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Holger Kraft and Mogens Steffensen

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Holger Kraft

University of Kaiserslautern, Department of Mathematics,
Mathematical Finance Group, D-67653 Kaiserslautern, Germany,
eMail: kraft@mathematik.uni-kl.de

and

Fraunhofer ITWM, Institute for Industrial Mathematics,
Department of Finance, Kaiserslautern, Germany

Mogens Steffensen

Laboratory of Actuarial Mathematics, Institute of Mathematical Sciences,
University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø,
Denmark, eMail: mogens@math.ku.dk

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ABSTRACT: In this paper, we analyze the impact of default risk on the portfolio decision of an investor wishing to invest in corporate bonds. Default risk is modeled via a reduced form approach and we allow for random recovery as well as joint default events. Depending on the structure of the model, we are able to derive almost explicit results for the optimal portfolio strategies. It is demonstrated how these strategies change if common default factors can trigger defaults of more than one bond or different recovery assumptions are imposed. In particular, we analyze the effect of beta distributed loss rates.

1 Introduction

Credit risk has been one of the hot topics in Finance over the past years. There are mainly two approaches to model credit risk. The first approach is the so-called structural approach and goes back to Black/Scholes (1973) and Merton (1974). In this approach a corporate bond is modeled as a contingent claim on the value of a firm which is assumed to be endogenously given. Starting with Black/Cox (1976), there are several extensions in which default is modeled as the first hitting time of the firm value on a predefined barrier. The advantage of these models is also their main drawback: One needs to model the whole capital structure of a firm. Besides, if the firm value is assumed to follow a diffusion process, the default time is predictable implying that the spreads of short term debt are almost zero which is in sharp contrast to empirical short term spreads. The second approach is the so-called reduced form approach and was developed in the papers by Jarrow/Turnbull (1995), Duffie/Singleton (1999a), and Lando (1998), among others. In this approach default is modeled as the first jump of a (compound) Poisson process and therefore default comes as a sudden surprise. Seen from a mathematical point of view, this means that the corresponding stopping time is not predictable. This has the nice economic implication that spreads of short term debt are greater than zero. Furthermore, the capital structure needs not to be explicitly modeled in this approach.[1]

Portfolio optimization has been a heavily researched field of Finance as well. Although Korn/Kraft (2003) and Kraft/Steffensen (2004) analyzed portfolio problems with defaultable assets in a firm value framework, to the best of our knowledge, there has not been much work on portfolio problems with default risk where default is modeled applying a reduced-form approach. A first exception is the original work by Merton (1971) who solves a portfolio problem in the special case of a reduced form model with constant interest rates and constant default intensity. Of course, it is relevant to study problems where both assumptions are relaxed. A second exception are the papers by Hou/Jin (2002), Hou (2003), and Walder (2001). These authors use a diversification argument presented in an insightful paper by Jarrow/Lando/Yu (2005) implying that event risk factors (formally counting processes) do not show up in well-diversified portfolios of corporate bonds and the impact of random recovery on portfolio management cannot be studied. Consequently, the solution of these simplified portfolio problem can be found analogously to a problem with stochastic interest rates but without default risk.

Let us briefly comment on the afore-mentioned diversification argument which holds only in the limit when the number of corporate bonds goes to infinity. In this case, it can be shown that, roughly speaking, only a finite number of corporate bonds admit non-zero market prices of risk for default-timing risk, otherwise the market offers so-called asymptotic arbitrage opportunities, a notion studied by Kabanov/Kramkov (1998). Since investors can usually invest in default-free bonds or

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[1]In an insightful paper by Duffie/Lando (2001) it is demonstrated how these approaches can be unified by assuming that the firm value is not observable all the time.

stocks as well, investment in defaultable bonds adds no risk to a portfolio beyond the individual corporate default-timing risk. Therefore, one would expect the risk adverse investor not to invest in a defaultable bond unless he receives a positive market price of default-timing risk. In the following, we shall see that this intuition holds true. But as just explained, a positive market price of default-timing risk is not at all contradicting the no asymptotic arbitrage condition, even if the number of corporates tends to infinity. For all these reasons, in this paper we consider a portfolio problem with stochastic interest rates and default risk which is modeled by a reduced-form approach. We analyze the effect of default-timing risk and the risk of random recovery on portfolio management. Furthermore, it is demonstrated how the possibility of joint defaults affects portfolio decisions. All these points were not discussed in the above-mentioned papers.

The paper is structured as follows: In Section 2, we introduce a generalized version of the multiple default framework by Schönbucher (1998) where we allow for joint default events as introduced by Duffie/Singleton (1999b). Section 3 describes a portfolio problem with corporate bonds. For the general framework, Section 4 derives a characterization of the optimal solution and the optimal portfolio strategy. Section 5 analyzes the first-order conditions of the portfolio problem. In Section 6, we illustrate our results in the model of Jarrow (2001). Section 7 considers a situation where the investor cannot hedge against shifts in the state variables. Section 8 conludes. Technical details are relegated to the Appendix.

2   A Multiple Default Framework

Let $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ be a filtered probability space. The economy is driven by a K-dimensional state process $Z = (Z_1, \ldots, Z_K)$ consisting of economic variables like the interest rates, stock indices, or additional macroeconomic factors. The k-th entry of Z follows the diffusion process:

$$dZ_k(t) = \alpha_k(t)dt + \overline{\beta}_k(t)d\overline{W}_k(t),$$

where $\overline{W}$ is a K-dimensional correlated Brownian motion with $< \overline{W}_i, \overline{W}_j >_t = \rho_{i,j}t$ and α_k as well as $\overline{\beta}_k$ are real-valued functions of t and $Z(t)$. Here and in the following we assume that the coefficients of all stochastic differential equations (SDEs) are predictable processes which are sufficiently integrable such that the SDEs possess unique solutions.[2]   The state diffusions can be rewritten using a K-dimensional standard Brownian motion W with uncorrelated coordinates:

$$dZ_k(t) = \alpha_k(t)dt + \beta_k(t)'dW(t),$$

where β is a triangle matrix and β_k is its k-th row.[3]   In the economy, the short rate is a function of time t and state $Z(t)$. Of course, the short rate itself may be

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[2]See, e.g., Protter (2004), pp. 249ff.

[3]For instance, if $\overline{W}$ is two-dimensional we obtain $\overline{W}_1 = W_1$ and $\overline{W}_1 = \rho_{1,2}W_1 + \sqrt{1 - \rho_{1,2}}W_2$. This is a called a Cholesky decomposition.

one of the components of Z. Let $\eta^Z = (\eta_1^Z, \ldots, \eta_K^Z)$ be a predictable process being a function t and $Z(t)$. Assuming sufficient integrability,[4] η^Z defines a risk-neutral measure Q via Girsanov's theorem such that

$$dW^Q(t) = dW(t) + \eta^Z(t)dt,$$

is the increment of a K-dimensional Brownian motion under Q. The process η^Z is said to be the market price of risk of the diffusion part of the model which is represented by the state process Z.

In this economy, there exist firms having issued I corporate zero-coupon bonds (defaultable bonds). The bond payoffs are affected by J different kinds of credit events each modeled by a series of stopping times $\{\tau_{jn}\}_{n \in \mathbb{N}}$, $j = 1, \ldots, J$. The impact of a credit event triggered by factor j on the bonds is given by a series of I-dimensional markers $\{Y_{jn}\}_{n \in \mathbb{N}}$ taking values in a mark space $(E, \mathcal{E})$ such that $Y_{jn} = (Y_{1jn}, \ldots, Y_{Ijn})$. The corresponding counting processes are defined by

$$N_j(t, A) := \sum_{n \geq 1} 1_{\{\tau_{jn} \leq t\}} 1_{\{Y_{jn} \in A\}}$$

and the associated counting measures by $\mu_j((0, t], A) := N_j(t, A)$, $t \geq 0$, $A \in \mathcal{E}$. The process $N_j(\cdot, A)$ counts the credit events being triggered by factor j and leading to an impact of $A \in \mathcal{E}$ on the corporate bonds. For instance, if one assumes that a marker represents a fractional loss of the bonds' notional, the event $A = [0.2; 0.4] \times \ldots \times [0.2; 0.4]$ stands for defaults leading to losses between 20% and 40%. Furthermore, we define the counting processes $N_j(t) := N_j(t, E)$, $j = 1, \ldots, J$, counting all credit events triggered by factor j. The payoff of the i-th bond is defined as follows:

$$B_i(T_i, T_i) = \prod_{j=1}^{J} \prod_{n=1}^{N_j(T_i)} (1 - L_{ij}(\tau_{jn})), \tag{1}$$

The loss rate L_{ij} is a predictable function of the default time τ_{jn}, the corresponding state $Z(\tau_{jn})$, and the marker Y_{jn}, i.e. there exists a predictable function l_{ij} such that $L_{ij}(t) = l_{ij}(t, Z(t), Y_j(t))$, where the process $Y_j(t)$ denotes the piecewise constant, left-continuous time interpolation of $\{Y_{jn}\}_{n \geq 1}$. It is assumed that L_{ij} takes values in $[0, 1)$ almost surely, i.e. total losses are excluded. To shorten notations, we will in the following omit the functional dependency of l_{ij} on the state $Z(t)$.

This framework is inspired by Duffie/Singleton (1999b) and allows us to model various kinds of joint credit events.[5] In applications, however, one will impose more structure on the model by assuming that the counting processes trigger firm specific, sectoral, or global default events. Besides, it may be a reasonable assumption that the i-th element of the marker affects only the i-th bond, i.e. L_{ij} depends only on Y_{ij}. Even in this case it makes sense to distinguish between L_{ij} and Y_{ij} because there are in general two ways to model the loss rate: Firstly, one can assumes that

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[4]A sufficient condition would be that η^Z satifies Novikov's condition. See, e.g., Protter (2004).

[5]We will further comment on this point at the beginning of Section 5.3.

Y_{ij} is the loss rate itself possessing some distribution on $[0, 1]$ such as the beta distribution. In this case, we have $l_{ij}(t, z, y) = y$. Secondly, one can assume that each Y_{ijn} has some arbitrary distribution and then transform the realizations to $[0, 1]$. If, for instance, this transformation is carried out by a logit transformation, then $l_{ij}(y) = \frac{e^y}{1+e^y}$, where we disregard possible time- and state-dependencies.

The above described recovery regime is known as multiple-default model (MD) introduced by Schönbucher (1998). It can be shown that, under certain conditions, MD is equivalent to the assumption of recovery of market value (RMV). Furthermore, the assumption of recovery of treasury (RT) can be considered as special case of RMV because under the RT-assumption a corporate bond can be interpreted as a portfolio of a default-free bond an a defaultable bond with zero recovery.

