Discussion Papers
Department of Economics
University of Copenhagen

No. 18-12

Volatility-Induced Stationarity and Error-Correction in Macro-Finance Term Structure Modeling

Anne Lundgaard Hansen

Øster Farimagsgade 5, Building 26, DK-1353 Copenhagen K., Denmark
Tel.: +45 35 32 30 01 – Fax: +45 35 32 30 00
http://www.econ.ku.dk

ISSN: 1601-2461 (E)

Volatility-Induced Stationarity and Error-Correction in Macro-Finance Term Structure Modeling †

Anne Lundgaard Hansen ‡

November 30, 2018

Abstract

It is well-known that interest rates and inflation rates are extremely persistent, yet they are best modeled and understood as stationary processes. These properties are contradictory in the workhorse Gaussian affine term structure model in which the persistent data often result in unit roots that imply non-stationarity. We resolve this puzzle by proposing a macro-finance term structure model with volatility-induced stationarity. Our model employs a level-dependent conditional volatility that maintains stationarity despite presence of unit roots in the characteristic polynomial corresponding to the conditional mean. Compared to the Gaussian affine term structure model, we improve out-of-sample forecasting of the yield curve and estimate term premia that are economically plausible and consistent with survey data. Moreover, we show that volatility-induced stationarity affects the error-correcting mechanism in a system of interest rates, inflation, and real activity.

Keywords: Yield curve, error-correction, unit root, volatility-induced stationarity, macro-finance term structure model, level-dependent conditional volatility.

JEL classification: E43, E44, G12.

†I thank Torben G. Andersen, Heino Bohn Nielsen, Anders Rahbek, Viktor Todorov, and seminar and conference participants at the 11th International Conference on Computational and Financial Econometrics; the 2017 Society of Financial Econometrics Workshop at Kellogg School of Management, Northwestern University; Danmarks Nationalbank; and University of Copenhagen.

‡University of Copenhagen, Department of Economics and Danmarks Nationalbank.
Address: University of Copenhagen, Department of Economics, Øster Farimagsagde 5, 1353 Copenhagen K, Denmark. E-mail address: anne.lundgaard.hansen@econ.ku.dk.

The views of this paper are my own and do not necessarily express those of Danmarks Nationalbank.
Declarations of interest: none.

1 Introduction

Interest rates and inflation rates are extremely persistent data that can be described as unit-root processes. Moreover, there is strong co-movement between interest rates of different maturities. These stylized facts have motivated the use of cointegrated vector autoregressions (VARs) in term structure modeling starting with Engle and Granger (1987). An error-correcting model also provides a framework for testing the expectations hypothesis (Campbell and Shiller, 1987, Engle and Granger, 1987, Hall et al., 1992, Shea, 1992) and for understanding interactions between the yield curve and the macroeconomy. However, empirics as well as theory suggest that interest rates are bounded and stationary (Beechey et al., 2009), hence challenging the I(1) assumption that underpins the cointegrated VAR.

In this paper, we present a novel model of the term structure that at the same time allows for unit roots and maintains stationarity. Thus, our model embraces important features of the yield curve. We improve term structure modeling in terms of out-of-sample forecasting and term premia estimation compared to models based on VAR processes.

Rather than ensuring stationarity by assuming away unit roots from the conditional mean, the proposed model is stationary due to a level-dependent conditional volatility. Intuitively, large realizations of the process are followed by periods in which the stochastic component is weighted by a large conditional variance. The stochastic term eventually dominates the conditional mean allowing the process to be stationary despite presence of unit roots. As lagged levels of the process appear in both the conditional mean and the conditional volatility, the model has been coined the double autoregression (DAR). Ling (2004) and Nielsen and Rahbek (2014) provide conditions under which the univariate and multivariate DAR processes are stationary. The concept of volatility-induced stationary is discussed in further details in Albin et al. (2006) and Nicolau (2005).

We present an empirical macro-finance application that considers the short and long ends of the yield curve along with measures of inflation and real activity. There is evidence of volatility-induced stationarity in the interest rates and particularly in the short rate. We show that volatility-induced stationarity affects the error-correcting mechanism: in the DAR model, the yield spread is maintained as an equilibrium due to the long interest rate suggesting that market mechanisms enforce no-arbitrage. The cointegrated VAR, however, identifies the short rate as an error-correcting variable. Also, there is a second equilibrium between the short rate, inflation, and real activity, which mimics the dual mandate of the Federal Reserve. Here, volatility-induced stationarity determines whether the short rate participates in the error-correcting mechanism.

From this first set of empirical results, we conclude that the short rate is key for generating stationary dynamics. In the cointegrated VAR, this property is realized as an error-correcting behavior that is economically puzzling. In contrast, the DAR allows the short rate to induce stationarity through the conditional volatility.

Next, we ask if volatility-induced stationarity matters for term structure modeling. For this purpose, we build a no-arbitrage term structure model based on the DAR model. The model is akin to the discrete-time Gaussian affine term structure model (GATSM) studied in Ang and Piazzesi (2003), which is considered to be the workhorse in macro-finance term structure modeling. The difference is that the usual linear autoregressive dynamics are replaced by the DAR process. Since the model does not admit closed-form computation of zero-coupon bond yields, we propose an approximation in which model-implied yields are quadratic in the factor variables. In this sense, the model is related to the class of quadratic term structure models studied in Ahn et al. (2002), Leippold and Wu (2002), and Realdon (2006). However, whereas the quadratic component in these models comes from a quadratic specification of the short rate, the model considered here generates a quadratic term structure from the conditional variance. The approximation is sufficiently accurate for the purposes pursued in this paper.

We find that our DAR term structure model outperforms the GATSM in out-of-sample forecasting and obtains a closer match to survey expectations. The latter finding emphasizes that modeling unit roots and error-correction while maintaining stationarity is important for the decomposition of interest rates into expectations about future short rates and term premia. Thus, the paper contributes to the ongoing discussion of term premia estimation in interest rates that are more persistent than stationary VARs can capture, which has been dubbed the persistence problem.[1]

Volatility-induced stationarity in interest rate data was first studied by Conley et al. (1997), who consider Markov diffusion models with constant volatility elasticity as in the CKLS model in Chan et al. (1992). Conley et al. (1997) apply these models to overnight

[1]The persistence problem was first discussed in Shiller (1979) and has been approached by reducing small-sample bias as in Bauer et al. (2014) and Kim and Orphanides (2007,0); fractional cointegration in Jardet et al. (2013), Osterrieder (2013); and long-memory models in Abbritti et al. (2013), Goliński and Zaffaroni (2016).

effective federal funds rates and conclude that *"when interest rates are high, local mean reversion is small and the mechanism for inducing stationarity is the increased volatility"*. Nicolau (2005) also shows that the federal funds rate can be modelled by a process that exhibits volatility-induced stationarity. Nielsen and Rahbek (2014) extend these analyses by modeling two interest rates, namely the one- and three-month Treasury bill rates, allowing for reduced rank. Their implementation, however, does not enforce no-arbitrage restrictions. This paper contributes to this literature by (i) proposing a no-arbitrage model for the entire term structure and (ii) allowing for more than two factors, e.g., the usual level, slope, and curvature factors of the yield curve as suggested by Litterman and Scheinkman (1991) and macroeconomic factors as in Ang and Piazzesi (2003). Finally, this paper is the first to empirically study the impact of volatility-induced stationarity on error-correction analysis.

