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Abstract

We formulate a new theory of expected utility under uncertainty based on the notion of an event-lattice, which is a natural generalization of a Savage state space. The modelling of uncertainty is based on the idea that the decision maker for each group of related decisions to be taken creates a "small world" which functions as a local state space. We introduce a set of preference axioms similar in spirit to the Savage axioms, and show that they lead to a generalization of the standard von Neumann-Savage theory of expected utility. The generalization allows for an intuitive distinction between risk and uncertainty. In each "small world" risk is described by an additive probability measure; and these local risk measures all appear as restrictions of a common integrated additive expectation functional which is defined on the "grand world", thereby providing numerical expressions to the notion of uncertainty. We illustrate the use of the theory for the Ellsberg paradox and for some portfolio decisions which cannot be captured by the standard von Neumann-Savage theory.

JEL classification: D8 and G12.

Key words: Expected utility, decision making under uncertainty.

*We would like to dedicate this paper to the memory of Birgit Grodal and Karl Vind.

1 Introduction

In this paper we formulate a new theory of expected utility. The modelling of risk and uncertainty, and the differentiation between these two concepts, is based on the idea that the decision maker for each group of similar or related decisions to be taken creates a ”small world” or ”preparation” of the event-lattice such that only events relevant for the specific actions are considered. The notion of small worlds and their use in modelling decision making is described in Savage (1954)[1]:

> *In the sense under discussion a smaller world is derived from a larger by neglecting some distinctions between states, not by ignoring some states outright. The latter sort of contraction may be useful in case certain states are regarded by the person as virtually impossible so that they can be ignored.* *(page 9)*
>
> *The point of view under discussion may be symbolized by the proverb ”Look before you leap”, and the one to which it is opposed by the proverb, ”You can cross that bridge when you come to it”. ... One must indeed look before he leaps, in so far as the looking is not unreasonably time-consuming and otherwise expensive; but there are innumerable bridges one cannot afford to cross, unless he happens to come to them.* *(page 16)*

The starting point of our investigation is Savage’s notion of the ”states of the world” and its associated event structure which is a lattice satisfying very natural conditions. We propose to make this ”lattice of events” the primitive datum of decision theory under uncertainty rather than the state space from which it is derived.

This approach gives us the possibility to study decision theory even when there is no underlying state space and the event-lattice is introduced only as an abstract object specified in a certain way. The event-lattices constructed from a state space may then be characterized by an additional orthogonality condition.

The main contribution of this paper is the development of a theory of expected utility based on the notion of an event-lattice rather than on the notion of a state space. We achieve this by formalizing the notion of a ”small world” and then providing suitable axioms on preferences which in each ”small world” are indistinguishable from Savage’s axioms but provide for a consistent integration into a ”grand world” without compromising linearity of ”integrals”.

[1] The page numbers refer to the 1972 edition of the book.

Our approach has the advantage that we maintain Savage's axioms and the use of ordinary probability measures in each "small world", while we still are able to capture decision making under uncertainty in the "grand world" using an expected utility formulation. This is in contrast to the notable contribution by Schmeidler (1989) who, while maintaining the use of a state space in the "grand world", shows that by weakening the Savage axioms it is possible to formulate an expected utility theory which allows for a distinction between risk and uncertainty aversion in a consistent way. Schmeidler's result is however achieved at the "cost" of using non-additive probability measures in the formulation of expected utility.

Our theory does not break with the expected utility methodology currently used in economics, and in addition it allows us to capture and give numerical expressions to the notions of ambiguity or uncertainty in personal decisions, while maintaining the use of standard measures of probability in each "small world". Thus we are able to follow Knight (1921) and distinguish between "risk" involving events with clearly defined underlying probabilities and "uncertainty" involving vague (or ambiguous) events for which ordinary probabilities are not defined.

Savage (1954) demonstrated that preferences satisfying a plausible set of axioms correspond to the situation where the decision maker is having a subjective probability distribution over a state space. These subjective probabilities are then used, together with a fixed utility function defined on the set of consequences, to represent the given set of preferences.

However, Savage was keenly aware that the axioms he uses may be more appropriate when decision making takes place in a small world:

> *Though the "Look before you leap" principle is preposterous if carried to extremes, I would none the less argue that it is the proper subject of our further discussion, because to cross one's bridges when one comes to them means to attack relatively simple problems of decision by artificially confining attention to so small a world that the "Look before you leap" principle can be applied there. I am unable to formulate criteria for selecting these small worlds and indeed believe that their selection may be a matter of judgment and experience about which is it impossible to enunciate complete and sharply defined general principles, though something more will be said in this connection in § 5.5.* (page 16)

> *Any claim to realism made by this book - or indeed by almost any theory of personal decision of which I know - is predicated on the idea that some of the individual decision situations into*

which actual people tend to subdivide the simple grand decision do recapitulate in microcosm the mechanism of the idealized grand decision. *(page 83)*

Nevertheless, despite his hesitations and the extensive discussion of the notion of "small worlds" and the link between these and the "grand world" Savage uses a framework in which the states of the world are partitioned, without considering alternative descriptions.

