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Rolf Poulsen† and Kourosh Marjani Rasmussen‡

Abstract

In the basic mean/variance framework, a stock's weight in efficient portfolios goes up if its expected rate of return goes up. In more complicated, realistic portfolio choice problems, surprising effects can occur.

1 Introduction

A Giffen good is one for which demand goes down if its price goes down. At first, it is counter-intuitive that such goods exist at all. But most introductory text-books in economics will tell you that they do; some with stories about potatoes and famine in Ireland, some with first order conditions for constrained optimization and partial derivatives, see Kohli (1986) for review.

We show that in a basic one-period mean/variance-optimization framework (a *Markowitz model* in the following), there are no Giffen goods in the following sense: If a stock's expected rate of return goes up, its weight in any efficient portfolio goes up. This sounds reasonable and seems like a text-book result. We have, however, not been able to find it anywhere.

We use this absence of Giffen goods to give a much shorter (albeit less elementary) proof of a result from Zhang (2004) on when a new asset moves the capital market line.

Skeptics would say that Giffen goods exist in *and only in* economic text-books. Our second contribution is an illustration that Giffen-effects do occur in real-life portfolio

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†Department of Applied Mathematics and Statistics, Institute for Mathematical Sciences, Universitetsparken 5, University of Copenhagen, DK-2100, Denmark. Email: rolf@math.ku.dk

‡Informatics and Mathematical Modelling, Technical University of Denmark, Bldg. 305, DK-2800 Lyngby, Denmark. Email: kmr@imm.dtu.dk

choice problems. Our example considers a mortgager choosing his loan portfolio. We show that for a certain realistic – and completely rational – type of mortgagor, cheaper long-term financing causes him to use more short term financing.

2 (No) Giffen Goods in a Markowitz Model

Let us first consider a model with n risky assets whose expected rate of return is the vector μ and (invertible) covariance matrix Σ. The case with a risk-free asset follows later. The mean/variance efficient portfolios (a portfolio is represented by its weights, ie. by a vector in $\mathbb{R}^n$ whose coordinates sum to 1) can be found by solving

$$\max_w w^\top \mu - \frac{1}{2}\gamma w^\top \Sigma w \ \text{ st } \ w^\top \mathbf{1} = 1,$$

where $\mathbf{1}^\top = (1, \ldots, 1)$.

The efficient portfolios are *parametrized* by $\gamma \in]0; \infty[$. This is mathematically convenient,[1] and economically appealing as γ can be interpreted as a risk-aversion parameter.

For a specific γ, the optimal portfolio is

$$\widehat{w} = \gamma^{-1}\Sigma^{-1}\left(\mu - \eta(\gamma; \mu, \Sigma)\mathbf{1}\right)$$

where

$$\eta(\gamma; \mu, \Sigma) = \frac{\mathbf{1}^\top \Sigma^{-1}\mu - \gamma}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}}$$

can be interpreted as the expected rate of return on $\widehat{w}$'s zero-beta portfolio.

A sensible definition of a Giffen good in this context would an asset, say the i'th, for which $\partial\widehat{w}_i/\partial\mu_i < 0$ for some γ, this meaning that the when the asset's expectede rate of return goes up, its weight in some optimal portfolio goes down. Let us show that there are no such assets and γ's.

As a small trick, we do as Best & Grauer (1991) and consider the problem with the modified expected return vector $\mu + te_i$, where $t \in R$ and is e_i the i'th unit vector. This gives

$$\widehat{w} = \ldots + t\, \underbrace{\gamma^{-1}(\Sigma^{-1}e_i - \frac{e_i^\top \Sigma^{-1}\mathbf{1}^\top}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1})}_{h_1},$$

[1]This paramterizes only the efficient portfolios, not the minimum variance portfolios with expected rates of return below that of the global minimum variance portfolio.

where ... represents a vector that doesn't depend on t. Note that showing that $\partial w_i/\partial \mu_i > 0$ amounts to proving positivity of the ith coordinate of h_1, which we can write as

$$e_i^\top h_1 = \gamma^{-1}\left(e_i^\top \Sigma^{-1} e_i - \frac{(e_i^\top \Sigma^{-1} \mathbf{1}^\top)^2}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}}\right).$$

The positivity then boils down to

$$(e_i^\top \Sigma^{-1} e_i)(\mathbf{1}^\top \Sigma^{-1} \mathbf{1}) - (e_i^\top \Sigma^{-1} \mathbf{1}^\top)^2 \overset{?}{>} 0.$$

Because Σ^{-1} is strictly positive definite and symmetric, we may think of it as a covariance matrix of a random variable, say Z. But the left hand side in the inequality above is then the determinant of the covariance matrix of $(Z_i, \sum_{j=1}^{n} Z_j)$, and thus strictly positive.

The inclusion of a risk-free asset with rate of return r_0 is handled in the same way (easier, actually) with η replaced by r_0 (intuitively because the risk-free asset is any portfolio's zero-beta portfolio).