We assume that N is a multi-dimensional Cox process with risk-neutral intensity $\lambda^Q = (\lambda_1^Q, \ldots, \lambda_J^Q)$ being a function of time t and state $Z(t)$. As a consequence, the coordinates of N are independent if one conditions on the history of Z. Denoting the risk-neutral compensator measure of μ_j by ν_j^Q, we further obtain:

$$\nu_j^Q(dt, dy) = K_j^Q(t, dy)\lambda_j^Q(t)dt,$$

where $K_j^Q(t, dy)$ is a risk-neutral probability measure on $(E, \mathcal{E})$ providing the risk-neutral distribution of the marker $\{Y_{jn}\}_{n \geq 1}$. For fixed $y \in E$, we allow K_j^Q to be a predictable function of time t and state $Z(t)$. For notational convenience, we again omit the dependency on the state $Z(t)$. The bond dynamics are modeled as follows:

$$dB_i(t, T_i) = B_i(t-, T_i)\left[r(t)dt + \sum_{k=1}^K \sigma_{ik}(t)dW_k^Q(t) - \sum_{j=1}^J dM_{ij}^Q(t)\right], \qquad (2)$$

where σ_{ik} is a function of time t and state $Z(t)$ and

$$
\begin{aligned}
dM_{ij}^Q(t) &= \int_E l_{ij}(t, y)\big(\mu_j(dt, dy) - \nu_j^Q(dt, dy)\big) \\
&= l_{ij}(t, Y_j(t))dN_j(t) - \bar{l}_{ij}^Q(t)\lambda_j^Q(t)dt
\end{aligned}
$$

is a (local) Q-martingale. Here $\bar{l}_{ij}^Q(t) = \int_E l_{ij}(t, y)K_j^Q(t, dy)$ is the risk-neutral local expected loss rate which, by assumption, is a function of time t and state $Z(t)$. Changing the measure to the physical measure, we arrive at

$$dB_i(t, T_i) = B_i(t-, T_i)\left[(r(t) + \chi_i(t))dt + \sum_{k=1}^K \sigma_{ik}(t)dW_k(t) - \sum_{j=1}^J L_{ij}(t)dN_j(t)\right],$$
$$(3)$$

with

$$\chi_i = \sum_{j=1}^J \hat{l}_{ij}\lambda_j\eta_j + \sum_{k=1}^K \sigma_{ik}\eta_k^Z,$$

where $\hat{l}_{ij}(t) = \int_E l_{ij}(t, y)\phi_j(t, y)K_j(t, dy)$ and $\lambda_j = \lambda_j^Q/\eta_j$. The probability measure K_j describes the distribution of the marker Y_j under the physical measure and $\phi_j > 0$ is a Girsanov kernel with $K_j^Q(dt, dy) = \phi_j(t, y)K_j(dt, dy)$ and $\int_E \phi_j(t, y)K_j(t, dy) =$

1. The process λ_j is the intensity under the physical measure and $\eta_j - 1 > -1$ is the associated Girsanov kernel. Both Girsanov kernels can be interpreted as risk premiums and $\hat{l}_{ij}$ is the risk-adjusted expected loss rate under the physical measure. For risk averse investors, $\hat{l}_{ij}$ is usually greater than the expected loss rate under the physical measure, $\bar{l}_{ij}(t) = \int_E l_{ij}(t,y) K_j(t,dy)$.[6] It is assumed that the kernels are predictable functions of t, $Z(t)$, and, in the case of ϕ_j, of $y \in E$. Therefore, χ_i is a predictable function of t and $Z(t)$ as well. Note that, by assuming sufficient integrability, η^Z, η, and ϕ uniquely define the risk-neutral measure Q via its Girsanov density.

3   Portfolio Problem

Our goal is to analyze portfolio problems with corporate bonds. Additionally, investors may be able trade in other securities such as default-free bonds or bond indices as well as stocks or stock indices with dynamics[7]

$$dS_h(t) = S_h(t) \big[(r(t) + a_h(t))dt + b_h(t)'dW(t) \big],$$

where the processes a_h and b_h are functions of t and state $Z(t)$, $h = 1, \ldots, H$. We refer to these securities as indices. Consequently, the investor of our portfolio problem can allocate his funds between a (locally risk-free) money market account M, I defaultable bonds, and H indices. He maximizes utility from intermediate consumption and terminal wealth at final time $T \leq \min\{T_i, i = 1, \ldots, I\}$ with respect to the following utility function

$$U(t,x) = \frac{1}{\gamma} \psi(t)^{1-\gamma} x^\gamma, \qquad \gamma < 1,$$

where $\psi(t) = \psi(t, Z(t))$ is a non-negative state-dependent discount process reflecting the investor's time preferences. Disregarding consumption for the moment, the investor's time-t wealth is given by

$$X(t) = \sum_{h=1}^{H} \varphi_h^S(t) S_h(t) + \sum_{i=1}^{I} \varphi_i(t) B_i(t, T_i) + \varphi_M(t) M(t),$$

where $\varphi_h^S(t)$ denotes the number of shares of index h held in the investor's portfolio at time t. The processes φ_i and φ_M denote the number of shares invested in the i-th corporate bond and the money market account. Restricting our considerations to self-financing strategies $(\varphi^S, \varphi, \varphi_M)$ and applying Ito's formula yields the dynamics

$$dX(t) = \sum_{h=1}^{H} \varphi_h^S(t) dS_h(t) + \sum_{i=1}^{I} \varphi_i(t-) dB_i(t, T_i) + \varphi_M(t) dM(t),$$

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[6] See, e.g., Bakshi/Madan/Zhang (2001).

[7] More general, the market may consist of contingent claims on the state variables and the point processes. We implicitly assume that the maturity of the claims is greater than the investor's horizon.

The proportions invested in the h-th index and the i-th bond are given by $\pi_h^S = \varphi_h^S S_h / X$ and $\pi_i = \varphi_i B_i / X$. Therefore, we obtain the so-called wealth equation:

$$dX = X^- \big[(r + \sum_h \pi_h^S a_h + \sum_i \pi_i \chi_i) dt + \sum_k \tilde{\pi}_k dW_k - \sum_j \sum_i \pi_i^- L_{ij} dN_j \big],$$

where $\tilde{\pi}_k := \sum_h \pi_h^S b_{hk} + \sum_i \pi_i \sigma_{ik}$, $X^- := X(t-)$, and $\pi_i^- := \pi_i(t-)$. To shorten notations, here and in the following we deliberately omit functional dependencies whenever this causes no confusion. Since

$$\sum_h \pi_h^S a_h + \sum_i \pi_i \chi_i = \sum_i \pi_i \bar{\lambda}_i + \sum_k \tilde{\pi}_k \eta_k^Z$$

with $\bar{\lambda}_i = \sum_j \hat{l}_{ij} \lambda_j \eta_j$, rewriting the wealth equation gives

$$dX = X^- \big[(r + \sum_i \pi_i \bar{\lambda}_i + \sum_k \tilde{\pi}_k \eta_k^Z) dt + \sum_k \tilde{\pi}_k dW_k - \sum_j \sum_i \pi_i^- L_{ij} dN_j \big]. \tag{4}$$

If we disregard the counting processes, our problem coincides with the classical continuous-time portfolio problem of Merton (1969, 1971, 1973). In a complete market setting with stochastic short rate, the candidate for the value function, G, can be factorized using the separation $G(t, x, z) = \frac{1}{\gamma} x^\gamma f(t, z)^{1-\gamma}$, where f is given via a Feynman-Kac representation. Unfortunately, if we have both incompleteness and a stochastic short rate, this nice representation result breaks down.[8]

It is convenient to distinguish between completeness of the continuous part of the model represented by the K-dimensional Brownian motion and completeness of the (pure) jump part represented by the J-dimensional counting process N. For instance, if at least K securities are traded which depend on the diffusion factors only, i.e. $H \geq K$, the continuous part of the model is complete given that the corresponding volatility matrix is regular. Depending on the structure of the loss rate matrix $l = (l_{ij})$, there may be situations where one can achieve the same result with defaultable bonds only. Roughly speaking, this is possible if one can set up trading strategies such that the relevant counting processes cancel out.

As above mentioned, the literature so far suggests that closed-form solutions for portfolio problems with stochastic interest rates and unhedgeable state variables are not available.[9] This is even valid if we disregard default risk. The focus of our paper, however, are portfolio problems with default risk and we wish to characterize the solutions as explicit as possible. For this reason, in Sections 4 through 6 we assume that the diffusion part of the model is complete. From the investor's point of view this assumption means that he is able to hedge his portfolio against shifts in the state process separately from his investment in corporate bonds. Section 7 discusses the consequences of dropping this assumption and explicitly solves a corresponding portfolio problem where the unhedgeable state variable is the short rate.

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[8] See, e.g., Zariphopoulou (2001). One can verify that the Feynman-Kac representation result does not work if we have both incompleteness and default-free stochastic interest rates.

[9] Clearly, this statement depends on the opinion about what one is willing to accept as a closed-form solution.

Completeness of the continuous part allows us to treat the processes $\tilde{\pi}_k$ as control variables. To see this, let the first K indices span the continuous part of the model. Furthermore, assume that in a first step we have already computed optimal controls $\tilde{\pi}_k^*$ and π_i^*. By completeness of the continuous part, the system of linear equations $\tilde{\pi}_k^* = \sum_h \pi_h^S b_{hk} + \sum_i \pi_i^* \sigma_{ik}$, $k = 1, \ldots, K$ can then uniquely be solved for π_h^S, $h = 1, \ldots, K$, establishing a one-to-one correspondence between $(\tilde{\pi}_1, \ldots, \tilde{\pi}_K)$ and $(\pi_1^S, \ldots, \pi_K^S)$. In view of Merton (1973), this is a consequence of his n-fund theorem. In our problem one can think of $\tilde{\pi}_k$ as a proportion invested in a portfolio of assets being affected only by the Brownian motion W_k. Therefore, the $(K+I)$-dimensional process $(\tilde{\pi}, \pi)$ can be considered as the control variable of the problem. In view of the wealth equation (4), this leads to a clean separation of the continuous and the jump part of the model. If the continuous part of the model is incomplete, the variables $\tilde{\pi}$ and π cannot be chosen independently and the two step approach discussed above breaks down.

Since we have discussed the completeness assumption from a more general perspective so far, let us finally comment on some economic implications of the completeness assumption in the context of our particular portfolio problem with corporate bonds. For this reason, we recall the meaning of the state variables in our portfolio problem: Although the state variables can in principle affect all parameters of the model including risk premiums, their main purpose is to capture randomness of default intensities via a Cox process approach, i.e. these intensities are assumed to depend on time and state. From this point of view, assuming the continuous part of the model to be complete means that randomness in intensities is allowed, but needs to be hedgeable. To illustrate this point, we briefly consider the model by Jarrow (2001) which actually inspired us to analyze the implications of the completeness assumption for portfolio management. A detailed discussion of his model is however postponed to Section 6. Jarrow's state process consists of the short rate and an equity index. Therefore, his model satisfies the completeness assumption given that the investor can trade in both default-free bonds and the equity index. This seems to be a reasonable requirement and is in line with Jarrow's assumptions. To summarize, the completeness assumption of the continuous part implies that randomness in intensities does not add additional incompleteness to the model. Note that, whatever assumption is imposed on the continuous part (completeness as in Section 4 through 6 or incompleteness as in Section 7), the model is in general an incomplete one stemming from random recovery rates or common default factors.

