\hat{p}_1). \quad (16)$$

With $\eta_1 = 1 - G(\hat{p}_1|1)$, the right-hand side H is a continuous function of $\hat{p}_1 \in (0, 1)$. It tends to infinity as \hat{p}_1 tends to 0, and tends to $(\tau - u) / [(1 - \tau)(N - 1)] < 0$ as \hat{p}_1 tends to 1. Moreover, it can be checked that H is a decreasing function whenever it is positive. It follows that for every $\hat{p}_2 \in [0, 1]$ there is a unique solution $\hat{p}_1 \in (0, 1)$ to equation (14). Thus equation (14) defines a continuous curve in the space $(\hat{p}_2, \hat{p}_1) \in [0, 1]^2$. Using the implicit function theorem, it can be verified that this curve is downward sloping. Likewise, for every $\hat{p}_1 \in [0, 1]$ there is a unique solution $\hat{p}_2 \in (0, 1)$ to (15), and the set of solutions defines a downward sloping curve. The solution is unique since curve (15) is steeper than curve (14) wherever the two curves cross, as a consequence of the monotonicity of $g(p|1)/g(p|2)$.

For the comparative statics result, note that an increase in u implies a lower value of $H(\hat{p}_1)$ for every given \hat{p}_1 . The curve defined by (14) therefore shifts down: for every given \hat{p}_2 , the resulting \hat{p}_1 is lower than before. Likewise, the curve defined by (15) shifts up: for every given \hat{p}_1 , the resulting \hat{p}_2 is larger than before. Since the (15) is steeper, the intersection of the two curves must move in the direction where \hat{p}_2 rises and \hat{p}_1 falls.

Proof of Proposition 8. Under the symmetry assumptions, we have $\hat{p}_2 = 1 - \hat{p}_1$ and $\eta(1|1) = \eta(2|2) > \eta(2|1) = \eta(1|2)$. Suppose that the realized bet amounts are $b_2, b_1 > 0$. The market implied probability for outcome 1 is $\pi_1 = b_1 / (b_1 + b_2)$. The bet distribution for outcome 1 is

$$p(b_2, b_1|1) = \frac{N!}{b_2!b_1!(N - b_2 - b_1)!} \eta(2|1)^{b_2} \eta(1|1)^{b_1} (1 - \eta(1|1) - \eta(2|1))^{N - b_2 - b_1},$$

and likewise for $k = 2$. Hence, $(1 - \beta_1) / \beta_1 = p(b_2, b_1|2) / p(b_2, b_1|1) = (\eta(1|2) / \eta(1|1))^{b_1 - b_2}$. If $\pi_1 < 1/2$, or $b_2 > b_1$, the desired FLB inequality is $(1 - \pi_1) / \pi_1 < (1 - \beta_1) / \beta_1$, i.e., $[\pi_1 / [b_1(1 - 2\pi_1)]] \log[(1 - \pi_1) / \pi_1] < \log(\eta(1|1) / \eta(1|2))$. This inequality holds at the fixed realization b_2, b_1 , once the ratio $\eta(1|1) / \eta(1|2)$ is sufficiently large. We have

$$\frac{\eta(1|1)}{\eta(1|2)} = \frac{1 - G(\hat{p}_1|1)}{1 - G(\hat{p}_1|2)} = \frac{\int_{\hat{p}_1}^1 g(p|1) dp}{\int_{\hat{p}_1}^1 g(p|2) dp} = \frac{\int_{\hat{p}_1}^1 \frac{p}{1-p} g(p|2) dp}{\int_{\hat{p}_1}^1 g(p|2) dp} > \frac{\hat{p}_1}{1 - \hat{p}_1},$$

where the last expression tends to infinity as u tends to τ , according to Proposition 7.

Proof of Proposition 9 A symmetric equilibrium is defined by the fraction $y^* \in [0, 1]$ of money that each bettor places on the most likely outcome given the signal received. If bettor 2 follows this rule, then the expected payoff of bettor 1 when placing y on the outcome believed to be most likely is

$$\pi \left[\pi \frac{2y}{y^* + y} + (1 - \pi) \frac{2y}{1 - y^* + y} \right] + (1 - \pi) \left[\pi \frac{2(1 - y)}{y^* + 1 - y} + (1 - \pi) \frac{2(1 - y)}{1 - y^* + 1 - y} \right].$$

Taking the derivative with respect to y , it is easy to see that the symmetric equilibrium is interior, $y^* \in (0, 1)$. Evaluating the derivative at $y = y^*$, the equilibrium condition is

$$\pi^2 \frac{1}{4y^*} - \pi(1 - \pi)y^* = (1 - \pi)^2 \frac{1}{4(1 - y^*)} - \pi(1 - \pi)(1 - y^*), \quad (17)$$

where the left-hand side is decreasing in y^* while the right-hand side is increasing. Since the left-hand side exceeds the right-hand side at $y^* = \pi$ whenever $\pi > 1/2$, we have $y^* > \pi$.

