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Ralf Korn* and Mogens Steffensen†

Abstract

We formulate a portfolio optimization problem as a game where the investor chooses a portfolio and his opponent, the market, chooses some market crashes. The asymmetry of the opponents' decision processes leads to a new and delicate generalization of the classical Hamilton-Jacob-Bellman equation in stochastic control. We characterize the optimal controls in general and specify them further in the cases of HARA, logarithmic, and exponential utility of the investor.

Keywords: Continuous-time game, asymmetric decisions, market crash, utility optimization.

1 Introduction

The problem of finding an optimal investment strategy for an investor with given utility function and a fixed initial endowment - the so-called portfolio optimization problem - is one of the classical problems in financial mathematics and its applications in insurance mathematics. The corresponding modern continuous-time approach is pioneered by Merton (1969,1971) who applied classical stochastic control methods to reduce the portfolio problem to a matter of solving a Hamilton-Jacobi-Bellman partial differential equation (for short: HJB-equation).

Since Merton's pioneering work many attempts have been made to solve the portfolio optimization problem in a framework that allows for more realistic models of stock prices, in particular for models that can explain large price movements. Examples where portfolio optimization problems are treated in more general settings are portfolio optimization in jump-diffusion models (see e.g. Aase (1984)) or in a general semimartingale framework (see e.g. Kramkov and Schachermayer (1999)).

A different portfolio problem that includes dramatic negative changes of the stock prices (so-called crashes) has been introduced by Korn and Wilmott (2002). Their main idea consists of two aspects, the separation between normal times where the stock price behaves as a geometric Brownian motion, and crash times where it jumps downwards, and the introduction of a worst-case functional that resembles the form of a game-theoretic max-min approach. While in Korn and Wilmott (2002) the problem is only solved for the choice of the logarithmic utility function, a more general problem of the worst-case form is treated in Korn and Menkens (2005). At first sight, their approach seems to be an approach similar to the HJB-equation approach of stochastic control. However, their arguments are based

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on equilibrium and indifference considerations and they derive differential equations for the optimal portfolio processes and not for the value function. Even more, they could prove optimality of their proposed portfolio processes only within the class of (piecewise) deterministic control strategies.

The main purpose of this paper is to put the worst-case portfolio optimization in the HJB-equation framework, and thus to connect it with the mainstream of stochastic control theory. This leads us to a type of continuous-time game problem that, to our knowledge, is new in control theory. It is the asymmetry of the opponents' decision processes that makes the game so interesting and challenging from a control theoretical point of view: The investor decides about the portfolio process whereas the opponent, the market, decides when the stock market crashes. One could fear that this very asymmetry prevents solutions to the game problem but we show that the problem, indeed, has a solution and that the solution can be characterized by a generalized HJB-equation.

A related financial game problem is approached by Talay and Zheng (2000). They solve a problem where the opponent of the investor, the market, decides about the parameters in a diffusion model. Thus, the idea of seeing the market as an opponent is exactly the same as ours. But since their price processes are continuous, the decisions of both the investor and the market affect only the coefficients of the continuous portfolio process. Therefore, from a financial modelling point of view their problem is completely different from ours.

Our main result is a verification theorem asserting that a system consisting of an HJB-type inequality, a relation between value functions before and after an action of the market, final conditions and a complementarity condition determine the value function. This result and its consequences are highlighted by some explicit examples. Note in particular that we do not have to restrict ourselves to (piecewise) deterministic controls. Therefore, the existing work on worst-case portfolio optimization is substantially generalized.

One can imagine other areas where the structure of the problem and its solution can find applications. E.g. an insurance company decides about reinsurance against large claims e.g. triggered by storm or earth quakes. In its battle against the merciless Mother Earth, the company could adopt a worst case basis for making certain decisions, e.g. the extent of reinsurance protection. The structure of the HJB-equation for such a problem is similar to here and the relative explicit characterization of the optimal decision obtained here holds out every promise of success in other areas.

2 The Model and the Preferences

Take as given a probability space $(\Omega, \mathcal{F}, P)$. Let W be a standard Brownian motion defined on this probability space. Let us consider an agent over a fixed time interval $[0,T]$. At time 0 the agent is endowed with initial wealth x_0 and his problem is to allocate investments over the given time horizon. We assume that the agent's investment opportunities are given by the following financial market,

$$\begin{aligned} dB(t) &= r(t)\,B(t)\,dt, \\ B(0) &= 1, \\ dS(t) &= S(t-)\left(\alpha(t)\,dt + \sigma(t)\,dW(t) - \beta(t-)\,dN(t)\right), \\ S(0) &= s_0, \end{aligned}$$

where r, α, σ, and β are assumed to be bounded deterministic functions. For $N(t) = 0$, this market is a classical Black-Scholes market. We introduce, however, jumps in the Black-Scholes market and let N be a counting process counting the number of jumps such that

$$N(t) = \#\{0 < s \leq t : S(t) \neq S(t-)\}.$$

In usual jump-diffusion models the counting process is now assumed to follow some probability law on $(\Omega, \mathcal{F}, P)$. One could e.g. let N be a Poisson process or a Cox process. Here, however, we take the counting process to be chosen by the market, which from the point of view of the agent, is considered as an opponent. I.e., the market decides when the stock jumps and N is to be considered as a decision variable held by the market. We speak of jumps in N as interventions.

We assume that the market is able to decide on an only limited number of interventions and denote the maximal number of interventions over $[0, T]$ by n_0. Figure 1 illustrates the process of interventions. The market can, however, also choose not to exercise its intervention options, so at time T the process of interventions can be in any of the states in Figure 1. Further, to avoid technical complications, we assume that the market never intervenes more than once in one time instant (a multiple intervention of the market in one time instant is also not reasonable from an intuitive point of view).

n_0 int. left $N = 0$	$\rightarrow$	$n_0 - 1$ int. left $N = 1$	$\rightarrow \cdots \rightarrow$	1 int. left $N = n_0 - 1$	$\rightarrow$	0 int. left $N = n_0$

Figure 1: Process of interventions

The investment behavior of the agent is modelled by a predictable portfolio strategy π denoting the proportion of wealth invested in S, i.e. π is a decision variable held by the agent. Restricting ourselves to self-financing portfolio strategies, the wealth process follows the differential equation

$$\begin{aligned} X(0) &= x_0, \\ dX(t) &= X(t-)\left(r(t)\,dt + \pi(t)\left((\alpha(t) - r(t))\,dt + \sigma(t)\,dW(t) - \beta(t-)\,dN(t)\right)\right). \end{aligned}$$

The differential equation can be considered as a controlled differential equation with a pair of controls being a pair of portfolio strategies and the interventions, i.e. (π, N). The agent is allowed to choose $\pi \in \mathcal{A}$ and the market is allowed to choose an intervention $N \in \mathcal{B}$ such that the pair of controls (π, N) leads to a well-posed optimization problem below. Even more, we consider $\mathcal{A}$ to be the set of all predictable processes (with respect to the $\sigma-$algebra generated by the stock price process) such that we have

$$\begin{aligned} E\int_0^T |\pi(t)|^m\,ds &< \infty \quad for \quad m = 1, 2, ..., \\ \pi(t)\,\beta(t) &< 1 \quad for\ all\ t \in [0, T]. \end{aligned}$$

These requirements in particular ensure that the wealth process stays non-negative and has finite moments of all order.