4   Characterization of the Optimal Solution

In this section, we derive a solution of the investor's portfolio problem under the assumption that he is able to hedge his portfolio against shifts in the state process. If we allow for consumption as well, the wealth equation (4) reads

$$dX = X^- \big[(r + \sum_i \pi_i \bar{\lambda}_i + \sum_k \tilde{\pi}_k \eta_k^Z) dt + \sum_k \tilde{\pi}_k dW_k - \sum_j \sum_i \pi_i^- L_{ij} dN_j \big] - c\,dt.$$

The investor faces the following optimization problem:

$$\max_{\tilde{\pi},\pi,c} \mathrm{E}\left[\int_0^T \frac{1}{\gamma}\psi(s)^{1-\gamma}c(s)^{\gamma}ds + \frac{1}{\gamma}\psi(T)^{1-\gamma}\left(X^{\tilde{\pi},\pi,c}(T)\right)^{\gamma}\right]. \qquad (5)$$

Calculations detailed in Appendix A.1 lead to the following first-order conditions determining the optimal portfolio strategy and consumption

$$\bar{\lambda}_i = \sum_j \lambda_j \int_E l_{ij}(y)(1-\sum_\nu \pi_\nu^* l_{\nu j}(y))^{\gamma-1} K_j(dy), \qquad i=1,\ldots,I, \qquad (6)$$

$$\tilde{\pi}_k^* = \frac{1}{1-\gamma}\eta_k^Z + \sum_m \beta_{mk}\frac{f_m}{f}, \qquad k=1,\ldots,K, \qquad (7)$$

$$c^* = x\frac{\psi}{f},$$

where the function f satisfies the PDE

$$0 = f_t - \tilde{r}f + \sum_m \tilde{\alpha}_m f_m + 0.5\sum_n\sum_m\sum_k \beta_{nk}\beta_{mk}f_{nm} + \psi \qquad (8)$$

with

$$\tilde{r} = -\frac{\gamma}{1-\gamma}\left(r + \sum_i \pi_i^*\bar{\lambda}_i + 0.5\frac{1}{1-\gamma}\sum_k(\eta_k^Z)^2\right.$$
$$\left. + \frac{1}{\gamma}\sum_j \lambda_j\left\{\int_E (1-\sum_i \pi_i^* l_{ij})^{\gamma} K_j(dy) - 1\right\}\right),$$

$$\tilde{\alpha}_m = \alpha_m + \frac{\gamma}{1-\gamma}\sum_k \eta_k^Z \beta_{mk}$$

and f_m denotes the first partial derivative of f with respect to the m-th state variable. The value function measuring the optimal utility for a given inital value of the state process, (t,x,z), is given by $G(t,x,z) = \frac{1}{\gamma}x^{\gamma}f(t,z)^{1-\gamma}$.[10] In the sequel, we refer to the first-order conditions for the optimal corporate bond proportions (6) as FOC. Due to (7), the portfolio proportions $\tilde{\pi}_k$ are of Merton-Breeden type, i.e. each consists of a myopic term and Merton-Breeden terms taking the random changes in the state variables into account. Furthermore, under mild regularity conditions, the I equations (6) implicitly define the optimal proportions π_i^* for the corporate bonds. In general, these equations cannot be solved explicitly for π_i^*. If, however, the jumps are deterministic and the jump part of the model is complete as well, then this is possible. We will discuss the FOC in the next section. In any case, under mild technical conditions, we get solutions for π_i^*, either implicitly or explicitly. Most importantly, the corporate bond proportions π_i^* depend on the state variables, but not on f or its derivatives. For this reason, the PDE (8) for f is of Black-Scholes-type. Assuming (8) to possess a classical solution,[11] we can thus call upon the Feynman-Kac theorem to give the following representation for f:

$$f(t,z) = \tilde{\mathrm{E}}^{t,z}\left[\int_t^T e^{-\int_t^s \tilde{r}(u)\,du}\psi(s)\,ds + e^{-\int_t^T \tilde{r}(u)\,du}\psi(T)\right], \qquad (9)$$

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[10]See Appendix A.1 for a formal definition of the value function.

[11]See Appendix A.2 for more details.

where the expectation $\tilde{\mathrm{E}}$ is calculated with respect to the measure $\tilde{\mathcal{P}}$ defined by the Girsanov density

$$\left. \frac{d\tilde{\mathcal{P}}}{d\mathcal{P}} \right|_{\mathcal{F}_t} = \exp\left(-0.5\left(\tfrac{\gamma}{1-\gamma}\right)^2 \int_0^t |\eta^Z(s)|^2 \, ds + \tfrac{\gamma}{1-\gamma} \int_0^t \eta^Z(s)' \, dW(s) \right). \qquad (10)$$

The dynamics of the state process under $\tilde{\mathcal{P}}$ are given by

$$dZ_n = \tilde{\alpha}_n dt + \sum_k \beta_{nk} d\tilde{W}_k$$

where $\tilde{W}$ is a Brownian motion under $\tilde{\mathcal{P}}$ defined via $d\tilde{W}_k = dW_k - \tfrac{\gamma}{1-\gamma}\eta_k^Z dt$.

This completes our characterization of the optimal solution. Depending on the specification of the model, it may be possible to compute the function f explicitly. In Section 6, we will discuss an example where this is possible. Otherwise, one needs to resort to numerical methods. Our analysis shows, however, that the FOC does not depend on the function f. Furthermore, if we stand at time t, the intensities can be treated as if they were constant. Therefore, one can solve the FOC without specifying the intensity processes and taking their time-t value as given. These properties are consequences of the assumption that the continuous part of the model is complete, i.e. the investor is able to hedge his portfolio against shifts in the state variables. This leads to a kind of separation result: The optimal proportions invested in corporate bonds are characterized independently of the value function of the model. Consequently, we can analyze these proportions without having calculated the value function. The decision process thus follows a two-step procedure: In a first step, the optimal proportion invested in the corporate bonds are calculated. In a second step, the optimal proportion of the indices are determined. The indices serve as hedging instruments against shifts in the state variables. To discuss this second step, we need to specify the state processes.

5   Solving the FOC

Under the completeness assumption of the diffusion part, we can characterize the optimal portfolio decision concerned with corporate bonds independently of other available investment opportunities. In this section, we consider three different situations which are of particular interest.

5.1   Completeness

If both the continuous and the jump part of the model are complete, then there exists a convenient procedure to compute the optimal proportions. It is one of the core results of Finance that, loosely speaking, a financial market is complete if the number of traded assets is equal to (or greater than) the number of sources of risk. For our particular problem with a finite number of traded assets, the market can thus be complete only if finitely many jump sizes possess strictly positive

probabilities. For this reason, let all loss rates be deterministic functions of time, i.e. l_{ij} depends on time t only. Then a necessary condition for the market to be complete is that the number of traded corporate bonds is greater than the number of different sources of credit events, $I \geq J$. Without loss of generality, assume $I = J$. Setting $k_j := (1 - \sum_\nu \pi_\nu^* l_{\nu j})^{\gamma-1}$ leads to a system of linear equations for k_j:

$$\bar{\lambda}_i = \sum_j l_{ij} \lambda_j k_j, \qquad i = 1, \ldots, I,$$

which has then unique solutions k_j^*, $j = 1, \ldots, J$. In a second step, one can solve the following system of linear equations for π_i^*, $= 1, \ldots, I$:

$$(k_j^*)^{\frac{1}{\gamma-1}} = 1 - \sum_\nu \pi_\nu^* l_{\nu j}, \qquad j = 1, \ldots, J.$$

Hence, the problem reduces to solving successively two systems of linear equations.

5.2 No Joint Defaults

One can assume that each default factor is associated with exactly one corporate bond, i.e. joint default events cannot occur. In this case, the factor N_i counts the number of defaults of that firm which issued the i-th corporate bond and the payoff (1) of this bond simplifies into

$$B_i(T_i, T_i) = \prod_{n=1}^{N_i(T_i)} (1 - Y_{in}),$$

where, for simplicity, the loss rate is assumed to be $l_i(t, Y_i(t)) = Y_i(t)$. Therefore, the first-order condition for the optimal proportion invested in the i-th corporate bond reads

$$\hat{l}_i \eta_i = \int_{[0,1]} y(1 - \pi_i y)^{\gamma-1} K_i(dy) \tag{11}$$

with $\hat{l}_i = \int_{[0,1]} \phi_i(y) y K_i(dy)$. We wish to stress that the default intensity cancels out meaning that our result holds independently of a specific choice of this intensity.[12] In general, the optimal proportion π_i is state- and time-dependent because ϕ_i, η_i, and K_i can be state- and time-dependent. Otherwise, the left-hand side is simply a constant implying that the proportion π_i is a constant as well. If the loss rate is also constant implying $\hat{l}_i = \bar{l}_i = l_i = Y_i = const.$, we immediately get

$$\pi_i^* = \frac{1 - \eta_i^{\frac{1}{\gamma-1}}}{l_i}, \tag{12}$$

which is a special case of the complete case discussed in Subsection 5.1. In the sequel, we will frequently compare the optimal proportions of more involved models with this result. It can be considered as a rule of thumb for an investor deliberately ignoring the randomness of recovery rates. We refer to this proportion as naive strategy. Figure 5.1 demonstrates how this benchmark result varies with different

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[12]Note that η does not cancel out. Since η is the ratio of risk-neutral default intensity to physical default intensity, this ratio between default intensities plays an important role.

values of the risk premium η_i and different degrees of risk aversion γ. The loss rate l_i is equal 0.5. It can be seen that the optimal proportion increases (decreases) with the risk premium (with the degree of risk aversion). The shape of the hyperplane looks very similar for different choices of the loss rate, but the absolute values of the optimal proportions are greater (smaller) for smaller (greater) loss rates.

[INSERT FIGURE 5.1 ABOUT HERE]

To model recovery risk, in applications loss rates are frequently assumed to be beta distributed. This is a reasonable assumption because the probability mass of a beta distribution is concentrated on the interval $[0, 1]$. Besides, the class of beta distributions is rather flexible allowing for various kinds of loss distributions. Under this assumption, K_i has a density given by $K_i(dy) = y^{p_i-1}(1 - y)^{q_i-1}dy/\mathcal{B}(p_i, q_i)$, where p_i and q_i are positive constants and $\mathcal{B}(p_i, q_i) = \int_0^1 s^{p_i-1}(1 - s)^{q_i-1}ds$ is the beta function. Loosly speaking, p_i controls for the shape of the distribution of large losses and q_i for the shape of the distribution of small losses. Some examples can be found in Figure 5.4. Note that, in general, we allow the loss distribution K_i to depend on time and state. In the particular case of a beta function, this means that p_i and q_i may depend on time and state. Since for a given time point t, both variables can be considered as constants, we disregard possible dependencies in this subsection. The FOC reads:

$$\hat{l}_i \eta_i = \frac{1}{\mathcal{B}(p_i, q_i)} \int_0^1 y^{p_i}(1 - \pi_i y)^{\gamma-1}(1 - y)^{q_i-1} \, dy. \tag{13}$$

Using the hypergeometric function h, this can be rewritten in a compact way:

$$\frac{\hat{l}_i}{\bar{l}_i} \eta_i = h(p_i + 1, 1 - \gamma, p_i + q_i + 1, \pi), \tag{14}$$

where we refer the reader to Appendix A.3 for further details on the hypergeometric function. The ratio between the risk-adjusted expected loss rate and the actual expected loss rate, $\hat{l}_i/\bar{l}_i$, can be interpreted as a risk premium due to recovery risk. Hence, the left-hand side is the product of two risk premiums: The risk premium η_i stems from the fact that generally a default can happen (default-timing risk premium) and the risk premium $\hat{l}_i/\bar{l}_i$ stems from the fact that the impact of a default event is random (premium for recovery risk). According to the FOC (14) the optimal proportion needs to be chosen such that the product of risk premiums is matched by the hypergeometric function on the right-hand side.