Since Ellsberg (1961) a substantial empirical literature has documented that subtle differences between sources of risk or uncertainty can lead a decision maker to treat them differently. Reflecting this, there has in the last 10 to 20 years been a growing theoretical literature which has focused on modelling decision making under uncertainty. For surveys of this literature see Karni and Schmeidler (1991) and more recently Wakker (2004).

The objective of most of this literature has been to modelling decision making which allows for a distinction between risk and uncertainty in the spirit of Knight (1921). Early contributions include Fellner (1961) and Quiggin (1982). In general this literature, including in particular Schmeidler (1989), has focused on weakening the Savage axioms. Others including Vind (2003) has chosen to abandon the Savage axioms - in the case of Vind (2003) totality of preferences. The objective has been to make it possible to formulate a more general decision theory which allows for a distinction between risk and uncertainty aversion in a consistent way. Schmeidler's result is however achieved at the "cost" of using non-additive probability measures in the formulation of expected utility which to some may appear difficult to justify. For a recent survey of applications of this approach see Mukerji and Tallon (2004).

However, to our knowledge, there is only a limited collection of papers that do not rely on the use of a Savage state space when modelling decision making under uncertainty.

Gilboa and Schmeidler (2001) models subjective distributions without relying on a state space by modelling preferences over acts conditional on bets. However, they assume the existence of an outcome-independent linear utility on bets and derive subjective probabilities on the outcome that is consistent with expected value maximizing behavior. Karni (2004) develops an axiomatic theory of decision making under uncertainty that dispenses with the Savage state space. The results are subjective expected utility models with unique, action-dependent, subjective probabilities. Conditions are provided under which the decision-making process may be decomposed into two cognitive sub-processes. The first is the assessment of the likely realization of different "effects" conditional on the actions. The second is the evaluation of the consequences, that is, effect-payoff pairs, that may result from the implementation of those actions. The two processes are then integrated to produce a value, that is, the subjective expected utility corresponding to each action-bet pair. The result is a subjective expected utility theory that does not invoke the notion of states of the world to resolve uncertainty. However, the modelling does not rule out that decision makers may mentally construct a state space to help organize their thoughts - but it does not require that they do. Thus, "traditional" theory can be embedded in this framework by defining the action-bet pairs as random variables on the state space and, for every given action, assigning to the effects the probabilities of the events in the state space in which these effects are realized under the given action.

Sagi and Hong (2003) assumes a Savage state space, but provide a set of axioms which allows for "small worlds" or domains of events that arise endogenously according to the preferences of the decision maker and the manner in which distinct sources of uncertainty are treated. The authors also show that, given weak assumptions, preferences restricted to a domain exhibit probabilistic sophistication. This allows for an endogenous formulation of the two-stage approach and a distinction between risk and uncertainty in a setting with a Savage state space. However, as opposed to Savage the question of consistent extension of decision making across distinct "small worlds" is deferred, as the approach taken is to model decisions as generally taking place at the "small world" level.

Our work also has links to discussions of the foundation of quantum physics, in particular quantum-mechanical derivations of probability which is closely modelled on the classical theory. In particular Wallace (2003a) and Wallace (2003b) discuss and explain how decision theory can be applied to quantum-mechanical contexts.

The paper is structured as follows: In the next section the "lattice of events" is introduced and the links to the Savage state space are discussed. Section 3 contains the main results of the paper. The final section contains two examples, the Ellsberg paradox and a portfolio decision problem, that illustrate the use of the theory.

2 The lattice of events

When uncertainty is modelled by a state space an event is represented by a subset, and the event occurs, or obtains in the language of Savage, if and only if the true state of nature is contained in the subset representing the event. More precisely, the events are represented by the elements in a σ-algebra $\mathcal{F}$ of subsets of the state space Ω.

The set of events naturally form a lattice by introducing the majorant event $A \vee B = A \cup B$ and the minorant event $A \wedge B = A \cap B$ for arbitrary events $A, B \in \mathcal{F}$. The lattice is ordered by setting $A \leq B$ if $A \subseteq B$. Following Savage we introduce the vacuous event $0 = \emptyset$ and the universal (sure) event $1 = \Omega$ and summarize:

(1) There are elements 0 and 1 in $\mathcal{F}$ such that $0 \leq A \leq 1$ for all $A \in \mathcal{F}$.

