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Abstract

A new method for static hedging of barrier options under general asset dynamics is introduced. The method unifies previous approaches and nests their extensions. Using a finite set of hedge instruments the method is directly implementable and it is shown how to operationalize the hedge in a jump-diffusion model with correlated stochastic volatility. The performance of the hedge is thoroughly studied and generic sources of hedge errors are addressed.

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1 Introduction

Static hedging of barrier options has received considerable attention in a number of studies. The motivation has been the widespread use of these options, both as individual contracts and as components of more complicated structures. Work on static hedges of barrier options has followed two main approaches, either using calendar-spreads starting with Derman, Ergener & Kani (1995) or using strike-spreads as suggested in Carr & Chou (1997) and Carr, Ellis & Gupta (1998). Extensions include an accommodation to deterministic local volatility functions and the presence of jumps in the process of the underlying by Andersen, Andreasen & Eliezer (2002). An approach to handle stochastic volatility is suggested in Fink (2003). These extensions have utilized the calendar-spread approach. This paper develops a method for applied static hedging barrier options in the presence of leverage, correlated stochastic volatility and jumps in the dynamics of the underlying. The method nests applied versions of almost all extensions in both approaches as special cases, hence unify and extend previous methods. By developing the method using a finite set of hedge options the hedge directly implementable, contrary to most of the results in the literature. Extending the existing literature, it is taken into account that hedging is done under the objective measure and shown that this is important for the performance of the hedge. Furthermore, it is explicitly shown how to incorporate information obtained under the objective measure as well as that implicit in option prices to optimize the performance of the hedge. The performance of the hedge is compared to dynamic strategies.

The rest of the paper is structured as follows. The unifying static hedging method is developed under general asset dynamics in section 2. In that section it is also shown how several previous methods are special cases of our approach. In section 3 we show how to operationalize the hedge in a parsimonious jump-diffusion model with correlated stochastic volatility, investigate the performance of the hedge and compare it to that of dynamic strategies. Section 4 discusses extensions and concludes.

2 A Unifying Static Hedge

Empirical evidence suggests that the dynamics of assets exhibit both stochastic volatility and discontinuities as found e.g. by Bakshi, Cao & Chen (1997), Eraker, Johannes & Polson (2003) and Eraker (2004). In this section a method for constructing static hedges of barrier options is developed under such asset dynamics by unifying and extending previous methods suggested in the literature.

Most of the existing literature have developed theoretically perfect hedges under various assumptions on the dynamics of the underlying. In many cases, the resulting hedges require positions in a continuum or even a double continuum of plain vanilla options. Although theoretically satisfying, this is clearly inappropriate for practical applications. To apply the results, the hedges have to be approximated by finite hedge portfolios. Discretization of the theoretical results has not been the focus of the previous literature and hence the practical applicability of the results is not known. To avoid this, we work directly in an applied setting and construct an approximate hedge using a finite set of hedge instruments. This means that the hedge is not theoretically perfect,[1] but the hedge is directly implementable and the performance results are interpretable.

In this section, we first derive the hedge under general asset dynamics. We then show how to obtain a number of previously proposed methods as special cases.

2.1 Static Hedging under General Dynamics

The hedging method is described under the following general asset dynamics (under a real-world probability measure $\mathbb{P}$)

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma(S(t),t)\sqrt{V(t)}dW_S(t) + J(t)dN(t), \tag{1}$$

[1] Indeed, it is shown later that such a perfect hedge cannot be constructed.

where $V(t)$ is an Ito process, which may be correlated with $dW_S(t)$ and where $dN(t)$ is an independent compensated Poisson process. Depending on the choices of $\sigma(S(t),t)$, the process for instantaneous variance $V(t)$ and the distribution of the jump size $J(t)$, this model specification encompasses most of the popular models of asset dynamics.

Being able to price plain vanilla options is necessary for any static hedge method to work. For many special cases of the general model, such prices are available in semi-closed form, and in the following we let

$$C(s,t,v;K,T)$$

denote the call-price function, i.e. $C(S(t),t,V(t);K,T)$ is the time-t price of a strike-K, expiry-T call. Note that, as the model in equation (1) is generically incomplete, the call-price function depends non-trivially on risk-neutral ("$\mathbb{Q}$") parameters.

The method is developed using a zero-rebate up-and-out call as an example to ease the exposition, but can easily be adapted for all other single barrier options.

The problem of hedging a barrier option consists in matching the payoffs along the barrier and at expiry. Matching the payoff at expiry can be done in a model-independent way by a position in the underlying option. The problem is thus reduced to matching the payoff along the barrier. In the generic model the state variable is two-dimensional. Furthermore, the jump component means that the barrier may be passed discontinuously, resulting in an overshoot beyond the barrier. Thus, to construct a perfect hedge, in general payoffs for a double continuum of state variable values at each point in time until expiry need to be matched. In effect, payoffs should be matched in a cube of state variable and time combinations, as depicted in figure 1. However, even in theory, at most a double continuum of hedge instruments are available, namely vanilla options distinguished by strike and expiry. This shows that a perfect hedge cannot be constructed, there simply are not enough distinct hedge instruments available.[2] Indeed, the previous literature is

[2]This conclusion was reached by a different line of reasoning by Andersen et al. (2002).

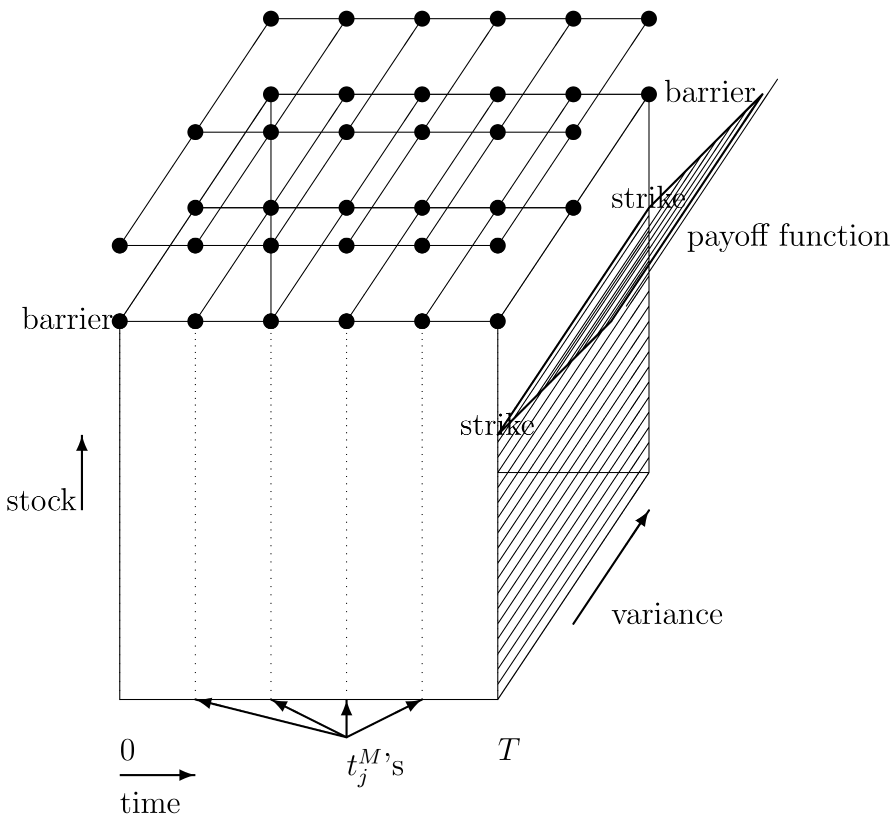

Figure 1: The shaded area is the expiry payoff function for the barrier option (an up-and-out call in this case). The aim of the static hedge construction is to design a portfolio whose value is 0 on the the grid; the bullets thus represent ((time) match points, asset (jump) hedge levels, variance hedge levels).

characterized by constraining either state variable to be deterministic at the first passage time and thus having enough hedge instruments available. This is done by Andersen et al. (2002), where jumps induce an overshoot, but the volatility is a deterministic function of time and asset level and e.g. by Fink (2003) where the volatility is stochastic, but where continuity of the process means that the first passage time will be a hitting time and hence, the value of the asset is known.

Implementation of any of the previously proposed hedge strategies involve an approximation due to discretization of the hedge. The key idea for construction of an approximate hedge of a barrier option against both stochastic volatility and overshoot is to use a discretization of the cube of time, volatility and overshoot and hedge against the most likely points of the resulting grid. One way to do this is described in the next section. For now, the finite set of hedge options is taken as given, and indexed or parameterized it as follows.

Denote the points in time where the hedge is constructed to match the barrier option payoff by $t^M_1(= t_0 + \Delta t^M_0), \dots, t^M_{N^M}(< T)$ and call these match points. Furthermore, denote the times of expiry of the hedge options by $t^E_1(= t_0 + \Delta t^E_0), \dots, t^E_{N^E}(= T)$ and call these expiry points. Taking the finite set of hedge options as given corresponds to the situation where exchange-traded options are used to set up the hedge. In the following the natural restriction that each hedge option can only be included in the hedge once is imposed. This allows identification of a hedge option by the pair $(t^H_{j,i}, K^i_j)$. Here $t^H_{j,i} \in \{t^E_{\cdot} | t^E_{\cdot} > t^M_j\}$ is the expiry point associated with the i'th hedge option used in the hedge at the match point t^M_j and K^i_j is the strike of that hedge option. At each expiry point a number of options with strikes at or beyond the barrier may be available, call these hedge options with associated hedge strikes. Denote the hedge strikes used for setting up the hedge at time t^M_j by $K^0(= K) < K^1(= B) < \{K^i_j\}^{I_j}_{i=2}$. Note, that the first two strike-levels are kept fixed as they relate to the barrier contract specification, although at any match point they may be excluded, whereas the remaining strikes, if any, are allowed to vary over expiry points as are their numbers. Furthermore, note that the remaining strikes need not be an ordered set when indexed by i.

The construction of the hedge is a two-step procedure, where the steps correspond to the strike-spread approach and the calendar-spread approach respectively.

The first step, corresponding to the strike-spread approach, constructs a hedge, denoted Λ, at expiry of the barrier option. Naturally, the underlying option is included to hedge against those paths that have not touched the barrier. The key insight of the strike-spread approach is that a portfolio of options with strikes beyond the barrier can be constructed, such that the value resembles that of the underlying along the barrier. This can then be used to set up a portfolio with the same payoff as the barrier option at expiry and with the same value as the barrier option along the barrier. Under the model by Black & Scholes (1973) and certain special cases of stochastic volatility, a theoretically perfect hedge can be constructed. This is not possible under the general model. But the idea is important and can be used in the following ways. Take as given a set of hedge strikes available for hedge options with expiry equal to the expiry of the barrier option.

