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Emergence of an Urban Traffic Macroscopic Fundamental Diagram

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Abstract

This paper establishes mild probability-theoretical conditions under which observations of space-averaged speed and occupancy in some area concentrate with low scatter around a well-defined curve. These conditions are validated against empirical data from Stockholm and Geneva. No equilibrating process is required to be in operation.

Keywords: Congestion; Macroscopic Fundamental diagram.

1 Introduction

Traffic flow theory is concerned with fundamental indicators such as speed, occupancy (or density), and flow. Speed, flow and occupancy at a point on a road are related through an identity whereby the flow, measured as the number of vehicles passing a point per time unit, is equal to the speed, in distance per time unit, multiplied by occupancy, which is the number of vehicles per unit of road length. The fundamental diagram of traffic flow, due to Greenshields (1935), provides another relationship between these three variables. It may be expressed as a relationship between flow and occupancy, according to which flow increases with the occupancy up to the capacity of the road and then decreases to zero due to congestion. The fundamental diagram may equivalently be expressed as a

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decreasing relationship between speed and occupancy, or as a backward-bending two-valued relationship between speed and flow.[1]

The macroscopic fundamental diagram (MFD) is similar to the fundamental diagram, but rather than referring to a point on a road, it relates space-averages of speed, flow and occupancy for some wider area. According to this theory, an MFD should have a well-defined maximum and be invariant to changes in demand over the day and between days. The MFD was first investigated by Godfrey (1969), and has recently generated a lot of excitement due to its promise that the complications of traffic networks can be ignored for many purposes. With the MFD, traffic in large urban areas can be modeled dynamically at an aggregate level.

Early work investigated the macroscopic characteristics of traffic flow based on data from lightly congested real-world networks Godfrey (1969); Ardekani and Herman (1987); Olszewski et al. (1995), while other contributions have used simulation data with artificial routing rules and static demand Mahmassani et al. (1987); Williams et al. (1987); Mahmassani and Peeta (1993). However, the existence of an invariant MFD for a real congested urban network was not demonstrated mainly due to the difficulty in obtaining sufficient traffic data for a large network.

With the help of the recent rapid development of intelligent transportation systems, it has become possible to obtain traffic data for urban networks at a large scale. Geroliminis and Daganzo (2008) analyzed the relation between the average flow and average occupancy of Yokohama (Japan) with data collected from both loop detectors and GPS equipped taxis, and found that the data revealed a well-defined MFD. More specifically, they plotted the average flow against the average occupancy with data points representing different points in time. Data points were found to be concentrated with low scatter around a well-defined function that relates each level of the average occupancy to a single value for the average flow.

In response to the empirical MFD literature, a theoretical literature is emerging that explores the implications of the existence of an MFD for the regulation of urban congestion through, e.g., metering and gating (Keyvan-Ekbatani et al., 2012; Ramezani et al., 2015), pricing (Fosgerau, 2015; Simoni et al., 2015; Daganzo and Lehe, 2015), route guidance (Hajiahmadi et al., 2013; Xiong et al., 2016), and provision of road and transit capacity (Geroliminis et al., 2014; Loder et al., 2017).

To ascertain the universality of a well-defined urban MFD, Daganzo and Geroliminis (2008) and Helbing (2009) have developed analytical theories. These studies build on the assumption of a mechanism that equilibrates traffic density across space. Daganzo

[1] In traffic engineering, congested conditions are said to occur when the occupancy is above the capacity level. In transportation economics, the term congestion refers to the phenomenon that the speed decreases below the free-flow speed due to high occupancy. The economists use the term hypercongestion for the situation where the occupancy is above the critical level and an increase in speed is (somewhat paradoxically) associated with an increase in flow. This paper uses the terminology from engineering.

and Geroliminis (2008) formulate sufficient regularity conditions for the existence of a well-defined MFD, requiring a slowly varying and distributed demand, a network with redundancy (many route options), a homogeneous network of links with similar fundamental diagrams, and links whose fundamental diagrams are not significantly influenced by varying route choices. These conditions imply that density and speed will be evenly distributed in the network.

Subsequently, Geroliminis and Sun (2011a) used data from Yokohama and found that a well-defined MFD may exist even if density is not uniformly distributed among links. On the other hand, analysis for a freeway network in Minnesota showed that the spatial distribution of vehicle densities in the network is a key component affecting the scatter of an MFD and its shape. Geroliminis and Sun (2011b) showed that the freeway network not only has curves with high scatter, but also exhibits hysteresis phenomena. Thus, higher network flows are observed in the onset and lower in the offset of congestion, for the same average network density, which leads to a two-valued relationship from average density to average flow. Geroliminis and Sun (2011a) found that in the Yokohama network, unlike the freeway network, the density distribution among links is invariant over time conditional on the average network density. The authors concluded that this is an important property for the existence of an MFD.

Several studies have further investigated the network and behavioral conditions under which a well-defined MFD does and does not emerge. Daganzo et al. (2011) use idealized networks and simulations to study bifurcations in aggregate flow-density relationships. Gayah and Daganzo (2011) extend this work to explain the phenomenon of clockwise hysteresis loops observed in some MFDs through network dynamics and instability. Mazloumian et al. (2010) use simulation without equilibration to show that spatial heterogeneity in density can lead to wide variation in average network flow at constant average density. Along similar lines, Knoop et al. (2013) and Knoop and Hoogendoorn (2013) propose and investigate a two-dimensional generalization of the MFD, relating the average flow to both the average density and the (spatial) heterogeneity of density. Thus, spatial heterogeneity of density may preclude the existence of a single-valued MFD.

This paper establishes conditions that ensure that an MFD emerges, even in the absence of an equilibration effect. First, we impose mild conditions on the statistical dependency of traffic variables across space which are sufficient to ensure the convergence of the space-averages of traffic variables to fixed values at any point in time. Second, we show that the space-averages of speed/flow are time-invariant conditional on the average occupancy, provided that the spatial distribution of the speed/flow is the same conditional on each value of the average occupancy. That is, if we consider a time slice and pool the speed from different locations into one distribution, then we require that this distribution is the same whenever the average occupancy is the same. This allows for traffic patterns that may be different at different times of day, as long as the pooled distribution is constant conditional on the number of vehicles in motion. Weather and other factors can also affect the pattern, but are not included in our analysis due to limitations of the available data.