(2) To arbitrary events $A, B \in \mathcal{F}$ there is a minorant event $A \wedge B \in \mathcal{F}$ with the property that $C \leq A \wedge B$ for any event $C \in \mathcal{F}$ with $C \leq A$ and $C \leq B$.

(3) To arbitrary events $A, B \in \mathcal{F}$ there is a majorant event $A \vee B \in \mathcal{F}$ with the property that $A \vee B \leq C$ for any event $C \in \mathcal{F}$ with $A \leq C$ and $B \leq C$.

It is natural to consider B a larger or more comprehensive event than A if $A \leq B$, since we then know for sure that B obtains if A obtains.

The lattice of events has even more structure if we also consider the bijective mapping $A \to A^\perp$ of $\mathcal{F}$ onto itself defined by letting $A^\perp$ be the set complementary to A. We may then list the following additional properties:

(4) $A \leq B \;\Rightarrow\; B^\perp \leq A^\perp \quad$ for all $A, B \in \mathcal{F}$.

(5) $A^{\perp\perp} = A \quad$ for all $A \in \mathcal{F}$.

(6) $A \wedge A^\perp = 0 \quad$ for all $A \in \mathcal{F}$.

(7) $A \vee A^\perp = 1 \quad$ for all $A \in \mathcal{F}$.

The mapping $A \to A^\perp$ is sometimes called an orthocomplementation of $\mathcal{F}$.

One realizes that $\mathcal{F}$ is closed under both finite and countable infinite formation of majorants and minorants, but it is conceptually unattractive to put any restrictions on the formation of majorants and minorants. We will endeavour to be able to consider the union of any family of events as an event in itself. To this end we need a little more structure and suppose that $(\Omega, \mathcal{F}, \mu)$ is a probability space such that Ω is a locally compact, second countable Hausdorff space and μ is the completion of the Riesz representation of a Radon measure. With these extremely weak extra conditions in place we may appeal to Lebesgue's theorem of dominated convergence and obtain:

(8) To any family $(A_i)_{i \in I}$ of events in $\mathcal{F}$ there is a minorant event $\wedge_{i \in I} A_i$ and a majorant event $\vee_{i \in I} A_i$ in $\mathcal{F}$.

The introduction of an event-lattice may at first seem to be an excessive generalization, since an analysis based on counting naturally leads to a probabilistic description on a state space. However, the state space is an artifact of the counting process itself, and counting may not be appropriate when agents are confronted with unique choices without the possibility of regret. It takes a leap of faith to assume that the probabilistic description so suitable for the analysis of the past can be extended to a valid description of the uncertain future.

It is the belief of the authors that the lattice of events, rather than the state space from which it is constructed, should be the primitive datum of decision theory. We demonstrate that the theory of expected utility can be formulated uniquely in terms of an event-lattice, hence the theory covers also cases where the event-lattice is not induced by a state space. We underpin our approach by demonstrating that we readily and naturally can explain the behaviour of the agents in Ellsberg's paradox and uncertainty aversion without compromising additivity of "integrals". A formulation of arbitrage free valuation using an event-lattice approach can be found in Hansen (2003).

Definition 1 Let $\mathcal{F}$ be an ordered lattice equipped with an orthocomplementation map $A \to A^\perp$ satisfying conditions (1) to (8). We say that events A and B in $\mathcal{F}$ are mutually exclusive if $A \wedge B = 0$, and orthogonal if $A \leq B^\perp$.

The two notions introduced in the definition above coincide if the lattice is induced by a state space. For a general lattice orthogonal events are mutually exclusive, cf. Hansen (2003, Proposition 3.1 (iv)), but it may happen that mutually exclusive events are not orthogonal. The second author proved the following theorem, cf. Hansen (2003, Theorem 4.3 and Corollary 4.5).

Theorem 1 Let $(\mathcal{F}, H)$ be a pair consisting of a separable Hilbert space H and a family $\mathcal{F}$ of (self-adjoint) projections on H satisfying:

(i) The zero projection on H (denoted 0) and the identity projection on H (denoted 1) are both in $\mathcal{F}$.

(ii) $1 - P \in \mathcal{F}$ for arbitrary $P \in \mathcal{F}$.

(iii) $P \wedge Q \in \mathcal{F}$ for arbitrary $P, Q \in \mathcal{F}$.

(iv) $\sum_{i \in I} P_i \in \mathcal{F}$ for any family $(P_i)_{i \in I}$ of mutually orthogonal projections in $\mathcal{F}$.

If we set $P^\perp = 1 - P$ for arbitrary $P \in \mathcal{F}$ then $\mathcal{F}$ satisfies the conditions (1) through (8).

We recall that the minorant projection $P \wedge Q$ is the projection on the intersection of the ranges of P and Q, and that the majorant projection $P \vee Q$ is the projection on the closure of the sum of the ranges of P and Q. The order relation $P \leq Q$ is the natural order relation for projections.