The paper is structured as follows. Section 2 introduces the DAR model and discusses how unit roots can be reconciled with stationary dynamics through volatility-induced stationarity. The data is described in Section 3. Section 4 presents the first set of empirical results and analyzes the error-correcting mechanism under volatility-induced stationarity. In Section 5, we develop a no-arbitrage term structure model whose time-series structure is given by the DAR process. Section 6 uses this model to assess the implications of volatility-induced stationarity on the term structure of interest rates. Section 7 concludes.

2 Double Autoregressive Models

The DAR model includes lagged levels of the process in both the conditional mean and the conditional variance. The conditional mean is equivalent to that of the VAR, while the conditional variance can be specified based on various multivariate GARCH models. We consider a specification that resembles the BEKK ARCH model in Engle and Kroner (1995) to obtain a DAR model for which a stationarity condition has been established. To fix ideas, let X_t be a vector of p variables and consider the simplest form of a DAR process:

$$\begin{aligned} X_{t+1} &= \mu + \Phi X_t + \Omega_{t+1}^{1/2} z_{t+1}, \\ \Omega_{t+1} &= \Sigma_0 \Sigma_0' + \Sigma_1 X_t X_t' \Sigma_1', \\ z_{t+1} &\sim \text{i.i.d. } \mathcal{N}_p\left(0, \mathrm{I}_p\right), \end{aligned} \tag{1}$$

where Σ_0 is lower triangular with strictly positive elements on the diagonal. This unique Cholesky factor ensures that the conditional variance matrix is positive definite without imposing further parameter restrictions. The parameter Σ_1 determines the sensitivity of the conditional volatility to the level of the process realized in the previous period. In the special case where all elements of this matrix are zero, the model reduces to a VAR.

2.1 Volatility-Induced Stationarity

The DAR process in (1) is stationary despite presence of unit roots. Nielsen and Rahbek (2014, Theorem 1) show that stationarity and geometric ergodicity is ensured if the top-Lyapunov exponent is strictly negative. Formally,

$$\gamma = \lim_{\xi \to \infty} \left[\mathbb{E} \left(\log \| \prod_{t=1}^{\xi} (\Phi + e_t) \| \right) \right] < 0,$$

where e_t is a $p \times p$ matrix that is i.i.d. normally distributed with mean zero and covariance matrix given by $\Sigma_1 \otimes \Sigma_1$. Thus, the interaction between the autoregressive parameter, Φ, and Σ_1 determines the stationarity properties of the model.

To understand the intuition behind this result, consider a univariate example with $\Phi = 1$. If $\Sigma_1 = 0$, the process is a random walk and non-stationary. With $\Sigma_1 \neq 0$, however, a large value of X_t results in a large conditional variance of X_{t+1}. Thus, extended periods of increasing realizations eventually imply that the stochastic component dominates the unit-root conditional mean. In result, when a negative innovation is realized to a process that is far above zero, the process is pushed downwards. In this sense, the conditional volatility can induce stationarity in unit-root processes.

The level-dependent conditional volatility also allows for extended periods of low variance when levels of the process are near zero. In such regimes, however, the variance does not vanish and the process as well as the variance will rise when a large shock arrives. These properties are attractive when considering US Treasury bond yields over the past decade, which has been stuck at the zero-lower bound although with a positive probability of a lift-off. Indeed, we shall see that the model is successful at explaining historical interest rates near the zero-lower bound.

2.2 Reduced Rank, Error-Correction, and Lag Structure

The empirical work in this paper considers two generalizations of (1). First, since interest rates and macroeconomic variables are likely to co-move, we allow for reduced rank in the autoregressive coefficient Φ. Assuming that the characteristic polynomial has q unit roots and $p-q$ roots outside the unit circle, the rank of $\Phi - I_p$ in (1) is $r = p-q$. Then, Φ can be parameterized by $\Phi = I_p + \alpha\beta'$, where α and β are $p \times r$ matrices. This structure facilitates an error-correction interpretation of the dynamics, where α determines the adjustment to long-run equilibria given by $\beta' X_t$. Second, to achieve a well-specified model we allow for an extensive lag structure in the conditional mean. We find that this generalization is sufficient to match the data and thus we leave the conditional variance as specified in (1). The resulting generalized DAR model is given by

$$\begin{aligned} X_{t+1} &= \mu + \Phi X_t + \sum_{k=1}^{K} \Gamma_k \Delta X_{t+1-k} + \Omega_{t+1}^{1/2} z_{t+1}, \\ \Omega_{t+1} &= \Sigma_0 \Sigma_0' + \Sigma_1 X_t X_t' \Sigma_1', \\ z_{t+1} &\sim \text{i.i.d. } \mathcal{N}_p(0, \mathrm{I}_p) \end{aligned} \tag{2}$$

for $K \geq 1$ and $\Phi = \mathrm{I}_p + \alpha\beta'$. The associated top-Lyapunov exponent is given by

$$\gamma = \lim_{\xi \to \infty} \left[\mathbb{E} \left(\log \| \prod_{t=1}^{\xi} (\tilde{\Phi} + \tilde{e}_t) \| \right) \right] < 0,$$

where $\tilde{e}_t$ has dimension $p(K+1) \times p(K+1)$ and is i.i.d. normal with mean zero and covariance matrix equal to $\tilde{\Sigma}_1 \otimes \tilde{\Sigma}_1$. $\tilde{\Phi}$ and $\tilde{\Sigma}_1$ are defined by

$$\tilde{\Phi} = \begin{pmatrix} \Phi + \Gamma_1 & \Gamma_1 - \Gamma_2 & \dots & \Gamma_K - \Gamma_{K-1} & -\Gamma_K \\ \mathrm{I}_p & 0_{p\times p} & \dots & 0_{p\times p} & 0_{p\times p} \\ 0_{p\times p} & \mathrm{I}_p & & 0_{p\times p} & 0_{p\times p} \\ \vdots & & \ddots & & \vdots \\ 0_{p\times p} & 0_{p\times p} & \dots & \mathrm{I}_p & 0_{p\times p} \end{pmatrix}, \quad \tilde{\Sigma}_1 = \begin{pmatrix} \Sigma_1 & 0_{p\times pK} \\ 0_{pK\times p} & 0_{pK\times pK} \end{pmatrix}.$$

3 Data

Consider a system of four variables: the short rate, or one-period interest rate (r_t); a long-term interest rate (R_t); and two macroeconomic measures interpreted as respectively inflation (π_t) and real activity (g_t). Let X_t be the vector containing these variables, $X_t = [r_t, R_t, \pi_t, g_t]'$.