The fact that an MFD may emerge without equilibration also means that the existence of an equilibration process cannot be inferred from the observation that an MFD exists. In other words, merely observing a stable relationship between space-averages of speed or flow versus occupancy is not sufficient to infer the existence of a robust relationship that can be used for traffic control. This insight is consistent with the suggestion by Ji and Geroliminis (2012) that clustering should be applied to identify regions where an MFD could be applied.

The paper is organized as follows: Section 2 explains the assumptions of our model and their validity, and presents our main result regarding the existence of the MFD. Section 3 validates the model assumptions using a small simulation exercise and then using empirical data from Stockholm and the Geneva region. Section 4 concludes the paper. Proofs are given in the Appendix.

2 The model and the main result

We consider a road network partitioned into an infinite number of segments $i \in \mathbb{N}$. The assumption that there are infinitely many segments is a mathematical idealization that allows us to use certain asymptotic results. We observe traffic variables at each segment and at each time t in a set T.

Indexing of the road segments should begin in the center, and the index should increase as we go away from the center. It does not matter that we use a one-dimensional index over two-dimensional space: in practice it is possible to spiral out from the center. The sequence of indexing is important since some of our assumptions require almost no positive local correlation for traffic variables away from the center (with higher index). A city with multiple congested areas can be included as long as there are sub-urban regions (higher index) with no significant positive local correlation.

Let o_i^t be a positive random variable denoting the occupancy on segment $i \in \mathbb{N}$ at time $t \in T$. For every segment i, there is a local speed-occupancy function $v_i : \mathbb{R}^+ \to \mathbb{R}^+$ and a local flow-occupancy function $q_i : \mathbb{R}^+ \to \mathbb{R}^+$. The local speed-occupancy function is non-increasing, i.e., speed decreases or remains constant as the occupancy increases.

We assume that local occupancies, speeds and flows are uniformly bounded. This is clearly a realistic assumption as actual occupancies, speeds and flows are bounded physically. A uniformly bounded sequence converges in mean, which ensures that the average expected occupancy $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=1}^{n} \mathbb{E}(o_i^t)$ exists and similarly for speed and flow. Denote the limiting values

$$\hat{o}_t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left(o_i^t\right), \quad \hat{v}_t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left(v_i^t\right), \quad \hat{q}_t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left(q_i^t\right)$$

We recall two convergence concepts for sequences of random variables.

Definition 1. *A sequence $\{X_i\}$ of random variables on a sample space Ω is said to converge almost surely to a random variable X defined on Ω if $Pr(\omega \in \Omega \mid \lim_{i\to\infty} X_i(\omega) = X(\omega)) = 1$.*

Definition 2. *A sequence $\{X_i\}$ of random variables on a sample space Ω is said to converge in probability to a random variable X defined on Ω if $\lim_{i\to\infty} Pr(\omega \in \Omega \mid |X_i(\omega) - X(\omega)| > \epsilon) = 0 \; \forall \, \epsilon > 0$.*

Almost sure convergence implies convergence in probability, but the converse is not true. Sections 2.1 and 2.2 establish sufficient conditions for convergence in probability and almost surely, respectively, of the space-mean values of the traffic variables.

2.1 Assumptions for convergence in probability

2.1.1 Asymptotic boundedness and distant convergence

We introduce concepts of asymptotic boundedness and distant convergence relevant for a two-dimensional array a_{ij} of real numbers. The concept of distant convergence is a contribution of this paper.

Definition 3. *An array a_{ij} is said to be asymptotically bounded by 0 if there exists a non-negative sequence $q(i) \to 0$ such that for all $i, k \geq 1$: $a_{i,i+k} < q(i)$.*

Definition 4. *An array a_{ij} is said to be distant convergent to a if for every $\epsilon > 0$ there exists N so that if $|i-j| > N$, then $|a_{ij} - a| < \epsilon$.*

Further, it is important to note that asymptotically boundedness by 0 is not the same as distant convergence to 0. Definition 3 says that a_{ij} is asymptotically bounded by 0 if $\lim_{i\to\infty} a_{ij} = 0$, whereas a_{ij} is distant convergent to 0 if $\lim_{|i-j|\to\infty} a_{ij} = 0$ by definition 4. The asymptotically bounded property implies convergence with respect to the first index only, and the distant convergent property implies convergence with the difference of the two indices. The following lemmas are proven in the Appendix:

Lemma 1. *Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of random variables with uniformly bounded means μ_i and covariances σ_{ij} that are asymptotically bounded by* 0*. Then,* $\frac{1}{n}\sum_{i=1}^{n} X_i$ *converges in probability to* $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mu_i$.

Lemma 2. *Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of random variables with uniformly bounded means μ_i and covariances σ_{ij} that are distant convergent to* 0*. Then,* $\frac{1}{n}\sum_{i=1}^{n} X_i$ *converges in probability to* $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mu_i$.

2.1.2 Application to the model

We will use the following assumptions.

Assumption 1. *One of the following holds:*

1. *For every $t \in T$, $Cov(o_i^t, o_j^t)$, $Cov(v_i(o_i^t), v_j(o_j^t))$, and $Cov(q_i(o_i^t), q_j(o_j^t))$ are asymptotically bounded by 0.*
2. *For every $t \in T$, $Cov(o_i^t, o_j^t)$, $Cov(v_i(o_i^t), v_j(o_j^t))$, and $Cov(q_i(o_i^t), q_j(o_j^t))$ are distant convergent to 0.*

Assumption 1.1 requires that the covariance between the occupancy of adjacent streets decreases with an increase in index. Assumption 1.2 holds when the covariance between the occupancy of segments decreases with increasing distance between two streets, i.e., the occupancy on adjacent segments may have higher covariance whereas occupancy on distant segments will have smaller covariance. The same assumption is also introduced for speed and flow.