Definition 2 Let $\mathcal{F}$ be a lattice of projections on a separable Hilbert space H as specified in Theorem 1. We say that $\mathcal{F}$ is induced by a state space if

(i) The projections in $\mathcal{F}$ commute.

(ii) There exists a probability space $(\Omega, \mathcal{S}, \mu)$ such that Ω is a locally compact, second countable Hausdorff space, and μ is the completion of the Riesz representation of a Radon measure.

(iii) There is an isomorphism

$$\Phi : L(\mathcal{F}) \rightarrow L^\infty(\Omega, \mathcal{S}, \mu)$$

from the norm closed linear space $L(\mathcal{F})$ generated by $\mathcal{F}$.

(iv) An operator $P \in L(\mathcal{F})$ is in $\mathcal{F}$ if and only if $\Phi(P)$ is (the equivalence class of) the indicator function for a (measurable) set in $\mathcal{S}$.

We realize that if an event-lattice $(\mathcal{F}, H)$ is induced by a state space Ω, then the set of events cannot be distinguished from the lattice of measurable sets on $(\Omega, \mathcal{S}, \mu)$. We finally consider an additional condition for an event-lattice $(\mathcal{F}, H)$.

(9) $A \wedge B = 0 \Rightarrow A \leq B^\perp$ for all $A, B \in \mathcal{F}$.

The condition expresses that mutually exclusive events are orthogonal. It is a corollary to the theory to be exhibited in the next section that axiom (9) prevents us from considering mutually exclusive events which cannot be fitted into the same "small world". The second author proved the following characterization, cf. Hansen (2003, Corollary 4.5):

Theorem 2 *An event-lattice $(\mathcal{F}, H)$ as specified in Theorem 1 is induced by a state space if and only if condition (9) is satisfied.*

3 Expected utility theory with "small worlds"

Let $(\mathcal{F}, H)$ be an event-lattice. To avoid technical difficulties we assume in this paper that the Hilbert space H is of finite dimension. This corresponds to a finite state space in the standard model.

Definition 3 *A preparation or context of $(\mathcal{F}, H)$ is a set $\{P_1, \ldots, P_n\}$ of projections in $\mathcal{F}$ with sum $P_1 + \cdots + P_n = 1$. The set of preparations of $(\mathcal{F}, H)$ is denoted by $P(H)$.*

One may consider a preparation (or context) to be a subdivision of the sure event into those constituent parts (risky events) which are pertinent for a particular set of acts. If the event-lattice is induced by a state space (the standard model), a preparation is nothing but a partition of the state space. The notion of a preparation thus fits neatly into Savage's concept of "neglecting some distinctions between states".

The events in a preparation are mutually exclusive and their majorant event is the sure event. Therefore exactly one of the events obtains. A preparation may in this way function as a state space for a "small world" simply by letting the states of the "small world" be the events appearing in the preparation. The obtaining event then plays the role of the true state of nature in the "small world". The set of preparations $P(H)$ thus becomes a set of state spaces, each describing a certain part of the "grand world" specified by the event-lattice $(\mathcal{F}, H)$.

Note that the lattice operations $\vee$ (majorant) and $\wedge$ (minorant) may join together events coming from different "small worlds".

Definition 4 *An act is a pair (α, f) consisting of a preparation $\alpha \in P(H)$ and a mapping $f : \alpha \to C$, where C is the set of consequences.*

The set of consequences C is assumed to have an affine structure. This implies that we may define the convex combination (α, h) of acts (α, f) and (α, g) with the same preparation $\alpha \in P(H)$ by setting

$$h(P) = tf(P) + (1-t)g(P) \qquad t \in [0, 1]$$

for each event $P \in \alpha$.

To a given event-lattice $(\mathcal{F}, H)$ and set of consequences C, we consider a set L of actions. For each preparation $\alpha \in P(H)$ we assume that the subset $L_\alpha \subseteq L$, consisting of the acts in L with preparation α, is convex and includes the set of constant actions. The primitive datum of the utility theory is a binary preference relation $\succeq$ over the set L.

We may for any consequence $c \in C$ and preparation $\alpha \in P(H)$ consider the constant action $(\alpha, c) \in L_\alpha$ defined by setting $c(P) = c$ for every $P \in \alpha$. Note that for each preparation $\alpha \in P(H)$, the preference relation on L induces a preference relation $\succeq_\alpha$ on C by setting

$$c \succeq_\alpha d \qquad \text{if} \quad (\alpha, c) \succeq (\alpha, d)$$

for consequences c and d in C.

We shall discuss a number of possible axioms for a given binary preference relation $\succeq$ over L.

(i) **Totality**: For all (α, f) and (β, g) in L we have either $(\alpha, f) \succeq (\beta, g)$ or $(\beta, g) \succeq (\alpha, f)$.