We use monthly data between January 1985 and December 2016. The short rate is measured by the one-month US Treasury Bill rate from the Fama Treasury Bills Term Structure Files available at CRSP. We define the long rate by the ten-year US Treasury bond yield from Gürkaynak et al. (2007). These data are observed end-of-month.

	Explained (pct)	Empirical correlation coefficients					
		CPI	PPI	UNEMP	EMP	PROD	HELP
π_t	85.49	0.92	0.92	-	-	-	-
g_t	65.75	-	-	-0.71	0.93	0.71	0.87

Table 1: The percentage of variation in respectively inflation data (CPI, PPI) and data related to real activity (unemployment rates, UNEMP, employment growth rates, EMP, the production index, PROD, and the HELP index, HELP) explained by respectively the inflation measure (π_t) and the real-activity measure (g_t). Empirical correlation coefficients between these measures and the underlying variables are shown as well.

The macroeconomic variables are constructed following the approaches in Ang and Piazzesi (2003) and Goliński and Zaffaroni (2016). The inflation measure, π_t, is thus given by the first principal component of standardized series of CPI and PPI data from the US Bureau of Labor Statistics. Analogously, the measure of real activity, g_t, is the first principal component of standardized data on the unemployment and employment growth rates from the US Bureau of Labor Statistics; the industrial production index from Federal Reserve Economic Data; and the help-wanted-advertising-in-newspapers (HELP) index from Barnichon (2010).

Table 1 details how π_t and g_t correlate with the underlying observed data as well as the fraction of total variation they capture. The inflation variable, π_t, is highly correlated with both inflation measurements and explains 85 pct of the total variation in these data. The variable measuring real activity, g_t, correlates strongest with employment growth rate and the HELP index. Correlation with the unemployment rate is negative as expected. Our measure of real activity captures 66 pct of the variation in the underlying observables.

The data for $X_t = [r_t, R_t, \pi_t, g_t]'$ are exhibited in Figure 1. The series appear extremely persistent and use of conventional unit root and stationarity tests indeed identify unit roots, see Table 9 in Appendix C. Therefore, modeling this data using a linear VAR involves the implicit assumption that interest rates are generated by non-stationary processes, which is puzzling from both theoretical and empirical standpoints. Therefore, we propose the DAR model for these persistent and stationary data.

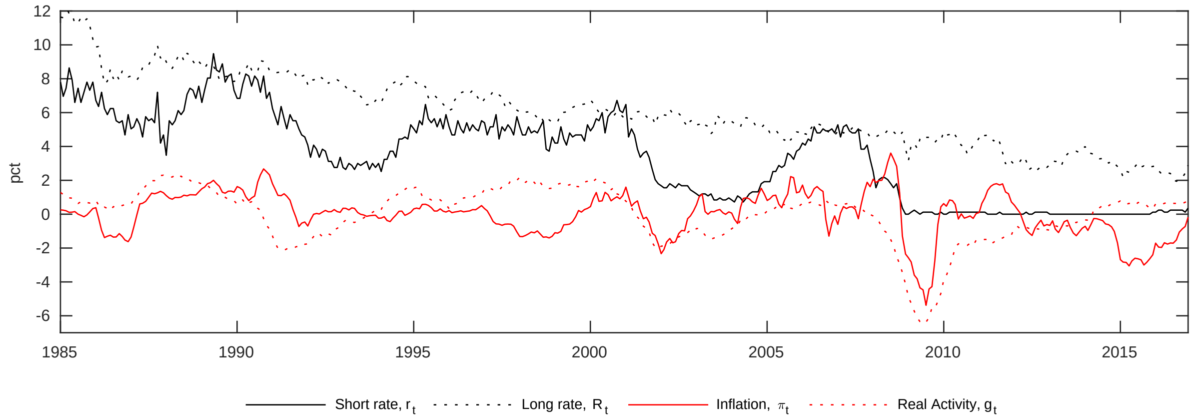

Figure 1: Data for $X_t = [r_t, R_t, \pi_t, g_t]'$. The interest rates, r_t and R_t, are the 1-month Treasury bill rate and the 10-year Treasury bond yield. The inflation measure, π_t, is the first principal component of CPI and PPI rates. Real activity, g_t, is measured the first principal component of unemployment and employment growth rates, and the industrial production and HELP indices.

4 Volatility-Induced Stationarity and Error-Correction

This section presents the empirical analysis of the DAR model in (2) for the data, $X_t = [r_t, R_t, \pi_t, g_t]'$. In particular, we study the mechanisms related to volatility-induced stationarity and error-correction.

4.1 Estimation Results

The data suggest a reduced rank of $r = 2$ and lag-length of $K = 3$, when specification testing is conducted with use of conventional methods for VAR models. With these choices, the DAR model appears to be well-specified. In fact, the DAR model removes autocorrelation and improves normality tests of the standardized residuals in comparison with the VAR. Details are provided in Appendix A.

Thus, we estimate the DAR model with $r = 2$ and $K = 3$ along with identifying restrictions also described in the appendix. For reference, we also provide estimates of the cointegrated VAR which corresponds to the DAR with restriction $\Sigma_1 = \mathbf{0}_{4\times 4}$. The DAR obtains the lowest AIC value, see Table 2, and the likelihood values of the models are significantly different when compared by a LR test. Moreover, we note that the estimated top-Lyapunov exponent in the DAR is strictly negative.[2] Therefore, the process

[2]The Lyapunov exponents are obtained by the efficient and numerically stable algorithm described in Nielsen and Rahbek (2014).

	DAR	VAR
Log-likelihood	6986	6823
Akaike information criteria	−13806	−13507
Top-Lyapunov exponent	−0.004	

Table 2: Log-likelihood and Akaike information criteria for the DAR and VAR models. Top-Lyapunov exponent is reported for the DAR.

is stationary despite presence of unit roots. This feature contrasts the VAR dynamics in which unit roots result in a non-stationary model. We shall see in Section 4.3 that this difference carry implications for the long-run adjustments implied by the models.

4.2 Evidence and Sources of Volatility-Induced Stationarity

The estimated model-implied conditional variances are exhibited in Figure 2. In the DAR model, conditional covariances are highly time-varying and fluctuate around the constant levels estimated by the VAR. There are pronounced spikes at the outbreak of the financial crisis in 2007/08 and variances are small and near constant during the zero-lower bound regime in the aftermath of the crisis.

