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Exotic Options: Proofs Without Formulas

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Abstract

We review how reflection results can be used to give simple proofs of price formulas and derivations of static hedge portfolios for barrier and lookback options in the Black-Scholes model.

1 Introduction

One way to view this paper is as a survey of static hedging results for first generation exotic options: Single-barrier, double-barrier, and lookback options. But there is also a pedagogical point. We use the Black-Scholes model throughout, so pricing formulas are well-known, but this approach yields self-contained and considerably simpler proofs than the long, tedious, and often omitted (hence the title of the paper) ones in many articles and text-books. The driving force behind static hedging results is the decomposition of exotic options into plain vanilla products, (possibly many) piecewise linear functions of powers of Geometric Brownian motion. This makes the implementation easier to structure and less prone to coding-errors, and since the formulas are closed-form, this weighs heavily compared to raw numerical efficiency.

The rest of the paper is organized as follows. In Section 2 we formulate and prove a reflection theorem that we will draw heavily on. Section 3 shows how the reflection theorem gives price formulas for zero-rebate single-barrier options, finds the (often useful) joint distribution of (Geometric) Brownian motion and its minimum as a special case, and shows that static hedges are an easy by-product. The last three sections deal with extensions where closed-

*Rolf Poulsen (`rolf@math.ku.dk`), Department of Applied Mathematics and Statistics, Institute for Mathematical Sciences, Universitetsparken 5, University of Copenhagen, DK-2100, Denmark. This version: September 18, 2004 form solutions and static hedges can still be found with minimal calculations: Section 4 reviews rebates, Section 5 looks at lookbacks, and Section 6 describes double barriers.

2 A reflection theorem in the Black-Scholes model

We consider the Black-Scholes model with constant short term interest rate r and a risky asset whose price is a Geometric Brownian motion,

$$dS(t) = \mu S(t)dt + \sigma S(t)dW^Q(t),$$

under the equivalent martingale measure Q. Using $\mu = r$ gives the classic model for a non-dividend paying stock, $\mu = r - d$ models a stock with dividend yield d, $\mu = 0$ models forwards/futures, and in a currency model μ is the difference between the domestic and the foreign interest rate.

Put

$$p = 1 - \frac{2\mu}{\sigma^2},$$

and consider a simple claim with a pay-off at time T specified by a pay-off function g (a 'g-claim' for short). Its arbitrage-free time-t price is

$$\pi^g(t) = e^{-r(T-t)}\mathbf{E}_t^Q(g(S(T))) = e^{-r(T-t)}f(S(t), t),$$

where of course $f(S(t), t) = \mathbf{E}_t^Q(g(S(T)))$, and the Markov property of S ensures that this is non-deceptive notation. Let $H > 0$ be a constant and define a new function $\widehat{g}$ by

$$\widehat{g}(x) = (x/H)^p\, g\left(H^2/x\right).$$

We call this *the reflection of g through H*. (Putting $g(x) = x$, and making log/log plots of g and $\widehat{g}$ for different p's gives some explanation for term 'reflection'; but generally it's just a word.)

The next theorem shows that g- and the $\widehat{g}$-claims are very closely connected and it is the main source of the subsequent results in the paper.

Reflection theorem *Let the set-up be as above and consider a simple claim with pay-off function $\widehat{g}$. The arbitrage-free time-t price of this $\widehat{g}$-claim is*

$$\pi^{\widehat{g}}(t) = e^{-r(T-t)}(S(t)/H)^p f(H^2/S(t), t). \tag{1}$$

Proof Using the Ito formula (or the well-known form of Geometric Brownian motion) on the process Z defined by

$$Z(t) = \left(\frac{S(t)}{H}\right)^p$$

tells us that

$$dZ(t) = p\sigma Z(t) dW^Q(t),$$

so $Z(t)/Z(0)$ is a positive, mean-1 Q-martingale. Here the exact form of p is needed. The result would not hold if σ were time-dependent or stochastic. (That is, unless it happens that $\mu = 0$, as is the case in Andreasen (2001).) This means that

$$\frac{dQ^Z}{dQ} = \frac{Z(T)}{Z(0)}$$

defines a probability measure $Q^Z \sim Q$. Now use the abstract Bayes formula for conditional means (Karatzas & Shreve (1992, Lemma 3.5.3)) to write the price of the $\widehat{g}$-claim as

$$\pi^{\widehat{g}}(t) = e^{-r(T-t)} \mathbf{E}_t^Q \left(\left(\frac{S(T)}{H}\right)^p g\left(\frac{H^2}{S(T)}\right)\right) = e^{-r(T-t)} \left(\frac{S(t)}{H}\right)^p \mathbf{E}_t^{Q^Z} \left(g\left(\frac{H^2}{S(T)}\right)\right).$$