To derive implications for the FLB, we compare the market probability with the corresponding posterior probability. When the bettors obtain opposite signals, the market probability and the posterior probability are both equal to $1/2$. When the bettors obtain the same signal, the market probability for the favorite is y^* , while the posterior probability is $\pi^2 / [\pi^2 + (1 - \pi)^2]$. Substituting this posterior probability for y^* into (17), we obtain that the right-hand side exceeds the left-hand side. Hence the FLB always results.

Appendix B: Dirichlet Example

We derive the equilibrium probabilities (3) and (4) in the example introduced in Section 3.1. By symmetry, it suffices to derive $\eta(1|1)$. Bayes' rule implies $g(p|1) = Kp_1g(p)$, hence p is Dirichlet distributed with parameters $(\theta + 1, \theta, \dots, \theta)$. By Proposition 2, a bettor bets on outcome 1 when $p_1 > p_k$ for all $k > 1$.

If θ is an integer, let X_1, \dots, X_K be independent χ^2 distributed random variables, where the degrees of freedom are $2(\theta + 1)$ for X_1 and 2θ for X_2, \dots, X_K . The distribution function for X_k when $k > 1$ is

$$H(x) = 1 - e^{-x/2} \sum_{t=0}^{\theta-1} \frac{x^t}{2^t t!}. \quad (18)$$

Letting $r_k = X_k / \sum_{l=1}^K X_l$, Section 40.5 in Johnson and Kotz (1972) notes that $r = (r_1, \dots, r_K)$ follows the same Dirichlet distribution as p . The chance that $p_1 > p_k$ for all $k > 1$ is then equal to the chance that $X_1 > X_k$ for all $k > 1$. Given X_1 , this chance is $H(X_1)^{K-1}$. Using the expression for the density of the χ^2 distribution, we obtain

$$\eta(k|k) = \frac{1}{2^{\theta+1}\theta!} \int_0^\infty \left(1 - e^{-x/2} \sum_{t=0}^{\theta-1} \frac{x^t}{2^t t!}\right)^{K-1} x^\theta e^{-x/2} dx. \quad (19)$$

If $\theta = 1$, (18) gives $H(x) = \int_0^x e^{-y/2}/2 dy = 1 - e^{-x/2}$. From (19), we have

$$\begin{aligned} \eta(k|k) &= \frac{1}{4} \int_0^\infty (1 - e^{-x/2})^{K-1} x e^{-x/2} dx = \frac{1}{4} \int_0^\infty \left(\sum_{j=0}^{K-1} \binom{K-1}{j} (-1)^j e^{-jx/2} \right) x e^{-x/2} dx \\ &= \frac{1}{4} \sum_{j=0}^{K-1} \binom{K-1}{j} (-1)^j \int_0^\infty x e^{-(j+1)x/2} dx = \sum_{j=0}^{K-1} \binom{K-1}{j} (-1)^j \frac{1}{(j+1)^2}, \end{aligned}$$

proving (3).

When $K = 2$, $\eta(1|1)$ is the chance that $p_1 > 1/2$. Using integration by parts and $\frac{\Gamma(\theta+1)\Gamma(\theta)}{\Gamma(2\theta+1)} = \int_0^1 p_1^\theta (1-p_1)^{\theta-1} dp_1$, this chance satisfies

$$\frac{\Gamma(\theta+1)\Gamma(\theta)}{\Gamma(2\theta+1)} \eta(1|1) = \int_{1/2}^1 p_1^\theta (1-p_1)^{\theta-1} dp_1 = \frac{4^{-\theta}}{\theta} + \int_{1/2}^1 p_1^{\theta-1} (1-p_1)^\theta dp_1.$$

The substitution $p_2 = 1 - p_1$ yields

$$\int_{1/2}^1 p_1^{\theta-1} (1-p_1)^\theta dp_1 = \int_0^{1/2} p_2^\theta (1-p_2)^{\theta-1} dp_2 = \frac{\Gamma(\theta+1)\Gamma(\theta)}{\Gamma(2\theta+1)} - \int_{1/2}^1 p_1^\theta (1-p_1)^{\theta-1} dp_1.$$

Collecting terms, we obtain

$$2 \int_{1/2}^1 p_1^\theta (1-p_1)^{\theta-1} dp_1 = \frac{4^{-\theta}}{\theta} + \frac{\Gamma(\theta+1)\Gamma(\theta)}{\Gamma(2\theta+1)}$$

and hence (4).