2.1 On a Result by Zhang

Zhang (2004) considers introducing a new risky asset, the $(n+1)$'st. Assuming that the expected rates of return and covariances of the old risky assets do not change (the new asset *small* in some sense), he then shows that the capital market line will not be moved precisely if this CAPM-type equation holds

$$\mu_{n+1} - r_0 = \frac{\text{cov}(r_{n+1}, r_{tg})}{\text{var}(r_{tg})}(\mu_{tg} - r_0),$$

where 'tg'-subscripts denote quantities related to the tangent portfolio, ie. the mean/variance efficient portfolio with full investment in risky assets ($\mathbf{1}^\top w_{tg} = 1$).[2] Zhang further shows[3] that the $(n+1)$'st asset will have positive weight in the tangent portfolio in the economy if and only if

$$\mu_{n+1} - r_0 > \frac{\text{cov}(r_{n+1}, r_{tg})}{\text{var}(r_{tg})}(\mu_{tg} - r_0).$$

[2]Such a portfolio exists if the risk-free rate is lower than the expected rate of return on the old economy's global minimum variance portfolio, $\mu^\top \Sigma^{-1}\mathbf{1}/\mathbf{1}^\top \Sigma^{-1}\mathbf{1} > r_0$, which is a reasonable economic assumption. Under equilibrium assumptions the tangent portfolio is the market portfolio.

[3]In Theorem 1 in Zhang (2004) the "necessary" condition is under the implicit equilibrium assumption that there is a net demand (among investors) for the new asset, or in other words that it enters the new tangent portfolio with positive weight.

Zhang's proof is elementary, but runs several pages. A shorter one goes like this: It is well-known, see Constantinides & Malliaris (1995, Theorem 4) but it dates back to Roll (1977), that a portfolio w is mean/variance efficient precisely if for any individual asset i we have

$$\mu_i - r_0 = \frac{\text{cov}(r_i, r_w)}{\text{var}(r_w)}(\mu_w - r_0).$$

The capital market line is unchanged if the new asset has weight 0 in the new tangent portfolio. So we must show that the equation above holds for $i = 1 \ldots n+1$ with the role of w played by $(w_{tg}^\top, 0)^\top$. But for $i = 1, \ldots, n$ the equality holds because w_{tg} is mean/variance efficient in the old economy, and for $i = n+1$, it is exactly Zhang's statement. The statement about positive weights for the $(n+1)$'st asset follows immediately from the absence of Giffen goods.

3 A Giffen-Effect in Mortgage Choice

When choosing how to finance your mortgage the basic risk/return trade-off is that short rates are typically lower than long rates (return), but short term financing means that you don't know how much you'll have to pay in the future (risk). That, however, is where the similarity with the Markowitz model from Section 2 ends. This is a highly complex – yet still very real to the individual homeowner – dynamic problem, of which the portfolio choice on the liability side is really only a subproblem. The underlying stochastic model is *complicated* (a fair one word summary of the enormous literature on interest rates), there is a large number of available loan contracts some of which are "non-linear" (embedded options, possibly of American type), dynamic portfolio adjustments are possible, but there are transaction costs and budget/liquidity constraints.

Recently a number of articles and working papers have appeared that take the individual home-owner's point of view on the mortgage market, thus making it an optimal investment/consumption/portfolio choice problem. One strand of literature uses stochastic optimal control theory techniques. The significant progress in interest rate modelling and asset allocation theory means that many interesting problems can be solved analytically, see Hempert, de Jong & Driessen (2005) and the references therein. A different approach is taken in Nielsen & Poulsen (2004) and Rasmussen & Clausen (2005) who use modern multi stage stochastic programming techniques to solve mortgage choice problems numerically; "sophisticated brute-force" one could