Using Lemma 1 and 2, Assumption 1 guarantees the convergence in probability of the space-averaged occupancy, speed and flow. This means that spatial averages have a limiting value and the probability that spatial averages deviate by some fixed amount from their limiting values becomes arbitrarily small as the number of observations increases in the spatial dimension.

2.2 Assumptions for almost sure convergence

2.2.1 Asymptotically almost negative association (AANA)

Chandra and Ghosal (1996a,b) introduced the following dependence concept, which allows for local positive correlation while still being sufficient for obtaining the convergence results that we will state below.

Definition 5. *A sequence $\{X_i\}_{i\in\mathbb{N}}$ of random variables defined on a fixed probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is called asymptotically almost negatively associated (AANA) if there is a nonnegative sequence $q(i) \to 0$ such that for all $i, k \geq 1$,*

$$Cov(f(X_i), g(X_{i+1}, \cdots, X_{i+k})) \leq q(i)(Var(f(X_i))Var(g(X_{i+1}, \cdots, X_{i+k})))^{\frac{1}{2}} \tag{2.1}$$

for all coordinate-wise non-increasing continuous functions f and g such that the right-hand side of eq. (2.1) is finite.

Thus, an AANA sequence $\{X_i\}$ may have positive local correlations but the correlations must become small as i increases. The sequence $q(i)$ in the definition of AANA is called the mixing coefficient. We note that AANA implies asymptotic boundedness of the covariance. It is thus a strictly stronger assumption.

The following example is relevant for our model.

Example 1. *Let $\{Y_i\}_{i=1}^{\infty}$ be independent identically distributed standard normal random variables. For any non-negative integer p, define $X_i = (1 + a_i^2 + a_{i+1}^2 + \cdots + a_{i+p}^2)^{-\frac{1}{2}}(Y_i + a_iY_{i+1} + \cdots + a_{i+p}Y_{i+p+1})$ with $a_i > 0$ and $a_i \to 0$ as $i \to \infty$. Then $\{X_i\}_{i=1}^{\infty}$ is AANA. (Proof is given in the Appendix.)*

In Example 1, X_i can be considered as occupancies at various locations in a road network. Subscript i indexes road segments, beginning from inside a congested city center. Then the example describes a case where occupancies are positively correlated with occupancies of p neighboring road segments and where the correlations decrease as we go away from the congested region of the city.

The following Lemma, proved by Yuan and An (2009), states that the AANA property is invariant under monotonic transformations of the random variables in an AANA sequence. This is extremely useful since it means that the AANA property carries over from local occupancies to local speeds through monotone local speed-occupancy relationships.

Lemma 3. *Let $\{X_i\}$ be a sequence of AANA random variables with mixing coefficient $q(i)$, and let $f_1, f_2, \cdots$ be all nondecreasing (or nonincreasing) functions, then $\{f_i(X_i)\}$ is also a sequence of AANA random variables with mixing coefficient $q(i)$.*

Using AANA sequences of random variables enables us to use the following theorem, which was established by Wang et al. (2010).

Theorem 1. *Let $\{X_i, i \geq 1\}$ be a sequence of AANA random variables with mixing coefficient satisfying $\sum_{i=1}^{\infty} q^2(i) < \infty$. Denote $Q_n = \max_{1\leq i\leq n} \mathbb{E}(X_i)^2$ for $n \geq 1$ and $Q_0 = 0$. Then $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}(X_i)) = 0$ almost surely if $\sum_{n=1}^{\infty} \frac{Q_n}{n^2} < \infty$.*

Theorem 1 establishes that the average of a sequence of random variables converges to the average of the expected value of the random variables under weak conditions that allow these variables to be dependent. In particular, the random variables may be positively correlated locally.

2.2.2 Application to the model

Assuming asymptotically almost negative association between the occupancies on different segments enables us to derive stronger results for the space-averaged speed and occupancy.

Assumption 2. *For every t, the sequence $\{o_i^t\}_{i\in\mathbb{N}}$ is AANA with mixing coefficient satisfying $\sum_{i=1}^{\infty} q^2(i) < \infty$.*

Assumption 2 allows us to establish almost sure convergence of space-averaged occupancy and speed as stated in the following lemma:

Lemma 4. *Under Assumption 2, at any time $t \in T$, space-averaged occupancy $\bar{o}_{nt} = \frac{1}{n}\sum_{i=1}^{n} o_i^t$ and space-averaged speed $\bar{v}_{nt} = \frac{1}{n}\sum_{i=1}^{n} v_i(o_i^t)$ converge almost surely.*

2.3 Distributions of speed and flow

Lemma 1, 2 and 4 show that empirical averages of speed or flow against occupancy will be close to their expected values, where the meaning of closeness depends on whether we have convergence in probability or convergence almost surely. It may still happen, however, that the average speed or average flow converge on different values for the same value of the average occupancy at different times of day. A further assumption is required to guarantee that the limiting values are the same across times in the set T where the limiting value for the average occupancy is constant. The assumption that we will make is the weakest we can find that achieves this purpose. Importantly, it allows the distribution across space to vary over time. Thus, we do *not* require that the distribution of traffic variables at a location i is independent of time: it is sufficient to put a restriction on the distribution of speeds and flows, pooling them across space.

Let $T_{\hat{o}}$ be the set of times t where $\hat{o}_t = \hat{o}$.

Assumption 3. *For all $\hat{o}$, for all $s \geq 0$, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1_{\{v_i^t \leq s\}}$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1_{\{q_i^t \leq s\}}$ are constant a.s. independent of $t \in T_{\hat{o}}$.*

Assumption 3 implies that the distribution of speed and flow is independent of time, and depends only on the space-averaged occupancy.

Using data from Yokohama, Geroliminis and Sun (2011a) compare the distribution of occupancies across space at different times, conditional on the average occupancy. They find distributions that are highly dispersed. Nevertheless, they look quite stable and this evidence seems to be in good agreement with the present Assumption 3.

2.4 Emergence of a well-defined MFD

We can thus state our main result:

Theorem 2. *Under Assumptions 1 and 3, the limiting value for the space-averaged flow and speed is unique given the limiting value for space-averaged occupancy. In other words,* $\hat{o}_{t_1} = \hat{o}_{t_2} = \hat{o}$ *implies that* $\hat{v}_{t_1} = \hat{v}_{t_2}$ *and* $\hat{q}_{t_1} = \hat{q}_{t_2}$.