Girsanov's theorem (Karatzas & Shreve (1992, Theorem 3.5.1)) tells us that

$$dW^{Q^Z}(t) = dW^Q(t) - p\sigma dt$$

defines a Q^Z-Brownian motion. Put $Y(t) = H^2/S(t)$. Then the Ito formula and the definition of W^{Q^Z} gives us that (again, the particular form of p is needed)

$$dY(t) = \mu Y(t) dt + \sigma Y(t) \left(-dW^{Q^Z}(t)\right),$$

which means the law of Y under Q^Z is the same as the law of S under Q. Therefore

$$\mathbf{E}_t^{Q^Z} \left(g(Y(T))\right) = f(Y(t), t) = f(H^2/S(t), t),$$

and the result follows. ■

Reflection principles/results date a long way back in the literature on physics, partial differential equations and stochastic processes, and with Joshi (2003, Theorem 10.2) the formula above can be regarded as finance text-book result. Despite that, it is probably fair to attribute the formula, and certainly the demonstration of its usefulness for option pricing, to

Peter Carr, see Carr & Chou (1997a) and Carr, Ellis & Gupta (1998). The proof given here differs from Carr's original one by having more abstract probability and less (partial) integrations, and is thus more in the vein of Andreasen (2001). What you prefer is a matter of taste, but in a teaching context, it can make a nice example of how general concepts (Markov, Ito, measure-change, Girsanov) can lead a concrete formula.

3 Zero-rebate barrier options: Prices and static hedges

Price formulas for single-barrier zero-rebate options were given in Merton (1973) (where a footnote covers non-zero rebates), so they are as old as the Black-Scholes formula itself. This explains why, in the words of Pelsser (2000), 'most derivatives firms view "single barrier" options nowadays more like vanilla than exotic options'.

But let us show how the reflection theorem makes the pricing very easy and has some interesting 'spin-offs'. To do this consider a zero-rebate knock-out version of a v-claim with barrier B; 'the barrier option' in the following. (We look only at knock-out options; knock-in options are handled by parity.) Such an option has the pay-off

$$v(S(T))\mathbf{1}_{m(T)>B} \text{ (down-and-out case) or } v(S(T))\mathbf{1}_{M(T)<B} \text{ (up-and-out case),}$$

where $m(T) = \min_{u \le T} S(u)$ and $M(T) = \max_{u \le T} S(u)$ denote the running minimum and maximum. The two cases are treated completely similarly, so let us focus on down-and-out options. As input to the reflection theorem we use the g-function defined by

$$g(x) = v(x)\mathbf{1}_{x>B}$$

(which is *not* the pay-off of the barrier option) and B in the place of H. With $\widehat{g}$ denoting the reflection of g through B as defined in Section 2, we can look at the simple claim with pay-off function $h = g - \widehat{g}$; the 'adjusted pay-off' in the language of Carr and Chou. Note that $h(x) = g(x) = v(x)$ if $x > B$ and Equation (1) tells us that the time-t price of the h-claim is

$$\pi^h(t) = e^{-r(T-t)} \left(f(S(t), t) - (S(t)/B)^p f(B^2/S(t), t) \right). \tag{2}$$

In particular, we see that if $S(t) = B$, then the h-claim has a price of 0. Suppose now that we buy the h-claim, sell it again if the stock price hits the barrier, and if this doesn't happen, simply hold it until expiry. If the stock price stays above the barrier, we receive $g(S(T))$

at expiry, otherwise we get 0. In other words, exactly the same as the barrier option, so we can read off its price directly from Equation (2). This result is of course only useful if the g-claim itself has been priced, ie. we know the f-function. But in the case of calls or puts we easily establish the formulas from (for instance) Björk (1998, Chapter 13.2-3). In some cases, one has to be a little careful, because the g-function is *not* the regular pay-off function for the option, v, but rather its truncated-at-B version. But if v is piecewise linear, then so is g, and although the f-function becomes more complicated, we never leave the realm of plain vanilla options. Specifically for a down-and-out put $g(x) = (K - x)^{+}\mathbf{1}_{x>B}$, which is the pay-off from a portfolio containing 1 strike-K put, -1 strike-B puts and $B-K$ (pay-off under) strike-B digital options.