(ii) **Transitivity**: If $(\alpha, f) \succeq (\beta, g)$ and $(\beta, g) \succeq (\gamma, h)$ for actions (α, f), (β, g) and (γ, h) in L, then $(\alpha, f) \succeq (\gamma, h)$.

A total and transitive order relation is also called a weak ordering. If $(L, \succeq)$ satisfies the axioms (i) and (ii), then it follows that the induced order relation $\succeq_\alpha$ on C, for each preparation $\alpha \in P(H)$, enjoys the same properties.

(iii) **Independence**: Let (α, f), (α, g) and (α, h) be acts in L_α for a preparation $\alpha \in P(H)$. Then

$$(\alpha, f) \succ (\alpha, g) \quad \text{implies} \quad (\alpha, tf + (1-t)h) \succ (\alpha, tg + (1-t)h)$$

for each $t \in (0, 1]$.

(iv) **Continuity**: Let (α, f), (α, g) and (α, h) be acts in L_α for a preparation $\alpha \in P(H)$. If $(\alpha, f) \succ (\alpha, g)$ and $(\alpha, g) \succ (\alpha, h)$, then

$$(\alpha, tf + (1-t)h) \succ (\alpha, g) \quad \text{and} \quad (\alpha, g) \succ (\alpha, sf + (1-s)h)$$

for some numbers $t, s \in (0, 1)$.

(v) **Monotonicity**: Let (α, f) and (α, g) be acts in L_α for a preparation $\alpha \in P(H)$. If $f(P) \succeq_\alpha g(P)$ for each event $P \in \alpha$, then $(\alpha, f) \succeq (\alpha, g)$.

(vi) **Non-degeneracy**: To each preparation $\alpha \in P(H)$ there exist acts (α, f) and (α, g) in L_α such that $(\alpha, f) \succ (\alpha, g)$.

Note that the axioms (iii) through (vi) apply to each preparation or "small world" at a time. Since each preparation is viewed as a context dependent state space, and the said axioms coincide with the axioms considered by Savage, we immediately obtain the following result.

Theorem 3 (Savage) *Assume that the preference relation $\succeq$ satisfies the axioms (i) through (vi). Then there exists for each preparation $\alpha \in P(H)$ a map $u_\alpha \colon C \to \mathbf{R}$, unique up to an affine transformation, such that*

$$c \succ_\alpha d \quad \textit{if and only if} \quad u_\alpha(c) > u_\alpha(d)$$

for consequences $c, d \in C$. Furthermore, there exists a unique subjective probability distribution E_α over α such that

$$(\alpha, f) \succ (\alpha, g) \quad \textit{if and only if} \quad U_\alpha(f) > U_\alpha(g)$$

for arbitrary acts (α, f) and (α, g) in L_α, where the expected utility function U_α is defined by setting

$$U_\alpha(h) = \sum_{i=1}^{n} E_\alpha(P_i) u_\alpha(h(P_i))$$

for preparations $\alpha = \{P_1, \ldots, P_n\} \in P(H)$ where n is some natural number depending on α, and acts $(\alpha, h) \in L_\alpha$.

For a proof see Fishburn (1970, Chapter 14), Mas-Colell and Whinston (1995, Chapter 6) or the discussion in Schmeidler (1989, page 578). The proof is facilitated by the fact that each local state space or preparation $\alpha \in P(H)$, in the present version of the theory, is a finite set.

Corollary 1 *Any act (α, f) in L with preparation $\alpha \in P(H)$ is under the conditions of Theorem 3 equivalent to a constant act (α, c).*

Proof: Set $c = E_\alpha(P_1) f(P_1) + \cdots + E_\alpha(P_n) f(P_n) \in C$ where the preparation $\alpha = \{P_1, \ldots, P_n\}$, and let (α, c) be the act with constant consequence c. Since $(\alpha, c) \in L_\alpha$ and u_α is unique up to an affine transformation we obtain $U_\alpha(\alpha, c) = u_\alpha(c) = U_\alpha(\alpha, f)$. **QED**

We shall now consider axioms that describe the relationship between acts with possibly different preparations.

(vii) **Indifference**: For each consequence $c \in C$ and preparations $\alpha, \beta \in P(H)$ the constant acts (α, c) and (β, c) are equivalent; $(\alpha, c) \sim (\beta, c)$.

With the indifference axiom in place (which we henceforth will assume), the induced order relations $\succeq_\alpha$ on C become equivalent for all preparations $\alpha \in P(H)$. We may therefore suppress the subscript in $\succeq_\alpha$ and just write

$$c \succeq d \qquad \text{if} \quad (\alpha, c) \succeq (\beta, d)$$

for consequences c and d in C, and preparations $\alpha, \beta \in P(H)$.