We have noted that AANA implies asymptotic boundedness by 0. Then the stronger Assumption 2 for occupancies together with Assumption 3 ensures Theorem 2 as well as the stronger almost sure convergence result of Lemma 4.

Theorem 2 ensures unique limiting values for $\hat{v}_t$ and $\hat{q}_t$ for all $t \in T_{\hat{o}}$. Extending this to all values of $\hat{o}$ implies that a scatter plot of space-averages of speed or flow against occupancy will converge to a single-valued function. This hence establishes the existence of a well-defined MFD.

3 Model validation

We have established mathematical conditions that are sufficient to ensure the emergence of an MFD. Assumption 3 is in agreement with the result in Geroliminis and Sun (2011a) that if the spatial distribution of link density is the same for two different time intervals with the same number of vehicles in the network, then these two time intervals should have the same average flows. In this section, we will provide further checks on the validity of this assumption by confronting some testable implications of these conditions with simulated and empirical evidence.

3.1 Simulated data

We consider a network with 300 road segments. Each segment has a piece-wise linear speed-occupancy relationship in which the speed decreases linearly from 1 to 0.75 between 0 and a randomly chosen occupancy level between 3 and 4; after this point speed linearly decreases to 0 at the occupancy level of 6. The local speed-occupancy function (Figure 1) is thus piece-wise linear and concave. The local flow is computed as the product of speed and occupancy.

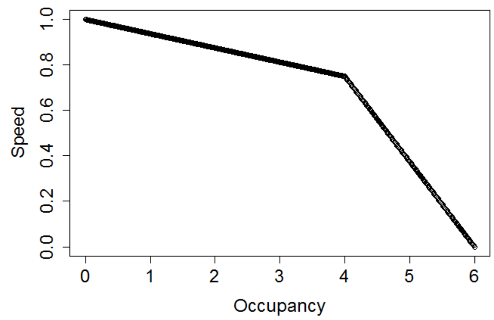

Figure 1: Example speed occupancy curve.

Segment occupancies are generated by drawing 301 normal random variables Y_i with mean randomly assigned between 3 and 4 and variance 1. We take a sequence $a_i = 1/i$, which converges to 0. Then the occupancy on the i^{th} segment is given by $X_i = Y_i + \frac{1}{i}Y_{i+1}$. This makes the occupancies AANA random variables (cf. Example 1). Furthermore, speed-occupancy is a monotonically decreasing function, and therefore speed is also AANA.

We replicate this 10,000 times with different means for occupancies (varying between 0 and 7) to obtain simulations over a range of values for the average occupancy. This leads to the speed-occupancy, flow-occupancy, and flow-speed scatter plots shown in Figure 2. We notice that a well-defined MFD emerges. This illustrates the basic insight underlying this paper, namely that the averages of random speeds, occupancies and flows converge to well-defined values under weak conditions, even when there is no process to equilibrate speeds across space, and even when local speed-occupancy relationships vary completely at random.

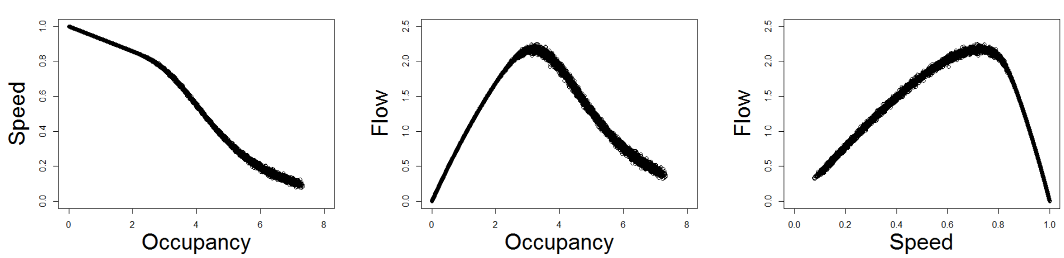

Figure 2: MFD for simulated data. Left-most picture represent the relation between speed and occupancy, middle picture represents flow and occupancy, and right-most picture represents flow and speed.

3.2 Stockholm taxi data

Our first empirical test of the model assumptions utilizes GPS traces from taxis in Stockholm. The data are used to estimate the distribution of speed across space and time. We can then test our assumptions related to speed, but we do not attempt to estimate occupancy or flow since the distribution of taxis in space and time may be quite different from that of the overall traffic.

We focus on the network on Södermalm island, which covers about 2×5 km^2 and is shown in Figure 3. The data consist of location reports with coordinates and timestamps from a fleet of about 1500 taxis. When active, each taxi reports its location once every 1-2 minutes. The data source is described in more detail in Rahmani et al. (2010); Jenelius et al. (2017). We use observations from weekdays between September 29 and October 10, 2014 and all hours of the day are used.

Figure 3: The network and region of Södermalm, Stockholm used in the model validation.

For each pair of consecutive reports from the same taxi, the speed is computed based on the Euclidean distance between the locations and the time difference between the timestamps. The speed observations are filtered to discard very low or high values that represent of stopped, waiting and parking vehicles. After filtering, the data set contains 113,823 speed observations. Each speed observation is then spatially associated with the mid-point between the start and end locations of the pair.

3.2.1 Test of distant convergent speed assumption

To study the relation between distance and the covariance of speed, the region is spatially discretized in a square grid with 0.25 km distance between grid points. The speed observations are binned spatially according to the nearest grid point, and temporally according to 15-minute clock-time intervals and days. There are 119 grid points covering the region, 96 clock-time intervals and 10 days, generating in total 114,240 time-space bins. The local average speed is computed for each bin based on the total distance and time travelled according to Edie's definitions (Edie, 1965). For each grid point and clock-time interval the mean value across days is subtracted from the average speed to remove the effect of varying speeds during the day. Since the taxis do not cover the whole network at all times, values are missing for 61% of the bins.