Then, either take the portfolio specified by the strike-spread approach, found by solving a system of linear equations as illustrated in section 2.2. The resulting expiry hedge is model-dependent. Alternatively, use the portfolio as inspiration for a model-independent expiry hedge as illustrated in section 3. For now, assume that the expiry hedge has been found, i.e. that Λ has been specified.

The second step corresponds to the calendar-spread approach. The basic calendar-spread approach proceeds to include positions, γ_j^1 in the strike K^1 call expiring at a later time $t_{j,1}^H$, such that the value of the total hedge is equal to the rebate at time t_j^M. Here computation of option prices is necessary. This procedure is then repeated back to time t_1^M.

If stochastic volatility is included in the model then one must use some value for the state variable V_j. The main contribution in Fink (2003), which is also one of the points in the nice paper by Allen & Padovani (2002), is that several volatility levels can be hedged at time t_j^M by including more options in the hedge at each matched point. In general, the hedge levels of the instantaneous variance can be chosen as some function h of the joint distribution of the instantaneous variance and the asset level, $P(S_{t_j^M}, V_{t_j^M})$, conditional on $t_j^M = \tau_B$ being the first passage time of the asset across the barrier $\tau_B = \inf\{t > 0 | S_t = B\}$. Since hedging is done under the objective measure, the relevant distribution to use is P. The evolution of vega could also be used to specify other choices for the variance hedge levels, in particular their number and dispersion

$$V_j^i = h\left(P\left(V_{t_j^M} | S_{t_j^M} = K^1, \tau_B = t_j^M, \mathcal{F}_0\right), \frac{\partial C}{\partial V}, i\right), \quad i = 1, \ldots, N_j^V \tag{2}$$

Note, that h depends on the distribution of $V_{t_j^M}$ given that $S_{t_j^M} = K^1$. If there was no jump component in the asset dynamics, then this would be the relevant conditional distribution since the asset would have continuous paths and hence the equality would hold at the first passage time.[3]

In the following a hedge for the overshoot due to the jump component is developed. The probability of the asset jumping exactly onto the barrier is 0. Thus, in this setting equation (2) specifies the relevant variance hedge levels for those cases where the asset path is continuous at the first passage time. Hence, the h-function is still taken of the relevant conditional distribution.

For hedging against discontinuities in the asset dynamics the ideas in Andersen et al. (2002) are now utilized. Since only a finite number of options are available for inclusion in the hedge the general results derived in that paper cannot be implemented.[4] Here, an applied method for hedging against discontinuities in the asset dynamics is proposed. As for the hedging of stochastic volatility, the possible overshoot due to the jump component is hedged by using hedge options with strikes above the barrier. Again, the challenge is to choose state variable levels to hedge against. In general, the asset levels used for hedging the jump component, depend on the joint distribution of the overshoot of the asset process and the first passage time,

$$S_{t_j^M}^{J,i} = g\left(P\left(S_{t_j^M-}(1 + J_{t_j^M}) | (1 + J_{t_j^M}) > \frac{B}{S_{t_j^M-}}, \tau_B = t_j^M, \mathcal{F}_0\right), i\right), \quad i = 1, \ldots, N_j^J. \tag{3}$$

This means that the hedge can accommodate hedging the jump component differently at the different match points, as was the case for the stochastic volatility component. To each jump

[3] Also note, that since $\sigma(S_t, t)$ is a deterministic function of S_t and t, the variance hedge levels will also reflect the local volatility factor of the volatility coefficient.

[4] The general results do not appear entirely practical since they depend on the Δ of a barrier option under under a jump diffusion. This is not known in (semi-) closed form. Furthermore, a double continuum of hedge options is needed.

hedge level we need to associate at least one level for the instantaneous variance. This can simply be done similarly to (2) as

$$V_j^{N_j^V+i} = h\left(P\left(V_{t_j^M} | S_{t_j^M} > K^1, \tau_B = t_j^M, \mathcal{F}_0\right), \frac{\partial C}{\partial V}, N_j^V + i\right), \quad i = 1, \dots, N_j^J. \tag{4}$$

At each match point t_j^M, $j = N^M, \dots, 1$, the corresponding portfolio weights can now be found by solving the following $(N_j^V + N_j^J) \times (N_j^V + N_j^J)$ system of linear equations

$$\begin{bmatrix} C(K^1, t_j^M, V_j^1;, K^1, t_{j,1}^H) & \dots & C(K^1, t_j^M, V_j^1; K_j^{N_j^V+N_j^J}, t^H_{j,N_j^V+N_j^J}) \\ \vdots & \ddots & \vdots \\ C(K^1, t_j^M, V_j^{N_j^V};, K^1, t_{j,1}^H) & \dots & C(K^1, t_j^M, V_j^{N_j^V}; K_j^{N_j^V+N_j^J}, t^H_{j,N_j^V+N_j^J}) \\ C(S_j^{J,1}, t_j^M, V_j^{N_j^V+1}; K^1, t_{j,1}^H) & \dots & C(S_j^{J,1}, t_j^M, V_j^{N_j^V+1}; K_j^{N_j^V+N_j^J}, t^H_{j,N_j^V+N_j^J}) \\ \vdots & \ddots & \vdots \\ C(S_j^{J,N_j^J}, t_j^M, V_j^{N_j^V+N_j^J}; K^1, t_{j,1}^H) & \dots & C(S_j^{J,N_j^J}, t_j^M, V_j^{N_j^V+N_j^J}; K_j^{N_j^V+N_j^J}, t^H_{j,N_j^V+N_j^J}) \end{bmatrix} \begin{bmatrix} \gamma_j^1 \\ \vdots \\ \gamma_j^{N_j^V} \\ \gamma_j^{N_j^V+1} \\ \vdots \\ \gamma_j^{N_j^V+N_j^J} \end{bmatrix}$$

$$= \begin{bmatrix} \Pi(K^1, V_j^1) \\ \vdots \\ \Pi(K^1, V_j^{N_j^V}) \\ \Pi(S_j^{J,1}, V_j^{N_j^V+1}) \\ \vdots \\ \Pi(S_j^{J,N_j^J}, V_j^{N_j^V+N_j^J}) \end{bmatrix} \tag{5}$$

where the elements on right hand side are the values of the options in the hedge that have already been included, i.e.

$$\Pi(x, V_j^i) = \begin{cases} -\Lambda(x, t_j^M, V_j^i; K^0, T) & j = N^M \\ -\Lambda(x, t_j^M, V_j^i; K^0, T) - \sum_{n=j+1}^{N_M} \sum_{k=1}^{I_n} \gamma_n^k C(x, t_j^M, V_j^i; K_n^k, t_{n,k}^H) & j < N^M. \end{cases} \tag{6}$$

Note, that all the portfolio weights can be found simultaneously by combining the above recursive sequence of linear equation systems into a block matrix structure.

This approach extends naturally to the case where each jump hedge level is hedged using more than one variance hedge level.[5]

If the LHS matrix in equation (5)is not square, then an approximate solution can be obtain by standard methods. Note, that even if the matrix is square an approximate solution may be appropriate in applications. This is discussed further in section 3 and a related investigation is performed in subsection 2.2.

The model-dependent specification of the expiry hedge in the first step, can use the functions $h(\cdot)$ from equations (2) and (4) and $g(\cdot)$ from equation (3), similarly to the second step. In fact, the first step can be seen as the simultaneous solution in the second step in the special case where all hedge options have expiry T.

2.2 Some Special Cases

First, assume a standard Black-Scholes model, i.e. in the notation of equation (1) $\sigma(\cdot, \cdot) \equiv 1$, $V(t) \equiv \sigma_{BS}^2$ and $J(t) \equiv 0$.

The calender-spread approach of Derman et al. (1995) uses the underlying call-option as expiry hedge, and (shorter-expiry, strike-on-barrier) calls to create a static hedge. In the notation of section 2 this amounts to using $\Lambda(S(t), t, \sigma_{BS}^2; K, T) = \mathrm{C}^{BS}(S(t), t, \sigma_{BS}^2; K, T)$ in the first step, and for the second step fixing some N^M, put $t_j^M = T \times j/N^M$, $t_j^E = t_j^M + T/N^M$, $I_j \equiv 1$, $N_j^V \equiv 1$ (and $V_j^1 \equiv \sigma_{BS}^2$) and $N_j^J \equiv 0$. Notice that in this case the LHS matrix in equation (5) is always 1×1, and the portfolio weights, γ_j's, are found by a particularly simple backward recursive algorithm.

The strike-spread hedge of Carr & Chou (1997) is achieved by a more careful/elaborate con-

[5] The extension is obvious, but to bound an already horrendous notation it is left out.

struction of the expiry-hedge portfolio Λ. Specifically, think of N^M hedge points, t_j^M's, as given, and suppose N^M expiry-T calls with strikes, K^i's, beyond the barrier are to be included in the expiry hedge. Let β_i be the number of units used of the i'th hedge instrument. By first including the underlying option, a reasonable choice of β is the solution to

$$A\beta = d$$

where $d_j = -\mathrm{C}^{BS}(B, t_j^M, \sigma_{BS}^2; K, T)$ and $A_{j,i} = \mathrm{C}^{BS}(B, t_j^M, \sigma_{BS}^2; K^i, T)$. This means that $\Lambda =$ (underlying call, β) matches the payoff of the barrier option at expiry when $S(T) \leq B$, and at the t_j^M points along the barrier. Because Λ is a simple T-claim, it is characterized by its payoff *function*. Letting N^M tend to infinity, this (purely numerically determined) function converges to the (analytically determined) *adjusted payoff function* of Carr & Chou (1997), see figure 2.

Suppose now, that a stochastic volatility extension of the Black-Scholes model is made. A natural way to augment the calender-spread approach, as suggested in Fink (2003), is to include (for each t_j^M) a number, N_j^V, of expiry t_{j+1}^M-options and then use these to create a portfolio whose value equals that of the barrier options for N_j^V levels for the local variance V. In compact notation equation (5) is then

$$A_j \gamma_j = \Pi_j$$

where A is an $N_j^V \times N_j^V$ matrix.

A problem encountered is that the A_j-matrix on the LHS of equation (2.2) quickly becomes ill-conditioned when N_j^V increases. A_j is close to, although not exactly, singular, so while a direct solution for γ_j is possible, its entries, i.e. portfolio weights, become unreasonably large for practical purposes. Without some sort of regularization the portfolio price and hedge accuracy can even diverge.