In general, equation (13) needs to be solved by numerical integration techniques. If, however, $p_i, q_i \in \mathbb{N}$, which already allows for a wide range of loss distributions, we obtain

$$\hat{l}_i \eta_i = \frac{(p_i + q_i - 1)!}{(p_i - 1)!(q_i - 1)!} \sum_{k=0}^{q_i-1} \binom{q_i - 1}{k}(-1)^k \int_0^1 y^{p_i+k}(1 - \pi_i y)^{\gamma-1} \, dy.$$

The integral on the right-hand side has a closed form solution which can be found by partial integration. For instance, given that the loss rates are uniformly distributed

$(p_i = q_i = 1)$, this simplifies into the following non-linear equation for π_i:

$$0 = 1 - (1 + \gamma)\pi_i(1 - \pi_i)^\gamma - (1 - \pi_i)^{1+\gamma} - \hat{l}_i\eta_i\gamma(1 + \gamma)\pi_i^2.$$

Note again that we are able to derive these results without specifying the default intensities or the term structure of default-free bonds. We end this subsection by discussing some numerical examples. From the FOC (14) it is obvious that for given γ, p_i, and q_i only the product of the risk premiums, $\hat{l}_i\eta_i/\bar{l}_i$, is relevant for computing the optimal proportion. For instance, the choices $\hat{l}_i/\bar{l}_i = 1.5$ and $\eta_i = 1$ as well as $\hat{l}_i/\bar{l}_i = 1$ and $\eta_i = 1.5$ lead to the same optimal proportion. Therefore, in the following numerical examples we set $\phi_i = 1$ and vary η_i only, i.e. we use η_i as the scaling factor of the overall risk premium. Firstly, we fix the investor's risk aversion $\gamma = -5$ and set $\eta_i = 1.5$.[13] The goal is now to analyze how the optimal proportion changes with different values for p_i and q_i of a beta distribution. The results are detailed in Figures 5.2 and 5.3. For the convenience of the reader, Figure 5.4 plots some characteristic shapes of a beta distribution. In principle, large values of p_i (q_i) increase the probability of large (small) losses. For $p_i = q_i$ the distribution is symmetric including the uniform distribution as a special case for $p_i = q_i = 1$. For $p_i > q_i$ ($p_i < q_i$) the distribution is left-skewed (right-skewed). If both values are greater than one, the distribution has a single hump. If, however, one of the values is smaller than one, then the corresponding tail value goes to infinity. Having these properties of the class of beta distributions in mind, the results shown in Figure 5.2 are pretty intuitive. If p_i (q_i) is increased, i.e. the probability of large losses increases (decreases), then the investor reduces (increases) his exposure to default risk. In general, the optimal proportion behaves rather moderately. Only for high values of q_i the proportion increases heavily. Loosely speaking, in this case the loss given default is small compared to the risk premium and, therefore, the corporate bond is a good deal for the investor. In Figure 5.3 we compare the optimal strategy satisfying (14) with the naive strategy (12) where the constant loss rate l_i is set to be equal to the expected value of the beta distribution, $p_i/(p_i + q_i)$. The figure shows the relative deviation $(\pi_i^* - \pi_i^c)/\pi_i^c$, where π_i^c denotes the naive strategy. Since the naive case does not take random losses into account, we have reduced the risk premium to $\eta_i^c = 1.3$. We emphasize that the shape of the plane remains almost the same if η_i^c is varied. The only consequence of varying η_i^c is that the whole hyperplane is moved up or down. For instance, setting $\eta_i^c = \eta_i$, the hyperplane is located below zero. For our parameter choice, the message of Figure 5.3 is clear: If the parameter p_i is large (small), i.e. large losses become more (less) likely, then the naive strategy underestimates (overestimates) the optimal solution. On the other hand, the deviation decreases with the parameter q_i. The main reason is that a constant loss rate approximated by the expected value cannot capture the randomness of the loss rate. Finally, Figure 5.5 shows how the optimal proportion varies with the risk aversion and the risk premium. The parameters of the beta

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[13]See Berndt/Douglas/Duffie/Ferguson/Schranz (2004) for an extensive analysis of the value of default risk premiums.

distribution are $p_i = 6$ and $q_i = 2$. We obtain the intuitive result that the optimal proportion increases with the risk premium and decreases with the investor's risk aversion.

[INSERT FIGURES 5.2, 5.3, 5.4, 5.5 ABOUT HERE]

5.3   Joint Defaults

Schönbucher (1998) points out that the default correlation induced by correlated intensities following diffusion processes may be too low for relatively high-quality entities. For this reason, in this section we also allow for joint defaults as introduced by Duffie/Singleton (1999b). They assume that upon certain credit events two or more firms may default on its debt. These default events could include for instants liquidity breakdowns or political events such as the acts of foreign governments. As noted in their paper, although correlations in the incidence of defaults within a given year are realistically captured, the model may imply an unrealistic amount of default within a given week or month. Since portfolio decisions are usually long-term decisions, this problem seems to be not so severe in our context.[14]

To formalize the idea of common default events, we add one additional default factor to the model of Subsection 5.2 such that every corporate bond depends on this factor as well. The corresponding counting process is denoted by N_M. When this factor is triggered, an I-dimensional vector of loss rates is drawn from some distribution $K_M(dy)$, $y = (y_1, \ldots, y_I) \in [0,1]^I$. If the loss rates are independently distributed, the distribution can be rewritten as $K_M(dy) = K_{1M}(dy_1) \ldots K_{IM}(dy_I)$, where $K_{iM}(dy_i)$ denotes the loss distribution of the i-th bond with respect to the common factor. Clearly, we can assume that some of the bonds do not depend on N_M implying that the corresponding distributions K_{iM} are Dirac measures concentrated at zero.

In order to solve the described problem, we need to consider the following first-order conditions for the optimal corporate bond proportions:

$$\bar{\lambda}_i = \lambda_i \int_{[0,1]} y_i(1 - \pi_i y_i)^{\gamma-1} K_i(dy_i) + \lambda_M \int_{[0,1]^I} y_i(1 - \sum_\nu \pi_\nu y_\nu)^{\gamma-1} K_M(dy)$$

where $\bar{\lambda}_i = \hat{l}_i \lambda_i \eta_i + \hat{l}_{iM} \lambda_M \eta_M$. In contrast to (11), the intensities do not cancel out implying that the optimal proportions π_i^* are in general state dependent. By specifying distributions for K_i and K_M, one can solve these conditions numerically. To concentrate on joint defaults only, we assume that the loss rates belonging to the bond specific factors are constant. This assumption allows us to use the concise result (12) as benchmark case and to abstract away from effects stemming from stochastic loss rates. Furthermore, we only consider a portfolio problem with two corporate bonds because this situation allows us to highlight the main results while possessing rather concise first-order conditions. The loss rates belonging to the

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[14]See Duffie/Singleton (2003), pp. 247ff, for a detailed discussion of this issue.

common factor are modeled in the following way:[15]

$$Y_{iM} = \begin{cases} l_{iM} \in (0,1) & \text{with} \quad p_{iM} = P(Y_{iM} = l_{iM}) \in (0,1), \\ \\ 0 & \text{with} \quad 1 - p_{iM} \end{cases} \tag{15}$$

leading to loss rates which are Bernoulli distributed. Note that l_{iM} has the interpretation of a loss rate given default which is assumed to be constant. The FOC for the optimal corporate bond proportions reads:

$$\begin{aligned} \bar{\lambda}_i &= \lambda_i l_i (1 - \pi_i l_i)^{\gamma-1} \\ &\quad + \lambda_M p_{iM} l_{iM} [p_{kM}(1 - \pi_i l_{iM} - \pi_k l_{kM})^{\gamma-1} + (1 - p_{kM})(1 - \pi_i l_{iM})^{\gamma-1}], \end{aligned}$$

where i, $k \in \{1,2\}$, $i \neq k$, and $\bar{\lambda}_i = l_i \lambda_i \eta_i + p_{iM} l_{iM} \phi_{iM} \lambda_M \eta_M$. Recall that the loss rate l_i and the loss rate given default l_{iM} are constants. Besides, the risk premiums η_i, η_M, and ϕ_{iM} are assumed to be constants as well. For this reason, $\bar{\lambda}_i$ is weighted average of the intensities λ_i and λ_M which are state-dependent. Therefore, the optimal proportions π_i^* are state-dependent although l_i and l_{iM} are constants. To calculate the optimal proportions, we need estimates for the time-t values of the intensities. Although these values are typically estimated by specifying the dynamics of the intensities, the exact specification is irrelevant for computing π_i^* and for a fixed point in time t we can treat the intensities as if they were constant. Let us now demonstrate our results by a numerical example.

[INSERT FIGURES 5.6, 5.7, 5.8, 5.9 ABOUT HERE]

Figures 5.6 and 5.7 show how the optimal proportions π_1^* and π_2^* vary with the probabilities p_{1M} and p_{2M}. The relevant parameters are chosen as follows: $\eta_M = 1.5$, $\lambda_M = 0.02$, $\gamma = -5$, $l_i = l_{iM} = 0.5$, $\eta_i = 1.5$, $\phi_{iM} = 1.2$, $\lambda_i = 0.05$ for $i \in \{1, 2\}$. It can be seen that due to the choice of parameters the figures are symmetric. Note that the benchmark case without a common default factor corresponds to the point in the hyperplane where $p_{1M} = p_{2M} = 0$. At first glance, it is somewhat counter-intuitive that for small p_{2M} the proportion π_1^* increases with p_{1M}. The reason, however, is that the investor is rewarded with a higher risk premium if p_{1M} increases because p_{1M} serves as a scaling factor in the excess return of the corporate bond. For the behavior of the optimal proportions it is thus crucial how large the default-timing risk premium of the common factor, η_M, is compared to the other risk premiums in the model. This becomes clear from Figure 5.8 where η_M is reduced to 1.2. In this case, the optimal proportions invested in the first bond decreases in all directions. On the other hand, for $\eta_M = 2.0$ the proportion π_1^* increases in all directions (except for p_{2M} close to one). This is detailed in Figure 5.9. We wish to remark that a similar behavior can be observed for the risk premiums ϕ_{1M} and ϕ_{2M}. The message of this numerical example is thus as follows: If a common

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[15]With a slight abuse of notation, we denote the constant loss rate given default by l_{iM} although the unconditional loss rate is not constant but Bernoulli distributed. In the same way, we denote the corresponding realization of the risk premium by ϕ_{iM} instead of $\phi_{iM}(l_{iM})$.

default factor is relevant, the behavior of the optimal proportions (compared to the benchmark case) critically hinges upon the structure of the risk premiums. In general, it is not clear in advance whether the proportions increase or decrease. As a rule of thumb, the proportions decrease with the corresponding probabilities if the risk premiums associated with the common factor are close to one. How large these risk premiums need to be such that the proportions increase basically depends on the values of the other risk premiums.

We end this section by a comment on the model behind the numerical results. We have here worked with constant individual loss rates and Bernoulli distributed common loss rates as suggested by Duffie/Singleton (1999b). However, this is not a unique specification of the loss rate distribution within our setup. Another model which specifies the same distribution is the following: A constant common loss rate triggered with intensity $\lambda_M p_{1M} p_{2M}$ and two individual default factors for each corporate with constant loss rates triggered with intensities λ_1 and $\lambda_M p_{1M}(1 - p_{2M})$ and with intensities λ_2 and $\lambda_M p_{2M}(1 - p_{1M})$, respectively. Then we have no Bernoulli distributions but instead five default factors. A third alternative is to collect the four individual default factors into two individual factors with intensities $\lambda_1 + \lambda_M p_{1M}(1 - p_{2M})$ and $\lambda_2 + \lambda_M p_{2M}(1 - p_{1M})$. Then one could keep the constant common loss rate and instead introduce individual Bernoulli distributed loss rates with probabilities $\lambda_1/(\lambda_1 + \lambda_M p_{1M}(1 - p_{2M})$ and $\lambda_M p_{1M}(1 - p_{2M})/(\lambda_1 + \lambda_M p_{1M}(1 - p_{2M})$ for the first corporate and similarly for the second. All these specifications lead to the same distribution and the number of market prices of risk are, of course, also the same.