For each pair of grid points, the covariance of local speeds across all clock-time intervals and days is estimated, taking missing values into account. Kernel regression is then used to estimate the mean covariance of local speeds as a function of the Euclidean distance between the grid points. A Gaussian kernel with bandwidth 0.25 km is used, and a 95% confidence interval is estimated with bootstrapping.

The speed covariance shows large variability between grid points, which is partly an effect of the noisy nature of the taxi data. Still, the average covariance decays with distance, and the confidence interval shows that it is not statistically significantly positive for distances beyond 0.75 km (Figure 4). This supports the assumption that the covariance of speed is distant convergent to 0 (Assumption 2.2).

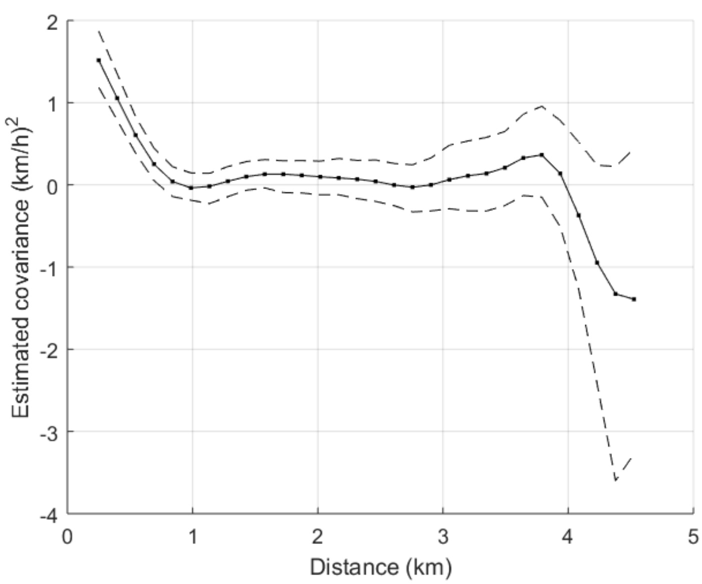

Figure 4: Covariance of local speed vs. distance for Södermalm, Stockholm based on binning of taxi data. Solid line: Covariance function estimated with kernel regression. Dashed line: Bootstrap 95% confidence interval.

3.3 Geneva loop-detector data

To further validate our assumptions, we study loop-detector data for the city of Geneva.[2] The Geneva network is shown in Figure 5. The data consists of occupancy, flow and speed measured by 254 detectors during September, 2014. These detectors are not physical detectors, but aggregated measurements used by the DGT (the transport authority La Direction Générale des Transports). The DGT typically aggregates different loops at the same level on different lanes into "one counting point".

Figure 5: The network and region of Geneva city used in the model validation.

The measurements are recorded every 3 minute and 20 seconds, and some of the detectors work only during daytime. There is a known bug in the flow measurements: During certain time intervals the sampling period is much shorter (about 1 min instead of 3 min 20 s): this bug results in normal occupancy and speed measurement but flows are 3 times smaller than expected.

Occupancy is measured as the number of cars on a street, whereas the speed and the

[2] We are thankful to Nikolas Geroliminis, EPFL Switzerland for providing us with access to the data.

flow are measured in kilometers per hour and number of cars passing per hour, respectively. The macroscopic fundamental diagram observed from the data is shown in Figure 6. We can notice two curves (flow), one of which is due to the measurement error in the flow. This error is also observed when we test Assumption 3.

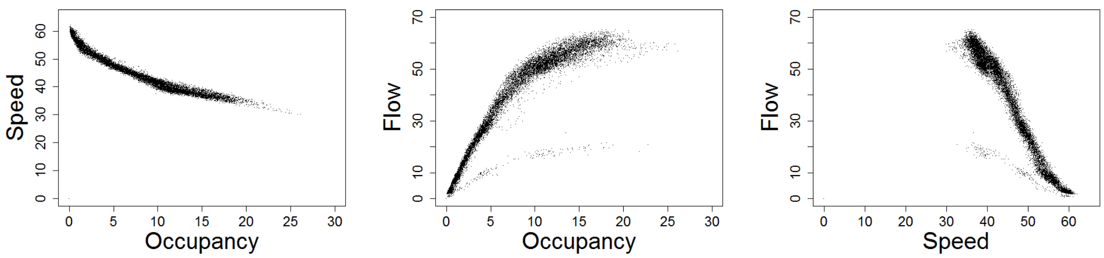

Figure 6: MFD for Geneva city. Left most picture represent the relation between speed and occupancy, middle picture represents flow and occupancy and the right most picture represents speed and flow relationship for the month of September 2014

3.3.1 Test of distant convergent speed, occupancy and flow

To study the relation between distance and the covariance of speed, we calculate the spatial distance between detectors. For each detector and clock-time interval the mean value across days is subtracted from the average speed to remove the effect of varying speeds during the day. For each pair of detectors, the covariance of speed, flow and occupancy is calculated, taking missing values in account. In Figure 7, we show the covariance function estimated with the kernel regression, and the shaded region shows the 95% confidence interval.

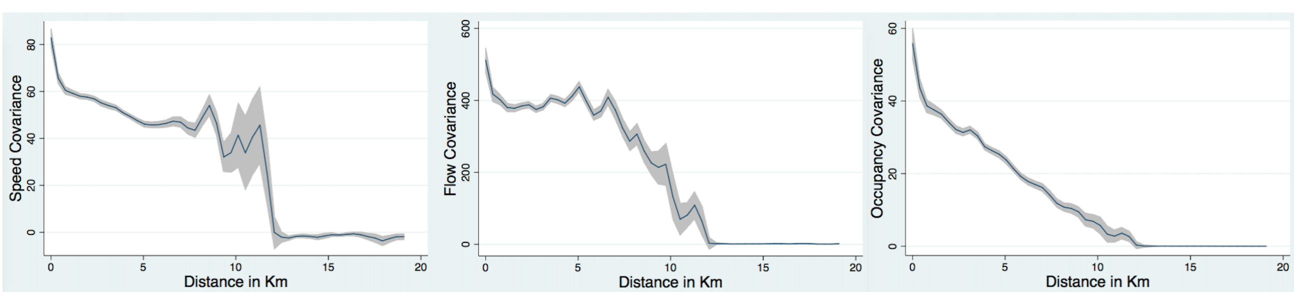

Figure 7: Covariances against distance for traffic variables: speed(left), flow(center) and occupancy(right).