As an alternative to direct solution, singular value decomposition (SVD) is used. This method is described e.g. in Press, Teukolsky, Vetterling & Flannery (1992). The use of SVD corresponds

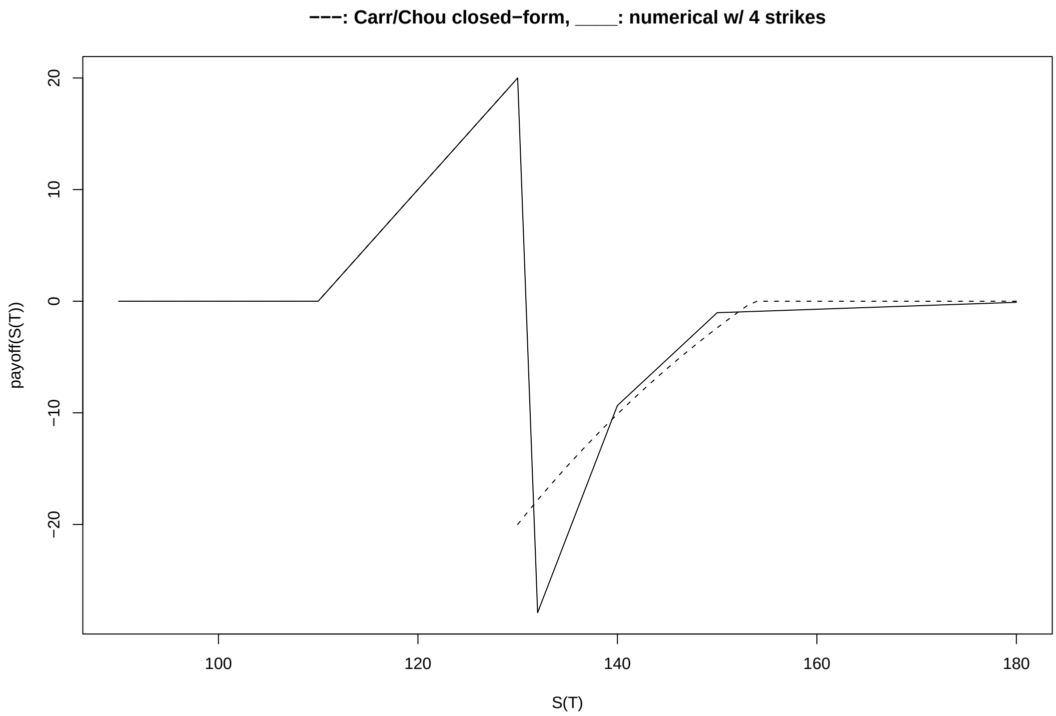

Figure 2: Payoff function of the expiry hedge portfolio, Λ, for an up-and-out call option with strike 110 and barrier 130. Λ is constructed with 4 evenly spaced match points and strikes 130, 132, 140 and 150. The dotted line is Carr/Chou's analytical adjusted payoff function; $(S(T)/B)^{(1-R/\sigma_{BS}^2)} \times (B^2/S(T) - K)^+$ above the barrier.

to minimizing the length of the solution vector at the cost of an approximate solution, i.e. an expected hedge error. Given that any hedge using a finite set of hedge options will not be perfect, an expected error is an acceptable trade-off for consistently small hedge positions. This singularity problem is also noted by Fink (2003) who suggests another regularization approach: The volatility hedge levels and the strikes of the hedge instruments are chosen to maximize the determinant of A_j. It seems more natural to let hedge levels depend on the probabilistic nature of the model, let the hedge instruments be what the market offers, and then do the adjustments on what the hedger actually chooses, namely the portfolio.

Experiments with the SVD approach have shown it to be far superior to direct solution when the number of volatility hedge levels increases, table 1 reports results. The number of match

N^V	Direct Solution max$\lvert\gamma_j^i\rvert$	price	μ	σ	Singular Value Decomposition max$\lvert\gamma_j^i\rvert$	price	μ	σ
1	0.5	101.1	-0.1	40.8	0.5	101.1	-0.1	40.8
3	6.2	100.7	3.8	37.3	3.0	99.2	2.4	37.7
5	1172.4	123.9	38.7	221.6	1.5	99.3	2.0	37.2

Table 1: Performance of static hedges constructed using direct solution or singular value decomposition. Varying numbers of volatility hedge levels. By μ and σ denote we, respectively, the (sample) mean and standard deviation of (discounted) hedge errors. All numbers except positions are in percent of the barrier option price which is 1.3150.

points is $N^M = 10$, the reflected butterfly (described in detail in the next section) is used for construction of the Λ portfolio and for the SVD solution, 0.2 is used as cut-off point for the singular values.[6] From the table it is clear that using a direct solution method is inappropriate. The size of the hedge positions explodes as more variance hedge levels are included, and prices and hedge accuracy diverge. This is because large positions mean large errors when the barrier is triggered at a point that is not on the (time points, volatility hedge level)-grid. Similar results (worse, in fact for direct solution) hold when there is a jump component in the model.

In this section an applied methodology for the construction of static hedges for barrier options on assets with general dynamics exhibiting stochastic volatility and jumps has been introduced. The methodology unifies previous methods in the literature, nests their extensions and extends the literature by handling more general dynamics and by addressing aspects relevant for applied static hedging. In the next section, the method is operationalized in a parsimonious model and the performance of the hedge is investigated.

[6] Other choices are possible and one could even optimize to satisfy a certain tolerance for the expected hedge error.

3 Performance of an Operationalized Static Hedge

This section first shows how the general formulation of the unifying static hedge from section 2 can be operationalized. This is done using a parsimonious model with correlated stochastic volatility and jumps. Then the performance of the operationalized hedge is investigated.

3.1 Operationalizing the Static Hedge in a Stochastic Volatility Jump-Diffusion Model

The formulation of the method so far has reflected the generality of the asset dynamics. In principle, construction of the hedge is as easy as computing vanilla option prices and solving a system of linear equations. The difficulty lies in the specification of variance and jump hedge levels, i.e. the choice of the functions $h(\cdot)$ and $g(\cdot)$. In the following explicit choices of these functions are made under the slightly restricted asset dynamics of the stochastic volatility jump-diffusion (SVJ) model used by Bakshi et al. (1997) among others.

The assumed model takes the following form under $\mathbb{P}$, where all parameters are constants,

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= \mu dt + \sqrt{V(t)}dW_S(t) + J(t)dN(t) \\
dV(t) &= \left(\theta_V - \kappa_V V(t)\right) dt + \sigma_V \sqrt{V(t)}dW_V(t) \\
dW_S(t)dW_V(t) &= \rho dt \\
dN_t &\sim \text{Poisson}(\lambda dt), \\
\ln\left(1 + J(t)\right) &\sim N\left(\log(1+\mu_J) - \frac{\sigma_J^2}{2}, \sigma_J^2\right).
\end{aligned}$$

The short rate, which is allowed to be stochastic in the general model of Bakshi et al. (1997), is assumed to be constant. The motivation for this, is that the importance of a stochastic interest rate is found to be minor in that paper. In the SVJ model prices of European options are known in semi-closed form as are Δs and vegas, see appendix A where default parameter values, chosen as those estimated in Eraker (2004), are also reported in table 6.

The most reasonable first jump hedge level to include is

$$S^{J,1}_{t^M_j} = E^{\mathbb{P}}_{t_0}\left[S_{t^M_j-}(1+J_{t^M_j}) | (1+J_{t^M_j}) > \frac{B}{S_{t^M_j-}}, \tau_B = t^M_j \right], \quad j = 1,\ldots,N^M. \tag{7}$$

In principle, this can be calculated from the joint distribution of S_t, V_t, J_t and τ_B, which determines the distribution of the overshoot $S_{t^M_j-}(1+J_{t^M_j}) - B > 0$. If the asset dynamics is a Levy process, then Schoutens (2003) discuss how to obtain this joint distribution in principle. Note, that in the papers by Kou & Wang (2003) and Kou & Wang (2004), a closed form solution for the joint distribution is obtained under a jump diffusion model with constant volatility, i.e. under a special case of a Levy process. A related result, allowing for regime-switching volatility, where the volatility is constant in each regime, is treated by Jacobsen (2005). Since empirical studies find that stochastic volatility is essential in describing asset dynamics, the following investigation is not to these cases. Unfortunately, the dependence introduced by the stochastic volatility means that the asset dynamics is not a Levy process and hence the results in the literature do not apply. Generalizations to jump-diffusions with stochastic volatility is a current research area.

Here, the following approach is chosen. With an assumption on the model dynamics under $\mathbb{P}$ and parameter estimates, the process can be simulated. From the simulations an estimate of the conditional expectation in (7) can be obtained. Note, that even under Levy processes, the joint distribution is obtained through its Laplace transform. Hence, in a multidimensional case similar to the one considered here, the use of the results mentioned above could involve numerical inversion of multidimensional integrals. Therefore, a simulation based approach is not necessarily less efficient than results obtained as a generalization of the literature.

Auxiliary studies show, that for reasonable parameter values $N^J_j \in \{0,1\}, \quad j = 1,\ldots,N^M$ suffices and hence specification of further jump hedge levels is not considered. A reasonable extension could use quantiles of the simulated distributions.

A similar approach is taken for the specification of the corresponding $V_j^{N_j^V+1}$

$$V_j^{N_j^V+1} = E_{t_0}^{\mathbb{P}}[V_{t_j^M}|S_{t_j^M} > K^1, \tau_B = t_j^M], \quad j = 1, \ldots, N^M.$$

Similar to the jump hedge levels, the expected instantaneous variance at the match point conditional on the asset passing the barrier for the first time is the most reasonable first variance hedge level to use.

$$V_j^1 = E_{t_0}^{\mathbb{P}}\left[V_{t_j^M}|S_{t_j^M} = K^1, \tau_B = t_j^M\right], \quad j = 1, \ldots, N^M.$$

Again, the conditional expectation can in principle be calculated from the joint distribution of S_t, V_t, J_t and τ_B, but this is not known in closed form. As in the case of the jump hedge level specifications, simulations are used to determine the variance hedge levels. However, more than one variance hedge level will be needed. Quantiles of the conditional distribution under $\mathbb{P}$ of $V_{t_j^M}$ are estimated for each match point.