The DAR model allows all variables to exhibit volatility-induced stationarity and furthermore, the conditional heteroskedasticity can be driven by all variables. This general setting allows us to make statements about (i) which variables exhibit volatility-induced stationarity? and (ii) what variables drive these effects? From Figure 2, the short rate stands out with a highly volatile conditional variance that ranges from high and rapidly changing to low and stable. Conditional volatility tends to be low and stable during recessions, e.g., during the current zero-lower bound regime. From the estimation results in Table 10, Appendix C the conditional variance of the short rate is given by

$$\mathrm{Var}_t\left(r_{t+1}\right) = \left(95.7 r_t + 18.9 R_t + 15.2 \pi_t\right)^2,$$

where insignificant coefficients are suppressed. Thus, volatility-induced stationarity in the short rate is mainly driven by the short rate itself, but also by the long rate and inflation.

Figure 2 also shows the conditional correlations in the DAR model. The short rate correlates positively with the long rate through the majority of the sample implying that the monetary transmission mechanism from short to long rates works in normal times.

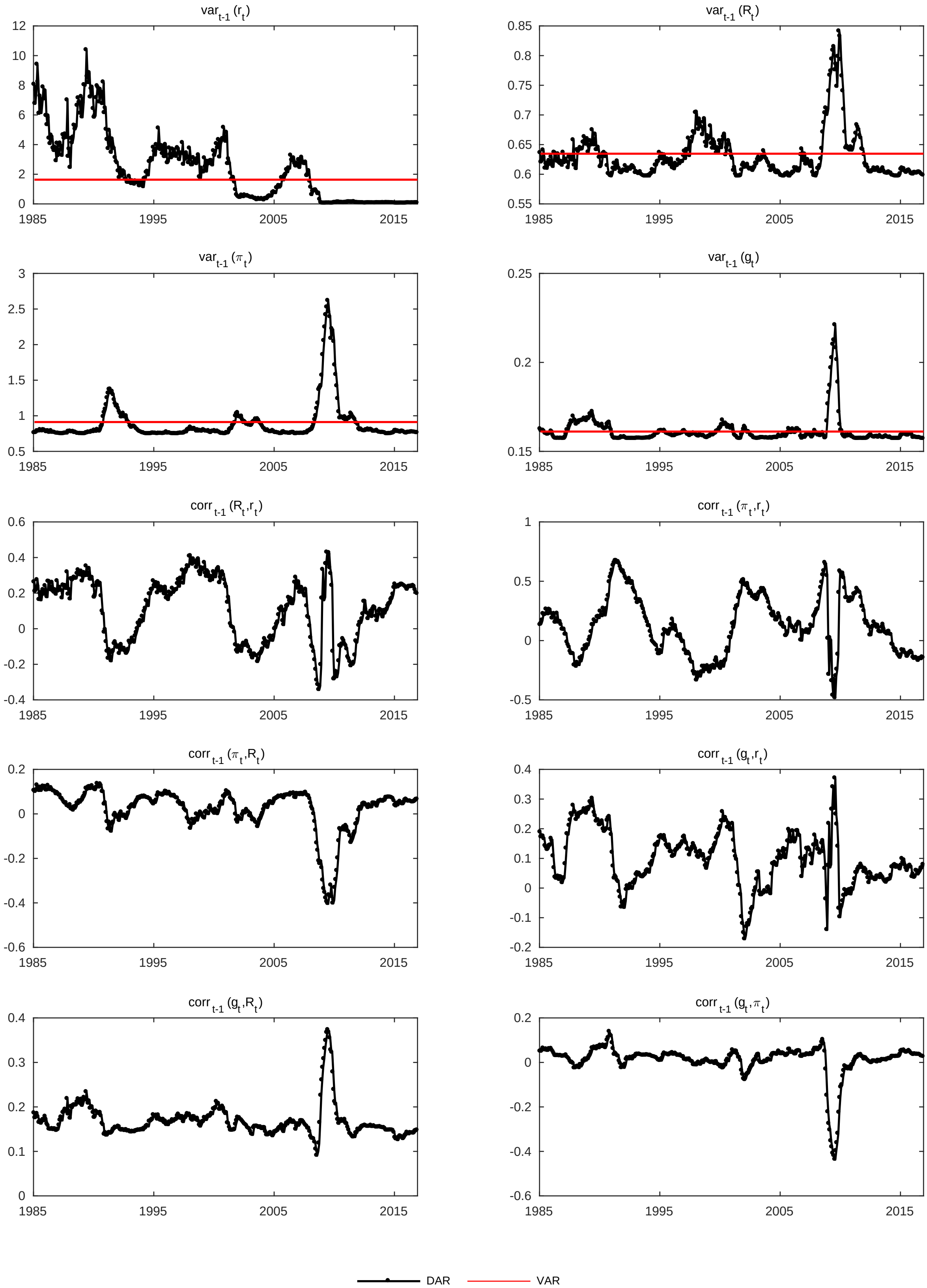

Figure 2: Estimated conditional volatilities and correlations. Reported in bps.

	DAR				VAR			
	r_t	R_t	π_t	g_t	r_t	R_t	π_t	g_t
α'	0	0	0.017 (0.008)	0.008 (0.003)	−0.016 (0.008)	0	0.017 (0.009)	0.009 (0.003)
	0	−0.031 (0.012)	0	0	0.050 (0.027)	0	−0.033 (0.017)	0
β'	1	0	−3.672 (1.150)	−1.681 (0.611)	1	0	−3.097 (0.911)	−2.174 (0.715)
	−1	1	0	0	−1	1	0	0

Table 3: Estimates of the coefficients in the long-run relations, β, and adjustment speeds, α, in the DAR (left panel) and VAR (right panel). Standard errors are in parentheses. Remaining estimation results are reported in Table 10 in Appendix C.

However, in times of crisis in which the short rate is lowered to low levels with stable variance, the correlation becomes negative.

4.3 Long-Run Dynamics and Error-Correction

Table 3 summarizes the long-run dynamics of the models. The DAR and VAR models carry similar predictions concerning the long-run relations, $\beta' X_t$: one relation is the spread between long and short rates as in Hall et al. (1992), and the short rate, inflation, and real activity constitute another relation. Since the short rate follows the Federal Funds rate closely as illustrated in Figure 8 in Appendix C, this relation mimics the dual mandate of the Federal Reserve (Fed). In addition, the signs of the estimates are intuitive: a low interest rate is associated with high levels of inflation and real activity.

The models, however, produce different estimates of the adjustment matrix. In the DAR, adjustment to the yield spread is maintained by the long rate, whereas the VAR predicts that the short rate adjust towards this equilibrium. To the extend that the yield spread relation reflects no-arbitrage, we should expect the long rate, i.e., market mechanisms, to erode short-lived arbitrage opportunities rather than the short rate, which is primarily controlled by the Fed.[3] As for the second relation, the models disagree on whether the short rate enters the error-correcting mechanism jointly with the macro variables. The fact that the Fed determines the policy discretionally rather than mechanically

[3]While the FOMC does use information on developments in the long-term yield to make decisions about the discount rate (Bernanke, 2006), it is not concerned with maintaining a stable spread.