Lemma 1 *Assume that the preference relation $\succeq$ satisfies the axioms (i) through (vii). Then there exists a common utility function $u : C \to \mathbf{R}$, unique up to an affine transformation, such that*

$$c \succ d \quad \textit{if and only if} \quad u(c) > u(d)$$

for consequences $c, d \in C$.

Proof: The statement follows since all the preference relations $\succeq_\alpha$, for $\alpha \in P(H)$, are equivalent. **QED**

(viii) **Separation**: Let $\alpha, \beta \in P(H)$ be preparations with a common event $P \in \alpha \cap \beta$. There exist equivalent actions (α, f) and (β, g) in L and non-equivalent consequences $a, b \in C$ such that

$$f(P) \sim g(P) \sim a \quad \text{and} \quad f(Q) \sim g(R) \sim b$$

for every $Q \in \alpha \backslash \{P\}$ and $R \in \beta \backslash \{P\}$.

Lemma 2 *Assume that the preference relation $\succeq$ satisfies the axioms (i) through (viii). If two preparations $\alpha, \beta \in P(H)$ have a common event $P \in \alpha \cap \beta$, then $E_\alpha(P) = E_\beta(P)$.*

Proof: Consider two preparations $\alpha, \beta \in P(H)$ with a common event $P \in \alpha \cap \beta$. By the separation axiom there exist equivalent actions (α, f) and (β, g) in L and non-equivalent consequences $a, b \in C$ such that

$$f(P) \sim g(P) \sim a \quad \text{and} \quad f(Q) \sim g(R) \sim b$$

for every $Q \in \alpha \backslash \{P\}$ and $R \in \beta \backslash \{P\}$. Since a and b are non-equivalent we may assume $u(a) < u(b)$. We set

$$d = E_\alpha(P)a + (1 - E_\alpha(P))b \in C$$

and calculate the α-utility

$$E_\alpha(\alpha, d) = u(d) = E_\alpha(P)u(a) + (1 - E_\alpha(P))u(b) = E_\alpha(\alpha, f)$$

of the constant action (α, d), and observe that (α, f) is equivalent to (α, d). Since the constant actions (α, d) and (β, d) are equivalent by the indifference axiom, we conclude that (β, g) and (β, d) are equivalent. Therefore,

$$u(d) = E_\beta(\beta, d) = E_\beta(\beta, g) = E_\beta(P)u(a) + (1 - E_\beta(P))u(b).$$

We have thus written $u(d)$ as two convex combinations of $u(a)$ and $u(b)$. Since $u(a) < u(b)$ we conclude that $E_\alpha(P) = E_\beta(P)$. **QED**

Lemma 2 ensures that we unambiguously may define a function

$$E : \mathcal{F} \rightarrow [0, 1]$$

by setting $E(P) = E_\alpha(P)$ for any preparation $\alpha \in P(H)$ containing P. The function furthermore has the property

$$E(P_1) + \cdots + E(P_n) = 1$$

for any sequence $P_1, \ldots, P_n$ of projections in $\mathcal{F}$ with sum $P_1 + \cdots + P_n = 1$ (note that such projections automatically are mutually orthogonal). A function with this property is called a frame function, and such functions were studied by Mackey (1957), Gleason (1957), Varadarajan (1968), Piron (1976) and others. The following remarkable result was conjectured by Mackey and proved by Gleason.

Gleasons' theorem *Let $\mathcal{F}$ be the lattice of (self-adjoint) projections on a (real or complex) separable Hilbert space H of dimension greater than or equal to three, and let $F : \mathcal{F} \to [0,1]$ be a frame function. Then there exists a positive semi-definite trace class operator h on H with unit trace such that*

$$F(P) = \mathrm{Tr}\,(hP)$$

for any $P \in \mathcal{F}$.

A frame function is thus automatically continuous under the conditions of the above theorem. It is a non-trivial and interesting question to study conditions on a general event-lattice $(\mathcal{F}, H)$ which ensures Gleasons' theorem to hold, but such a study is outside the scope of the present article. It is however easy to verify that Gleasons' result fails to be valid if $\mathcal{F}$ can be written as a direct sum of two components, one of which is chosen as the lattice of projections on $\mathbf{R}^2$.