Then, given a pair of controls (π, N), the controlled differential equation describing the wealth is given by

$$\begin{aligned} X^{\pi N}(0) &= x_0, \qquad (1) \\ dX^{\pi N}(t) &= X^{\pi N}(t-)\left(r(t)\,dt + \pi(t)\left((\alpha(t) - r(t))\,dt + \sigma(t)\,dW(t) - \beta(t-)\,dN(t)\right)\right). \end{aligned}$$

For a fixed time s and given $X^{\pi,N}(s) = x_s$, if τ is the first intervention time after s, we can obviously write

$$\begin{aligned} X^{\pi N}(s) &= x_s, \\ dX^{\pi N}(t) &= X^{\pi N}(t)(r + \pi(t)(\alpha - r))\,dt + \pi(t)\,\sigma dW(t), \ \ s < t < \tau, \\ X^{\pi N}(\tau) &= X^{\pi N}(\tau-)\left(1 - X^{\pi N}(\tau-)\,\pi(\tau)\,\beta\right). \end{aligned}$$

If we only need to consider X until the first intervention time, we can just as well denote the argument N by τ and we do so below.

We assume that the investor chooses a portfolio process to maximize worst case expected utility of terminal wealth in the sense of the following optimization problem:

$$\sup_{\pi \in \mathcal{A}} \inf_{N \in \mathcal{B}} E\left[U\left(X^{\pi N}(T)\right)\right].$$

For each function $v \in C^{1,2}$ we define the differential operator $\mathcal{L}^{\pi} v$ by

$$\mathcal{L}^{\pi} v(t,x) = v_t(t,x) + v_x(t,x)(r + \pi(\alpha - r))\,x + \frac{1}{2} v_{xx}(t,x)\,\pi^2 \sigma^2 x^2.$$

3 The Bellman system and the Verification Theorem

In this section we present and prove the Bellman system connected with the control problem described in the previous section.

We define the value function $\mathcal{J}^n(t,x,\pi)$ by

$$\mathcal{J}^n(t,x,\pi) = E_{t,x,n}\left[U\left(X^{\pi}(T)\right)\right],$$

where $E_{t,x,n}$ denotes conditional expectation given that $X(t) = x$ and that there are at most n possible jumps left. We define the optimal value function $V^n(t,x)$ by

$$V^n(t,x) = \sup_{\pi \in \mathcal{A}} \inf_{N \in \mathcal{B}} \mathcal{J}^n(t,x,\pi,\tau).$$

We can now present a Bellman system in a verification theorem.

Theorem 1 (Verification Theorem) *0. Assume that $v^0(t,x)$ is a classical solution of*

$$\begin{aligned} 0 &= \sup_{\pi \in \mathcal{A}} \left[\mathcal{L}^{\pi} v^0(t,x)\right], \\ v^0(T,x) &= U(x), \end{aligned}$$

which is polynomially bounded and that

$$p(t) = \arg\sup_{\pi \in \mathcal{A}} \left[\mathcal{L}^{\pi} v^0(t,x)\right]$$

is an admissible control function.

Then we have

$$V^0(t,x) = v^0(t,x),$$

and the optimal control function exists and is given by

$$\pi^{0*}(t) = p(t).$$

1. For $n \in \mathbf{N}$ and every function $v^n \in C^{1,2}$, define the sets $\mathcal{A}'_n(t)$ and $\mathcal{A}''_n(t)$ by

$$\begin{aligned} \mathcal{A}'_n(t) &= \{\pi : \pi \in \mathcal{A},\, 0 \le \mathcal{L}v^n(t,x)\}, \\ \mathcal{A}''_n(t) &= \left\{\pi : \pi \in \mathcal{A},\, 0 \le v^{n-1}(t, x(1-\beta\pi)) - v^n(t,x)\right\}. \end{aligned}$$

Assume that there exists a polynomially bounded $C^{1,2}-$solution of

$$\begin{aligned} 0 &\le \sup_{\pi \in \mathcal{A}''_n(t)} \left[\mathcal{L}^\pi v^n(t,x)\right], \\ 0 &\le \sup_{\pi \in \mathcal{A}'_n(t)} \left[v^{n-1}(t, x(1-\beta\pi)) - v^n(t,x)\right], \\ 0 &= \sup_{\pi \in \mathcal{A}''_n(t)} \left[\mathcal{L}^\pi v^n(t,x)\right] \sup_{\pi \in \mathcal{A}'_n(t)} \left[v^{n-1}(t, x(1-\beta\pi)) - v^n(t,x)\right], \\ v^n(T,x) &= U(x). \end{aligned}$$

and that

$$\begin{aligned} p(t) &= \arg \sup_{\pi \in \mathcal{A}''_n(t)} \left[\mathcal{L}^\pi v^n(t,x)\right], \\ \theta &= \inf_s \left[v^{n-1}(s, x(1-\beta\pi)) - v^n(s,x) \le 0\right], \end{aligned}$$

is a pair of admissible control functions.

Then

$$V^n(t,x) = v^n(t,x),$$

and the optimal control functions exist and are given by

$$\begin{aligned} \pi^{n*}(t) &= p(t), \\ \tau^{n*} &= \theta. \end{aligned}$$

For proving the verification theorem and for this we need the following lemma.

Lemma 2 *The value function can be represented in the following ways*

$$\begin{aligned} V^n(t,x) &= \sup_\pi \inf_N E_{t,x,n}\left[U\left(X^{\pi N}(T)\right)\right] \\ &= \inf_N \sup_\pi E_{t,x,n}\left[U\left(X^{\pi N}(T)\right)\right] \\ &= \sup_\pi \inf_\tau E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi\tau}(\tau-)\left(1-\beta\pi(\tau)\right)\right)\right] \\ &= \inf_\tau \sup_\pi E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi\tau}(\tau-)\left(1-\beta\pi(\tau)\right)\right)\right]. \end{aligned}$$

Proof of Lemma. Let $\varepsilon > 0$. Then, for a given first intervention time τ we can choose a portfolio strategy π^* which is $\varepsilon/4$-optimal until time τ and a portfolio strategy π^{**} which is arbitrary until time τ and $\varepsilon/4$-optimal after time τ in the sense that the following two inequalities hold (note that we cannot yet assume that V^n is indeed the value function)

$$\begin{aligned} & \sup_\pi E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi\tau}(\tau-)\left(1-\beta\pi(\tau)\right)\right)\right] && (2) \\ \le\; & E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi^*\tau}(\tau-)\left(1-\beta\pi^*(\tau)\right)\right)\right] + \varepsilon/4, \\ & \sup_\pi \inf_N E_{\tau,x,n}\left[U\left(X^{\pi N}(T)\right)\right] && (3) \\ \le\; & \inf_N E_{\tau, X^{\pi^{**}N}(\tau), n-1}\left[U\left(X^{\pi^{**}N}(T)\right)\right] + \varepsilon/4. \end{aligned}$$