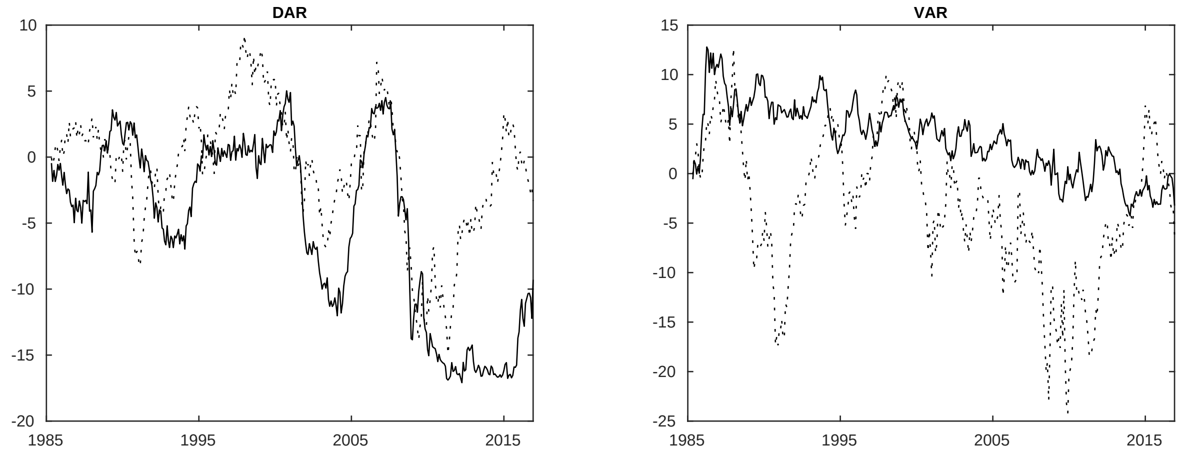

Figure 3: Common stochastic factors computed by $\hat{\alpha}'_{\perp} \sum_{s=1}^{t} \hat{\Omega}_s^{1/2} \hat{z}_s$ for the DAR and VAR.

from macroeconomic variables (Bernanke, 2015) is an argument for the plausibility of the DAR results in which the short rate is not significant in the adjustment matrix.

In the analysis of the cointegrated VAR, one is often interested in the common stochastic factors determined by the orthogonal complement of the adjustment matrix, $\alpha_{\perp}$, as $\hat{\alpha}'_{\perp} \sum_{s=1}^{t} \hat{\Omega}^{1/2} \hat{z}_s$. These are plotted in the right panel of Figure 3. By comparing with the data plot in Figure 1, we identify that the state vector is partly driven by shocks to the long rate according to the VAR model. A similar conclusion is reached by Hall et al. (1992) also in an affine model but with yield data only. They interpret these results as evidence of the expectations hypothesis: the term structure is driven by the long rate to which current short rates adjust using the information contained in the yield spread. We shall see in Section 6 that the VAR is indeed consistent with the expectations hypothesis as it implies almost constant term premia.

The left panel of the figure shows the equivalent to stochastic common trends in the DAR model. In this framework, the state vector appears to be driven by the short rate rather than the long rate. Hence, the DAR predicts that the Fed can affect the dynamics of the entire yield curve by controlling the short rate and that the market will adjust to such monetary policy shocks.

In conclusion, we find that the short rate is an important factor for ensuring well-defined long-run dynamics. In the cointegrated VAR, this property is realized as an error-correcting behavior that is economically puzzling. In contrast, the DAR allows the short rate to induce stationarity through the conditional volatility.

5 No-Arbitrage Term Structure Model with DAR Dynamics

This section casts the DAR dynamics analyzed thus far into a macro-finance term structure model. Thus, consider a four-factor term structure model with the observable state vector $X_t = [r_t, R_t, \pi_t, g_t]'$ whose dynamics is given by (2). We specify the term structure model such that it is comparable with the discrete-time Gaussian affine term structure model (GATSM) studied in Ang and Piazzesi (2003). This choice has two advantages. First, the GATSM is considered to be the workhorse model in the macro-finance term structure literature. Second, since the VAR process is nested in the DAR model, the GATSM will be nested in the specified term structure model with DAR dynamics.

5.1 Stochastic Discount Factor

We adopt the standard linear-exponential stochastic discount factor given by

$$\mathcal{M}_{t+1} = \exp\left(-r_t - \frac{1}{2}\Lambda_t'\Omega_{t+1}\Lambda_t - \Lambda_t'\Omega_{t+1}^{1/2}z_{t+1}\right), \tag{3}$$

where Λ_t is the market price of risk with risk measured by the conditional variance, Ω_{t+1}. We specify the market price of risk such that the factor dynamics under the risk-neutral $\mathbb{Q}$-measure follows a DAR model. Moreover, to reduce the number of parameters we treat lagged variables as unspanned factors.[4] Thus, the lag structure determines the dynamics under the real-world measure but is not priced in the term structure cross-section. The resulting market price of risk is given by

$$\Lambda_t = \Omega_{t+1}^{-1}\left(\lambda_0 + \lambda_1 X_t\right), \tag{4}$$

which is time-t measurable as Ω_{t+1} depends on X_t.

Under the real-world $\mathbb{P}$-measure, X_{t+1} follows a conditional normal distribution given the past, $\mathcal{F}_t = \{X_t,\ X_{t-1},\ \ldots,\ X_1\}$. Therefore, the change of measure implied by the stochastic discount factor in (3) is given by a discrete-time analogue to the Girsanov theorem. In particular, the conditional mean is risk-adjusted by $-\Omega_{t+1}\lambda_t$, while the conditional variance is equal to that under the $\mathbb{P}$-measure. The risk-neutral $\mathbb{Q}$-dynamics

[4] Models with unspanned factors are studied in Duffee (2011), Joslin et al. (2014) and Ludvigson and Ng (2009) among others.

is given by the following DAR model:

$$
\begin{aligned}
X_{t+1} &= \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t + \sum_{k=0}^{K} \Gamma_k \Delta X_{t-k} + \Omega_{t+1}^{1/2} z_{t+1}, \\
\Omega_{t+1} &= \Sigma_0 \Sigma_0' + \Sigma_1 X_t X_t' \Sigma_1', \\
z_{t+1}^{\mathbb{Q}} &=\sim \text{i.i.d. } \mathcal{N}_p(0, \mathrm{I}_4),
\end{aligned}
\tag{5}
$$

where $\mu^{\mathbb{Q}} = \mu - \lambda_0$, $\Phi^{\mathbb{Q}} = \Phi - \lambda_1$. Finally, note that per construction of the state vector, the short rate and the state vector are related by $r_t = \iota_1' X_t$ where ι_1' is a unit vector with one in the first entry.

These assumptions complete the term structure model with DAR dynamics. The GATSM is contained as a special case for $\Sigma_1 = \mathbf{0}_{4\times 4}$, i.e., when the factor dynamics reduces to a VAR.