Static hedging

Any simple claim, particularly the h-claim, can be statically hedged by a portfolio of plain vanilla puts and calls (an idea/observation dating back to Breeden & Litzenberger (1978)), so we can devise static hedges for the barrier option. We simply buy puts and calls at time 0 such that the pay-off function is matched at expiry. Note, however, that for a general p, the h-function is not piecewise linear, so the static hedge portfolio involves a continuum of options. In fact, h could be discontinuous at the barrier level; we saw this happen for down-and-out put. Then matching the pay-off becomes problematic in practice. Further, the hedge strictly only semi-static, because it must be unwound (the portfolio of puts and calls sold) if the barrier is hit.

If we assume that $\mu = 0$ and consider a call option, $v(x) = (x - K)^{+}$, then we find that

$$\widehat{g}(x) = (x/B)(B^2/x - K)^{+} = (K/B)(B^2/K - x)^{+},$$

which is the pay-off of K/B puts with strike B^2/K. So the knock-out call can be hedged by buying the plain vanilla call and shorting K/B strike-B^2/K puts. This result, a version of the put/call-symmetry, was a starting point static hedging, see Bowie & Carr (1994) and Carr et al. (1998). The ideas have been extended in Carr & Chou (1997*a*) and Carr & Chou (1997*b*) (the results of which we look at in Sections 5 and 6). A different approach to static hedging based on partial differential equations and calender spreads is presented in Derman, Ergener & Kani (1995), and has since been considered in for instance Chou & Georgiev (1998) (where the two approaches are connected) and Andersen, Andreasen &

Eliezer (2002) (extensions and connection).

Example: Geometric Brownian motion and its minimum

As a special case consider the pay-off function $v(x) = \mathbf{1}_{x \geq K}$, put $g(x) = v(x)\mathbf{1}_{x>B}$ for some B, where $0 \leq B \leq \min(K, S(0))$ (so actually, $v = g$). Then clearly

$$f(x,t) = \Phi\left(\frac{\ln(x/K) + (\mu - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Let π^h denote the price of the associated h-claim option. On the one hand

$$e^{rT}\pi^h(0) = \mathbf{E}^Q(\mathbf{1}_{S(T)\geq K}\mathbf{1}_{m(T)>B}) = Q(S(T) \geq K, m(T) > B),$$

where $m(T) = \min_{u \leq T} S(u)$. On the other hand Equation (2) tells us that

$$e^{rT}\pi^h(0) = \Phi\left(\frac{\ln\left(\frac{S(0)}{K}\right) + (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}\right) - \left(\frac{S(0)}{B}\right)^p \Phi\left(\frac{\ln\left(\frac{B^2}{S(0)K}\right) + (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}\right).$$

Combining these two equations and viewing the right hand side as a function of K and B, determines the joint distribution of Geometric Brownian motion and its minimum. The joint density, say $\phi_{S,m}$ is obtained by differentiating wrt. K and B. Because the logarithm is monotone, we immediately get the distribution of Brownian motion with drift and its minimum, and by symmetry the distribution of Brownian motion and its maximum, see Musiela & Rutkowski (1997, Appendix B.3) or Hunt & Kennedy (2000, Section 2.4.1) for formulas. (Note that we have *not* determined the joint distribution of the 3-dimensional variable $(m(T), S(T), M(T))$, but only two of its 2-dimensional marginals. The simultaneous distribution is considerably more complicated, but can be found using the analysis in Section 6.) Putting $K = B$ gives $(1-)$ the distribution function for the minimum of Geometric Brownian motion. With $\tau = \inf\{u | S(u) = B\}$ denoting the first hitting time to the level B we have that $Q(\tau \leq x) = 1 - Q(\tau > x) = 1 - Q(m(x) > B)$, and by differentiating (wrt. x) we get the first hitting time density.