6 Jarrow's Model

So far we have not specified the state process Z and the dependencies of the asset dynamics on Z. In a series of papers,[16] Robert Jarrow and co-authors have investigated a tractable model integrating market and credit risk with correlated defaults. Correlated defaults arise due to the fact that the firms' default intensities depend on common macro-factors, namely the short rate and an equity market index. The short rate is assumed to follow a Vasicek model,

$$dr(t) = (\theta - \kappa r(t))dt + \beta dW(t)$$

with θ and β being constants. The dynamics of a default-free bond with maturity T_f read

$$dP(t, T_f) = P(t, T_1) \left[(r(t) + \bar{\eta}^r d_P(t, T_f))dt - \beta d_P(t, T_1)dW(t) \right],$$

where $d_P(t, s) = \frac{1}{\kappa}(1 - e^{-\kappa(s-t)})$ denotes the duration of the default-free bond and $\bar{\eta}^r$ is a constant. This specification of the excess return, $\bar{\eta}^r d_P(t)$, implies a

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[16]See, e.g., Janosi/Jarrow/Yildiray (2000), Jarrow/Turnbull (2000), Jarrow (2001), and Jarrow/Yu (2001).

constant market price of risk $\eta^Z = -\bar{\eta}^r/\beta$. We emphasize that our results hold for deterministic and bounded market prices of risk as well (i.e. $\bar{\eta}^r$ can be bounded and deterministic instead of constant). As in Jarrow's papers, one can introduce a time-dependent θ in the short rate dynamics to make the model consistent with a given initial default-free term structure. In general, this implies a time-dependent $\bar{\eta}^r$ as well.

Since Janosi/Jarrow/Yildiray (2000) found that including an equity market index does not add additional explanatory power in the pricing of corporate debt, we assume that the default intensities depend on the short rate only. If the investor can trade in the equity index, it is straightforward to incorporate the equity index in our portfolio problem as well and the value function is still given in closed form. In Jarrow's model each default factor is associated with exactly one corporate bond, i.e. joint default events cannot occur. For this reason, we are in the situation described in Subsection 5.2. The default intensity of the firm which issued the corporate bond i is assumed to be linear in the short rate, i.e.

$$\lambda_i(t) = \lambda_i^0 + \lambda_i^r r(t),$$

where λ_i^0 and λ_i^r are constants. Alternatively, one can assume λ_i^0 to be a deterministic function which can then be used to calibrate the model to a given defaultable term structure (e.g. corporate bond prices or CDS quotes). The state process Z is one-dimensional and consists of the short rate only. For simplicity, we assume that the loss rates are i.i.d. and independent from all other random variables in the model. Besides, the risk premium η_i, which links together the default intensities under the physical and the risk-neutral measure, is assumed to be constant. The risk premium ϕ_i is assumed to depend on the loss rate Y_i only. Then the dynamics of i-th corporate bond read[17]

$$dB_i(t,T_i) = B_i(t-,T_i)[(r(t)+\chi_i(t))dt - \beta d_P(t,T_i)(1+c_i)dW(t) - Y_i(t)dN_i(t)],$$

where $c_i = \hat{l}_i \lambda_i^r \eta_i$ with $\hat{l}_i = \mathrm{E}[\phi_i(Y_{i1})Y_{i1}]$ is a constant[18] and the excess return is given by

$$\chi_i(t) = \hat{l}_i \lambda_i(t)\eta_i + \bar{\eta}^r d_P(t,T_i)(1+c_i).$$

Note that in this setting the expected risk-adjusted jump size $\hat{l}_i$ is constant. We consider a portfolio problem where the investor can put his funds into a money market account, defaultable bonds, and at least one default-free bond with maturity T_f greater than the investor's horizon T. Given our assumptions so far, the wealth equation (4) reads

$$dX = X^-\left[(r + \sum_i \pi_i \bar{\lambda}_i + \tilde{\pi}\eta^Z)dt + \tilde{\pi}dW - \sum_i \pi_i^- Y_i dN_i\right]$$

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[17]For constant loss rates see Jarrow/Turnbull (2000). The generalization to the setting with i.i.d. loss rates is straightforward.

[18]Since the series of loss rates $\{Y_{in}\}_{n\in N}$ is i.i.d., without loss of generality we can take expectations with respect to the first loss Y_{i1}.

with $\tilde{\pi} = -\beta\pi_f d_P(t, T_f) - \beta\sum_i \pi_i d_P(t, T_i)(1 + c_i)$ and $\bar{\lambda}_i = \hat{l}_i \lambda_i \eta_i$. The optimal proportions invested in the corporate bonds are given by equation (11) and are constant. It remains to determine the value function and the proportion $\tilde{\pi}$. For simplicity, we assume that the investor maximizes terminal wealth only and that the discount factor ψ is equal to one. By (9), the function f is given by $f(t, r) = \tilde{E}^{t,r}\left[\exp(-\int_t^T \tilde{r}(u)\,du)\right]$ with $\tilde{r} = b_1 r + b_0$, where

$$b_1 = -\frac{\gamma}{1-\gamma}\left(1 + \sum_i \pi_i^* \hat{l}_i \eta_i \lambda_i^r + \frac{1}{\gamma}\sum_i \lambda_i^r \left\{E[(1 - \pi_i^* Y_{i1})^\gamma] - 1\right\}\right),$$

$$b_0 = -\frac{\gamma}{1-\gamma}\left(\sum_i \pi_i^* \hat{l}_i \eta_i \lambda_i^0 + 0.5\frac{1}{1-\gamma}(\eta^Z)^2 + \frac{1}{\gamma}\sum_i \lambda_i^0 \left\{E[(1 - \pi_i^* Y_{i1})^\gamma] - 1\right\}\right)$$

with $E[(1 - \pi_i^* Y_{i1})^\gamma] = \int_{[0,1]}(1 - \pi_i^* y)^\gamma K_i(dy)$, $i = 1, \ldots, I$, are constants. By standard arguments, we obtain

$$f(t, r) = e^{A(t,T) - C(t,T)r} \tag{16}$$

with $C(t, T) = \frac{b_1}{\kappa}(1 - e^{-\kappa(T-t)}) = b_1 d_P(t, T)$ and A is a deterministic function being irrelevant for our further considerations. From (7) we have $\tilde{\pi}^*(t) = \frac{1}{1-\gamma}\eta^Z - \beta C(t, T)$, leading to the following optimal proportion invested in the default-free bond:

$$\pi_f(t) = \frac{1}{d_P(t, T_f)}\left(\frac{1}{1-\gamma}\frac{\bar{\eta}^r}{\beta^2} + b_1 d_P(t, T) - \sum_i \pi_i^* d_P(t, T_i)(1 + c_i)\right).$$

If we set $\pi_i = 0$ for all $i = 1, \ldots, I$, then we obtain the optimal portfolio strategy for a portfolio problem where the investor can put his funds only into a money market account and a default-free bond with maturity T_f.[19] Compared with this problem, the investor invests less in the default-free bond because, by investing in corporate bonds, he is already exposed to interest rate risk. The default-free bond is used to adjust this exposure such that his overall investment in the default-free interest rate market is optimal.

In the original version of Jarrow's model default correlation only arises due to the fact that a firms default intensities depend on common macro-factors. Clearly, we can also add joint default factors as in Subsection 5.3. If, however, the intensity is state-dependent, the optimal corporate bond proportions in general become involved functions of the state variables. Hence, there will not exist a closed-form solution for f such as the affine representation (16) in the case without joint defaults. Although the Feynman-Kac representation (9) for f still holds, one needs to solve the expectation numerically. If we assume that all intensities are deterministic, i.e. $\lambda_i^r = 0$, then the corporate bond proportions are deterministic as well and f is again given by (16), where b_1 is replaced by $-\frac{\gamma}{1-\gamma}$.

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[19]See, e.g., Korn/Kraft (2001).

7 Unhedgeable State Variables

So far we have assumed that the investor is able to hedge shifts in the state variables representing the continuous part of the model. As a consequence, the FOC does not depend on the function f being a central part of the value function. Therefore, we were able to analyze the FOC without specifying the state variables. If we drop the assumption that shifts in the state variables are hedgeable, then this separation result breaks down. We demonstrate this point for the case where the investor trades in corporate bonds only. We wish to remark that one gets a similar result if parts of the state variables are hedgeable. Calculations detailed in Appendix A.4 show that the corresponding FOC reads:

$$\bar{\lambda}_i + \sum_k \eta_k^Z \sigma_{ik} + (1-\gamma)\sum_m \sum_k \beta_{mk}\sigma_{ik}\frac{f_m}{f} \tag{17}$$

$$= (1-\gamma)\sum_k \sum_\nu \pi_\nu \sigma_{\nu k}\sigma_{ik} + \sum_j \lambda_j \int_E l_{ij}(y)(1 - \sum_\nu \pi_\nu l_{\nu j}(y))^{\gamma-1} K_j(dy).$$

In contrast, to the previous sections, the FOC also involves terms of Merton-Breeden type, $\beta_{mk}\sigma_{ik}\frac{f_m}{f}$. Whereas before the investor invested in corporate bonds to optimize exclusively his default risk exposure, he now tries to resolve the trade-off between this goal and the wish to hedge against shifts in the state variables. For this reason, the Merton-Breeden terms enter the FOC distorting the clean result (6) which can be reproduced by setting formally all σ_{ik} equal to zero in (17). On the other hand, setting all λ_j equal to zero, the credit risk vanishes and the corporate bonds become default-free. In this case, the FOC corresponds to the first-order condition of a portfolio problem with default-free bonds.

In abstract mathematical terms, the optimal proportions invested in corporate bonds now depend on f and an explicit Feynman-Kac representation for f is not available. In general, the only way out is thus to solve the HJB numerically. It is well-known that this is a non-trivial task. In this section, we will analyze a situation where despite of non-hedgeable state variables a closed-form solution still exists. We consider a problem where the investor maximizes expected utility from terminal wealth with $\psi \equiv 1$ and puts his funds into corporate bonds only. The dynamics of the bonds are modeled according to Jarrow's model with joint defaults and we allow for stochastic recovery rates being state-independent. Furthermore, we assume the default intensities to be deterministic, i.e. $\lambda_i^r = 0$. This requirement allows us to derive a closed solution. Note that in this setting default correlation arises due to joint default events only. In analogy to Section 6, the corporate bond dynamics read:

$$dB_i(t,T_i) = B_i(t-,T_i)[(r(t) + \chi_i(t))dt + \sigma_i(t)dW(t) - \sum_{j=1}^J Y_{ij}(t)dN_j(t)] \tag{18}$$

with $\sigma_i(t) = -\beta d_P(t,T_i)$ and $\chi_i(t) = \sum_j \hat{l}_{ij}\lambda_j(t)\eta_j + \bar{\eta}^r d_P(t,T_i)$. Recall that the marker Y_j consists of I components, where in this specification its i-th component, Y_{ij}, is equal to the factor-j loss rate of bond i. The risk premium ϕ_j is assumed to

be of the form $\phi_j(y) = \prod_{i=1}^{I} \phi_{ij}(y_i)$ such that $\hat{l}_{ij} = \text{E}[\phi_{ij}(Y_{ij1})Y_{ij1}]$, where each ϕ_{ij} depends on the loss rate Y_{ij} only (and not on time or state). The excess return χ_i is thus a deterministic function. In Appendix A.4 it is shown that the FOC of this problem reads

$$\chi_i - \sigma_i \beta C = (1-\gamma)\sigma_i \sum_\nu \pi_\nu \sigma_\nu + \sum_j \lambda_j \int_E y_i (1 - \sum_\nu \pi_\nu y_\nu)^{\gamma-1} K_j(dy),$$

where $C(t,T) = -\frac{\gamma}{\kappa}(1-e^{-\kappa(T-t)}) = -\gamma d_P(t,T)$. Although the optimal proportions depend on C which is a part of the value function, in our setting the FOC completely characterizes the optimal proportions invested in corporate bonds and we are not in the same trouble as in the general case of equation (17). This is due to the fact that the relevant part of the value function, C, can be calculated explicitly. Since C is deterministic, the optimal proportions are deterministic as well. It is straightforward to solve the FOC numerically for the optimal proportions invested in the corporate bonds.