It could be tempting to conjecture that the conditional expectation of the variance at a match point on the barrier is close to the conditional expectation of the variance at the match point on the barrier when the match point is the first passage time. That is

$$E_{t_0}^{\mathbb{P}}\left[V_{t_j^M}|S_{t_j^M} = K^1\right] \approx E_{t_0}^{\mathbb{P}}\left[V_{t_j^M}|S_{t_j^M} = K^1, \tau_B = t_j^M\right], \quad j = 1, \ldots, N^M.$$

This approach has the advantage that the conditional expectation can be calculated in semi-closed form. The conjectured approach extends the literature on local volatility functions and uses the methodology in a novel way by recovering conditional expectations under $\mathbb{P}$. The potential benefits warrant an investigation which is performed in appendix B. The conclusion is, however, that the conjecture does not hold and hence the approximation is not used.

Throughout, it has been emphasized that since hedging is done under the objective measure, the conditional expectations should be taken under this measure. This point is rarely addressed in the literature on static hedging of barrier options. Most of the previous approaches formulate a hedge by constructing a portfolio, which matches the barrier option value at each point on the barrier and at expiry. This means that the price of the hedge portfolio and the barrier option must be equal by no-arbitrage. Hence, these results are correctly obtained under the pricing measure. Unfortunately, these results typically involve an infinite number of hedge options. To operationalize the results, the hedges must be discretized and a finite number of options are used. This means that the hedge is no longer perfect, hence the values are no longer necessarily equal. Thus, the pricing measure is no longer the relevant measure for the conditional distributions of state variables. To illustrate the relevance of using the correct measure, the differences of the estimated conditional expectations, obtained under $\mathbb{P}$ and $\mathbb{Q}$ are shown in figure 3. From the figure it can be seen that if $\mathbb{Q}$-estimates are used, then this corresponds to hedging against an instantaneous variance which is off by up to about 4 percentage points on average in the case of the lower barrier.

From the simulation results fitting techniques are used to obtain $\hat{h}(\cdot)$ and $\hat{g}(\cdot)$. Specifically, cubic polynomials are fitted to the conditional expectations and the corresponding 5, 25, 75 and 95 percentiles.[7] From a practical point of view, the fitted functions can be refitted whenever the parameter estimates are updated by rerunning the simulations. The fitted functions allow for fast specification of static hedges between recalibration times. Note further, that only one simulation run is needed to obtain fitted functions for a variety of barrier levels by simulating all paths to expiry of the barrier option with the longest time to expiry.

3.2 Investigation of Hedging Performances

Having operationalized the static hedge its performance is now investigated. First, the impact of different expiry hedges is considered. Then, the effects from varying the number of match

[7] Depending on the results from the simulations, other techniques may be more appropriate for other parameter values.

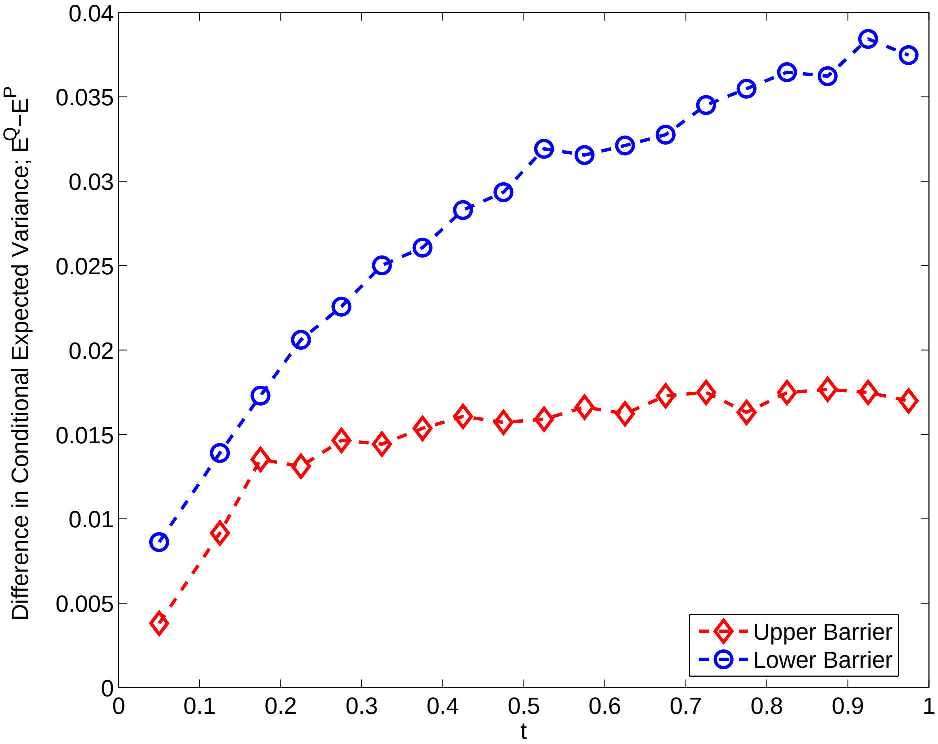

Figure 3: Difference in conditional expected instantaneous variances; $E^{\mathbb{Q}}[V(t)|S(t) = B \wedge \tau_B = t] - E^{\mathbb{P}}[V(t)|S(t) = B \wedge \tau_B = t]$.

points, variance hedge levels and jump hedge levels are investigated. Finally, the performance of the static hedge is compared to that of two dynamic hedging strategies. All investigations consider the case of a zero-rebate up-and-out call.

The investigations of the performance of the static hedge are conducted as simulation studies. They are all conducted in the following experimental design.

1. Initialize the hedge strategies.

2. Simulate the asset dynamics. If the asset price crosses the barrier, then record passage time and hedge values. If the barrier is crossed by a jump, then the post-jump asset level is used when liquidating the hedge portfolios, otherwise the barrier is used as level. If dynamic

strategies are considered, then only simulate to the next rebalancing point.

3. At the next rebalancing time, adjust the dynamic hedges.

4. Iterate until expiry and record terminal values.

The simulations are made using an Euler discretization of the asset dynamics, the details of which are given in appendix A. All simulations use 10^5 paths.

Each simulation path runs from time 0 to time τ, which is either expiry or the first time the barrier is hit, whichever comes first. At time τ all strategies are liquidated and the discounted hedge error is recorded. The discounted hedge error is

$$\epsilon_m^k = \exp(-R\tau_m)\left(\text{Value of hedge portfolio}_k(\tau_m) - \text{Value of barrier option}(\tau_m)\right),$$

for the k'th hedge strategy and the m'th stock-price path. Then the performance of the various hedges is evaluated, by the standard deviation of the discounted hedging error.[8]

Equidistant match points are used unless otherwise stated. The expiry points of the hedge options associated with a match point are that match point with Δt^M added, unless otherwise stated. An example illustrating this corresponds to monthly expiries being available. With $N^M = 10$ the match points are then $t_j^M = j/12, \quad j = 1, \ldots, 10$ and the corresponding expiry points are $t_j^E = (j+1)/12, \quad j = 1, \ldots, 10$. Finally, assume equidistant strikes for the hedge options and use the same strikes at each expiry point, unless otherwise stated.

With these standard specifications in place, the investigations are now conducted.

[8] Other measures of accuracy could be used. These include value-at-risk inspired quantiles or expected shortfall measures.

3.2.1 Hedge Performance

Applied hedging of options with discontinuous payoffs is hard and thus interesting. Therefore, a barrier option with exactly this feature is considered. A natural starting point is to investigate the effect of different hedges of the barrier option payoff at expiry, i.e. different choices of Λ. The focus will be on the model-independent approach for specifying the expiry hedge in the first step. The perfect expiry payoff replicating portfolio is a long position in the underlying call, a short position of $K^1 - K^0$ binary calls and a short position a strike K^1 call. This is termed the perfect onesided hedge. One problem is that binaries are not exchange-traded. The literature on the strike-spread suggests using a negative payoff above the barrier. Exact results for the specification are available for simpler asset dynamics, but they do not extend to the present case. Therefore, two hedges with the qualitatively important aspect of the strike-spread approach incorporated are considered. Consider the portfolio consisting of the underlying call, a short position of $2(K^1 - K^0)$ in a strike K^1 binary call and a short position of one in the strike $2K^1 + K^0$ call. This is termed the perfect twosided hedge. If binaries are not available, then one way to approximate this is to construct a reflected butterfly position. That is, the underlying call, $\alpha = (2K^0 - K^1 - K^2)/(K^2 - K^1)$ strike K^1 calls, $\beta = (-(K^3 - K^0) - \alpha(K^3 - K^1))/(K^3 - K^2)$ strike K^2 calls and $-1 - \alpha - \beta$ strike K^3 calls, where $K^2 = K^1 + \Delta K$ and $K^3 = K^2 + (K^1 - K^0)$.

Since the impact of different specifications has not yet been considered, a simple static hedge is chosen. At this point, the interest is in getting an indication of the relative performance of the three choices of expiry hedging. The static hedge uses $N^M = 10$ and $N^J = 0$ and the same random number sequences are used for all runs. The results are reported in table 2.

From the table, at may be concluded that, indeed, the choice of Λ is important and that a negative payoff beyond the barrier leads to better performance.

Analysis of the results shows that the main contribution to the hedge errors and their variation

Hedge	Mean error	Accuracy
$N^V = 0$		
Perfect Onesided	80.0	159.9
Perfect Twosided	-21.2	51.3
Reflected butterfly	-13.6	54.9
$N^V = 1$		
Perfect Onesided	24.7	114.0
Perfect Twosided	-8.0	39.9
Reflected butterfly	-1.6	45.4

Table 2: Hedge performance for varying expiry hedges. $N^M = 10$ and $N^J = 0$. All numbers are in percent of the barrier option price which is 1.2547.

arises from barrier passages in the interval from the last expiry point to expiry, i.e. $\tau_B \in]t^H_{N^M,\cdot}, T[$. The reason is that only the expiry hedge is in effect in this interval.

What happens is the following. Due to negative correlation[9] a barrier crossing will on average correspond to a low instantaneous variance. Furthermore, in the time interval considered, the time to expiry is short. Taken together, this means that the value of an option position will be close to its intrinsic value. The discontinuous payoff of the barrier option means that at the barrier the intrinsic value is at its maximum. The onesided perfect expiry hedge also attains its maximum intrinsic value at the barrier and since it has a strictly nonnegative payoff its value will be very high. This leads to large positive hedge errors. The perfect twosided hedge on the other hand has its maximum positive payoff at $K^1 - \epsilon$ and its maximum negative payoff at $K^1 + \epsilon$ with the absolute values of the two extreme payoffs being equal. Hence, at short times to expiry and low volatilities, so that the distribution is almost symmetric about its current level, the value

[9]Which is one of the few consistent findings of empirical estimation of parameters in option pricing models with stochastic volatility.

of the position will be close to zero, which is what is needed. The systematic positive errors are removed. It corresponds closely to the idea in the strike-spread literature. Approximating the perfect twosided hedge with a reflected butterfly is better than strictly nonnegative expiry hedges. The effectiveness of this type of expiry hedge depends critically on the width of the embedded bear-spread, $K^2 - K^1$. The explanation corresponds exactly to that noted for the perfect twosided hedge. With short times to expiry and low volatilities a wider bear-spread makes the reflected butterfly more like a nonnegative payoff than a more narrow bear-spread. The distribution of the asset is very narrow, with our standard parameters, a widening from 1 to 5 makes hedge performance deteriorate considerably.