With the joint density at hand, the density of the minimum conditional on $S(T) = x$ is simply $\phi_{S,m}(x,y)/\phi_S(x)$. This can also be interpreted as the minimum of a Brownian bridge, and can be quite useful for efficient computations, see Beaglehole, Dybvig & Zhou (1997) and Metwally & Atiya (2002).

4 Barrier options with rebates: Prices and static hedges

Using the distribution functions/densities just found, price formulas for barrier options with rebates (as in given in Rubinstein & Reiner (1991) or Haug (1997, Section 2.10.1)) can be derived by direct but sometimes cumbersome calculations. However, using the idea of Carr & Picron (1999), the formulas can be derived without any need for new calculations. This way also makes it clear how to statically hedge the options.

To this end assume that $r \geq 0$, let a be a constant and look at

$$X(t) = e^{-rt} S^a(t) = S^a(0) \exp\left((a(\mu - \sigma^2/2) - r)t + a\sigma W(t)\right).$$

If a solves

$$a(\mu - \sigma^2/2) - r = -a^2\sigma^2/2,$$

then X is a martingale. This quadratic equation has the two roots

$$a_{+/-} = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + \frac{2r}{\sigma^2}},$$

where as earlier $p = 1 - 2\mu/\sigma^2$. The roots are both real, and a_+ is strictly positive, a_- negative (strictly so, unless in the trivial case where $r = \mu = 0$).

Let us now look specifically at a down-and-out option that pays the rebate R when the barrier is hit. (The up-and-out case is treated similarly, except some inequalities must be reversed because $\alpha_- < 0$. Cases where the rebate is paid at expiry are easily dealt with.) Remembering that we have already priced the zero-rebate version, what has to be calculated is $R\mathbf{E}^Q(e^{-r\tau}\mathbf{1}_{\tau<T})$, where $\tau = \inf\{u|S(u) = B\}$. Optional stopping (Karatzas & Shreve (1992, Theorem 1.3.22)) with the bounded stopping time $\tau \wedge T$ gives us that

$$S^{a_+}(0) = X(0) = \mathbf{E}^Q(X_{\tau\wedge T}) = \mathbf{E}^Q(e^{-r\tau}B^{a_+}\mathbf{1}_{\tau<T}) + \mathbf{E}^Q(e^{-rT}S^{a_+}(T)\mathbf{1}_{\inf_{u\leq T} S(u)>B}),$$

meaning that (because $a_+ > 0$ we can rewrite the indicator function)

$$R\mathbf{E}^Q(e^{-r\tau}\mathbf{1}_{\tau<T}) = \frac{R}{B^{a_+}}\left(S^{a_+}(0) - e^{-rT}\mathbf{E}^Q(S^{a_+}(T)\mathbf{1}_{\inf_{u\leq T} S^{a_+}(u)>B^{a_+}})\right).$$

The last term looks new, but isn't. Note first that S^{a_+} is itself a Geometric Brownian motion with drift-rate r (follows immediately because $e^{-rt}S^{a_+}(t)$ is a martingale) and volatility $a^+\sigma$. Second, note that the last term is (minus) the price of a strike-0 call with transformed barrier B^{a_+} and rebate 0 written on the transformed process S^{a_+}. Therefore its price can be found from the zero-rebate down-and-out call with

$$\sigma \rightsquigarrow a_+\sigma, \ \mu \rightsquigarrow r, \ S(0) \rightsquigarrow S^{a_+}(0), \ K = 0, \ \text{ and } B \rightsquigarrow B^{a_+}.$$

So the rebate-R barrier-B call corresponds a long position in the rebate-0 call, R/B^{a_+} units of the a_+-security, and R/B^{a_+} units short in the strike-0, rebate-0 barrier-B^{a_+} call on the a_+-security. Each of these 3 components can be statically hedged by plain vanilla puts and calls.

5 Lookback options: Prices and somewhat static hedges

Barrier option pay-offs depend on the maximum or minimum of the underlying, but only through indicator-functions. Lookback options have pay-offs that depend more explicitly on the extreme value. For instance a lookback call-option pays $S(T) - m(T)$ at time T, where as before m denotes the running minimum. This makes the pricing and hedging more difficult, not least so because we have to be careful when determining the price at a time-point after initiation, where it will depend non-trivially on both the current stock price and the extremum to date. Equipped with the joint distribution of S and the extremum, price formulas can be found by what Björk (1998, Section 13.5) describes as 'a series of elementary, but extremely tedious, partial integrations' (and that's just for time-0 prices). But as shown in Carr & Chou (1997*b*, Section 8), there is an easier way that uses the previous ideas in this paper. To keep the analysis simple, we now look at a contract that pays $m(T)$ at time T, and because there is no real loss of generality in this, we refer to it as *the* lookback option.