Theorem 4 *Let $(\mathcal{F}, H)$ be the event-lattice consisting of all (self-adjoint) projections on a (real or complex) Hilbert space of finite dimension greater than or equal to three, let C be a set of consequences equipped with an affine structure, and let L be a set of actions. For each preparation $\alpha \in P(H)$ we assume that the subset $L_\alpha \subseteq L$, consisting of the acts in L with preparation α, is convex and includes the set of constant actions. The primitive datum of the utility theory is a binary preference relation $\succeq$ over the set L satisfying the axioms (i) through (ix). There exists then a map $u\colon C \to \mathbf{R}$, unique up to an affine transformation, and a positive semi-definite operator A on H with unit trace such that*

$$(\alpha, f) \succ (\beta, g) \quad \textit{if and only if} \quad U(\alpha, f) > U(\beta, g)$$

for arbitrary acts (α, f) and (β, g) in L, where the expected utility function U is defined by setting

$$U(\gamma, h) = \sum_{i=1}^{n} \mathrm{Tr}\,(AP_i) u(h(P_i))$$

for any act $(\gamma, h) \in L$ with preparation $\gamma = \{P_1, \ldots, P_n\} \in P(H)$, where n is some natural number depending on γ.

Proof: The two acts (α, f) and (β, g) in L are by Corollary 1 equivalent to constant acts (α, c) and (β, d) respectively, and since $U_\gamma = U$ for any preparation $\gamma \in P(H)$ we obtain

$$U(\alpha, f) = U(\alpha, c) = u(c) \quad \text{and} \quad U(\beta, g) = U(\beta, d) = u(d).$$

But since constant acts are ordered by u the statement follows. **QED**

4 Examples

4.1 Ellsbergs' paradox

As in the example by Ellsberg (1961) a decision maker is presented with an urn containing 90 balls. He is told that 30 of the balls are red and that the remaining 60 balls are either black or yellow, but he is given no information about the distribution of the black and yellow balls. The decision maker is first asked to state his preferences between three bets, each on the exact color of a single drawn ball. The decision maker is then asked to state his preferences between three bets in which he is given a choice between two colors of a single drawn ball. All six bets pay out the same amounts, conditional on the outcome of the draw.

Since the decision maker has exact information about the fraction of the red balls, he considers a bet on the red ball to be a simple lottery described by a probability distribution given the weight 1/3 to the event "the ball is red" and the weight 2/3 to the event "the ball is not red", and this last event is recognized to be the same event as "the ball is either black or yellow".

This may be modelled by letting the event "red ball" be represented by the projection R and the event "not red ball" or "black or yellow ball" be represented by the projection $1 - R$ where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 1 - R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The decision maker is in the absence of further information not able to subdivide the "black or yellow ball" event into two orthogonal single color events with a probability distribution. They are in Knight's words "ambiguous events for which ordinary probabilities are not defined". The two single color events "black ball" and "yellow ball" belong to different "small worlds" and may be represented by the projections B and Y given by

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

Note that the three single color events are mutually exclusive, that is

$$R \wedge B = 0, \quad B \wedge Y = 0, \quad R \wedge Y = 0,$$

and as required the majorant event $B \vee Y = 1 - R$. The two other majorant events are then easily calculated to be

$$R \vee B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R \vee Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

The three two color events $B \vee Y$, $R \vee B$ and $R \vee Y$ are thus endogenously given by the lattice operations once the one color events R, B and Y are specified. Assume that the decision maker prefers a bet on a "red ball" to a bet on a "black ball" and is indifferent between a bet on a "black ball" and a bet on a "yellow ball", that is

$$\text{Bet}(R) \succ \text{Bet}(B) \sim \text{Bet}(Y).$$

Assume furthermore, following Ellsberg (1961) and others, that the decision maker belongs to the group of people (appearing in numerous empirical experiments) which prefer a bet on a "black or yellow ball" to a bet on either a "red or yellow ball" or a bet on a "black or red ball", and is indifferent between these two last bets, that is

$$\text{Bet}(B \vee Y) \succ \text{Bet}(R \vee Y) \sim \text{Bet}(B \vee R).$$

We will show that these preferences may be expressed in terms of the expected utility theory developed in the last section. The set C of consequences is the unit interval $[0, 1]$ and each of the six bets is an act with a two-element preparation of projections on the Hilbert space $\mathbf{R}^3$ equipped with the scalar product (this is the simplest case covered by Gleason's theorem). The bet on a "red ball" is thus the act

$$\big(\alpha_R, \text{Bet}(R)\big) \quad \text{with preparation} \quad \alpha_R = \{R, 1 - R\}$$

such that $\text{Bet}(R)(R) = 1$ and $\text{Bet}(R)(1 - R) = 0$. The five other bets are in the same way defined as acts such that the consequence is 1 in the projection (the event) associated with the bet, and 0 in the orthogonal complement.