Further, for a given portfolio strategy introduce an $\varepsilon/4$-optimal strategy for the first intervention τ^* and, given an arbitrary first intervention time τ, an $\varepsilon/4$-optimal intervention strategy N^* after time τ, again in the sense that the following two inequalities are valid:

$$\begin{aligned} & \inf_{\tau} E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi\tau}\left(\tau-\right)\left(1-\beta\pi\left(\tau\right)\right)\right)\right] && (4) \\ \geq\ & E_{t,x,n}\left[V^{n-1}\left(\tau^*\left(\pi\right), X^{\pi\tau^*}\left(\tau^*\left(\pi\right)-\right)\left(1-\beta\pi\right)\right)\right]-\varepsilon/4, \\ & \inf_{N} E_{\tau,X^{\pi}(\tau),n-1}\left[U\left(X^{\pi N}\left(T\right)\right)\right] && (5) \\ \geq\ & E_{\tau,X^{\pi N^*}(\tau),n-1}\left[U\left(X^{\pi N^*}\left(T\right)\right)\right]-\varepsilon/4. \end{aligned}$$

Then we have the following list of inequalities (explanation follows after):

$$\begin{aligned} & \sup_{\pi}\inf_{N} E_{t,x,n}\left[U\left(X^{\pi N}\left(T\right)\right)\right] && (6) \\ \geq\ & \inf_{N} E_{t,x,n}\left[E_{\tau,X^{\pi^{**}N}(\tau),n-1}\left[U\left(X^{\pi^{**}N}\left(T\right)\right)\right]\right] \\ \geq\ & \inf_{\tau} E_{t,x,n}\left[\inf_{N} E_{\tau,X^{\pi^{**}N}(\tau),n-1}\left[U\left(X^{\pi^{**}N}\left(T\right)\right)\right]\right] \\ \geq\ & \inf_{\tau} E_{t,x,n}\left[\sup_{\pi}\inf_{N} E_{\tau,X^{\pi N}(\tau),n-1}\left[U\left(X^{\pi N}\left(T\right)\right)\right]\right]-\varepsilon/4 \\ \geq\ & \inf_{\tau} E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi\tau}\left(\tau-\right)\left(1-\beta\pi\left(\tau\right)\right)\right)\right]-\varepsilon/4 \\ \geq\ & E_{t,x,n}\left[V^{n-1}\left(\tau^*, X^{\pi\tau^*}\left(\tau^*-\right)\left(1-\beta\pi\left(\tau^*\right)\right)\right)\right]-\varepsilon/2. \end{aligned}$$

The 1st inequality follows from plugging in the portfolio strategy π^{**} and the tower property. The 2nd inequality follows from interchanging the first expectation and the infimum over intervention strategies after the first intervention. The 3rd inequality follows from (3). The 4th inequality follows from the definition of V. The 5th inequality follows from (4). Taking supremum on both sides gives

$$\begin{aligned} & \sup_{\pi}\inf_{N} E_{t,x,n}\left[U\left(X^{\pi N}\left(T\right)\right)\right] && (7) \\ \geq\ & \sup_{\pi} E_{t,x,n}\left[V^{n-1}\left(\tau^*, X^{\pi\tau^*}\left(\tau^*-\right)\left(1-\beta\pi\left(\tau^*\right)\right)\right)\right]-\varepsilon/2. \end{aligned}$$

We also have the following list of inequalities (explanation follows after):

$$\begin{aligned} & \inf_{N}\sup_{\pi} E_{t,x,n}\left[U\left(X^{\pi N}\left(T\right)\right)\right] \\ \leq\ & \sup_{\pi} E_{t,x,n}\left[E_{\tau,X^{\pi N^*}(\tau),n-1}\left[U\left(X^{\pi N^*}\left(T\right)\right)\right]\right] \\ \leq\ & \sup_{\pi} E_{t,x,n}\left[\inf_{N} E_{\tau,X^{\pi N}(\tau),n-1}\left[U\left(X^{\pi N}\left(T\right)\right)\right]\right]+\varepsilon/4 \\ \leq\ & \sup_{\pi} E_{t,x,n}\left[\sup_{\pi}\inf_{N} E_{\tau,X^{\pi N}(\tau),n-1}\left[U\left(X^{\pi N}\left(T\right)\right)\right]\right]+\varepsilon/4 \\ \leq\ & \sup_{\pi} E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi N}\left(\tau-\right)\left(1-\beta\pi\left(\tau\right)\right)\right)\right]+\varepsilon/4. \end{aligned}$$

The 1st inequality follows from plugging in the intervention strategy N^* and the tower property. The 2nd inequality follows from 5). The 3rd inequality is obvious. The 4th inequality follows from the definition of V. Taking infimum on both sides results in

$$\begin{aligned} & \inf_{N}\sup_{\pi} E_{t,x,n}\left[U\left(X^{\pi N}\left(T\right)\right)\right] && (8) \\ \leq\ & \inf_{\tau}\sup_{\pi} E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi\tau}\left(\tau-\right)\left(1-\beta\pi\left(\tau\right)\right)\right)\right]+\varepsilon/4. \end{aligned}$$

Finally we can gather the inequalities (explanation follows after):

$$
\begin{aligned}
& \sup_{\pi} \inf_{N} E_{t,x,n}\left[U\left(X^{\pi N}(T)\right)\right] \\
\geq\ & \sup_{\pi} E_{t,x,n}\left[V^{n-1}\left(\tau^{*}, X^{\pi\tau^{*}}(\tau^{*}-)\left(1-\beta\pi\left(\tau^{*}\right)\right)\right)\right]-\varepsilon/2 \\
\geq\ & \inf_{\tau} \sup_{\pi} E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi\tau}(\tau-)\left(1-\beta\pi(\tau)\right)\right)\right]-\varepsilon/2 \\
\geq\ & \inf_{\tau} E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi^{*}\tau}(\tau-)\left(1-\beta\pi^{*}(\tau)\right)\right)\right]-\varepsilon/2 \\
\geq\ & \inf_{\tau} \sup_{\pi} E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi\tau}(\tau-)\left(1-\beta\pi(\tau)\right)\right)\right]-3\varepsilon/4 \\
\geq\ & \inf_{N} \sup_{\pi} E_{t,x,n}\left[U\left(X^{\pi N}(T)\right)\right]-\varepsilon \\
\geq\ & \sup_{\pi} \inf_{N} E_{t,x,n}\left[U\left(X^{\pi N}(T)\right)\right]-\varepsilon.
\end{aligned}
$$

The 1st inequality is just (7). The 2nd inequality follows from giving up the specification of the first intervention. The 3rd inequality follows from plugging in the portfolio strategy π^{*}. The 4th inequality follows from (2). The 5th inequality follows from (8). The 6th inequality is the usual $\sup\inf \leq \inf\sup -$ relation.