A problem with the twosided expiry hedges in practice is that the expiry hedge option with the highest strike is very deep OTM. A highly probable situation is that the highest tradable strike is lower than required by the reflected butterfly. The sensitivity to this is investigated as follows. Fix $N^M = 10$, $N^J = 0$, $\Delta K = 1$ and vary N^V and K^3 using a skewed version of the reflected butterfly expiry hedge. Apart from α and β as above, this requires a position of $\delta = -1 - \alpha - \beta$ in the strike K^3 call. The accuracy of the static hedges is shown in figure 4. Several interesting points may be seen from the graphs. First, the performance of a hedge based solely on Λ performs almost as good as one which hedges one variance level along the barrier when the expiry hedge is chosen appropriately. Thus, decomposing the performance of the entire hedge, the expiry hedge and hence the strike-spread component is of primary importance. The skewed reflected butterfly has not been constructed based on model assumptions, but resembles the adjusted payoff specified in the strike-spread approach. The investigation thus illustrates and explains the robustness of the strike-spread approach to model misspecification, see Nalholm & Poulsen (2006) for an investigation of this. Second, observe that the performance does not improve below a certain threshold for the two hedges considered here. Lowering this threshold is possible by inclusion of more hedge options.

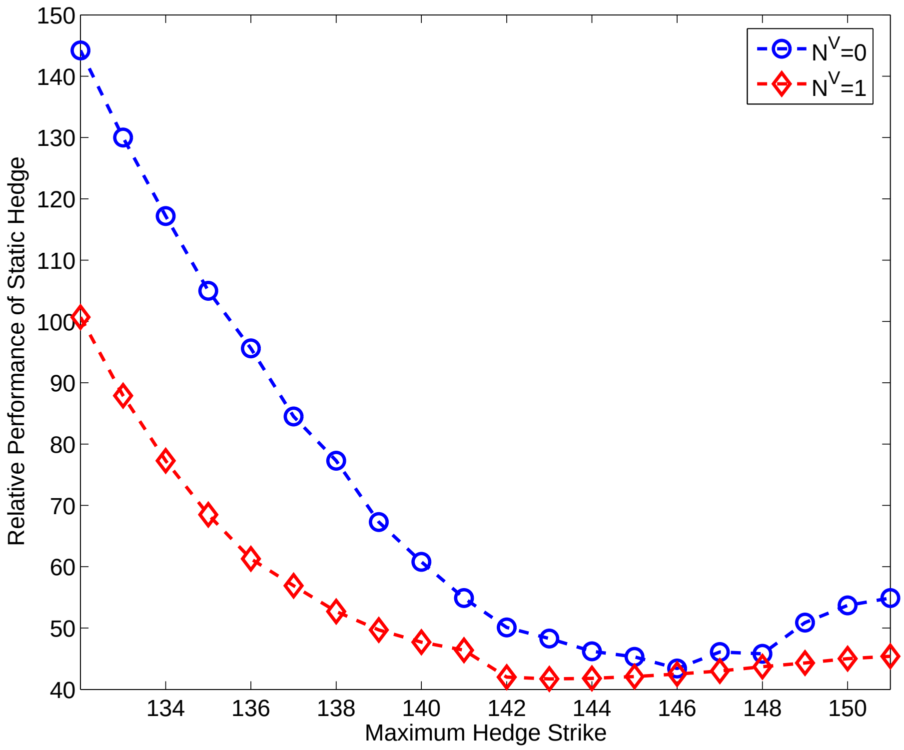

Figure 4: Accuracy of static hedges for varying maximum hedge strike when Λ is a skewed reflected butterfly.

On 6 October 2005, the maximum strike available for a call on the S&P 500 index at CBOE was 33.8% OTM.[10] In the setting of the present investigation this corresponds to a maximum hedge strike of about 134. Scaling the available strikes means that $\Delta K = 1$ is a conservative choice. In practice a barrier option will not have a barrier beyond the range of tradable strikes, since the writer typically needs to hedge at least some of the exposure.

In the interest of practical relevance of our study, the investigations are limited to using the skewed reflected butterfly with a maximum strike of $K^3 = 135$. With $\Delta K = 1$ this means that at any expiry point at most 6 hedge strikes are available, including the barrier. Thus,

[10] On this arbitrarily chosen date, the index level was 1196 while the maximum strike available was 1600. Data from www.cboe.com.

($N^V = 5, N^J = 1$) will be the maximum number of hedge levels used.

Next, the performance of the static hedge for varying specifications is investigated. Specifically, the number of match points and the number of volatility hedge levels are varied as is hedging of the jump component. The results are reported in table 3.

N^J	N^M	N^V	Mean error	Accuracy	N^J	N^M	N^V	Mean error	Accuracy
		1	30.4	86.6			1	37.5	90.0
	3	3	30.4	87.5		3	3	21.2	81.8
		5	25.3	81.8			5	25.7	81.9
		1	13.3	61.0			1	28.4	76.4
0	10	3	10.9	61.6	1	10	3	10.32	65.1
		5	11.5	59.0			5	13.6	62.5
		1	5.2	59.0			1	26.6	72.0
	24	3	4.7	52.4		24	3	6.9	67.5
		5	5.4	51.2			5	8.4	58.3

Table 3: Performance of static hedges of an up-and-out call. Varying numbers of match points and volatility hedge levels, with and without hedge of the jump component. All numbers are in percent of the barrier option price which is 1.2547.

From table 3, it may be seen that taking stochastic volatility into account when constructing static hedges does matter. This is in line with the results for dynamic hedging found by Bakshi et al. (1997). Furthermore, it may be seen that the hedge performance does not improve as more volatility levels are included and that it actually deteriorates when hedging the jump component. This is a combined effect stemming from the use of SVD and the small average overshoot when the barrier is crossed by a jump. Hedging the jump component means taking positions in more hedge options. Since the overshoot is small on average for the default parameters, the LHS matrix in equation (5) becomes closer to singular and a larger approximation is made. When other state variable values than those hedged against are realized, then a larger hedge portfolio contributes to the hedge error. Other parameter values could induce a larger overshoot making the jump hedge component necessary.

A natural way to address the concentration of large hedge errors close to expiry is to change the assumption of uniform match and expiry points. To investigate this, fix $N^V = 1$ and $N^J = 0$. The match points are placed at $t_j^M = j/12, \quad j = 1, \ldots, 11$ and the expiry points at $t_{j,1}^H = t_{j+1}^M, \quad j = 1, \ldots, 10$ and $t_{11,1}^H = T - x$. Then vary $x \in]0, 1/12[$, the performance of the resulting hedges is reported in table 4. This investigation is in the spirit of Geiss (2002), who considers non-equidistant placement of time points for improvement of convergence rates in simulations of stochastic integrals.

x	Mean error	Accuracy
1/24	5.3	57.3
1/52	4.9	51.6
1/252	4.3	49.7

Table 4: Effect of position of last expiry point. All numbers are in percent of the barrier option price which is 1.2547

From the table it can be seen that the performance is improved relative to the earlier experiments. The closer the last expiry point is to T, the better the hedge performs. In practice, however, exchange-traded options are usually available with equidistant expiries. Therefore, the use of uniform match and expiry points is continued. Note, however, that if trading in options expiring closer to expiry of the barrier option is possible, then it can improve the hedge performance significantly.

3.2.2 Comparison with Dynamic Strategies

Having investigated the performance of the static hedge under different specifications, it is now compared to the performance of two dynamic hedge strategies.

Two different dynamic strategies are considered. One is a hedge consisting of the underlying option and a Δ-hedge of the residual. This improves the traditional Δ-hedge by removing the kink in the payoff function of the underlying, a feature known to affect the performance of dynamic strategies adversely. The residual exposure is calculated based on the closed form greeks for options under the Black-Scholes model. The implied volatility of the underlying option is used. The second strategy is like the first, but uses the strike $K^{\text{vega}}(t) = S(t)\exp((R + \frac{1}{2}\sigma^2_{\text{imp}})(T-t))$ call to hedge the residual vega exposure. Since the $K^{\text{vega}}(t)$ changes with $S(t)$ and σ^2_{imp}, this means that positions are taken in a different call at each rebalancing point. The strike $K^{\text{vega}}(t)$ call is used because it has the maximal vega and hence induces the smallest hedge positions. To avoid unrealistic hedge strategies we truncate the interval of possible Δ's and vega's to $[-10, 10]$. Using bi-daily rebalancing of the dynamic strategies and comparing it with a static hedge with $N^M = 10$, $N^V = 1$ and $N^J = 0$, the results are reported in table 5.

Hedge	Mean error	Accuracy
Static hedge	13.2	67.5
Dynamic Δ-hedge	-23.8	145.9
Dynamic Δ- vega-hedge	-26.5	264.1

Table 5: Performance of static hedge and dynamic hedges. All numbers are in percent of the barrier option price which is 1.2547.

The performance of the static hedge is much better than what is found for the dynamic strategies. This result is encouraging for applications of the unifying static hedge, since the hedge used in the comparison is an almost minimal implementation of the general method. Even more so, because the dynamic strategies used are not entirely naive. A somewhat surprising finding, is that the vega-neutral dynamic strategy does not perform better than the just Δ-neutral strategy. This effect can be attributed to two sources. Firstly, just before expiry the positions in the strike $K^{\text{vega}}(t)$ call become large on some asset paths, resulting in large errors at expiry. Once again, it is the *curse of discontinuities* in the form of exploding greeks that shows up close to (B, T). Secondly, the vega used in the dynamic hedge is the Black-Scholes vega, and hence the wrong exposure is hedged. Taken together, this leads to the poor performance of the Δ- vega-hedge.