As a stepping-stone, we need to consider so-called 1-touch digital options. These are contracts that pay 1 at time T if the underlying touches a barrier B during the life of the option. In a diffusion setting the terms '1' and 'touch' are somewhat superfluous, because touching and crossing is the same thing, and if the underlying touches once, it does so infinitely often. Words aside, the pay-off is (in the relevant down-case)

$$\mathbf{1}_{m(T) \leq B},$$

and that depends on $m(T)$ in exactly the indicator-function way, that allows us to price and statically hedge the contract using previous results. In particular, the 1-touch digital is equivalent to a simple claim with pay-off function

$$h_{\text{OTD}}(x;B) = \begin{cases} 0 & \text{for } x > B \\ 1 + (x/B)^p & \text{for } x \leq B \end{cases}$$

(The 1-touch digital is an in-option, so we first find the adjusted pay-off of the out-version, and then use in/out-parity, which simply amounts to subtracting from 1.)

Next note that

$$m(T) = m(t) - \int_0^{m(t)} \mathbf{1}_{m(T)\leq B} dB, \tag{3}$$

so when viewed from time t, receiving the pay-off $m(T)$ is equivalent to receiving

$$h_m(S(T), m(t)) = m(t) - \int_0^{m(t)} h_{\text{OTD}}(S(T);B)dB = m(t) - \int_{\min(m(t),S(T))}^{m(t)} (1+(S(T)/B)^p)dB.$$

The last term is just an integral of a power function (two, if you're pedantic), and when $p \neq 1$ we get

$$h_m(S(T);m(t)) = \begin{cases} m(t) & \text{for } S(T) > m(t) \\ \frac{2-p}{1-p}S(T) + \frac{(m(t))^{1-p}}{p-1}(S(T))^p & \text{for S(T) } \leq m(t) \end{cases}$$

Viewed from time t — remember that $m(t)$ is known then (but not before) — this pay-off is that of a portfolio of digital options and two types of gap options, the last of which is written on the transformed stock-price S^p. This is easily priced.

In case $\mu = 0$, which corresponds $r = 0$ in no-dividend stock models (standard literature, such as Conze & Wiswanathan (1991), covers this case only in a limiting sense; understandably so), we have $p = 1$, and the h_m-function involves a term of the form $S(T)\ln S(T)$. One completion of the square when valuing this brings us back gap options on Brownian motion which is well-known territory, see Hunt & Kennedy (2000, Appendix 3) for instance. The resulting formula does appear different though; it involves the normal *density* function.

Equation (3) shows that the lookback option can be statically hedged by a portfolio with (a continuum of) 1-touch digitals. Each 1-touch digital can be statically hedged by puts and calls. There is a small complication though: The static hedge portfolio of a given 1-touch digital must be liquidated when its barrier is hit. This means that liquidation takes place every time the minimum changes. And that happens almost certainly on an uncountable set of measure 0. But at least we *have found* a hedge portfolio, the usual (stock, bank-account) hedge portfolio whose existence is ensured by martingale representation is quite tricky to find, see Bermin (2000).

6 Double-barrier options

A double-barrier (knock-out) version of a v-claim pays

$$v(S(T))\mathbf{1}_{m(T)>L, M(T)<U} \text{ at time } T,$$

where as before M and m are, respectively, the running maximum and minimum.
Inspired by the single-barrier analysis let us look at

$$g_0(x) = v(x)\mathbf{1}_{L<x<U}.$$

To price the double-barrier option an immediate idea is to apply the reflection theorem twice to g_0; once with the role of H played by L, once with U as H. This will not quite work, though, because the U-reflected claim will make a non-zero contribution (that varies with time to expiry) along the L-barrier, where the g_0 and the L-reflected claims net out. Therefore the combined portfolio does not have value 0 along the L-barrier (and likewise for the U-barrier). But it is a step in the right direction, because the U-reflected claim only pays off above U, so its price when $S(t) = L(< U)$ is typically small. To finish the job (ie. find a price formula and a static hedge) Carr & Chou (1997*b*, Section 6) has another trick: A series of reflections. (The same idea is used, although not with static hedging in mind, by Hui, Lo & Yuen (2000), and people with experience with partial differential equations will see it as nothing other than probabilistic sugar-coating of the age-old method of images.) Specifically, let us split the positive real line into the regions shown in Figure 1, ie.