We notice that covariance function decays with distance for all variables. Figure 7 helps us conclude that covariances are distant convergent to 0.

3.3.2 Test of conditionally invariant speed and occupancy distributions

We divide the average occupancy in different intervals of length 1 (car) and see the empirical cumulative distribution function for speed and flow when the average occupancy lies in that interval. The range of average occupancy is $(0, 26)$ and we round these values to integers. We present the results at average occupancy level $2, 9, 15, 19$, though we have checked for every rounded occupancy level. We choose these intervals based on the frequency table of the average occupancy, and display the results where the frequency of average occupancy has significant change.

Figure 8 shows that the distribution of speeds across the city of Geneva overlap at given average occupancy level. Furthermore, Figure 6 shows a very small variability in the speed-occupancy diagram. We also observe that at higher average occupancy, the overlapping of the speed distribution is clearly represented by the diagram, and therefore we have almost no variability in the occupancy-speed curve.

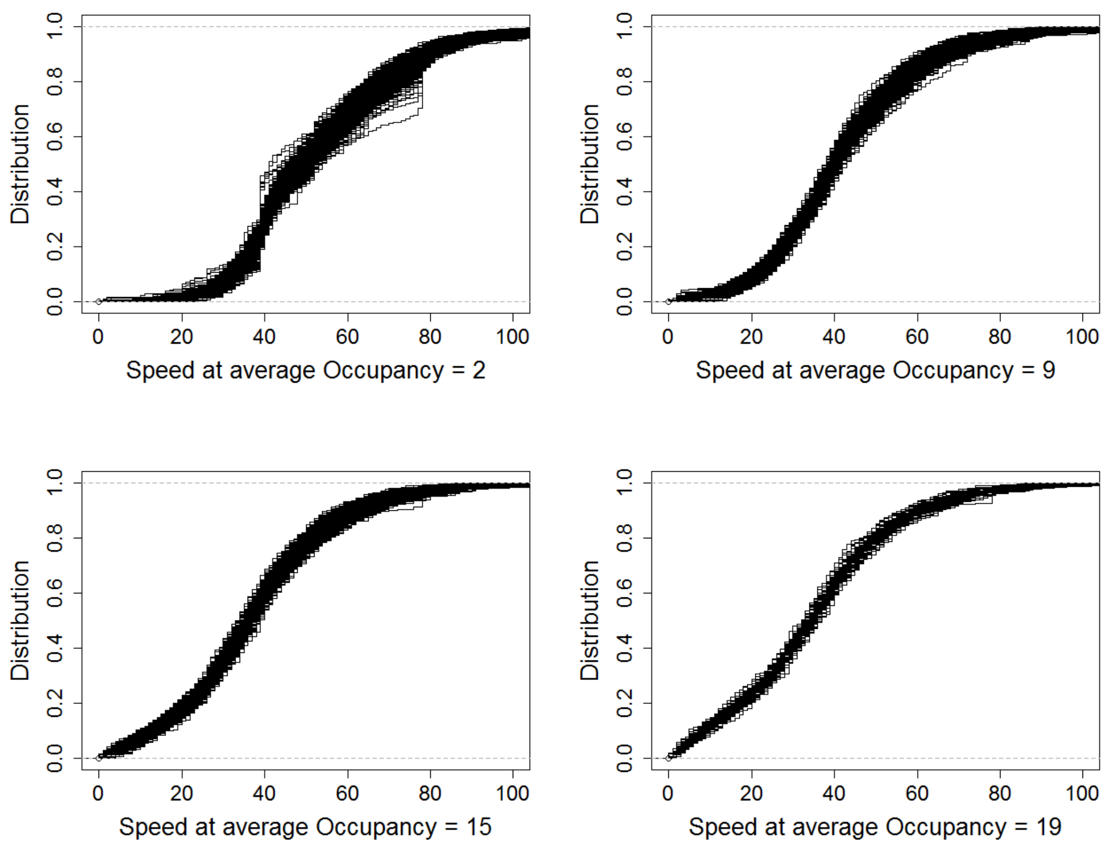

Figure 8: Distribution of speeds at average occupancy $2, 9, 15, 19$ respectively.

Figure 9 shows the distribution of flows across the city overlapping at a given average occupancy with few exceptions. We notice that flows have high variability in the distribution at lower occupancy levels ($\bar{o} = 2, 9$ in the figure), but it overlaps at higher occupancy levels. The higher variability in the empirical CDF at lower frequency is due to the measurement error in the flow.