In figure 5 the performance of each hedge is decomposed, by considering the standard deviation of the hedge errors resulting from first passage times in bins centered at the markers. The

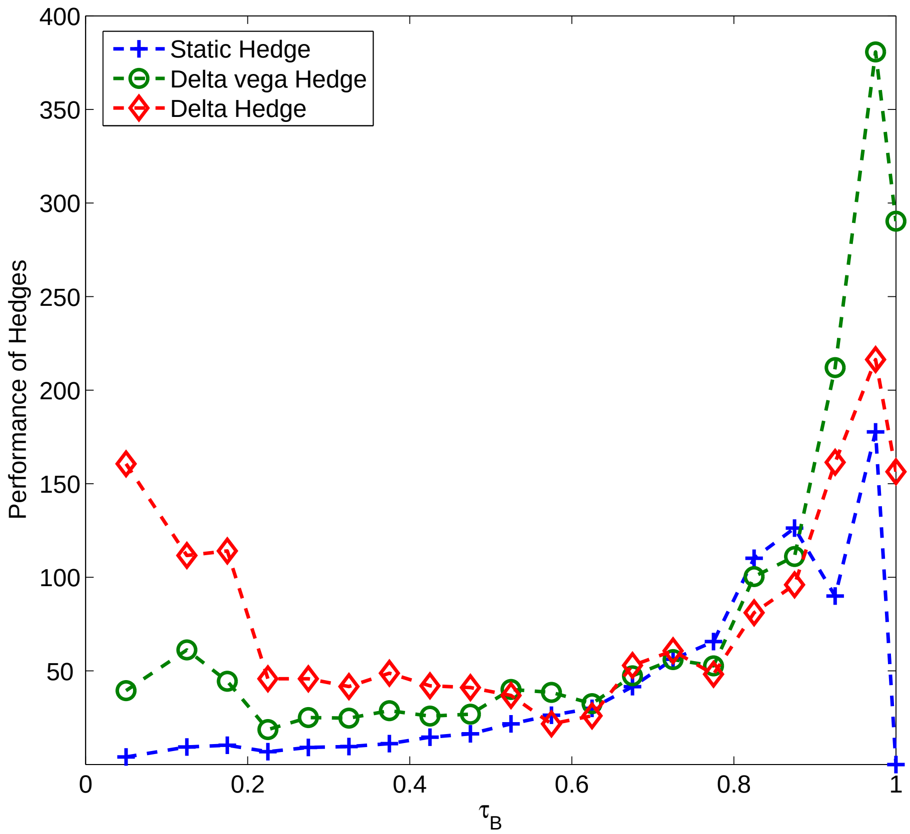

Figure 5: Performance of static and dynamic hedges along the barrier. Standard deviations are in percent of the barrier option price which is 1.2547

performance of the static hedge is better than or equal to that of the dynamic strategies along the barrier. Note that the performance of the static hedge can easily be improved by increasing N^V or N^M. Improving on the dynamic strategies is less obvious. What really makes the difference in performance is the perfect hedge at expiry in the static hedge. Most paths stay below the barrier, so a perfect expiry hedge is crucial. Contrary to the dynamic strategies, the static hedge does not have a residual hedge portfolio to unwind at expiry. This is one of the real benefits of static hedges compared to dynamic strategies when applied in a real world situation with an incomplete market and discrete hedging.

In this section it has been shown in detail how the unifying static hedge method can be operationalized in a model with correlated stochastic volatility and jumps. Investigation of the performance of the operationalized hedge has illustrated generic sources of hedge errors for static hedges of barrier options and shown the performance of the unifying method to be superior to that of dynamic strategies.

4 Extensions and Conclusion

The formulation and investigations of the unifying static hedge method has made simplifying assumptions for ease of exposition. Ways to relax some of these are now mentioned. Possible extended applications and lines of future research are also noted. Finally, the main results of the paper are summarized.

The method developed in this paper can, at the cost of more notation, be extended to incorporate time-varying rebates and barriers. To incorporate time-varying rebates, $H(t)$, one just needs to match that value at each match point. That is, one needs to add $H(t_j^M)$ to the RHS of the system of linear equations, i.e. to equation (6). To incorporate a time-varying barrier, $B(t)$, continue to require the hedge strikes to be beyond the barrier at the corresponding expiry point. For example, for an up and out contract it is required that $\{K_j^i\}_{j=2}^{I^j} > B(t_{j,i}^H)$ with $K_j^1 = B(t_{j,1}^H)$. The rest of the method remains unchanged. Naturally, a time-varying barrier affects the conditional densities of the instantaneous variance and the overshoot conditioned on the first passage time. This makes the operationalization of the hedge more cumbersome, but using simulations and fitting techniques remains feasible.

The barrier feature is present in a number of options that are more complicated than single barrier options. These include multiple barrier options and look-back options. Although, these have not elaborated upon, extending the present method to these options seem feasible. Essentially, the unifying static hedge can be applied to all the contracts dealt with in the literature on its special cases.

Finally, the ideas underlying the unifying method can be adapted to static hedging of discretely monitored barrier options. Specifically, at each monitoring date, the asset levels beyond the barrier and the variance dimension are discretized as in the present method. Now however, the trigger time need not be a first passage time. This affects the conditional distributions used for choosing the hedge levels. In particular, more asset hedge levels should be included, since not only an overshoot needs to be hedged, but also all the paths that have gone beyond the barrier level since the last monitoring time.

If there are non-negligible transaction costs associated with trading in the hedge options, then these should be reflected in the static hedge. Taking transaction costs into account is a non-trivial extension of the present method, but the principle remains unchanged. For barrier options with discontinuous payoff functions, the expiry hedge related to the strike-spread approach consists of relatively large positions. Thus, there is a trade-off between deteriorating hedge accuracy from smaller positions in the expiry hedge and smaller cost of setting up the hedge. One should then optimize this trade-off. The results are very dependent on the contract specifications and the cost structure faced by the hedger. Investigations of this type are left for future research.

In this paper, a new method for constructing static hedges of barrier option under general asset dynamics has been proposed. The method unifies almost all of the approaches previously suggested in the literature. The method extends the literature by considering general asset dynamics and by explicitly addressing the need for simultaneous use of both the real-world and pricing measures when applying static hedging approaches in general. Thorough investigations of the hedge has identified sources of hedge errors common to static hedges of barrier options, decomposed the performance to find the strike-spread component being of primary importance and shown that the hedge performs much better than traditional hedge strategies.

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A Option Pricing Formulas

In the SVJ model European call prices are known in semi-closed form. The expressions, derived in Bakshi et al. (1997), are repeated here for completeness

$$
\begin{aligned}
C(t,\tau,S(t),R,V(t),K) &= S(t)\Pi_1(t,\tau,S,R,V) - Ke^{-R\tau}\Pi_2(t,\tau,S,R,V) \\
\Delta(t,\tau,S(t),R,V(t),K) &= \Pi_1(t,\tau,S,R,V) \\
\frac{\partial C}{\partial V}(t,\tau,S(t),R,V(t),K) &= S(t)\frac{\partial \Pi_1(t,\tau,S,R,V)}{\partial V} - Ke^{-R\tau}\frac{\partial \Pi_2(t,\tau,S,R,V)}{\partial_V} \\
\Pi_j(t,\tau,S(t),R,V(t)) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left(\frac{e^{-i\phi\ln(K)}\hat{f}_j(t,\tau,S(t),R,V(t),\phi)}{i\phi}\right)d\phi, \quad j=1,2 \\
\frac{\partial \Pi_j}{\partial V} &= \frac{1}{\pi}\int_0^\infty \Re\left(\frac{e^{-i\phi\ln(K)}}{i\phi}\frac{\partial \hat{f}_j}{\partial V}\right)d\phi, \quad j=1,2
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{f}_1 &= \exp\left(i\phi(\ln(S(t)) - \ln(B(t,\tau)))\right. \\
&\quad -\frac{\theta_V}{\sigma_V^2}\left[2\ln\left(1 - \frac{(\xi_V - \kappa_V^{\mathbb{Q}} + (1+i\phi)\rho\sigma_V)(1-e^{-\xi_V\tau})}{2\xi_V}\right) + (\xi_V - \kappa_V^{\mathbb{Q}} + (1+i\phi)\rho\sigma_V)\tau\right] \\
&\quad +\lambda\tau\left((1+\mu_J^{\mathbb{Q}})((1+\mu_J^{\mathbb{Q}})^{i\phi}e^{i\phi(1+i\phi)\sigma_J^2/2} - 1) - i\phi\mu_J^{\mathbb{Q}}\right) \\
&\quad \left.+\frac{i\phi(i\phi+1)(1-e^{-\xi_V\tau})}{2\xi_V - (\xi_V - \kappa_V + (1+i\phi)\rho\sigma_V)(1-e^{-\xi_V\tau})}V(t)\right) \\
\hat{f}_2 &= \exp\left(i\phi(\ln(S(t)) - \ln(B(t,\tau)))\right. \\
&\quad -\frac{\theta_V}{\sigma_V^2}\left[2\ln\left(1 - \frac{(\xi_V^* - \kappa_V^{\mathbb{Q}} + (1+i\phi)\rho\sigma_V)(1-e^{-\xi_V^*\tau})}{2\xi_V^*}\right) + (\xi_V^* - \kappa_V^{\mathbb{Q}} + (1+i\phi)\rho\sigma_V)\tau\right] \\
&\quad +\lambda\tau\left((1+\mu_J^{\mathbb{Q}})((1+\mu_J^{\mathbb{Q}})^{i\phi}e^{i\phi(1+i\phi)\sigma_J^2/2} - 1) - i\phi\mu_J^{\mathbb{Q}}\right) \\
&\quad \left.+\frac{i\phi(i\phi+1)(1-e^{-\xi_V^*\tau})}{2\xi_V^* - (\xi_V^* - \kappa_V^{\mathbb{Q}} + (1+i\phi)\rho\sigma_V)(1-e^{-\xi_V^*\tau})}V(t)\right) \\
\frac{\partial \hat{f}_j}{\partial V} &= g\hat{f}_j, \quad j=1,2 \\
\xi_V &= \sqrt{(\kappa_V^{\mathbb{Q}} - (1+i\phi)\rho\sigma_V)^2 - i\phi(i\phi+1)\sigma_V^2} \\
\xi_V^* &= \sqrt{(\kappa_V^{\mathbb{Q}} - (1+i\phi)\rho\sigma_V)^2 - i\phi(i\phi-1)\sigma_V^2}.
\end{aligned}
$$

In this paper, standard quadrature methods are used to compute the integrals. Note, that the

FFT method (see Carr & Madan (1999)) or even the FRFT method (see Chourdakis (2004)) could have been used. Since only pricing of few options only differing by strike at a time is required, the choice of quadrature methods was made.