To understand the consequences of unhegeable state variables for the management of corporate bond portfolios, finally we are going to discuss a numerical examples. We analyze the situation of Subsection 5.2 where the firm specific loss rates are assumed to be constant. To keep the FOC as simple as possible, a portfolio problem with only two corporate bonds is considered. Our assumptions allow us to compare the results with the benchmark solution (12), where the loss rates are constant. The FOC simplifies into

$$\chi_i - \sigma_i \beta C = (1-\gamma)\sigma_i(\pi_1 \sigma_2 + \pi_2 \sigma_2) + \lambda_i l_i (1-\pi_i l_i)^{\gamma-1}$$

for $i \in \{1,2\}$. Figure 7.10 shows how the optimal proportion invested in one of the corporate bond varies with risk averion γ and risk premium η_i. The parameters of the corporate bonds are fixed as follows: $l_i = 0.5$, $\phi_{iM} = 1.2$, $\lambda_i = 0.05$, $T_i = 10$ for $i \in \{1,2\}$. The investor's time horizon is $T = 5$ years and he is standing at time zero. The parameters of the default-free term structure are $\bar{\eta}^r = 0.00075$, $\beta = 0.02$, and $\kappa = 0.25$.[20]

[INSERT FIGURES 7.10, 7.11 ABOUT HERE]

Due to our choice of parameters, we are in the same situation as in Figure 5.1 except for the fact that the investor cannot hedge against shifts in the short rate. It is obvious that the investor now invests in corporate bonds even if the risk premium is zero. This comes from his hedging demand against shifts in the risk-free term structure. In general, he thus puts more money into corporate bonds. Only for high values of the risk premium he invests less than before. Although a high risk premium makes a corporate bond a good deal, a big position in corporate bonds is always combined with a great exposure to default-free interest rate risk which cannot be hedged away. Therefore, his demand for corporate bonds increases more

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[20]The same parameters were used in Sørensen (1999) who analyzed a portfolio problem with default-free bonds.

slowly than before. Furthermore, for a risk premium close to one the hyperplane is not monotonously decreasing with risk aversion, but hump-shaped. The reason is that an logarithmic investor ($\gamma = 0$) does not wish to hedge against shifts in state variables. This well-known feature of a logarithmic investor distinguishes his so-called myopic portfolio strategy from investors with different risk aversions. Although an investor with $\gamma < 0$ reduces his investment in risky assets because of this more pronounced risk aversion, he does hedge his portfolio against shifts in the short rate. Since default-free bonds are not available for him, he increases his position in corporate bonds. Which of these two effects will dominate depends on the degree of risk aversion. Figure 5.1 demonstrates that this trade-off vanishes if the investor can trade in default-free bonds. To get an impression how the investor's intertemporal hedging demand behaves over time, Figure 7.11 details how the optimal proportion varies with time t. All parameters are the same as before except for the maturities of the corporate bonds which are $T_i = T = 5$ years. Due to this choice, the optimal bond proportions converge to the benchmark strategy (12) when time approaches the time horizon. Since this effect is especially pronounced for smaller degrees of risk aversion, Figure 7.11 only shows risk aversions with $\gamma \leq -2$. For simplicity, we keep the intensities constant over time. It can be seen that due to the hedging demand the investment in corporate bonds is three times greater than in the benchmark case if the investment horizon lies five years ahead.

8   Conclusion

The goal of this paper has been to analyze portfolio problems with corporate bonds. To model default risk, we have used a combination of the multiple default framework by Schönbucher (1998) and the joint default framework by Duffie/Singleton (1999b). Under the assumption that shifts in the state process, which above all drives the default intensities, are hedgeable, we have derived a Feynman-Kac representation for the solution of the portfolio problem. Besides, we have been able to analyze the first-order condition determining the optimal proportions invested in corporate bonds separately from the rest of the model. Using the optimal proportions for constant recovery without joint defaults as benchmark results, we have shown how random recovery rates (especially beta distributed loss rates) or joint default factors influence the investor's portfolio decision. We have pointed out that the structure of the risk premiums is crucial for the derivation from the benchmark case. In principal, assuming the recovery rates to be constant can lead to inferior portfolio decisions. The same is true if one ignores common default factors. Finally, we have demonstrated that dropping the aforementioned completeness assumption leaves the investor with a trade-off: He needs to find a balance between an optimal exposure to default risk and hedging against unfavorable shifts in the state variables.

A Appendix

A.1 Derivation of the Optimal Strategies

In this part of the Appendix we will solve the investor's problem (5). The value function of the problem is defined by

$$G(t,x,z) = \sup_{\tilde{\pi},\pi,c} \mathrm{E}^{t,x,z}\left[\int_t^T \tfrac{1}{\gamma}\psi(s)^{1-\gamma}c(s)^{\gamma}ds + \tfrac{1}{\gamma}\psi(T)^{1-\gamma}\left(X^{\tilde{\pi},\pi,c}(T)\right)^{\gamma}\right].$$

The Hamilton-Jacobi-Bellman equation (HJB) for the candidate G of the value function is given by

$$0 = \sup_{\tilde{\pi},\pi,c}\left\{A^{\tilde{\pi},\pi,c}G + \tfrac{1}{\gamma}\psi^{1-\gamma}c^{\gamma}\right\} \tag{19}$$

with terminal condition $G(T,x,z) = \tfrac{1}{\gamma}\psi^{1-\gamma}x^{\gamma}$ and

$$
\begin{aligned}
A^{\tilde{\pi},\pi,c}G &= G_t + \{x(r + \textstyle\sum_i\pi_i\bar{\lambda}_i + \sum_k\eta_k^Z\tilde{\pi}_k) - c\}G_x + \sum_k\alpha_k G_k\\
&\quad + 0.5x^2\textstyle\sum_k\tilde{\pi}_k^2 G_{xx} + x\sum_m\sum_k\tilde{\pi}_k\beta_{mk}G_{xm} + 0.5\sum_n\sum_m\sum_k\beta_{nk}\beta_{mk}G_{nm}\\
&\quad + \textstyle\sum_j\lambda_j\left\{\int_E G(t,x(1 - \sum_i\pi_i l_{ij}(y)),z)\,K_j(dy) - G\right\}
\end{aligned}
$$

where, to shorten notations,

$$
\int_E G(t,x(1 - \textstyle\sum_i\pi_i l_{ij}(y)),z)\,K_j(dy)
$$

$$
= \int_E G(t,x(1 - \textstyle\sum_i\pi_i(t)l_{ij}(t,y)),z)\,K_j(t,dy).
$$

Note that, except for $i = 1,\ldots,I$ and $j = 1,\ldots,J$, all sums run from 1 to K. Besides, $G_k = \partial G/\partial z_k$, $G_{xm} = \partial^2 G/(\partial z_m\partial x)$, and $G_{nm} = \partial^2 G/(\partial z_m\partial z_n)$. The first-order conditions for π_i, $\tilde{\pi}_k$, and c read

$$
\begin{aligned}
0 &= \bar{\lambda}_i G_x - \textstyle\sum_j\lambda_j\int_E l_{ij}(y)G_x(t,x(1 - \sum_\nu\pi_\nu^* l_{\nu j}(y)),z)\,K_j(dy),\\
\tilde{\pi}_k^* &= -\eta_k^Z\frac{G_x}{xG_{xx}} - \textstyle\sum_m\beta_{mk}\frac{G_{xm}}{xG_{xx}},\\
c^* &= \psi G_x^{\frac{1}{\gamma-1}},
\end{aligned}
$$

where $\nu = 1,\ldots,I$. Applying the separation $G(t,x,z) = \tfrac{1}{\gamma}x^{\gamma}f(t,z)^{1-\gamma}$ with $f(T,z) = \psi(T)$ yields a PDE for f in terms of the optimal proportions and optimal consumption:

$$
\begin{aligned}
0 &= \tfrac{1-\gamma}{\gamma}f_t + (r + \textstyle\sum_i\pi_i^*\bar{\lambda}_i + \sum_k\eta_k^Z\tilde{\pi}_k^*)f - \tfrac{c^*f}{x} + \tfrac{1-\gamma}{\gamma}\sum_k\alpha_k f_k \tag{20}\\
&\quad -0.5(1-\gamma)\textstyle\sum_k(\tilde{\pi}_k^*)^2 f + (1-\gamma)\sum_m\sum_k\tilde{\pi}_k\beta_{mk}f_m\\
&\quad -0.5(1-\gamma)\textstyle\sum_n\sum_m\sum_k\beta_{nk}\beta_{mk}f_m f_n f^{-1} + 0.5\tfrac{1-\gamma}{\gamma}\sum_n\sum_m\sum_k\beta_{nk}\beta_{mk}f_{nm}\\
&\quad + \tfrac{1}{\gamma}\textstyle\sum_j\lambda_j\left\{\int_E(1 - \sum_i\pi_i^* l_{ij}(y))^{\gamma}K_j(dy) - 1\right\}f + \tfrac{1}{\gamma}\psi^{1-\gamma}\left(\tfrac{c^*f}{x}\right)^{\gamma},
\end{aligned}
$$

Moreover, the FOC can be rewritten as in (6). Substituting the FOC for $\tilde{\pi}_k^*$ and c^* into the PDE (20) and simplifying, we arrive at (8). We emphasize that terms with

$f_n f_m f^{-1}$ which were present in (20) have canceled out in (8) leading to a Black-Scholes-type PDE. This results from the assumption that the continuous part of the model is complete. Otherwise, these terms would not cancel out completely and a Feynman-Kac representation for f would not hold. Remarkably, this representation is not affected by the completeness or incompleteness of the jump part.

Finally, we emphasize that this procedure is only meaningful if the operator $A^{\tilde{\pi},\pi,c}$ and the value function remain finite. Since for $\gamma < 0$ the value function is bounded from above by zero, in principle, problems can only occur for $\gamma > 0$. A positive γ, however, leads to a risk aversion being unrealisticly low (at least from a practical point of view) and we thus discuss this issue only briefly. The problems which may occur from explosions stemming from the continuous part are comprehensively discussed in Korn/Kraft (2004) and we will not repeat the discussion here. For the jump part of the model, one needs to ensure that

$$\int_E (1 - \textstyle\sum_i \pi_i l_{ij}(y))^\gamma \, K_j(dy) < \infty \tag{21}$$

for the optimal proportions π_i^* invested in corporate bonds. For $\gamma > 0$ this condition needs to be imposed for all admissible strategies as well, i.e. for all strategies which the investor is allowed to choose. In Subsection A.3, we will exemplify this point for the beta distribution.

A.2 Verification

The HJB (19) is of partial integro-differential type. The advantage of our approach is that we are able to transform this HJB into a Cauchy problem (8), where the "integro part" has vanished. Under suitable technical conditions, this problem can be solved by applying the Feynman-Kac theorem. The intention of this part of the Appendix is to sketch briefly why this method can be justified. The main questions to be asked are the following:

(i) Does the FOC (6) uniquely characterize the optimal proportions invested in the corporate bonds, π^*?