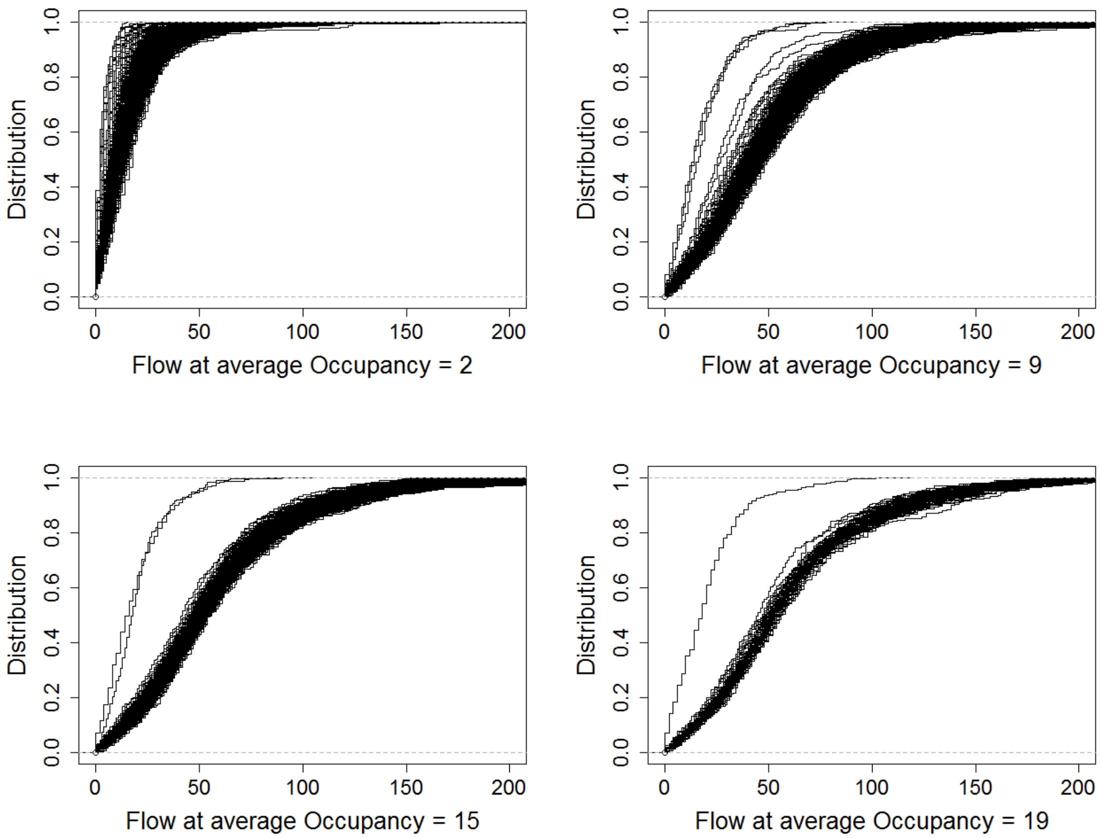

Figure 9: Distribution of flows at average occupancy $2, 9, 15, 19$ respectively.

4 Concluding remarks

The paper has established conditions that ensure that an MFD emerges in the limit as one increases the size of the traffic region in the analysis. The paper has shown that the law of large number applies in the case of traffic networks, even in the presence of positive spatial correlation, and ensures the emergence of an (unstable) MFD. The additional requirement that the spatial distribution of the speed/flow is the same conditional on each value of the average occupancy ensures a well-defined MFD. We have confirmed using empirical data that the correlation of speed across space decreases with distance, as required by the theory. In practice, this means that we would expect an MFD to emerge when the number of measurement locations and the size of the traffic region is sufficiently large. This occurs without any assumption of any equilibrating process. In particular, it is not required that speeds tend to be equal across space specially when the region of study is large.

The results have important implications for the use of an MFD for traffic control. For a drastic illustration, we can think of a city consisting of two independent parts with no connecting roads between them. Let us say that the conditions for the emergence of an MFD are satisfied in both parts and that these MFDs are different. Then, as this paper shows, an MFD will also emerge for the averages covering both parts of the city due to the assumed independence. If we now reduce traffic in one part of the city, the MFD for the whole city that would then be dominated by the MFD of the other part. Then the averaged MFD would change as a result of the reduction in traffic. This shows that metering would affect the observed shape of the MFD and that the MFD therefore does not predict the effect of metering. This example is a stark case in which an MFD emerges in a way that clearly contradicts the intuition behind the MFD.

The findings here imply that merely observing a stable relationship between space-averages of speed or flow versus occupancy is not sufficient to infer the existence of a robust relationship that can be used for traffic control. It seems that some kind of equilibrating mechanism is required, as assumed by the papers mentioned in Introduction. This raises the question of how the existence and strength of such an equilibrating mechanism can be validated with empirical data. This is an important topic for future research.

A Proofs

We refer to Yuan and An (2009) and Wang et al. (2010) for proofs of Lemma 3 and Theorem 1, respectively.

Proof of Lemma 1. Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and denote also $\sigma = \max_i Var(X_i)$, $\bar{\mu}_n = \frac{1}{n}\sum_{i=1}^{n}\mu_i$, and $\bar{\mu} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu_i$. Then $\mathbb{E}(\bar{X}_n) = \bar{\mu}_n$.

Given some $\epsilon > 0$, use the asymptotical bound on the covariances to choose k such that $q(i) < \epsilon$ for all $i > k$. Then

$$\begin{aligned}
Var(\bar{X}_n) &= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) + \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sigma_{ij} \\
&\leq \frac{\sigma}{n} + \frac{2}{n}\sum_{i=1}^{n} q(i) \\
&\leq \frac{\sigma}{n} + \frac{2}{n}(k\max_{i\leq k} q(i) + n\epsilon) \\
&\leq \frac{\sigma}{n} + \frac{2k\max_{i\leq k} q(i)}{n} + 2\epsilon,
\end{aligned}$$

which means that $Var(\bar{X}_n) \leq 3\epsilon$ for sufficiently large n. By Chebychev's inequality we then have

$$Prob(|\bar{X}_n - \mathbb{E}(\bar{X}_n)| \geq a) \leq \frac{Var(\bar{X}_n)}{a^2} \leq \frac{3\epsilon}{a^2}$$

for sufficiently large n. Then also

$$Prob(|\bar{X}_n - \bar{\mu}| \geq a) \leq Prob(|\bar{X}_n - \bar{\mu}_n| \geq a - |\bar{\mu}_n - \bar{\mu}|) \leq \frac{3\epsilon}{(a - |\bar{\mu}_n - \bar{\mu}|)^2}$$

for sufficiently large n. Since this is true for all $\epsilon > 0$, we conclude that $Prob(|\bar{X}_n - \bar{\mu}| \geq a)$ tends to zero as n tends to infinity. This is the required result that $\bar{X}_n$ converges in probability to $\bar{\mu}$. $\square$

Proof of Lemma 2. Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then $\mathbb{E}(\bar{X}_n) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i)$. Denote $\sigma = \max_i Var(X_i)$, and $\delta = \max_{i,j} Cov(X_i, X_j)$, $\bar{\mu}_n = \frac{1}{n}\sum_{i=1}^{n}\mu_i$, and $\bar{\mu} = \frac{1}{n}\sum_{i=1}^{\infty}\mu_i$.