For use in the Monte Carlo studies, the model has been discretized using the following Euler scheme. For pricing of the barrier option, simulations are made using $\mathbb{Q}$-parameters, while for the hedging studies, the $\mathbb{P}$-parameters are used, where these differ.

$$\begin{aligned}
S(t+\Delta t^{\text{sim}}) &= S(t)\left(1+\mu^{\mathbb{M}}\Delta t^{\text{sim}}+\sqrt{V(t)}\Delta W_S(t)+(\exp(M(t))-1)\right), \quad \mathbb{M}=\mathbb{P}\vee\mathbb{Q} \\
M(t) &= \begin{cases} 0, & \Delta N(t)=0 \\ (\ln(1+\mu_J^{\mathbb{M}})-\sigma_V^2/2)\Delta N(t)+\sigma_J\Delta W_J(t), & \Delta N(t)>0 \end{cases}, \quad \mathbb{M}=\mathbb{P}\vee\mathbb{Q} \\
V(t+\Delta t^{\text{sim}}) &= \max\left(0, V(t)(1+(\theta_V-\kappa_V^{\mathbb{M}}V(t))\Delta t^{\text{sim}})+\sigma_V\sqrt{V(t)}\Delta W_V(t)\right), \quad \mathbb{M}=\mathbb{P}\vee\mathbb{Q} \\
& \Delta W_S(t)\sim N(0,\Delta t^{\text{sim}}), \quad \Delta W_V(t)\sim N(0,\Delta t^{\text{sim}}), \quad \Delta W_S(t)\Delta W_V(t)=\rho\Delta t^{\text{sim}} \\
& \Delta N(t)\sim \text{Poisson}(\lambda\Delta t^{\text{sim}}), \quad \Delta W_J(t)\sim N(0,\Delta N(t))
\end{aligned}$$

.

Note, that $V(t+\Delta t)=\max(0,V(t+\Delta t))$ is used at the boundary at 0 for V. Reasons for not using a reflecting boundary are investigated in Asmussen, Glynn & Pitman (1995). We are aware, that using a discretized process introduces both a bias in the paths and in the frequency of barrier crossings, however by using a small time step this should be negligible.

The default parameter values used in the investigations are reported in table 6 along with contract specifications and implementation choices. The parameter estimates are taken from tables III and IV in Eraker (2004) and reproduced here for convenience.

Parameter	Value	Parameter	Value	99% C.I.
$S(0)$	100	$\kappa_V^{\mathbb{Q}}$	2.772	$(2.5200, 3.0240)$
K	110	$\kappa_V^{\mathbb{P}}$	4.788	$(1.2600, 8.5680)$
R	0.02	$\theta_V/\kappa_V^{\mathbb{P}}$	0.042	$(0.0384, 0.0448)$
$V(0)$	0.042	σ_V	0.512	$(0.4712, 0.5494)$
T	1	ρ	-0.586	$(-0.652, -0.526)$
ΔK	1	$\mu_J^{\mathbb{Q}}$	-0.020	$(-0.061, 0.020)$
$\mathrm{B}^{\mathrm{UOC}}$	130	$\mu_J^{\mathbb{P}}$	-0.004	$(-0.086, 0.076)$
$\mathrm{B}^{\mathrm{DOP}}$	80	σ_J	0.066	$(0.050, 0.095)$
$\mu^{\mathbb{Q}}$	$R - \lambda\mu_J^{\mathbb{Q}}$	λ	0.504	$(0.2520, 0.7560)$
Δt^{sim}	1/3150	$\mu^{\mathbb{P}}$	0.066	$(0, 000, 0.1260)$

Table 6: Default parameter values, implementation choices and contract specifications.

B Conditional Expected Instantaneous Variance in Jump-Diffusion with General Stochastic Volatility

This appendix shows how the expected instantaneous variance at a given asset level and time under both the objective and the pricing measure can be found from the implied local volatility surface and parametric assumptions on the jump process in a jump-diffusion with general stochastic volatility.

The result is derived under both the objective measure, $\mathbb{P}$ and the market determined pricing measure, $\mathbb{Q}$. In both cases the parametric assumptions on the model are used explicitly. When the expectation is taken under $\mathbb{Q}$ the information in the observed volatility surface is utilized directly. This is not the case when taking the expectation under the objective measure. However, this is not as problematic as it may appear, since parameters that are invariant under a change of measure could in practice be calibrated to the observed volatility surface, with the remaining parameters being obtained by other means. See Eraker (2004) for a recent example.

The derivation is inspired by the papers by Dupire (1994) and Derman & Kani (1998) and extensions such as that noted by Andersen et al. (2002). The present result extends those results by considering more general dynamics and deriving the result under both the objective and the pricing measures.

Denote the measure under which the expectation is taken by $\mathbb{M} \in \{\mathbb{P}, \mathbb{Q}\}$. The assumed $\mathbb{M}$-dynamics take the following form

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= \left(\tilde{R}^{\mathbb{M}}(t) - \lambda(t)\mu_J^{\mathbb{M}}(t)\right) dt + \sigma_S(t, S(t))\sqrt{V^{\mathbb{M}}(t)}dW_S^{\mathbb{M}}(t) + J^{\mathbb{M}}(t)dN(t) \\
V^{\mathbb{M}}(t) &= \alpha(t, V^{\mathbb{M}}(t))dt + \beta(t, V^{\mathbb{M}}(t))dW_V^{\mathbb{M}}(t), \\
dW_S^{\mathbb{M}}(t)dW_V^{\mathbb{M}}(t) &= \rho dt, \\
\mathcal{L}^2 &\ni \left\{\alpha(t, V(t)), \beta(t, V(t)), \sigma_S(t, S(t))\sqrt{V^{\mathbb{M}}(t)}\right\} \\
\sigma_S(t, x) &\neq 0 \quad \forall (t, x) \in [t_0, T] \times [0, \infty[, \\
dN_t &\sim \text{Poisson}(\lambda(t)dt), \\
\left(1 + J^{\mathbb{M}}(t)\right) &\sim \xi^{\mathbb{M}}(t) \in [L(t), U(t)] \subseteq [0, \infty[\\
E^{\mathbb{M}}[1 + J^{\mathbb{M}}(t)] &= \mu_J^{\mathbb{M}}(t), \\
\tilde{R}^{\mathbb{M}}(t) &= \begin{cases} R(t) & , \mathbb{M} = \mathbb{Q} \\ \mu^{\mathbb{P}}(t) + \lambda(t)\mu_J^{\mathbb{P}}(t) & , \mathbb{M} = \mathbb{P} \end{cases} .
\end{aligned}$$

That is, there is a deterministic drift component. Furthermore, there is a compensated Poisson component with deterministic intensity and stochastic jump sizes with a time-varying distribution $\xi^{\mathbb{M}}(t)$. The asset level after a jump is $S(t) = S(t-)(1 + J^{\mathbb{M}}(t))$. This means that $0 \leq L(t) \leq U(t) \leq +\infty$, which allows for fairly general jump size distributions, nesting e.g. the standard choice of a displaced log-normal distribution used by Bakshi et al. (1997). Finally, the volatility component nests leverage and local volatility approaches as well as fairly general stochastic volatility processes.

The dynamics must be such that the following relation holds

$$F^{\mathbb{M}}(t,S(t);T,K) = \exp(-\int_t^T \tilde{R}^{\mathbb{M}}(u)du)E_t^{\mathbb{M}}[S(T)-K)^+],$$

for $\mathbb{M} \in \{\mathbb{P},\mathbb{Q}\}$ and it must be possible to evaluate this expression. That is, call prices must be computable under the assumed dynamics. Note, that computing call prices corresponds to evaluating $F^{\mathbb{M}}$ given that $\mathbb{M} = \mathbb{Q}$. The mathematically equivalent scaled conditional expectation when $\mathbb{M} = \mathbb{P}$, does not have an economic interpretation,[11] but is computable, which is what allows the $\mathbb{P}$-expectation of the instantaneous variance to be computed.

Using the theory of distributions, the following two results are well known. Define $G(S,K) = (S-K)^+$, then

$$\begin{aligned}\frac{\partial}{\partial K}G(S(T),K) &= -\mathcal{H}(S(T)-K),\\ \frac{\partial^2}{\partial K^2}G(S(T),K) &= \delta(S(T)-K)\end{aligned}$$

Here $\mathcal{H}(\cdot)$ is the Heaviside function and $\delta(\cdot)$ is the Dirac delta function.

The Itô-Doeblin formula, see e.g. Theorem 11.5.1 in Shreve (2004), extends the usual Itô formula to the case where the continuous process has a pure jump component. However, the functions we take of the process here are generalized functions, thus, we need one of the Tanaka-Meyer formulas extending the result, see e.g. remark 8.4 in Cont & Tankov (2004), to derive the dynamics of G. The resulting $\mathbb{M}$-dynamics are

$$\begin{aligned}d(S(t)-K)^+ &= \mathcal{H}(S(t-)-K)(\tilde{R}^{\mathbb{M}}(t-)-\lambda(t-)\mu_J^{\mathbb{M}}(t-))S(t-)dt\\ &+\frac{1}{2}\delta(S(t-)-K)\sigma_S^2(t-,S(t-))V^{\mathbb{M}}(t-)S^2(t-)dt\\ &+\mathcal{H}(S(t-)-K)\sigma_S(t-,S(t-))\sqrt{V^{\mathbb{M}}(t-)}dW_S^{\mathbb{M}}(t)\\ &+[\big((1+J^{\mathbb{M}}(t))S(t-)-K\big)^+-(S(t-)-K)^+]dN(t).\end{aligned}$$

[11] Except for the obvious interpretation as the call price in a world where $\mathbb{P}$ is the pricing measure.