5.2 Approximation for Analytical Bond Pricing

The no-arbitrage time-t price of a zero-coupon bond with $n+1$ periods to maturity is given by

$$
P_{t,n+1} = \mathbb{E}_t \left(\mathcal{M}_{t+1} P_{t+1,n} \right),
$$

where $\mathbb{E}_t(\cdot)$ denotes the conditional expectation given $\mathcal{F}_t = \{X_t, X_{t-1}, \ldots, X_1\}$ under real-world probabilities. Our model does not admit a closed-form bond price expression that satisfies this equation. Instead, we propose an exponential-quadratic approximation that allows the conditional covariance matrix to affect bond yields. This is similar to the GATSM in which the closed-form solution depend on the constant conditional variance, see Ang and Piazzesi (2003). Also, we make sure that for $\Sigma_1 = \mathbf{0}_{4\times 4}$, bond yield computation must coincide with the solution of the GATSM. Appendix B shows that such an approximation can be obtained by controlling the dynamics of the conditional variance under the $\mathbb{Q}$-measure. The resulting approximation is given by

$$
P_{t,n} = \exp\left(A_n + B_n' X_t + C_n' \mathrm{vec}\left(X_t X_t' \right) \right), \tag{6}
$$

where

$$
\begin{aligned}
A_n &= A_{n-1} + B_{n-1}' \mu^{\mathbb{Q}} + C_{n-1}' \left(\mathrm{vec}\left(\mu^{\mathbb{Q}} \mu^{\mathbb{Q}'} \right) + \mathrm{vec}\left(\Sigma_0 \Sigma_0' \right) \right) + \frac{1}{2} B_{n-1}' \Sigma_0 \Sigma_0' B_{n-1} \\
B_n' &= -\iota_1 + B_{n-1}' \Phi^{\mathbb{Q}} + C_{n-1}' \left(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}} + \mu^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}} \right) \\
C_n' &= C_{n-1}' \left(\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}} + \Sigma_1 \otimes \Sigma_1 \right) + \frac{1}{2} \left([B_{n-1}' \Sigma_1] \otimes [B_{n-1}' \Sigma_1] \right)
\end{aligned}
$$

initiated at $n = 0$ with $A_0 = 0$, $B_0 = \mathbf{0}_{p\times 1}$, $C_0 = \mathbf{0}_{p^2\times 1}$.[5] The associated approximated bond yield is given by

$$Y_{t,n} = -\frac{1}{n}\log\left(P_{t,n}\right) = -\frac{1}{n}A_n - \frac{1}{n}B_n' X_t - \frac{1}{n}C_n' \mathrm{vec}\left(X_t X_t'\right). \tag{7}$$

The approximated bond yield expression is similar to the solution of the class of quadratic term structure models (QTSMs) studied in Leippold and Wu (2002), Ahn et al. (2002), and Realdon (2006). Thus, the model based on DAR dynamics and the QTSM can produce similar shapes of the yield curve. However, the source of the quadratic term and thus the loading recursions are highly different across the two model frameworks: whereas the quadratic bond yield stems from the variance specification in the DAR, the QTSM imposes this non-linearity through a quadratic specification of the short rate. This difference is particularly highlighted in the macro-finance model considered in this paper in which the short rate is a factor itself. In this setting, the short-rate specification is linear per construction and the QTSM reduces in this case to the GATSM.

6 Term Structure Results

6.1 Estimation and In-Sample Performance

We estimate $\mu^{\mathbb{Q}}$ and $\Phi^{\mathbb{Q}}$ by non-linear least squares given the parameters obtained for the factor dynamics in Section 4.1, $\hat{\Theta}^{\mathbb{P}} = \{\hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\Sigma}_0, \hat{\Sigma}_1, \hat{\Gamma}_1, \hat{\Gamma}_2, \hat{\Gamma}_3\}$.[6] The yield data are annualized, continuously compounded yields of US Treasury bonds from Gürkaynak et al. (2007) with maturities $n = 1, 2, \ldots, 10$ years. Let $Y^O_{t,n}$ denote the observed n-month yield and $Y_{t,n}$ the corresponding model-implied yield computed by (7). Then, the estimation problem is given by

$$\min_{\mu^{\mathbb{Q}},\Phi^{\mathbb{Q}}} \quad \sum_{t=1}^{T}\sum_{i=1}^{10}\left\{Y^O_{t,12i} - Y_{t,12i}\left(\mu^{\mathbb{Q}}, \Phi^{\mathbb{Q}}, \hat{\Theta}^{\mathbb{P}}\right)\right\}^2$$

To achieve convergence of this highly non-linear problem, we first estimate $\Phi^{\mathbb{Q}}$ with $\mu^{\mathbb{Q}} = \mathbf{0}_{4\times 1}$ and then, $\mu^{\mathbb{Q}}$ given the estimated value of $\Phi^{\mathbb{Q}}$. Using these estimates as starting values, the parameters are re-estimated jointly. The results are shown in Table 4.

[5] To evaluate the approximation error, we proxy the exact solution by averaging 10,000,000 paths of $\exp\left(-\sum_{i=0}^{n-1}\iota_1'\hat{X}_t\right)$, where $\hat{X}_t$ is simulated under the $\mathbb{Q}$-measure according to (5). Then, this simulated bond price is converted to yields. We repeat this procedure for all months of January in the sample using parameter values reported in Table 4. The approximation error is largest for the ten-year yield, for which the average absolute error is 38 bps corresponding to 6.94 pct of the average ten-year yield level.

[6] This two-step estimation method is a common approach in the macro-finance term structure literature, see for instance Ang and Piazzesi (2003) and Ang et al. (2006).

	Term structure model with DAR				GATSM			
$(\mu^{\mathbb{Q}})' \times 10^{-3}$	-2.591 (1.336)	-2.765 (1.212)	1.459 (0.731)	5.371 (2.490)	-0.140 (0.050)	0.066 (0.047)	-0.284 (0.284)	-0.052 (0.095)
$\Phi^{\mathbb{Q}}$	-1.336 (0.235)	1.615 (0.329)	1.413 (0.085)	-0.130 (0.087)	0.548 (0.011)	0.251 (0.015)	-0.010 (0.010)	-0.029 (0.010)
	-2.451 (0.197)	2.728 (0.178)	1.750 (0.213)	-0.059 (0.089)	-0.037 (0.008)	1.034 (0.011)	0.157 (0.008)	0.068 (0.006)
	1.374 (0.217)	-0.876 (0.157)	0.550 (0.195)	0.046 (0.068)	0.043 (0.044)	0.050 (0.066)	1.396 (0.045)	0.035 (0.037)
	4.635 (0.297)	-3.240 (0.458)	-2.961 (0.251)	1.613 (0.169)	0.014 (0.018)	0.022 (0.026)	0.058 (0.016)	1.217 (0.015)

Table 4: Estimated conditional mean parameters under the $\mathbb{Q}$-measure given the estimated parameters of the factor dynamics, $\hat{\Theta}^{\mathbb{P}} = \{\hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\Sigma}_0, \hat{\Sigma}_1, \hat{\Gamma}_1, \hat{\Gamma}_2, \hat{\Gamma}_3\}$. Standard errors in paranthesis.