$$\text{region}_i = \begin{cases} \left[\left(\frac{U}{L}\right)^{i-1} U; \left(\frac{U}{L}\right)^{i} U\right] & \text{for } i > 0, \\ [L; U] & \text{for } i = 0, \\ \left[\left(\frac{U}{L}\right)^{i} L; \left(\frac{U}{L}\right)^{i+1} L\right] & \text{for } i < 0. \end{cases}$$

Reflection of g_0 through L gives a claim with pay-off function

$$\begin{aligned} \widehat{g}_0(x) &= \left(\frac{x}{L}\right)^p g_0\left(\frac{L^2}{x}\right) = \left(\frac{x}{L}\right)^p v\left(\frac{L^2}{x}\right)\mathbf{1}_{L<L^2/x<U} \\ &= \left(\frac{x}{L}\right)^p v\left(\frac{L^2}{x}\right)\mathbf{1}_{\text{region }_{-1}}(x) := -g_{-1}(x). \end{aligned}$$

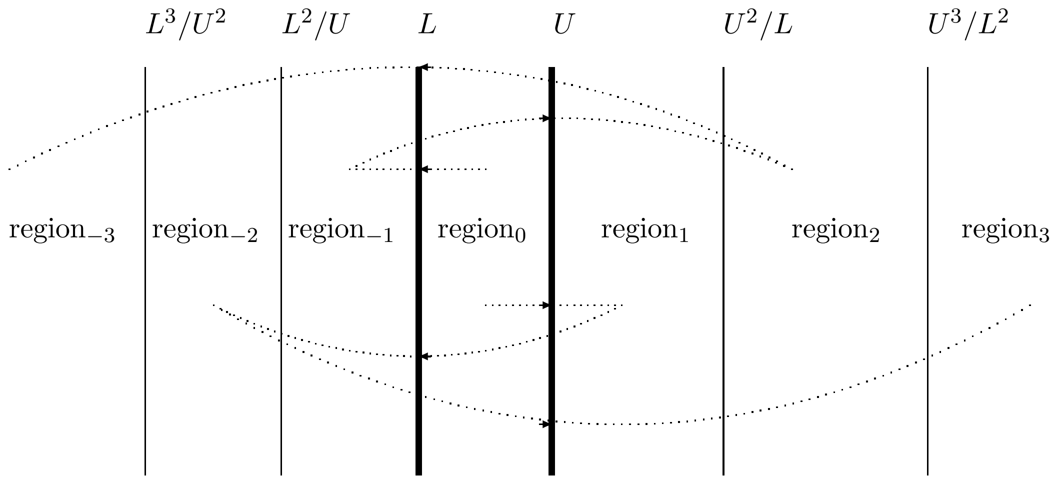

Figure 1: The reflection regions and the connections.

By the reflection theorem

$$\pi^{g_{-1}}(t) = -e^{-r(T-t)}\left(\frac{S(t)}{L}\right)^p f_0\left(\frac{L^2}{S(t)},t\right) := -e^{-r(T-t)} f_{-1}(S(t),t),$$

so as we want, the g_0- and the g_{-1}-claim net out when $S(t)=L$. To remove the g_{-1}-contribution along the U-barrier, we reflect its pay-off through U, ie. look at

$$\begin{aligned}\widehat{g}_{-1}(x) &= \left(\frac{x}{U}\right)^p g_{-1}\left(\frac{U^2}{x}\right) = \left(\frac{x}{U}\right)^p\left(\frac{U^2/x}{L}\right)^p v\left(\frac{L^2}{U^2/x}\right)\mathbf{1}_{L^2/U<U^2/x<L}\\ &= \left(\frac{U}{L}\right)^p v\left(x\frac{L^2}{U^2}\right)\mathbf{1}_{\text{region}_2}(x) := -g_2(x).\end{aligned}$$