The reversed line of arguments (left to the reader) gives that

$$
\begin{aligned}
& \inf_{N} \sup_{\pi} E_{t,x,n}\left[U\left(X^{\pi N}(T)\right)\right] \\
\leq\ & \sup_{\pi} \inf_{\tau} E_{t,x,n}\left[V^{n-1}\left(\tau, X^{\pi N}(\tau-)(1-\beta\pi)\right)\right]+\varepsilon/2 \\
\leq\ & \sup_{\pi} \inf_{N} E_{t,x,n}\left[U\left(X^{\pi N}(T)\right)\right]+\varepsilon \\
\leq\ & \inf_{N} \sup_{\pi} E_{t,x,n}\left[U\left(X^{\pi N}(T)\right)\right]+\varepsilon.
\end{aligned}
$$

Since all inequalities above hold for any ε, they must hold as equalities, and consequently the theorem is proved. ■

Proof of Verification Theorem. Concerning the part of the verification theorem numbered by 0 (corresponding to 0 interventions left) is classical and its proof can be found in any textbook on dynamic portfolio optimization, e.g. Korn (1997).

The second part is proved by induction. Firstly, we prove the verification theorem for $n = 1$. Here the control N and the control τ are equivalent and we can denote $X^{\pi,N}$ by $X^{\pi,\tau}$.

Choose an arbitrary control (π, τ) and fix a point (t, x). Let X follow the dynamics given in (1) with the time point 0 replaced by the time point t. Inserting $X^{\pi,\tau}$ in v^1 and using Itô's formula we obtain

$$
\begin{aligned}
v^{1}\left(t, X^{\pi,\tau}(t)\right) &= v^{1}(t, x), \\
dv^{1}\left(s, X^{\pi,\tau}(s)\right) &= \mathcal{L}v^{1}\left(s, X^{\pi,\tau}(s)\right) ds + v_{x}^{1}\left(s, X^{\pi,\tau}(s)\right)\sigma X^{\pi,\tau}(s)\, dW(s), \ t<s<\tau, \\
dv^{1}\left(\tau, X^{\pi,\tau}(\tau)\right) &= v^{1}\left(\tau, X^{\pi,\tau}(\tau-)\left(1-\pi(\tau)\beta\right)\right)-v^{1}\left(\tau-, X^{\pi,\tau}(\tau-)\right).
\end{aligned}
$$

such that

$$
\begin{aligned}
v^{1}\left(\tau-, X^{\pi,\tau}(\tau-)\right)-v^{1}(t, x) &= \int_{t}^{\tau} \mathcal{L}v^{1}\left(s, X^{\pi,\tau}(s)\right) ds \\
&\quad + \int_{t}^{\tau} v_{x}^{1}\left(s, X^{\pi,\tau}(s)\right)\sigma X^{\pi,\tau}(s)\, dW(s).
\end{aligned} \tag{9}
$$

Now fix the investment strategy $\pi(s) = p(s)$, $t \leq s \leq \tau$. Then we know from the Bellman system that for $t \leq s \leq \tau$,

$$0 \leq \mathcal{L}^p v^1 \left(s, X^{p,\tau}(s)\right),$$

and since $p(s) \in \mathcal{A}''_{v^1}(s)$, also

$$0 \leq v^0 \left(s, X^{p,\tau}(s)\left(1 - \beta p(s)\right)\right) - v^1 \left(s, X^{p,\tau}(s)\right).$$

But this means that inserting p in (9) gives the inequality

$$v^1(t,x) \leq v^0 \left(\tau, X^{p,\tau}(\tau-)\left(1 - \beta p(\tau)\right)\right) - \int_t^\tau v_x^1 \left(s, X^{p,\tau}(s)\right) \sigma X^{p,\tau}(s)\, dW(s).$$

Due to our requirements on the admissible controls and on the value function, the stochastic integral vanishes when taking expectation, leaving us with the inequality

$$v^1(t,x) \leq E_{t,x}\left[v^0 \left(\tau, X^{p,\tau}(\tau-)\left(1 - \beta p(\tau)\right)\right)\right]. \tag{10}$$

Then, on the one hand, we can immediately conclude that

$$v^1(t,x) \leq \sup_\pi E_{t,x}\left[v^0 \left(\tau, X^{\pi,\tau}(\tau-)\left(1 - \beta \pi(\tau)\right)\right)\right].$$

such that taking infimum over τ on both sides gives

$$v^1(t,x) \leq \inf_\tau \sup_\pi E_{t,x}\left[v^0 \left(\tau, X^{\pi,\tau}(\tau-)\left(1 - \beta \pi(\tau)\right)\right)\right]. \tag{11}$$

On the other hand, taking infimum over τ on both sides of (10) gives

$$v^1(t,x) \leq \inf_\tau E_{t,x}\left[v^0 \left(\tau, X^{p,\tau}(\tau-)\left(1 - \beta p(\tau)\right)\right)\right],$$

and then we can conclude that also

$$v^1(t,x) \leq \sup_\pi \inf_\tau E_{t,x}\left[v^0 \left(\tau, X^{\pi,\tau}(\tau-)\left(1 - \beta \pi(\tau)\right)\right)\right]. \tag{12}$$

Consider again the equation (9). Now fix the time $\tau = \theta$. Then we know that

$$v^0 \left(s, X^{\pi,\theta}(s-)\left(1 - \beta \pi(s)\right)\right) - v^1 \left(s-, X^{\pi,\theta}(s-)\right) > 0, \; t \leq s < \theta, \tag{13}$$

$$v^0 \left(\theta, X^{\pi,\theta}(\theta-)\left(1 - \beta \pi(\theta)\right)\right) - v^1 \left(\theta, X^{\pi,\theta}(\theta-)\right) \leq 0. \tag{14}$$

Now, either $0 > \mathcal{L}^\pi v^1 \left(s, X^{\pi,\tau}(s)\right)$ or $0 \leq \mathcal{L}^\pi v^1 \left(s, X^{\pi,\tau}(s)\right)$. But if $0 \leq \mathcal{L}^\pi v^1 \left(s, X^{\pi,\tau}(s)\right)$ then $\pi \in \mathcal{A}'_{v^1}$, and then (13) give us that

$$\sup_{\pi \in \mathcal{A}'_{v^1}(s)} \left[v^0 \left(s, X^{\pi,\theta}(s-)\left(1 - \beta \pi(s)\right)\right) - v^1 \left(s-, X^{\pi,\theta}(s-)\right)\right] > 0.$$

By complementarity, we then know that

$$\sup_{\pi \in \mathcal{A}''_{v^1}(s)} \left[\mathcal{L}^\pi v^1 \left(s, X^{\pi,\theta}(s)\right)\right] = 0,$$