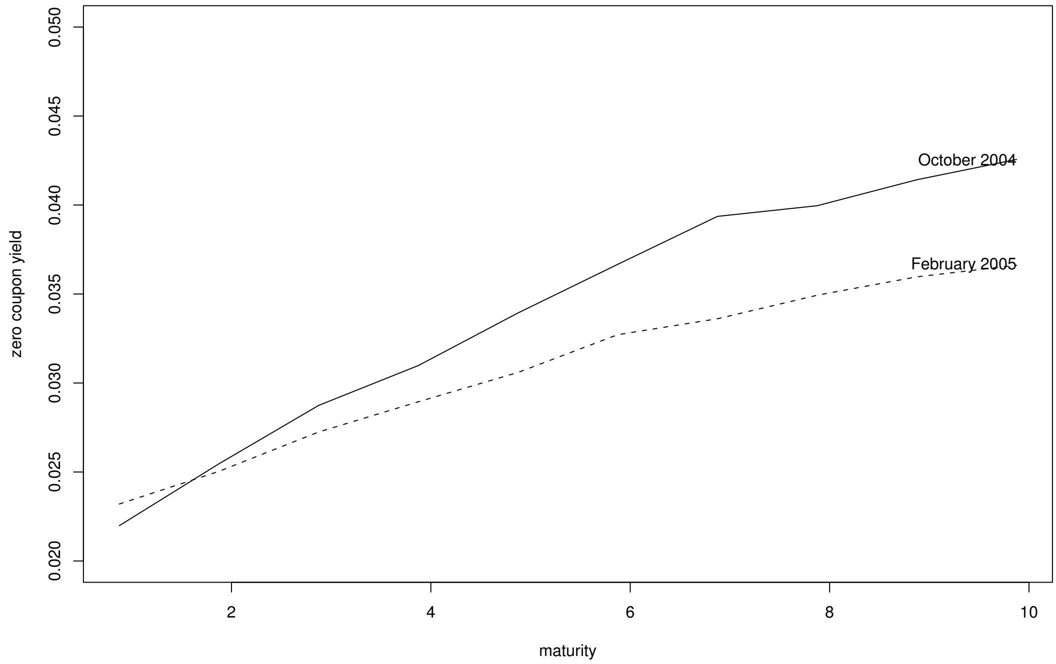

Figure 1: Danish zero coupon yield curves from October 2004 and February 2005.

say. While the first approach is quite elegant, the second is (at the moment) better suited for problems with the imperfections (liquidity constraints, transaction costs) and non-linearities (predominantly the American call-option embedded in most fixed rate mortgage bonds) of the real-life mortgage market, and it is therefore what the following results are based on.

Our aim is to illustrate a Giffen-effect. Let us consider three different types of mortgagors:[4]

- One that focuses solely on minimizing his discounted expected payments; a risk-neutral mortgagor in other words.
- A very risk-averse mortgagor, by which we mean one that minimizes his maximal (discounted aggregate) payments over all scenarios.
- A mortgagor who is risk-neutral but under budget/liquidity constraints.

[4]The first two mortgagor-types actually play no part for the Giffen-argument, but they make nice benchmarks.

Mortgagor type	Optimal initial loan portfolio			
	October 2004		February 2005	
	30Y callable %4	1Y ARM	30Y callable %4	1Y ARM
Risk neutral	0	100%	0	100 %
Constrained risk neutral	42%	58%	28%	72%
Very risk averse	57%	43%	66 %	34%

Table 1: Optimal loan portfolio compositions for three different mortgagor-types and with the two different initial yield curves shown in Figure 1. The only products used in the optimal portfolios are the 1-year adjustable rate loans (ARMs) and 30-year callable annuities with a 4% coupon rate.

For each type, we solved each mortgagor's portfolio problem with the two different initial yield curves shown in Figure 1, but all other quantities kept fixed. The yield curves are from Denmark in October 2004 and February 2005, respectively.[5] We see that the long rates fell, while the short rates were almost unchanged. Duffee (2002) among others document that this is a common type of shift when the yield curve is steep. Table 1 shows the composition of the (initial) optimal portfolios. Only the 1-year floating rate bond and the 30-year 4%-coupon callable annuity (the callable annuity with the lowest yield) are used in the optimal portfolio, although the numerical algorithm allowed for larger a universe of mortgage products.

Row-wise comparisons in Table 1 give few surprises; the risk neutral mortgagor uses full short-term financing, the budget-constrained mortgagor must act a little more conservatively, and the very risk averse mortgagor is prone to fixed rates. It is interesting though that even the very risk averse mortgagor uses a fair share of short-term financing. The reason is that short rates are (historically) low and yield curve quite steep, so short-term financing is very attractive – the mortgagor is risk-averse, not stupid – and the possibility to adjust portfolios dynamically means that he can flee into fixed rates if things take a turn for the worse. Further, the fact most mortgagors diversify their portfolios suggests that there might be a market for mixed loan products (such as a capped short rate loan), possibly with some "free parameter" (a

[5]At these times, one or the other author had a more than academic interest in the Danish mortgage market.

cap level) that individual mortgagor can set. Incidentally, Danish "manufacturers" of mortgage products have noticed that, and are currently introducing a variety of such products. However, for the purposes of this paper the main message lies in a column-wise comparison of the results in Table 1, ie. what the drop in long rates with unchanged short rates does to optimal portfolios. The very risk averse mortgagor uses a larger proportion of fixed rate loans after the drop, (short-term financing becomes relatively less attractive), and the risk-neutral mortgagor doesn't care. But along the "risk-neutral but constrained"-row, we see the promised Giffen-effect: The drop in long rates makes fixed rate financing cheaper, yet this mortgagor now uses more short-term financing. The reason is that he uses fixed rate loans not because he wants to (at heart, he is risk-neutral), but because he has to, just like the Irish peasants in the original Giffen-good example ate a lot of potatoes not because they liked them better than meat, but because surviving was a strong constraint. Now that the "life-necessity" (fixed rate loans) becomes cheaper, the mortgagor has more freedom to do what he really likes.

4 Conclusion

We showed that in a basic one-period mean/variance setting, an asset gets a higher weight in efficient portfolios if its expected rate of return goes up. In more complicated settings Giffen-effects may occur even when agents are completely rational. We illustrated that by showing that if long rates drop (and short rates don't) a liquidity constrained mortgagor should use more short-term financing.

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