$$\begin{aligned}
Var(\bar{X}_n) &= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) + \frac{1}{n^2}\sum_{1\leq i\neq j\leq n} Cov(X_i, X_j) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{0<|i-j|\leq k} Cov(X_i, X_j) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{|i-j|>k} Cov(X_i, X_j).
\end{aligned}$$

From the definition of distant convergence, for all $\epsilon > 0$, there exist k such that $|Cov(X_i, X_j)| < \epsilon$ for all i, j such that $|i - j| > k$. Note that k does not depend on n. Therefore,

$$\begin{aligned}
Var(\bar{X}_n) &\leq \frac{\sigma}{n} + \frac{k}{n^2}\sum_{i=1}^{n}\max_{0<|i-j|\leq k} Cov(X_i, X_j) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{|i-j|>k)}\epsilon \\
&\leq \frac{\sigma}{n} + \frac{k}{n}\delta + \epsilon \\
&\leq 3\epsilon.
\end{aligned}$$

By Chebychev's inequality we then have

$$Prob(|\bar{X}_n - \mathbb{E}(\bar{X}_n)| \geq a) \leq \frac{Var(\bar{X}_n)}{a^2} \leq \frac{3\epsilon}{a^2}$$

for sufficiently large n. Then also

$$Prob(|\bar{X}_n - \bar{\mu}| \geq a) \leq Prob(|\bar{X}_n - \bar{\mu}_n| \geq a - |\bar{\mu}_n - \bar{\mu}|) \leq \frac{3\epsilon}{(a - |\bar{\mu}_n - \bar{\mu}|)^2}$$

for sufficiently large n. Since this is true for all $\epsilon > 0$, we conclude that $Prob(|\bar{X}_n - \bar{\mu}| \geq a)$ tends to zero as n tends to infinity. This is the required result that $\bar{X}_n$ converges in probability to $\bar{\mu}$. $\square$

Proof of Lemma 4. The conditions of Theorem 1 are satisfied since occupancies and speeds are uniformly bounded. Then we have $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(o_{it} - \mathbb{E}(o_{it})) = 0$ almost surely. Since

$$\left|\frac{1}{n}\sum_{i=1}^{n} o_{it} - \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(o_{it})\right| < \left|\frac{1}{n}\sum_{i=1}^{n} o_{it} - \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(o_{it})\right| + \left|\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(o_{it}) - \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(o_{it})\right|$$

and both right-hand side terms tend to zero almost surely, we find that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} o_{it} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(o_{it})$$

almost surely.

By Lemma 3, speeds are also AANA with mixing coefficient $q(i)$. Then the conclusion for the space-averaged speed follows in the same way as for the space-averaged occupancy. $\square$

Proof of Theorem 2 . From Assumption 3, $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} 1_{\{v_i^t \leq s\}}$ is constant a.s. for all s, and independent of $t \in T_{\hat{o}}$. We denote the limiting function by $V(s)$. Then

$$V(s) = \mathbb{E}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} 1_{\{v_i^t \leq s\}}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} P\left(v_i^t \leq s\right)$$

by dominated convergence, noting that $\frac{1}{n}\sum_{i=1}^{n} 1_{\{v_i^t \leq s\}} \leq 1$.

That $\hat{v}_t$ is independent of t follows, again using dominated convergence, since

$$\begin{aligned}\hat{v}_t = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(v_i^t) &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\int\limits_0^\infty P(v_i^t > s))ds\\ &= \int\limits_0^\infty \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} P(v_i > s)ds\\ &= \int\limits_0^\infty (1 - V(s))ds,\end{aligned}$$

which is independent of t by assumption.

The proof for $\hat{q}_t$ is similar. □

Proof of Example 1. We shall show that the correlation coefficient between $U = f(X_i)$ and $V = g(X_{i+1}, \cdots, X_{i+k})$ is dominated in absolute value by a sequence b_i converging to 0. It is sufficient to prove this under the additional hypothesis $\mathbb{E}(U) = 0 = \mathbb{E}(V), \mathbb{E}(U^2) = 1 = \mathbb{E}(V^2)$. Then,

$$\begin{aligned}|(Cov(U,V))| &\leq Cov(U, \mathbb{E}(U \mid X_{i+1}, \cdots, X_{i+k})\\ &= \mathbb{E}(\mathbb{E}(U \mid X_{i+1}, \cdots, X_{i+k}))^2\\ &\leq \mathbb{E}(\mathbb{E}(U \mid Y_{i+1}, \cdots, Y_{i+k+p+1}))^2\\ &= \mathbb{E}(\mathbb{E}(U \mid Y_{i+1}, \cdots, Y_{i+p+1}))^2\\ &= \mathbb{E}\left(\mathbb{E}\left(U \mid Z_{i+1} = \frac{(a_i Y_{i+1} + \cdots + a_{i+p} Y_{i+p+1})}{(a_i^2 + a_{i+1}^2 + \cdots + a_{i+p}^2)^{\frac{1}{2}}}\right)\right)^2\end{aligned} \tag{A.1}$$

Clearly, Z_{i+1} is a standard normal random variable. Let $\psi_i(x,z)$ be the conditional density of X_i given Z_{i+1} and $\phi(x)$ be the density of standard normal random variable. Using the fact $\mathbb{E}(U) = 0$, we have

$$\mathbb{E}(\mathbb{E}(U \mid Z_{i+1}))^2 = \int\limits_{-\infty}^{\infty}\left(\int\limits_{-\infty}^{\infty} f(x)\left(\frac{\psi_i(x,z)}{\phi(x)} - 1\right)\phi(x)dx\right)^2\phi(z)dz$$

By using Cauchy-Schwartz inequality, the integral is at most

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left(\frac{\psi_i(x,z)}{\phi(x)} - 1\right)^2\phi(x)dx\phi(z)dz = a_i^2 + a_{i+1}^2 + \cdots + a_{i+p}^2 = b_i^2$$

Moreover, if $a_i \to 0$ then $b_i \to 0$. And $\sum\limits_{i=1}^{\infty} a_i^2 < \infty$ implies $\sum\limits_{i=1}^{\infty} b_i^2 < \infty$. □

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