But since $\pi(s) \in \mathcal{A}''_{v^1}(s)$ by (13), we then know that

$$\mathcal{L}^\pi v^1 \left(s, X^{\pi,\theta}(s)\right) \leq 0.$$

So, in any case, $\mathcal{L}^{\pi} v^1 \left(s, X^{\pi,\theta}(s)\right) \leq 0$, $s < \theta$. But this means that inserting θ in (9) and using (14) gives the inequality

$$v^1(t,x) \geq -\int_t^{\theta} v_x \left(s, X^{\pi,\theta}(s)\right) \sigma X^{\pi,\theta}(s)\, dW(s) + v^0 \left(\theta, X^{\pi,\theta}(\theta-)(1-\beta\pi(\theta))\right).$$

Due to our requirements on the admissible controls and on the value function, the stochastic integral vanishes when taking expectation, leaving us with the inequality

$$v^1(t,x) \geq E_{t,x}\left[v^0 \left(\theta, X^{\pi,\theta}(\theta-)(1-\beta\pi(\theta))\right)\right]. \tag{15}$$

Then, on the one hand, we can conclude that

$$v^1(t,x) \geq \inf_{\tau} E_{t,x}\left[v^0 \left(\tau, X^{\pi,\tau}(\tau-)(1-\beta\pi(\tau))\right)\right],$$

such that taking supremum over π on both sides gives

$$v^1(t,x) \geq \sup_{\pi} \inf_{\tau} E_{t,x}\left[v^0 \left(\tau, X^{\pi,\tau}(\tau-)(1-\beta\pi(\tau))\right)\right]. \tag{16}$$

On the other hand, taking supremum over π on both sides of (15) gives

$$v^1(t,x) \geq \sup_{\pi} E_{t,x}\left[v^0 \left(\theta, X^{\pi,\theta}(\theta-)(1-\beta\pi(\theta))\right)\right],$$

such that we can conclude that

$$v^1(t,x) \geq \inf_{\tau} \sup_{\pi} E_{t,x}\left[v^0 \left(\tau, X^{\pi,\tau}(\tau-)(1-\beta\pi(\tau))\right)\right]. \tag{17}$$

From (11), (12), (16), and (17), we conclude that

$$\begin{aligned} v^1(t,x) &= \inf_{\tau} \sup_{\pi} E_{t,x}\left[v^0 \left(\tau, X^{\pi}(\tau-)(1-\beta\pi(\tau))\right)\right] \\ &= \sup_{\pi} \inf_{\tau} E_{t,x}\left[v^0 \left(\tau, X^{\pi}(\tau-)(1-\beta\pi(\tau))\right)\right], \end{aligned}$$

and it only remains to be realized that V is characterized by this equation. ∎

Remark 3 *A careful inspection of the proof shows that the expectation requirements on the admissible controls and the polynomial growth condition for the value function are indeed only needed for the expectation of the stochastic integrals to vanish just before relations (10) and (15). Of course, these requirements are only sufficient for the proof to go through. The assumption of polynomial growth of the value function is not satisfied for our examples of the logarithmic utility and the exponential utility function below. However, one can directly check that the above proof still goes through for those special choices of the utility functions as it can be verified that the expectations of the two mentioned stochastic integrals vanish.*

4 Characterization of the Solution

To apply the verification theorem we are now going to construct (in a heuristic way) general candidates for the value functions V^n and the optimal controls along the lines of the theorem. In the following chapters it is shown that for special choices of the utility function U these heuristically derived candidates indeed satisfy all the requirements of the verification theorem and are thus solutions of the control problem. We assume in this section and the following examples that for some real constant β we have that

$$\beta(t) = \beta.$$

Let us start by considering the inequality

$$0 \leq \sup_{\pi \in \mathcal{A}'_n(t)} \left[V^{n-1}\left(t, x\left(1-\beta\pi\right)\right) - V^n\left(t,x\right) \right]. \tag{18}$$

As for $\beta > 0$ and a (strictly) increasing utility function U we have that $V^{n-1}\left(t, x\left(1-\beta\pi\right)\right)$ is a decreasing function of π, the supremum in (18) is obtained for the smallest π with

$$V_t^n\left(t,x\right) \geq -V_x^n\left(t,x\right)\left(r + \pi\left(\alpha - r\right)\right)x - \frac{1}{2}V_{xx}^n\left(t,x\right)\pi^2\sigma^2 x^2. \tag{19}$$

Under the assumption of a concave V^n we have that the supremum in (18) is attained for the smallest value of π for which (19) holds as an equality. We consider the obvious choice for the separation of the $(t,x)-$space into the set $\mathcal{M}$ where the right hand side of the inequality (18) is strictly positive, and its complement. Outside $\mathcal{M}$, π and V are determined by the set of equations

$$\begin{aligned} V^n\left(t,x\right) &= V^{n-1}\left(t, x\left(1-\beta\pi\right)\right), \\ V_t^n\left(t,x\right) &= -V_x^n\left(t,x\right)\left(r+\pi\left(\alpha-r\right)\right)x - \frac{1}{2}V_{xx}^n\left(t,x\right)\pi^2\sigma^2x^2. \end{aligned}$$

Note that the first equation has to hold by the complementarity condition in the verification theorem. Inside $\mathcal{M}$ we must have $\sup_{\pi\in\mathcal{A}''_n(s)}\left[\mathcal{L}^\pi v^n\left(s, X^{\pi,\theta}\left(s\right)\right)\right] = 0$, again by complementarity. Ignoring the constraint $\pi \in \mathcal{A}''_n\left(t\right)$ we can compute the usual candidate for an optimal portfolio process by the first order conditions as:

$$\pi = -\frac{V_x^n\left(t,x\right)}{V_{xx}^n\left(t,x\right)x}\frac{\alpha - r}{\sigma^2}. \tag{20}$$

If for the strategy (20), we have that

$$V^n\left(t,x\right) \leq V^{n-1}\left(t, x\left(1-\beta\pi\right)\right),$$

then (20) indeed satisfies the constraint $\pi \in \mathcal{A}''_n\left(t\right)$ and can be considered as the candidate optimal portfolio. If however for π as given in equation (20) we have that

$$V^n\left(t,x\right) > V^{n-1}\left(t, x\left(1-\beta\pi\right)\right),$$

then again (under suitable assumptions on U and β) we know that $V^{n-1,0}\left(t, x\left(1-\beta^{n,u}\pi\right)\right)$ decreases as a function of π. We further assume that

$$V^{n-1}\left(t, x\left(1-\beta\pi\right)\right) \to \infty$$

(this always has to be checked for concrete choices of the utility function U when even more explicit computations are performed in later sections!). As

$$V_x^n\left(t,x\right)\left(r+\pi\left(\alpha-r\right)\right)x + \frac{1}{2}V_{xx}^n\left(t,x\right)\pi^2\sigma^2x^2$$

is increasing for $\pi < -\frac{V_x^n(t,x)}{V_{xx}^n(t,x)x}\frac{\alpha-r}{\sigma^2}$, then $\sup_{\pi\in\mathcal{A}_n''(s)}\left[\mathcal{L}^\pi v^1\left(s, X^{\pi,\theta}(s)\right)\right]$ is obtained for the π for which

$$V^n(t,x) = V^{n-1}(t, x(1-\beta\pi))$$

holds and consequently π and V are determined by the set of equations

$$\begin{aligned} V^n(t,x) &= V^{n-1}(t, x(1-\beta\pi)), \\ V_t^n(t,x) &= -V_x^n(t,x)(r+\pi(\alpha-r))x - \frac{1}{2}V_{xx}^n(t,x)\pi^2\sigma^2x^2. \end{aligned}$$