(ii) Under which conditions does the PDE (8) possess a unique solution?

ad (i). We define the function $F = (F_1, \dots, F_I)$ by

$$F_i(\pi_1, \dots, \pi_I) = \textstyle\sum_j \lambda_j \int_E l_{ij}(y) \big\{ \eta_j \phi_j(y) - (1 - \textstyle\sum_\nu \pi_\nu l_{\nu j}(y))^{\gamma-1} \big\} K_j(dy)$$

such that (6) can be rewritten as $F(\pi) = 0$. The function F is continuous and increasing in every component of π. For $\pi \to -\infty^I$ the function $F(\pi)$ becomes positive in every component. On the other hand, due to the root $(\cdot)^{\gamma-1}$ one cannot choose π_i arbitrarily large. More precisely, π_i has to be smaller than $1/\inf_y l_{ij_i}(y)$. For this reason, it is crucial that the function F becomes negative in every component if we take the limits $\pi_i \to 1/\inf_y l_{ij_i}(y)$. In this case, by continuity, there

exists a strategy π' such that $F(\pi') = 0$. However, there are rare cases where F remains positive. From an economic point of view, the risk premium η is then too large compared to the risk aversion of the investor, i.e. such problems can occur if both η and γ are large. It is beyond the scope of this paper to consider such a problem in detail and we assume that F becomes negative for $\pi_i \to 1/\inf_y l_{ij_i}(y)$. In our applications, one can easily check if this is valid and it is true for all our numerical examples. To prove uniqueness, in general further conditions (e.g. on the loss rates) need to be imposed. However, if we are in the situation of Subsection 5.2, one can directly conclude that the solution is unique given that there exists a solution. Furthermore, in this case it is also immediately clear that the HJB is strictly concave in the controls π_i, $\tilde{\pi}_k$, and c implying that the first-order conditions are also sufficient. This is so because the corresponding Hessian matrix is a diagonal matrix with negative entries on the diagonal. We wish to remark that the HJB of the problem of Subsection 5.3 is strictly concave as well.

ad (ii). A Cauchy problem possesses a unique classical solution if one imposes certain continuity and boundedness conditions on the coefficients. The interested reader is referred to the monograph by Duffie (2001, p. 345) and the references therein.

Since the answers to (i) and (ii) are in general positive, one can then prove that $G(t, x, z) = \frac{1}{\gamma} x^\gamma f(t, z)^{1-\gamma}$ with f given by (9) is the value function of our problem. Because we are dealing with a Cauchy problem, one can carry out the verification procedure by applying similar methods as used in problems with stochastic interest rates, but without default risk. The interested reader is referred to Korn/Kraft (2001) or Kraft (2004).

A.3 Beta Distribution and Optimality

In this part of the Appendix we analyze condition (13) in more detail. For $c > a > 0$ we define the hypergeometric function by

$$h(a, b, c, x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 y^{a-1}(1 - xy)^{-b}(1 - y)^{c-a-1}\, dy, \qquad (22)$$

where Γ is the gamma function. Since $\Gamma(x+1) = x\Gamma(x)$, $\mathcal{B}(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$, and $\bar{l}_i = \mathrm{E}[Y_{i1}] = \frac{p_i}{p_i + q_i}$, one can easily show that (14) holds. Note that usually the hypergeometric function is defined by the series $h(a, b, c, x) = \sum_{k=0}^\infty \frac{(a)_k (b)_k}{(c)_k k!} x^k$ with $(a_0) = 1$ and $(a)_k = a(a + 1)\ldots(a + k - 1)$ being rising factorials. One can then show that on the convergence region of the series both representations coincide for $c > a > 0$. Furthermore, it can be verified that (22) is real-valued for $x \in (-\infty, 1)$. If additionally

$$c > a + b \qquad (23)$$

then (22) is also real-valued for $x = 1$. We now consider condition (21) which becomes

$$\int_0^1 y^{p_i-1}(1 - \pi_i y)^\gamma (1 - y)^{q_i-1} dy < \infty. \qquad (24)$$

Condition (23) equals $\gamma > -q_i$ which is always satisfied for $\gamma > 0$. Hence, for $\gamma > 0$ condition (24) is valid for all $\pi \in (-\infty, 1]$. As indicated in Appendix A.1, for $\gamma < 0$ we need to check (24) only for the optimal proportion π_i^*. As long as $\pi_i^* < 1$, this is valid in any case and we are done. Hence, in both cases we only get into trouble if the FOC does not possess a null. As pointed out in Appendix A.1, we do not analyze this situation in detail. Nevertheless, we wish to remark that in the case where the FOC does not possess a null $\pi_i^* = 1$ is indeed the optimal proportion if $\gamma > -q_i$. For $\gamma \leq -q_i$ the problem is ill posed. This, however, can only happen for $\gamma < 0$.

A.4 Optimality with Unhedgeable State Variables

The FOC (17) follows directly from (19) by replacing $\tilde{\pi}_k$ by $\sum_i \pi_i \sigma_{ik}$ and taking derivative with respect to π_i.

The HJB to the concrete problem of Section 7 where the bond dynamics are given by (18), reads

$$0 = \sup_\pi A^\pi G$$

with terminal condition $G(T, x, r) = \frac{1}{\gamma} x^\gamma$ and

$$
\begin{aligned}
A^\pi G &= G_t + x(r + \sum_i \pi_i \chi_i) G_x + (\theta - \kappa r) G_r + 0.5 x^2 (\sum_i \pi_i \sigma_i)^2 G_{xx} + 0.5 \beta^2 G_{rr} \\
&\quad + x \sum_i \pi_i \sigma_i \beta G_{xr} + \sum_j \lambda_j \left\{ \int_E G(t, x(1 - \sum_i \pi_i y_i), z) \, K_j(dy) - G \right\}.
\end{aligned}
$$

In contrast to Section 3 and Appendix A.1, we directly start with the more specific affine separation $G(t, x, r) = \frac{1}{\gamma} x^\gamma e^{A(t,T) - C(t,T) r}$ with $A(T, T) = C(T, T) = 1$, where A and C are supposed to be deterministic functions. A priori, it is not obvious that *deterministic* functions A and C exist such that the corresponding function G fulfils the HJB. Taking derivative with respect to π_i yields the FOC:

$$\chi_i - \sigma_i \beta C = (1 - \gamma) \sigma_i \sum_\nu \pi_\nu \sigma_\nu + \sum_j \lambda_j \int_E y_i (1 - \sum_\nu \pi_\nu y_\nu)^{\gamma - 1} K_j(dy).$$

Therefore, if a deterministic function C exists, the optimal proportion π^* is deterministic as well. Substituting our separation into the HJB leads to the following equation:

$$
\begin{aligned}
A_t + (\kappa C - C_t + \gamma) r &= -\gamma \sum_i \chi_i \pi_i + \theta B + 0.5 \gamma (1 - \gamma) (\sum_i \pi \sigma_i)^2 - 0.5 \beta^2 B^2 \quad (25) \\
&\quad + \gamma \beta B \sum_i \pi_i \sigma_i - \sum_j \lambda_j \left\{ \int_E (1 - \sum_i y_i)^\gamma K_j(dy) - 1 \right\}.
\end{aligned}
$$

The deterministic function $C(t, T) = -\frac{\gamma}{\kappa}(1 - e^{-\kappa(T-t)})$ solves the Riccati equation $\kappa C - C_t + \gamma = 0$. Integrating the remaining part of (25), we obtain the function A which is deterministic because π_i^* is indeed deterministic. Note that A does not show up in the FOC. Hence, for practical purposes, its exact form is not relevant. The only important fact is that A is deterministic. Finally, we give a (not rigorous) argument why stochastic intensities do not admit closed-form solutions. In this case, the optimal proportions π^* are state-dependent leading to a state dependent A which violates the assumption that A is deterministic.