The reflection theorem tells us that

$$\pi^{g_2}(t) = -e^{-r(T-t)}\left(\frac{S(t)}{U}\right)^p f_{-1}\left(\frac{U^2}{S(t)},t\right) = e^{-r(T-t)}\left(\frac{U}{L}\right)^p f_0\left(S(t)\frac{L^2}{U^2},t\right),$$

which exactly equals $-\pi^{g_{-1}}$ when $S(t)=U$. We now reflect g_2 through L ($\rightsquigarrow g_{-3}$), g_{-3} through U, and so on. A pattern quickly emerges:

$$\pi^{g_i}(t) = \begin{cases} \left(\frac{U}{L}\right)^{jp} f_0\left(S(t)\left(\frac{L}{U}\right)^{2j},t\right) & \text{for } j=0,2,4,\ldots \\ -\left(\frac{S(t)}{U}\right)^p\left(\frac{L}{U}\right)^{jp} f_0\left(\left(\frac{U^2}{S(t)}\right)\left(\frac{U}{L}\right)^{2j},t\right) & \text{for } i=-1,-3,-5,\ldots\end{cases}$$

We can repeat the procedure starting with a reflection of g_0 through U. This analysis is similar up to some changes of signs on indices and barriers.

This leads to a price formula in terms of an infinite sum ($\sum_{-\infty}^{\infty}$) that only involves f_0 evaluated at appropriate points. This is the best we can realistically hope for given the

L	U	KI-price	CC-price			
			Nterms			
			0	1	2	3
500	1500	66.1289	66.1289	66.1289	66.1289	66.1289
800	1200	22.0820	22.1128	22.0820	22.0820	22.0820
900	1100	1.78676	3.01120	1.78693	1.78676	1.78676
950	1050	0.00057	0.24683	0.07458	0.00207	0.00057

Table 1: Prices of double-barrier knock-out calls. KI-price denotes the true price (calculated from the formula in the Kunitomo-Ikeda paper with 8 terms), CC-price is the price calculated using the algorithm described in this paper ('CC' is for 'Carr-Chou'). The parameters (besides those indicated above) are the same as those used by Pelsser: $S(t) = K = 1000$, $T - t = \frac{1}{2}$, $r = \mu = 0.05$, and $\sigma = 0.2$.

results in the literature, Kunitomo & Ikeda (1992) and Pelsser (2000) for instance. The formula is most easily conveyed in pseudo-code form (to get the pay-off function of the simple claim needed for static replication put $t = T$, ie. call the function with `T=0`). The algorithm for a double knockout call is given below, but to change to a different option type (put, digital) all that has to be altered is the definition of `f0`.

```
DoubleBarrier=function(S , T , L, U , T,  r , mu , sigma,Nterms=5){
  p=1-2*mu/sigma^2
  f0=function(y){
    f0<-BSCall(y,K,T,r,mu,sigma)-BSCall(y,U,[same])-(U-K)*BSDigital(y,U,[same])
  }
  Price=0
  for (j in -Nterms:Nterms){
    EvenTerm=(U/L)^(j*p)*f0((L/U)^(2*j)*S)
    OddTerm=-(S/U)^p*(L/U)^(j*p)*f0((U/L)^(2*j)*(U^2/S))
    Price=Price+EvenTerm+OddTerm
  }
  DoubleBarrier=Price
}
```

Table 1 shows numerical results for the implementation of the algorithm. Only a few terms are needed in the $-\infty$ to $+\infty$ summation to obtain very precise result. The closer L and

U are, the closer the option is to expiry, the closer $S(t)$ is to a boundary, the more terms are needed.

Several extensions follow by handwaving. Exponentially curved boundaries are treated by viewing the options as written on processes with transformed drifts. (The two series of reflections are independent, so the curvatures don't have to be the same.) Transformation also establishes results for regular Brownian motion, for instance its probability of staying within a strip. Non-zero rebates at hit, say R_L and R_U, can be handled using a portfolio (b_{a_+}, b_{a_-}) of the two stationary securities from Section 4 chosen such that

$$b_{a_+} U^{a+} + b_{a_-} U^{a-} = R_U \quad \text{and} \quad b_{a_+} L^{a+} + b_{a_-} L^{a-} = R_L.$$

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