As this is the same case as outside $\mathcal{M}$, we realize that $\mathcal{M}$ is not the relevant set that decomposes the state space in an appropriate way. Instead, we consider now a set $\mathcal{N}$ where we have

$$\begin{aligned} \pi(t,x) &= -\frac{V_x^n(t,x)}{V_{xx}^n(t,x)x}\frac{\alpha-r}{\sigma^2}, \\ V_t^n(t,x) &= -V_x^n(t,x)(r+\pi(\alpha-r))x - \frac{1}{2}V_{xx}^n(t,x)\pi^2\sigma^2x^2, \end{aligned} \tag{21}$$

and its complement characterized by

$$\begin{aligned} V^n(t,x) &= V^{n-1}(t, x(1-\beta\pi)), \\ V_t^n(t,x) &= -V_x^n(t,x)(r+\pi(\alpha-r))x - \frac{1}{2}V_{xx}^n(t,x)\pi^2\sigma^2x^2. \end{aligned} \tag{22}$$

Note in particular that for $n = 0$ we have that $\mathcal{N}$ typically equals the whole possible $(t,x)-$space while for $n > 1$ it might be possible that $\mathcal{N}$ is empty as we show for the specific choices of the utility functions below.

5 Power utility

In this section we consider the case of power utility,

$$U(x) = \frac{1}{\gamma}x^\gamma, \quad \gamma < 1, \quad \gamma \neq 0,$$

and assume $\alpha > r$. Inspired by the solution of the usual portfolio problem we try a solution of the form

$$V^n(t,x) = \frac{1}{\gamma}f^n(t)\left(\frac{x}{f^n(t)}\right)^\gamma$$

leading to

$$\begin{aligned} V_t^n(t,x) &= \frac{1-\gamma}{\gamma}f_t^n(t)\left(\frac{x}{f^n(t)}\right)^\gamma, \\ V_x^n(t,x) &= \left(\frac{x}{f^n(t)}\right)^{\gamma-1}, \\ V_{xx}^n(t,x) &= -(1-\gamma)\frac{1}{f^n(t)}\left(\frac{x}{f^n(t)}\right)^{\gamma-2}. \end{aligned}$$

With these relations we obtain the optimal portfolio

$$\pi^{*n} = \begin{cases} \frac{1}{1-\gamma}\frac{\alpha-r}{\sigma^2}, & (t,x,n) \in \mathcal{N}, \\ \frac{1}{\beta}\left(1-\left(\frac{f^n(t)}{f^{n-1}(t)}\right)^{\frac{1-\gamma}{\gamma}}\right), & (t,x,n) \notin \mathcal{N}. \end{cases}$$

(here we have again implicitly assumed strict concavity of V^n!). The usual solution in the case of $n = 0$ is well-known and given by (see e.g. Kraft and Steffensen (200x))

$$\begin{aligned}\pi^{*0}(t,x) &= \frac{1}{1-\gamma}\frac{\alpha-r}{\sigma^2}, \\ f^0(t) &= e^{-\left(r+\frac{1}{2}\frac{\alpha-r}{\sigma^2}\frac{1}{1-\gamma}\right)\frac{(T-t)(1-\gamma)}{\gamma}}\end{aligned}$$

Under the assumption of $\beta > 0$ we now conjecture that $\mathcal{N}$ is indeed empty for $n > 0$, i.e. for $n > 0$ we assume that we are always in the situation of

$$\begin{aligned}\pi^{*n}(t,x) &= \frac{1}{\beta}\left(1-\left(\frac{f^n(t)}{f^{n-1}(t)}\right)^{\frac{1-\gamma}{\gamma}}\right), \\ \mathcal{L}^{\pi^{*n}}v^n(t,x) &= 0\end{aligned}$$

If we now plug in our guess for $\pi^{*n}(t,x)$ and for $v^n(t,x)$ into the second condition above and use the final condition of

$$v^n(T,x) = v^{n-1}(T,x) = ... = v^0(T,x) = \frac{1}{\gamma}x^\gamma$$

implying

$$\pi^{*n}(T,x) = 0,$$

we arrive at the following ordinary differential equation for f,

$$f_t^n(t) = f^n(t)\left(-\frac{\gamma}{1-\gamma}(r+(\alpha-r)\pi^{*n})+\gamma\frac{1}{2}(\pi^{*n})^2\sigma^2\right), f^n(T) = 1.$$

for $n = 1, 2, ...$ With the help of this equation and the definition of π^{*n} We can derive an ordinary differential equation for π^{*n} which holds for $(t,x,n) \notin \mathcal{N}$,

$$\begin{aligned}\pi_t^{*n}(t) &= \frac{1}{\beta}(1-\pi^{*n}\beta)\left((\alpha-r)\left(\pi^{*n}-\pi^{*n-1}\right)-\frac{1}{2}(1-\gamma)\sigma^2\left((\pi^{*n})^2-\left(\pi^{*n-1}\right)^2\right)\right), \\ \pi^{*n}(T) &= 0.\end{aligned}$$

Using the explicit form of this differential equation and its final condition, one can show via induction that its solution satisfies

$$0 \le \pi^{*n}(t) \le \pi^{*n-1}(t),$$

is unique, and for $n = 1$ it can be explicitly given as the solution of a non-linear equation (see Korn and Wilmott (2002), Korn and Menkens (2005)). A consequence of this is in particular that the solution of the differential equation for $f^n(t)$ is always positive which implies that V^n of the above form is a concave function in x, as desired. Thus, all assumptions of the verification theorem are satisfied and we have indeed computed the optimal portfolio. The form of these optimal portfolio processes are illustrated via Figure 2 below where the maximum number n of crashes that can still occur determines which of the five lines is relevant for the optimal portfolio process $\pi^{*n}(t)$.