Written on integral form, this becomes

$$\begin{aligned}(S(T)-K)^+ &= (S(t)-K)^+ + \int_t^T \mathcal{H}(S(u-)-K)(\tilde{R}^{\mathbb{M}}(u-)-\lambda(u-)\mu_J^{\mathbb{M}}(u-))S(u-)du \\ &+\frac{1}{2}\int_t^T \delta(S(u-)-K)\sigma_S^2(u-,S(u-))V^{\mathbb{M}}(u-)S^2(u-)du \\ &+\int_t^T \mathcal{H}(S(u-)-K)\sigma_S(u-,S(u-))\sqrt{V^{\mathbb{M}}(u-)}dW_S^{\mathbb{M}}(u) \\ &+\int_t^T \left[((1+J^{\mathbb{M}}(u))S(u-)-K)^+-(S(u-)-K)^+\right]dN(u).\end{aligned}$$

Assume that integration and expectation can be interchanged, i.e. that Fubini's Theorem holds. Then, taking expectations on both sides of the integral form, interchanging order of integration and rewriting, yields

$$\begin{aligned}& F^{\mathbb{M}}(t,S(t);T,K)\exp(\int_t^T \tilde{R}^{\mathbb{M}}(u)du) \\ =\; & (S(t)-K)^+ \\ +\; & \int_t^T E_t^{\mathbb{M}}[\mathcal{H}(S(u-)-K)S(u-)](\tilde{R}^{\mathbb{M}}(u-)-\lambda(u-)\mu_J^{\mathbb{M}}(u-))du \\ +\; & \frac{1}{2}\int_t^T E_t^{\mathbb{M}}[\delta(S(u-)-K)\sigma_S^2(u-,S(u-))V^{\mathbb{M}}(u-)S^2(u-)]du \\ +\; & \int_t^T E_t^{\mathbb{M}}\left[((1+J^{\mathbb{M}}(u))S(u-)-K)^+-(S(u-)-K)^+\right]\lambda(u)du.\end{aligned}$$

Now, split the conditional expectation involving $V(u-)$ by considering

$$\begin{aligned}& E_t^{\mathbb{M}}[\delta(S(u-)-K)\sigma_S^2(u-,S(u-))V^{\mathbb{M}}(u-)S^2(u-)] \\ =\; & E_t^{\mathbb{M}}[\delta(S(u-)-K)E_t^{\mathbb{M}}[\sigma_S^2(u-,S(u-))V(^{\mathbb{M}}u-)S^2(u-)|S(u-)]] \\ =\; & E_t^{\mathbb{M}}[\delta(S(u-)-K)]E_t^{\mathbb{M}}[V^{\mathbb{M}}(u-)|S(u-)=K]\sigma_S^2(u-,K)K^2.\end{aligned}$$

Because, the jump process jumps at time u with probability 0, it holds that $S(u-)\sim S(u)$. Furthermore, by continuity of the process it holds that $V^{\mathbb{M}}(u-)\sim V^{\mathbb{M}}(u)$. Hence,

$$\begin{aligned}& E_t^{\mathbb{M}}[\delta(S(u-)-K)\sigma_S^2(u-,S(u-))V^{\mathbb{M}}(u-)S^2(u-)] \\ =\; & E_t^{\mathbb{M}}[\delta(S(u-)-K)]E_t^{\mathbb{M}}[V^{\mathbb{M}}(u)|S(u)=K]\sigma_S^2(u,K)K^2.\end{aligned}$$

Substitute into the expression for the $F^{\mathbb{M}}$ and differentiate wrt. T to obtain

$$\begin{aligned} & \frac{\partial}{\partial T}F^{\mathbb{M}}\exp(\int_t^T \tilde{R}^{\mathbb{M}}(u)du) + \tilde{R}^{\mathbb{M}}(T)F^{\mathbb{M}}\exp(\int_t^T \tilde{R}^{\mathbb{M}}(u)du) \\ = \; & \exp(\int_t^T \tilde{R}^{\mathbb{M}}(u)du)(F^{\mathbb{M}} - K\frac{\partial}{\partial K}F^{\mathbb{M}})(\tilde{R}^{\mathbb{M}}(T) - \lambda(T)\mu_J^{\mathbb{M}}(T)) \\ + \; & \frac{1}{2}\exp(\int_t^T \tilde{R}^{\mathbb{M}}(u)du)\sigma_S^2(T,K)K^2 E_t^{\mathbb{M}}[V^{\mathbb{M}}(T)|S(T)=K]\frac{\partial^2}{\partial K^2}F^{\mathbb{M}} \\ + \; & \lambda(T)E_t^{\mathbb{M}}\left[((1+J^{\mathbb{M}}(T))S(T)-K)^+ - (S(T)-K)^+\right]. \end{aligned}$$

Collecting terms, this reduces to

$$\begin{aligned} \frac{\partial}{\partial T}F^{\mathbb{M}} \; = \; & -K\frac{\partial}{\partial K}F^{\mathbb{M}}(\tilde{R}^{\mathbb{M}}(T) - \lambda(T)\mu_J^{\mathbb{M}}(T)) - \lambda(T)(1+\mu_J^{\mathbb{M}}(T))F^{\mathbb{M}} \\ & +\frac{1}{2}\sigma_S^2(T,K)K^2 E_t^{\mathbb{M}}[V^{\mathbb{M}}(T)|S(T)=K]\frac{\partial^2}{\partial K^2}F^{\mathbb{M}} \\ & +\lambda(T)(1+\mu_J^{\mathbb{M}}(T))\int_{L(T)}^{U(T)} F^{\mathbb{M}}(K/z)\tilde{\xi}^{\mathbb{M}}(T,z)dz, \end{aligned}$$

where $\tilde{\xi}^{\mathbb{M}}(T,z) = z\xi^{\mathbb{M}}(T,z)/(1+\mu_J^{\mathbb{M}}(T))$ and $\xi^{\mathbb{M}}(T,z)$ is the density of $1+J^{\mathbb{M}}(T)$ with support between $L(T)$ and $U(T)$.

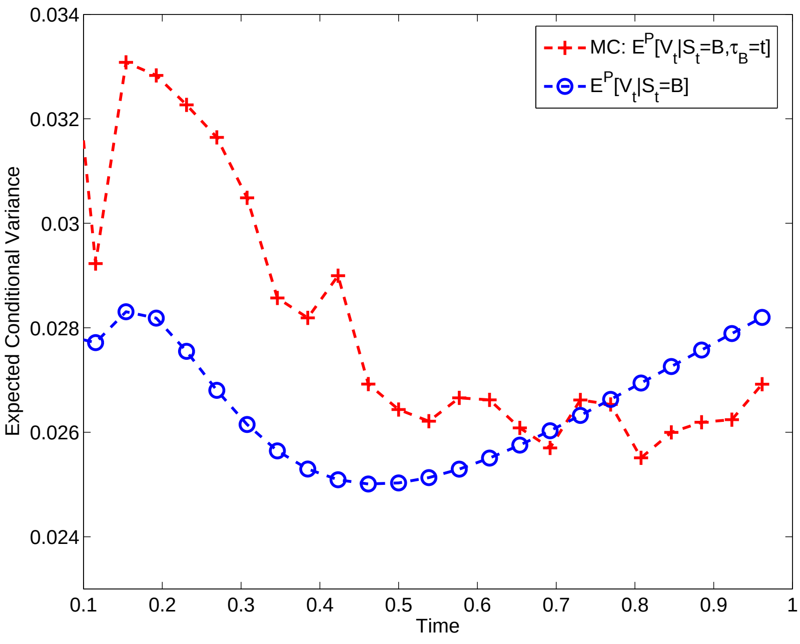
Now, one can infer the conditional expected instantaneous variance. The result is

$$\begin{aligned} E_t^{\mathbb{M}}[V^{\mathbb{M}}(T)|S(T)=K] \; = \; & \frac{2}{\sigma_S^2(T,K)K^2\frac{\partial^2}{\partial K^2}F^{\mathbb{M}}}\left[\frac{\partial}{\partial T}F^{\mathbb{M}} + K(\tilde{R}^{\mathbb{M}}(T) - \lambda(T)\mu_J^{\mathbb{M}}(T))\frac{\partial}{\partial K}F^{\mathbb{M}}\right. \\ & \left. -\lambda(T)(1+\mu_J^{\mathbb{M}}(T))\left(\int_{L(T)}^{U(T)} F^{\mathbb{M}}(K/z)\tilde{\xi}^{\mathbb{M}}(T,z)dz - F^{\mathbb{M}}\right)\right]. \end{aligned}$$

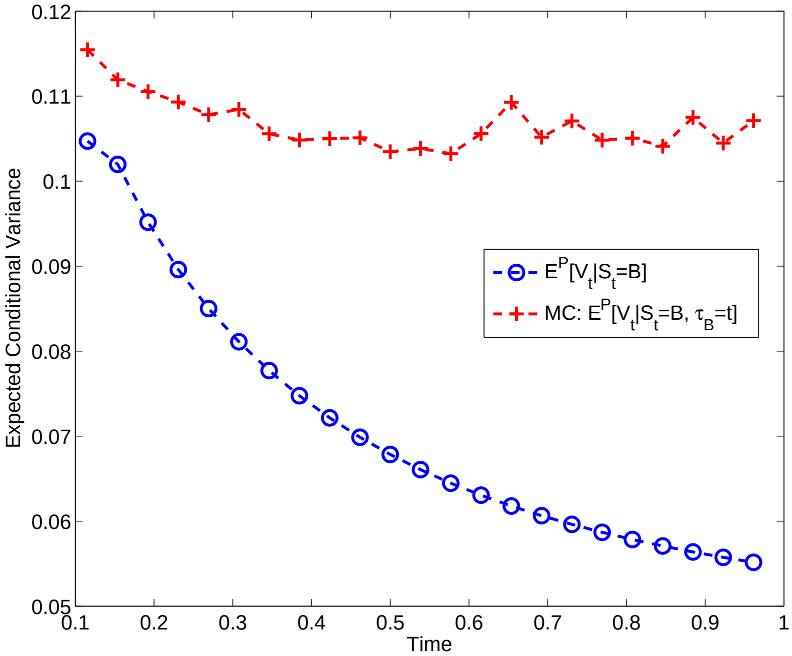
For the purpose of this paper the result has been derived for application in the conjecture

$$E_{t_0}^{\mathbb{P}}\left[V_{t_j^M}|S_{t_j^M} = K^1\right] \approx E_{t_0}^{\mathbb{P}}\left[V_{t_j^M}|S_{t_j^M} = K^1, \tau_B = t_j^M\right], \quad j = 1,\ldots,N^M.$$

The conjecture is investigated in figure 6. As is evident from the graphs, the conjecture does not hold. In particular, on the lower barrier the difference between the two conditional expectations is large. The reason for that the two conditional expectations are relatively close on the upper barrier, is the negative correlation between the Brownian motions in the asset and the variance processes. This correlation means that on average the variance will be below its long term level when the asset is at the upper barrier. Since the variance process is bounded by 0, this means that the difference between the two conditional expectations is bounded by the long term level of the variance. This is not the case on the lower barrier.



(a) Upper Barrier



(b) Lower Barrier

Figure 6: Comparison of $E^{\mathbb{P}}[V_t|S_t = B]$ and $E^{\mathbb{P}}[V_t|S_t = B, \tau_B = t]$ on upper and lower barriers.