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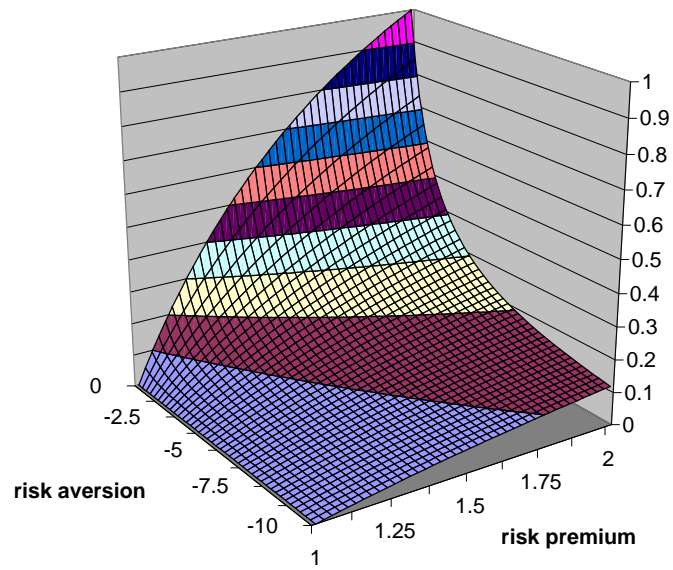
Figure 5.1: The figure details how the optimal portfolio proportion invested in a corporate bond varies with the risk premium η_i and the degree of risk aversion γ if the loss rate is constant with $\bar{l}_i = 0.5$ and joint defaults cannot occur.
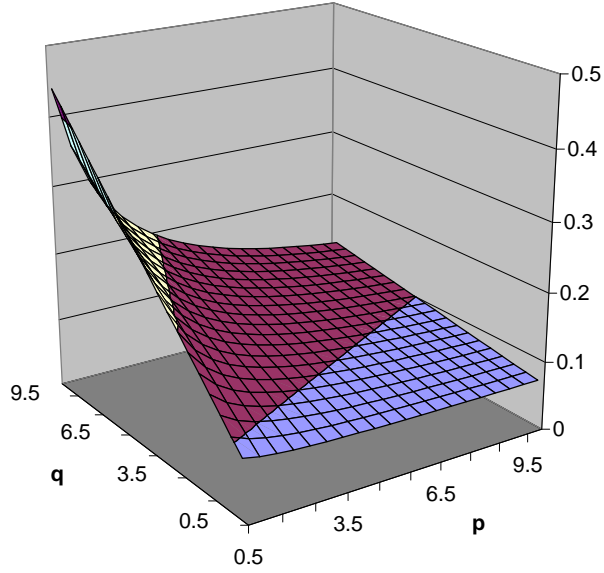
Figure 5.2: The figure details the optimal portfolio proportion invested in a corporate bond. The loss rate is beta distributed with parameters p and q. These parameters are varied in the figure. The risk aversion equals $\gamma = -5$ and the risk premiums $\eta = 1.5$ and $\phi = 1$.
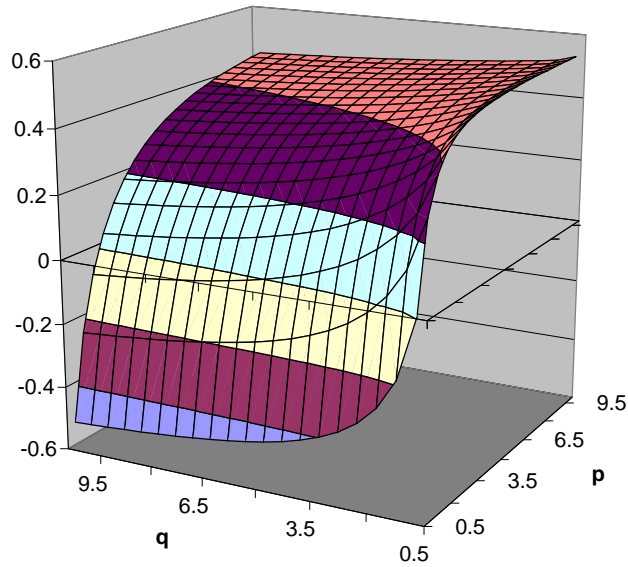
Figure 5.3: The figure compares the optimal strategy from Figure 5.2 with the optimal strategy if constant recovery is assumed. In the latter case the loss rate is set to be the expected value of the beta distribution.
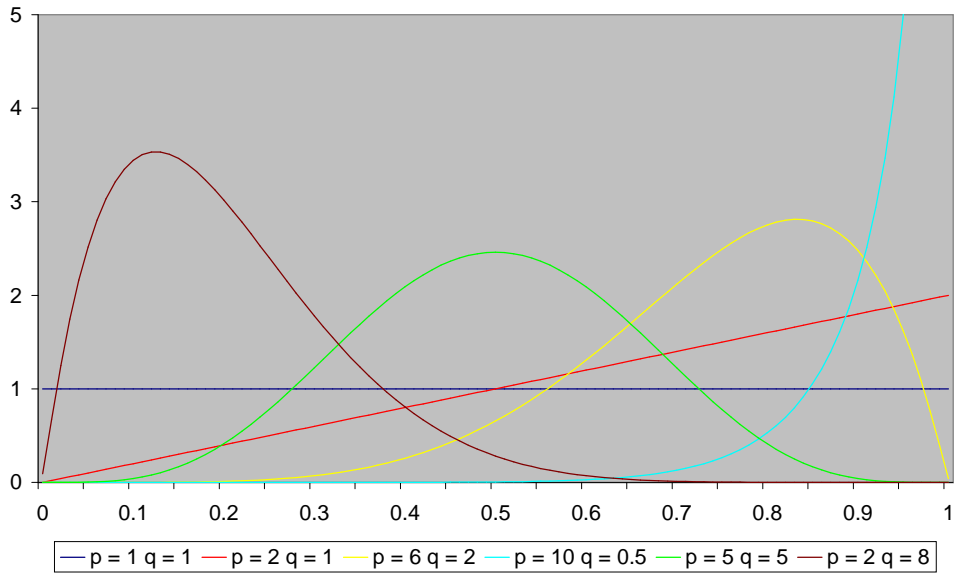
Figure 5.4: The figure details some characteristic shapes of the density of a beta distribution with parameters p and q.
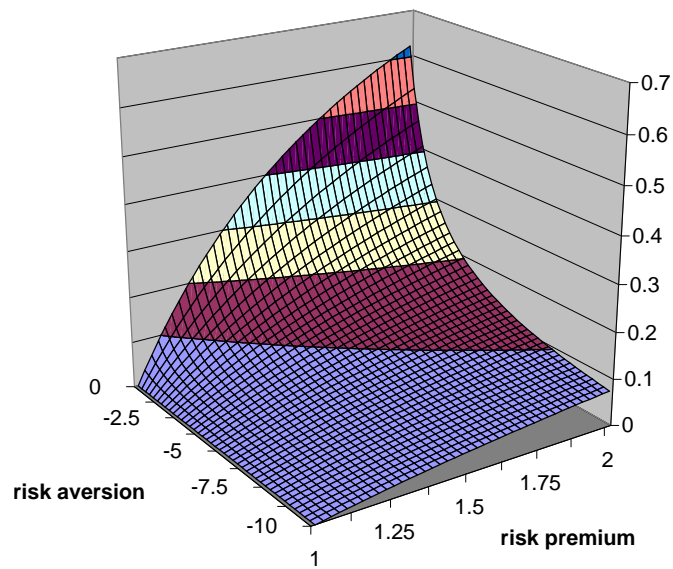
Figure 5.5: The figure shows how the optimal proportion varies with the risk premium and the investor's risk aversion. The parameters of the beta distribution are $p_i = 6$ and $q_i = 2$.
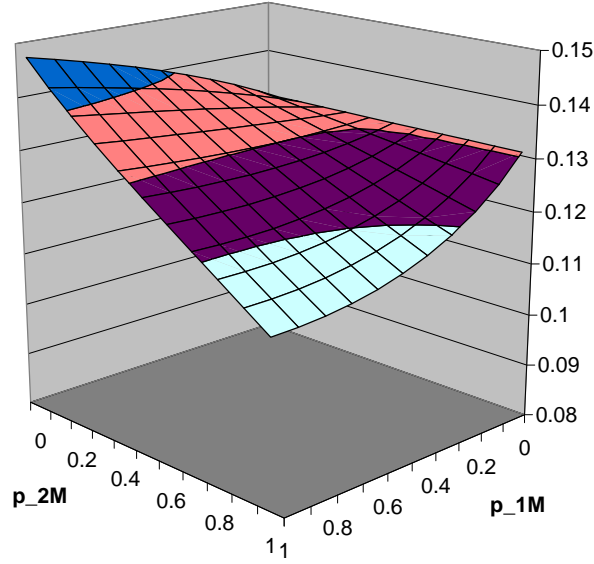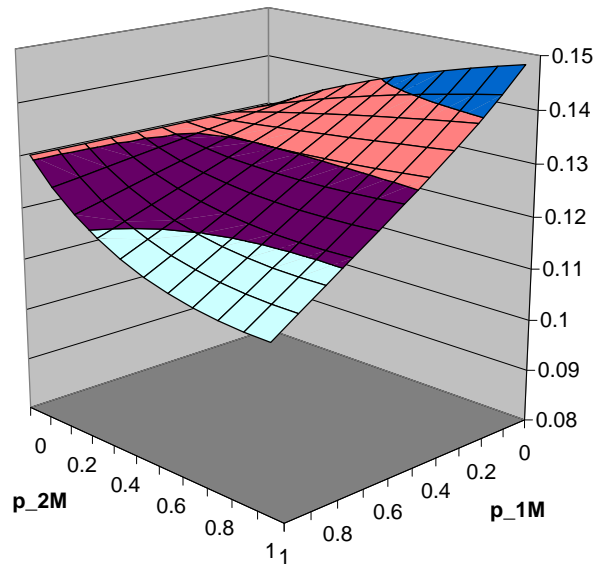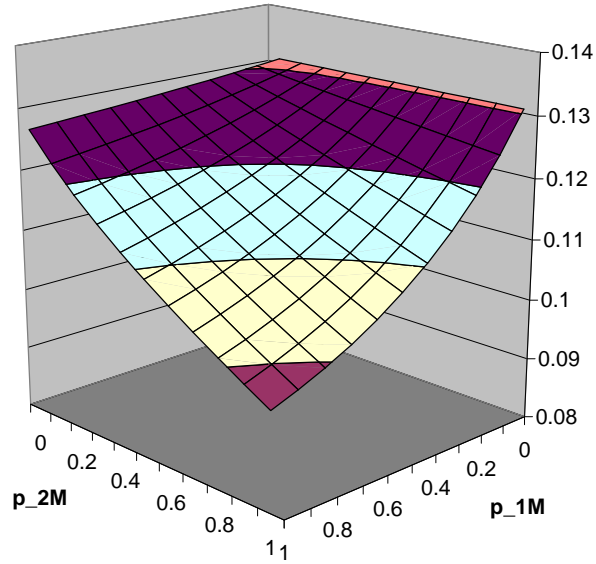
Figure 5.6: The figure shows the optimal proportion invested in the first corporate bond, π_1^*, when the probabilities p_{1M} and p_{2M} are varied. The parameters are chosen as follows: $\eta_M = 1.5$, $\lambda_M = 0.02$, $\gamma = -5$, $l_i = l_{iM} = 0.5$, $\eta_i = 1.5$, $\phi_{iM} = 1.2$, $\lambda_i = 0.05$ for $i \in \{1, 2\}$.

Figure 5.7: The figure shows the optimal proportion invested in the second corporate bond, π_2^*, when the probabilities p_{1M} and p_{2M} are varied. The parameters are identical to the parameters in Figure 5.6.

Figure 5.8: The figure shows the optimal proportion invested in the first corporate bond, π_1^*, when the probabilities p_{1M} and p_{2M} are varied. The parameters are identical to the parameters in Figure 5.6 except for $\eta_M = 1.2$.
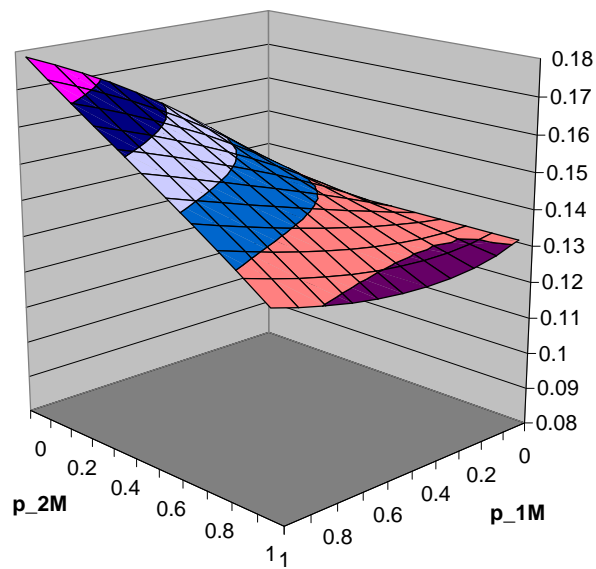

Figure 5.9: The figure shows the optimal proportion invested in the first corporate bond, π_1^*, when the probabilities p_{1M} and p_{2M} are varied. The parameters are identical to the parameters in Figure 5.6 except for $\eta_M = 2.0$.
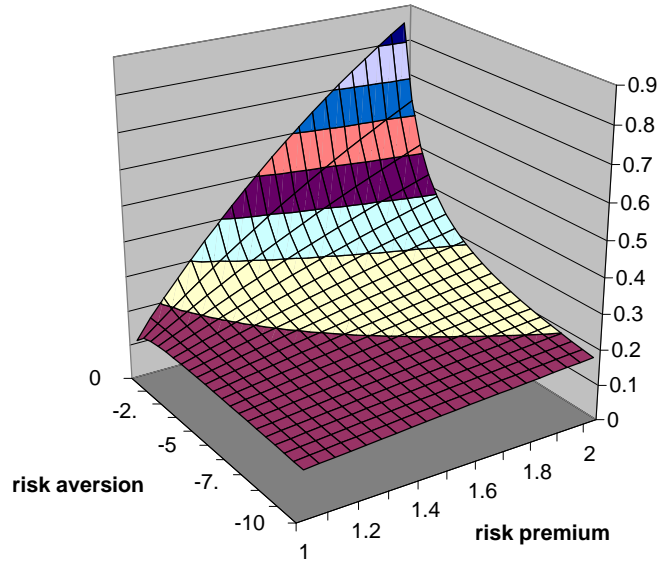
Figure 7.10: The figure shows how the optimal proportion invested in a corporate bond varies with risk aversion γ and risk premium η_i if the investor cannot hedge against shifts in the short rate. The parameters of the two available corporate bonds are $l_i = 0.5$, $\phi_{iM} = 1.2$, $\lambda_i = 0.05$, $T_i = 10$ for $i \in \{1, 2\}$. The investor's time horizon is $T = 5$ years and he is standing at time zero. The parameters of the default-free term structure are $\bar{\eta}^r = 0.00075$, $\beta = 0.02$, and $\kappa = 0.25$.
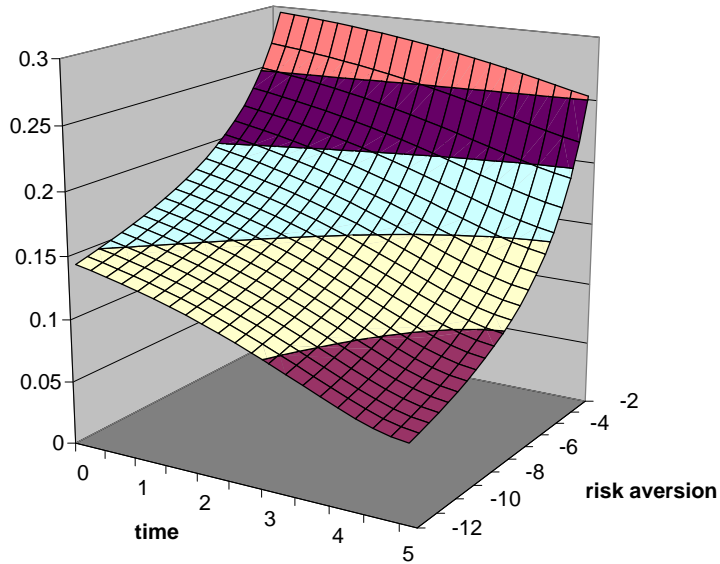

Figure 7.11: The figure shows how the optimal proportion invested in a corporate bond varies with time t and risk premium η_i if the investor cannot hedge against shifts in the short rate. The parameters are the same as in Figure 5.10 except for $T_i = T = 5$ for $i \in \{1, 2\}$.