In the case of $\beta < 0$ it can easily be verified that it is never optimal for the market to intervene if the investor uses the portfolio process $\pi^{*0} = \frac{1}{1-\gamma}\frac{\alpha-r}{\sigma^2}$. Consequently, in this setting we have

$$v^n(t,x) = v^{n-1}(t,x) = ... = v^0(t,x),$$

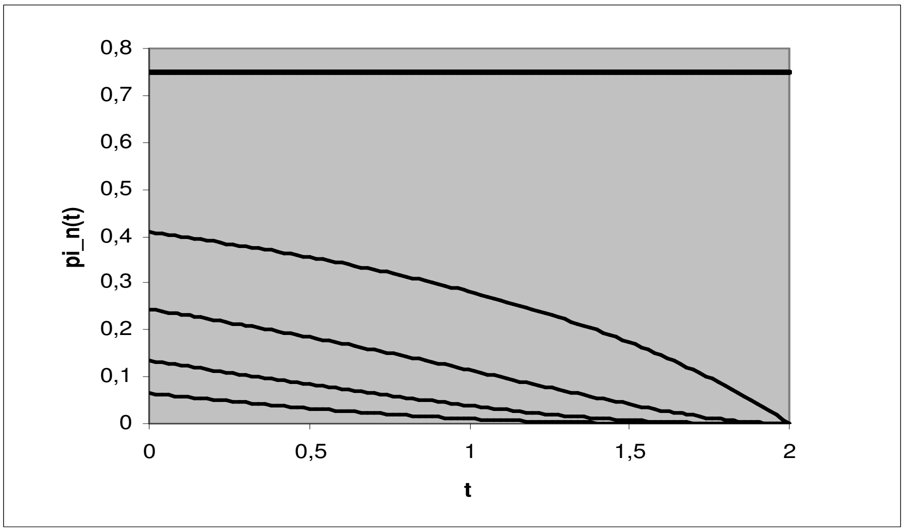

Figure 2: $\pi^{*n}(t)$ for $n = 0, 1, 2, 3, 4$ (from top to bottom), parameters: $\beta = 0.05,\ \alpha = 0.11,\ r = 0.05,\ \sigma = 0.4,\ T = 2,\ \gamma = 0.5$

and the optimality criteria of the verification theorem can only be satisfied for the strategy that consists of no intervention before time T at all. This is also intuitively clear, because this portfolio process leads to the highest expected utility in the standard market setting on one hand, and on the other hand a jump of positive size (which is the case for $\beta < 0$) would even make the situation of the investor better. Hence, the infimum over the intervention strategies is attained for the above mentioned no-jump strategy.

6 Log utility

The situation in the case of the logarithmic utility function is very similar to that of the Hara utility. In fact, it can mainly be solved by using the results of the foregoing section and setting $\gamma = 0$. We therefore shorten its presentation. Consider

$$U(x) = \log x.$$

and assume $\alpha > r$. The main difference to the Hara case is our guess of the form of the value functions (again inspired by the case $n = 0$):

$$\begin{aligned} V^n(t,x) &= \log x + f^n(t), \\ V_t^n(t,x) &= f_t^n(t), \\ V_x^n(t,x) &= \frac{1}{x}, \\ V_{xx}^n(t,x) &= -\frac{1}{x^2}. \end{aligned}$$

Inside $\mathcal{N}$ we obtain the form of π^{*n} as in the case of $n = 0$ while outside $\mathcal{N}$ the (candidate for the) optimal portfolio process is determined by the indifference requirement $V^n(t,x) = V^{n-1}(t, x(1-\beta\pi))$. This leads to

$$\pi^* = \begin{cases} \frac{\alpha - r}{\sigma^2}, & (t,x,n) \in \mathcal{N}, \\ \frac{1 - e^{f^n(t) - f^{n+1}(t)}}{\beta}, & (t,x,n) \notin \mathcal{N}. \end{cases}$$

Again, the case of $n = 0$ is well-known and given by (see e.g. Korn (1997))

$$\begin{aligned} \pi^{*0}(t,x) &= \frac{\alpha - r}{\sigma^2}, \\ f^0(t) &= \left(r + \frac{1}{2}\frac{\alpha - r}{\sigma^2}\right)(T - t). \end{aligned}$$

As above, for $\beta > 0$ we conjecture $\mathcal{N}$ to be empty for $n > 0$. As above we use form of the candidate for the optimal portfolio in this case, insert our guess into (21) and obtain a differential equation for f,

$$f_t^n(t) = -\left(r + \pi^{*n}(\alpha - r)\right) + \frac{1}{2}\left(\pi^{*n}\right)^2 \sigma^2, \quad f(T) = 0.$$

As above this leads to an ordinary differential equation for π which holds for $(t,x,n) \notin \mathcal{N}$,

$$\begin{aligned} \pi_t^{*n} &= \frac{1}{\beta}\left(1 - \pi^{*n}\beta\right)\left(f_t^{n+1}(t) - f_t^n(t)\right) \\ &= \frac{1}{\beta}\left(1 - \pi^{*n}\beta\right)\left((\alpha - r)\left(\pi^{*n} - \pi^{*n+1}\right) - \frac{1}{2}\sigma^2\left(\left(\pi^{*n}\right)^2 - \left(\pi^{*n+1}\right)^2\right)\right). \end{aligned}$$

Again, as shown in Korn and Menkens (2005) it has a unique solution which is bounded by 0 from below and by $\pi_t^{*(n-1)}$ from above for $n \geq 1$. Also for numerical examples, we refer to Korn and Menkens (2005).

Further, it is obvious that in the case of $\beta < 0$ the optimal intervention strategy consists of never doing a jump at all.

7 Exponential Utility

In this section we consider the case of exponential utility, i.e.

$$U(x) = -e^{-\theta x},$$

for some $\theta < 0$. Compared to the foregoing examples of the log-utility and the Hara case, the situation for the exponential is fundamentally different with respect to two aspects. First of all, a separation of the $t-$ and the $x-$variables in the HJB-equation is not possible, a property that is essentially due to the fact that the derivative of the exponential function is itself the exponential function. It is well-known from standard portfolio optimization (see e.g. Browne (1995)) that it is therefore more suitable to consider the amount of money invested in the risky stock at time, in our notation $\pi(t)X(t)$, as control variables as opposed to the portfolio process itself. As second difference, compared to the two utility functions considered above, note that the exponential utility function has a finite slope in $x = 0$ which results in the fact that the (unconstrained) optimal wealth process can attain negative values. Again, this is well known (compare again with Browne (1995)). To apply our main verification it would therefore be necessary to refine our definition of an admissible control. This can be done along the lines of Browne (1995), but details are left to the reader. Note that in contrast to Korn (2005) we do not have to restrict to deterministic strategies.
Keeping all these consideration in mind, we guess the following form of the value function, inspired by the case $n = 0$,

$$V^n(t,x) = -e^{-\theta f(t)x - g^n(t)},$$

leading to

$$\begin{aligned} V_t^n(t,x) &= -e^{-\theta f(t)x-g^n(t)}\left(-\theta f_t(t)\,x-g_t^n(t)\right), \\ V_x^n(t,x) &= \theta f(t)\,e^{-\theta f(t)x-g^n(t)}, \\ V_{xx}^n(t,x) &= -\theta^2 f(t)^2\,e^{-\theta f(t)x-g^n(t)}. \end{aligned}$$