In the DAR-based term structure model, $\Phi^{\mathbb{Q}}$ has eigenvalues outside the unit circle. However, due to the time-varying conditional volatility, the model remains stationary under the $\mathbb{Q}$-measure with a top-Lyapunov exponent of -0.017. The maximum eigenvalue of $\Phi^{\mathbb{Q}}$ in the GATSM is 0.9999.

Model-implied yields are plotted for the one- and five-year maturities in Figure 4. The immediate conclusion is that volatility-induced stationarity does not affect in-sample fit: the term structure model with DAR and the GATSM both match the observed yields closely. This is not surprising because the differences between the models are given in the conditional variance, which only affects bond yields through a convexity effect that is small (Joslin and Konchitchki, 2018).

6.2 Out-of-Sample Performance

We assess the out-of-sample performance through a rolling-window forecasting exercise. In particular, we estimate the models with one lag in the factor dynamics for the sample from January 1985 to December 2005 ($T = 252$). Using these estimated models, the yield curve is forecasted 3, 6, and 12 months ahead. We repeat this procedure by re-estimating the models based on a rolling-window sample of length $T = 252$ from January 2006 to December 2015. This period contains events that are difficult to forecast including the financial crisis of 2007/08 and the zero-lower bound regime.

Root mean squared errors from this exercise are presented in Table 5 along with ran-

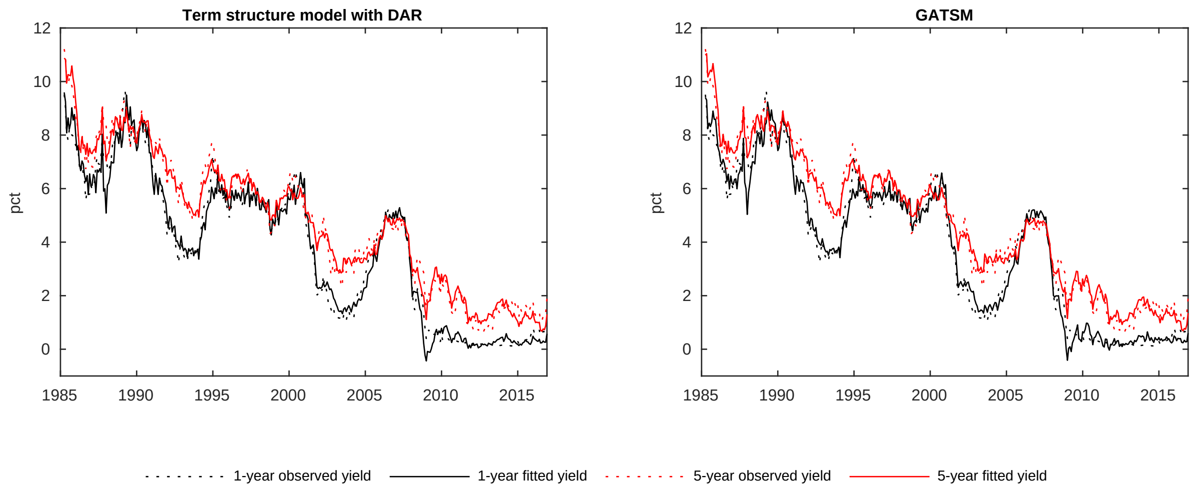

Figure 4: Observed and fitted yields with maturities 1 and 5 years.

dom walk forecasts. The term structure model with DAR outperforms both the GATSM and the random walk for all maturities but the 10-year yield. The differences between the models' forecasting performance are larger for shorter maturity reflecting that volatility-induced stationarity is generated by the short end of the yield curve. Thus, volatility-induced stationarity clearly improves out-of-sample forecasting of the yield curve.

6.3 Model-Implied Expectations and Survey Data

Estimation of term premia is an important application of term structure models, e.g., for understanding the reasons behind interest rate movements which serve as a guideline for policy makers (Bernanke, 2006). Term premia are defined by the difference between no-arbitrage bond yields and an expectations component given by the following conditional expectation under real-world probabilities:

$$\tilde{Y}_{t,n} = -\frac{1}{n} \log \mathbb{E}_t \left(\exp \left[-\sum_{i=0}^{n-1} r_{t+i} \right] \right).$$

While the term structure model based on DAR dynamics does not outperform the GATSM in matching yields, there are good reasons to expect that volatility-induced stationarity help estimating expectations under the $\mathbb{P}$-measure as the DAR process simultaneously allows for stationarity and unit roots in the short rate. In the cointegrated VAR dynamics underpinning the GATSM, the short rate is a random walk such that the expectations component is almost equivalent to current yields and term premia are almost constant.

To empirically evaluate the models in this respect, we compare model-implied expectations with survey expectations. The survey data is from the Survey of Professional

	Model with DAR			GATSM			Random Walk		
Horizon	3M	6M	12M	3M	6M	12M	3M	6M	12M
Average	**58.06**	**71.77**	**87.42**	69.07	88.21	102.74	63.37	79.48	125.61
1Y	65.90	**85.18**	**113.43**	99.57	131.34	168.83	**65.53**	102.71	175.08
2Y	**63.23**	**78.86**	**99.29**	86.35	111.69	136.38	64.60	91.40	156.03
3Y	**61.77**	**75.03**	**90.18**	77.96	99.04	115.84	64.20	83.29	140.76
4Y	**60.24**	**72.83**	**85.93**	72.16	90.75	103.14	63.82	78.08	129.29
5Y	**58.33**	**70.88**	**83.53**	67.52	84.74	94.88	63.45	75.12	120.86
6Y	**56.33**	**69.02**	**81.94**	63.50	79.94	89.03	63.12	73.64	114.60
7Y	**54.55**	**67.39**	**80.76**	59.96	75.87	84.56	62.80	73.00	109.85
8Y	**53.24**	**66.16**	**79.95**	56.90	72.36	80.96	62.47	72.73	106.11
9Y	**52.64**	**65.56**	**79.46**	54.35	69.37	78.05	62.07	72.54	103.05
10Y	54.36	**66.78**	79.72	**52.42**	66.95	**75.77**	61.60	72.28	100.44

Table 5: Root mean squared errors from forecasting the yield curve using the term structure model with DAR and the GATSM. The models are estimated on a rolling window starting with the sample from January 1985 to December 2005. Forecasts by the random walk are reported for reference. The minimum value obtained for each forecast horizon and maturity is boldfaced. Reported in bps.