Preferences satisfying axioms (i) through (ix) are by Theorem 4 represented by an expected utility function U constructed from a linear functional E on the form

$$E(A) = \text{Tr}\,(hA) \qquad A \in B(H),$$

where h is a positive semi-definite unit trace operator. We may calculate the expected utility of the bet on a "red ball",

$$U(\alpha_R, \mathrm{Bet}(R)) = E(R) \cdot 1 + E(1-R) \cdot 0 = E(R),$$

and find that it equals the expected value $E(R)$ of the event R associated with the red ball. The same result applies, mutatis mutandis, to the five other bets. If we choose

$$h = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/6 & -1/6 \\ 0 & -1/6 & 1/2 \end{pmatrix}$$

and calculate the expectations

$$\begin{aligned} E(R) &= \frac{1}{3} & E(B) &= \frac{1}{6} & E(Y) &= \frac{1}{6} \\ E(B \vee Y) &= \frac{2}{3} & E(R \vee B) &= \frac{1}{2} & E(R \vee Y) &= \frac{1}{2} \end{aligned}$$

we find that

$$E(R) > E(B) = E(Y) \quad \text{and} \quad E(B \vee Y) > E(R \vee Y) = R(B \vee R).$$

This shows that the decision maker's preferences may be represented by an expected utility function also when he is unable to fit the ambiguous events "black ball" and "yellow ball" into the same "small world" and accord ordinary probabilities.

The "non red ball" and the "black or yellow ball" events are well defined and identical, while the ambiguous events "non black ball" and "red or yellow ball" are represented by different projections. This may seem meaningless from the very set up of the experiment, but one should remember that we model not the physical system, but the decision maker's perception of the given situation.

Note that if the ambiguous "non black ball" and "red or yellow ball" events are perceived by the decision maker to be identical and therefore represented by the same projection, that is $1 - B = R \vee Y$ then B and Y are orthogonal and thus $1 = R + B \vee Y = R + B + Y$. This corresponds to the situation where the decision maker has created a 3 point state space and the linear expectation functional E becomes then an ordinary probability measure on that space. This description is not, as is well known, compatible with the decision maker's preferences.

4.2 Portfolio decisions

In this section we consider an example presented in Dow and Werlang (1992) which illustrates the portfolio decisions of an agent who faces uncertainty. The example has a single investor which has wealth $W > 0$, a risk free asset and a single risky asset in a one-period model. The price of the risky asset is p and the present value of an investment in the risky asset is either H (high) or L (low). To avoid arbitrage possibilities we assume $L < p < H$.

The investor may choose between going long (strategy 1) or going short (strategy 2) in the risky asset at the given price, or he may choose to invest his wealth in the risk free asset (strategy 3). The investor considers the two risky investment strategies to be qualitatively different and belonging to different "small worlds". In the language of the theory developed in the previous sections we say that the two acts have different preparations.

Assume that strategy 1 is given by (α, l) with preparation $\alpha = \{P, 1-P\}$ and strategy 2 is given by (β, s) with preparation $\beta = \{Q, 1-Q\}$ such that

$$l(P) = H - p \quad \text{and} \quad l(1-P) = L - p$$

together with

$$s(Q) = -H + p \quad \text{and} \quad s(1-Q) = -L + p.$$

Note that the investor does not question the outcomes of the two different strategies, but only considers the known consequences to be triggered by different events. The two preparations may be specified by setting

$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that P and Q are (self-adjoint) projections on the real Hilbert space $\mathbf{R}^3$ equipped with the scalar product (this is the simplest case covered by Gleason's theorem). Preferences satisfying axioms (i) through (ix) are by Theorem 4 represented by an expected utility function U constructed from a linear functional E on the form

$$E(A) = \mathrm{Tr}\,(hA) \qquad A \in B(H),$$

where h is a positive semi-definite unit trace operator. If for example

$$h = \begin{pmatrix} 1/3 & -1/6 & 1/12 \\ -1/6 & 1/6 & -1/6 \\ 1/12 & -1/6 & 1/2 \end{pmatrix}$$

we find

$$E(P) = \frac{1}{6} \quad \text{and} \quad E(Q) = \frac{1}{3}$$

and may calculate the expected utilities

$$U(\alpha, l) = E(P)(H - p) + E(1 - P)(L - p) = \frac{5}{6}L + \frac{1}{6}H - p$$

and

$$U(\beta, s) = E(Q)(-H + p) + E(1 - Q)(-L + p) = -\frac{2}{3}L - \frac{1}{3}H + p$$

of going either long or short in one unit of the risky asset. The investor has a third possible strategy which is to invest in the risk free asset, and this constant act has a utility level of 0. If the price p of the risky asset is confined to the smaller interval (of length $(H - L)/6$) given by

$$L < \frac{5}{6}L + \frac{1}{6}H < p < \frac{2}{3}L + \frac{1}{3}H < H,$$

we obtain both $U(\alpha, l) < 0$ and $U(\beta, s) < 0$. In this case the investor prefers to invest in the risk free asset to going either long or short in the risky asset. These preferences may be interpreted as a reflection of uncertainty aversion, which is modelled by choosing different preparations for the two risky investment strategies.

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