Assuming strict concavity of V^n in x - which is given if f is non-vanishing - (suitable) application of the verification theorem yields the following candidate for the optimal amount of money invested in the stock,

$$\pi^* x = \begin{cases} \frac{1}{\theta f(t)}\frac{\alpha-r}{\sigma^2}, & (t,x,n)\in\mathcal{N}, \\ \frac{1}{\theta f(t)}\frac{g^{n+1}(t)-g^n(t)}{\beta}, & (t,x,n)\notin\mathcal{N}. \end{cases}.$$

In the case of $n=0$ it is well-known that we have

$$\begin{aligned} f(t) &= \exp\left(r(T-t)\right), \\ g^0(t) &= \frac{1}{2}\left(\frac{\alpha-r}{\sigma}\right)^2(T-t), \\ \pi^{*0}(t)\,x &= \frac{1}{\theta}\frac{\alpha-r}{\sigma^2}e^{-r(T-t)}. \end{aligned}$$

Once again, assuming $\mathcal{N}$ to be empty for $n>0$, we use the above derived form of our candidate optimal strategy $\pi^* x$ and insert our guesses in (21) to obtain a differential equation for g^n,

$$g_t^n(t) = -\frac{g^{n+1}(t)-g^n(t)}{\beta}(\alpha-r)+\frac{1}{2}\frac{\left(g^{n+1}(t)-g^n(t)\right)^2}{\beta^2}\sigma^2.$$

This results in an ordinary differential equation for $\pi^* x$ for $(t,x,n)\notin\mathcal{M}$,

$$\begin{aligned} \pi_t^{*n}(t)\,x &= r\pi^{*n}(t)\,x-\left(\pi^{*n+1}(t)\,x-\pi^{*n}(t)\,x\right)\frac{\alpha-r}{\beta} \\ &\quad+\left(\left(\pi^{*n+1}(t)\,x\right)^2-\left(\pi^{*n}(t)\,x\right)^2\right)\frac{\frac{1}{2}\theta f(t)\,\sigma^2}{\beta}, \\ \pi_T^{*n}(t)\,x &= 0, \end{aligned}$$

for which we can show with standard arguments that a unique bounded and non-negative solution exists. Before we illustrate the form of the optimal strategy let us remark that, in the case of $r=0$, an explicit solution for $n=1$ exists which has the form (see Korn (2005) for a different derivation),

$$\pi^{*1}(t)\,x = \frac{\alpha}{\theta\sigma^2}+\frac{2\beta}{\theta\sigma^2\left[(T-t)-\frac{2\beta}{\alpha}\right]}.$$

The form of the optimal trading strategies are illustrated in Figure 3 below. They look very similar to the optimal portfolio processes of Figure 2 and of course the comments for their use depending on the maximum number n of crashes remain valid. However, note that if we would plot the portfolio processes we would have very irregular curves as they are inversely proportional to the actual wealth process curve, and for small values of the wealth process the portfolio process can grow above all limits (but the amount of money invested in the stock stays bounded!).

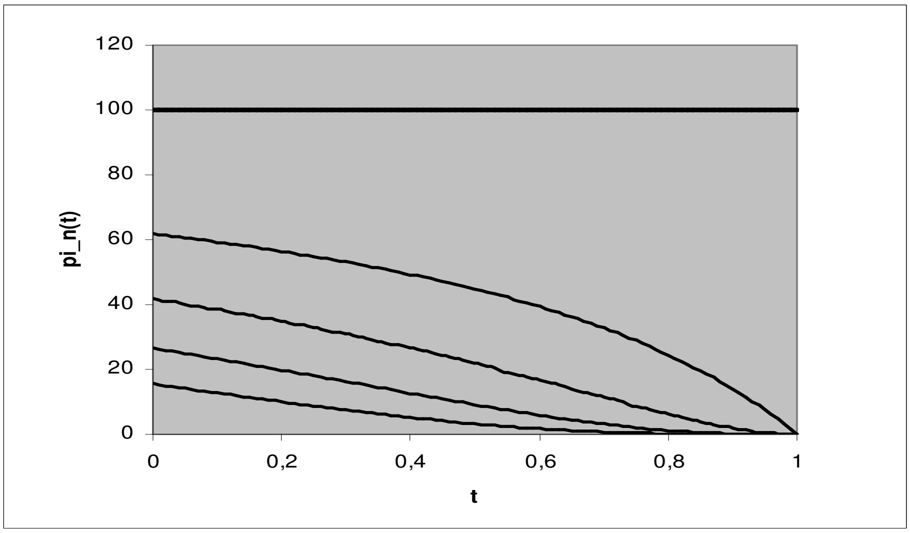

Figure 3: $\pi^{*n}(t)\,x$ for $n = 0, 1, 2, 3, 4$ (from top to bottom), parameters: $\beta = 0.05,\ \alpha = 0.16,\ r = 0,\ \sigma = 0.4,\ T = 1,\ \theta = 0.01$

8 Conclusion and Further Aspects

In this paper we have put the worst-case approach to portfolio optimization as developed by Korn and Wilmott (2002) into a generalized HJB-equation framework. This has the particular advantage that the restriction to deterministic control processes is no longer required. This framework can be used for a worst-case approach in other areas than finance. But even within the portfolio application, there remain many open problems and generalizations for future research such as

- Explicit solution of problems with many stocks (in contrast to the Korn and Wilmott (2002) approach this should be possible in a more explicit way using our new approach)
- Explicit solution of problems with non-constant β (this should again be possible in an easier way in our HJB-equation framework)
- Weakening the regularity assumptions of the verification theorem (maybe via the use of viscosity solution techniques).

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References

Aase, K. K. (1984). Optimum Portfolio Diversification in a General Continuous-Time Model. *Stochastic Processes and their Applications*, 18:81–98.

Browne, S. (1995). Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin. *Mathematics of Operations Research*, 20:937–957.

Korn, R. (1997). *Optimal Portfolios*. World Scientific.

Korn, R. (2005). Worst-Case Scenario Investment for Insurers. *Insurance: Mathematics and Economics*, 36:1–11.

Korn, R. and Menkens, O. (2005). Worst-Case Scenario Portfolio Optimization: A New Stochastic Control Approach. *Mathematical Methods of Operations Research*, 62(1):123–140.

Korn, R. and Wilmott, P. (2002). Optimal Portfolios under the Threat of a Crash. *International Journal of Theoretical and Applied Finance*, 5(2):171–187.

Kramkov, D. and Schachermayer, W. (1999). The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets. *Annals of Applied Probability*, 9(3):904–950.

Merton, R. C. (1969). Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case. *Review of Economics and Statistics*, 51:247–257.

Merton, R. C. (1971). Optimum Consumption and Portfolio Rules in a Continuous Time Model. *Journal of Economic Theory*, 3:373–413; Erratum 6 (1973); 213–214.

Talay, D. and Zheng, Z. (2002). Worst Case Model Risk Management. *Finance and Stochastics*, 6(4):517–537.