	Term structure model with DAR			GATSM		
	Short rate	Long rate	Inflation	Short rate	Long rate	Inflation
3M	**39.71**	**217.80**	**106.38**	44.09	221.18	115.33
6M	**49.14**	**200.23**	**98.49**	58.74	205.46	113.78
1Y	**74.56**	**167.48**	**92.66**	83.90	174.33	106.66

Table 6: Root mean squared errors of model-implied expectations under the real-world measure compared to survey data from Survey of Professional Forecasters. Reported in bps.

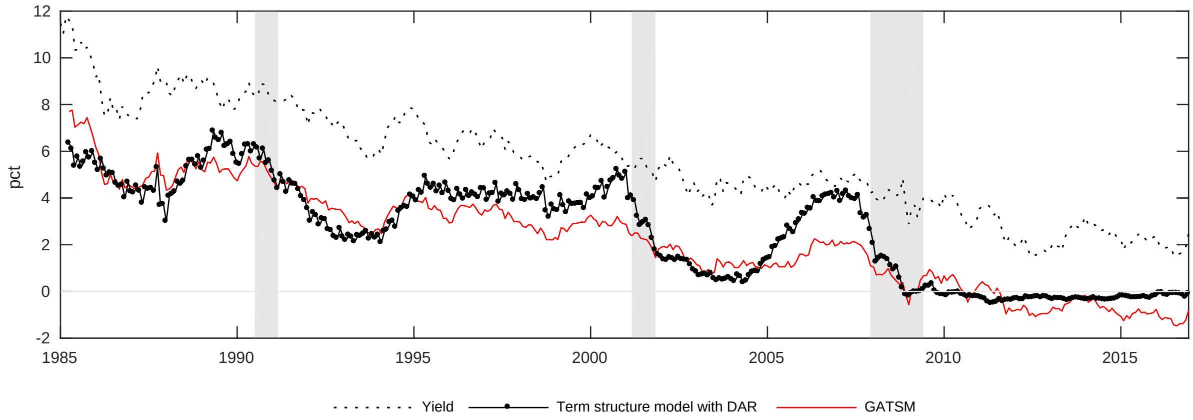

Figure 5: Expectations components for the 10-year maturity implied by the term structure model with DAR and the GATSM. The 10-year yield is plotted for reference. Shaded areas mark recessionary periods as defined by NBER.

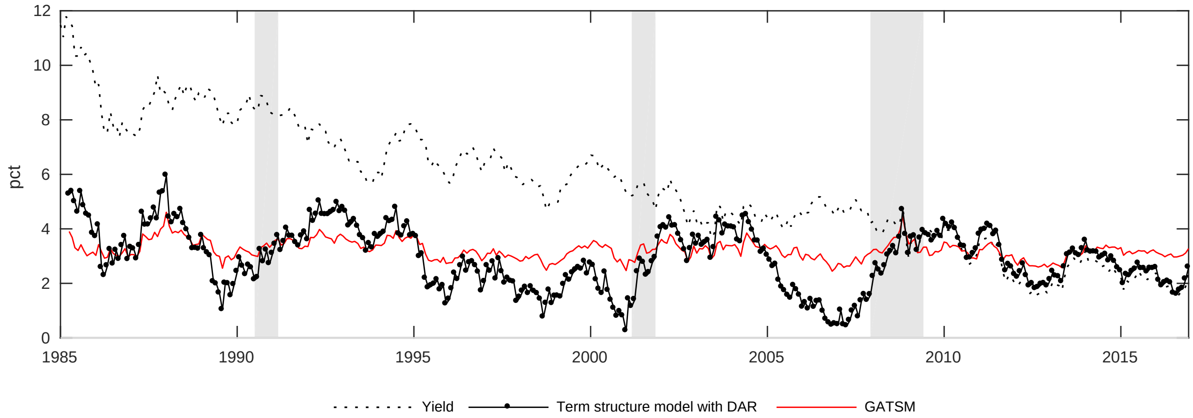

Figure 6: 10-year term premium implied by the term structure model with DAR and the GATSM. The 10-year yield is plotted for reference. Shaded areas mark recessionary periods as defined by NBER.

Forecasters (SPF) conducted by the Federal Reserve Bank of Philadelphia on a quarterly basis. We use median forecasts of the three-month Treasury bill rate as proxy for the short rate; the ten-year Treasury bond yield; and standardized CPI inflation following the construction of the inflation factor. We compare this data to expectations of the corresponding risk factors computed with respectively the model with DAR dynamics and the GATSM. Table 6 compares root mean squared errors in bps between model-implied and survey expectations for forecasting horizons of 3, 6, and 12 months ahead. The results unambiguously show that volatility-induced stationarity help matching market expectations.

The model-implied expectations components are plotted in Figure 5, which supports the a priori hypothesis: the GATSM assigns all variation of interest rates to the expectations component because the model does not allow unit-root processes to be stationary. The effect of volatility-induced stationarity is clearly illustrated in the figure by the fact that the term structure model based on DAR dynamics allows the expectations component to move along different patterns than the interest rate itself.

The implications for term premia estimates are shown in Figure 6. The DAR dynamics implies term premia that are time-varying, but stable, and countercyclical. In contrast, the GATSM estimates term premia that are almost constant and do not vary with business cycles. These differences carry over to different policy recommendations. For instance, in the period leading to the financial crisis of 2007/08, the DAR model suggests a downward trend in the term premium, which signals a movement towards more risk-taking among investors calling for a monetary policy tightening. The GATSM, however, does not predict a change in term premium in this period and, combined with a stable interest rate, does not suggest neither policy easing or tightening.

6.4 Conditional Volatilities

The DAR model distinguishes from the GATSM by a time-varying conditional volatility of the factor dynamics. Thus, a natural question is, how well does the term structure model with DAR dynamics capture the conditional volatility of the yield curve?

To compute model-implied conditional yield volatilities based on the approximative bond yield equation in (7), we apply a local linearization around X_{t-1} such that $\mathrm{Var}_{t-1}(Y_{t,n}) \approx [B_n' + C_n' f(X_{t-1})]\, \Omega_t\, [B_n' + C_n' f(X_{t-1})]' / n^2$, where the function $f(\cdot)$ is defined by $f(a) = \partial \mathrm{vec}\,(X_t X_t') / \partial X_t' |_{X_t = a}$. This measure is compared to two proxies of the unobserved conditional variance: realized variance computed from daily observations, available from Gürkaynak et al. (2007), and rolling-sample variance with a 6-month look-back. The results are shown for the one- and five-year yields in Figure 7. The graphs also show the constant conditional volatility of the GATSM given by $\mathrm{var}_{t-1}(Y_{t,n}) = B_n' \Sigma_0 \Sigma_0' B_n / n^2$.

Considering the one-year maturity, the DAR term structure model captures the low conditional variance during the zero-lower bound regime with a peak at the outbreak of the financial crisis of 2007/08. Apart from these observations, the model-implied volatility appears to be more correlated with levels of the risk factors than with the conditional volatility proxies. At the longer maturity, the level of the model-implied volatility is consistent with the proxies, but the fluctuations are not captured.

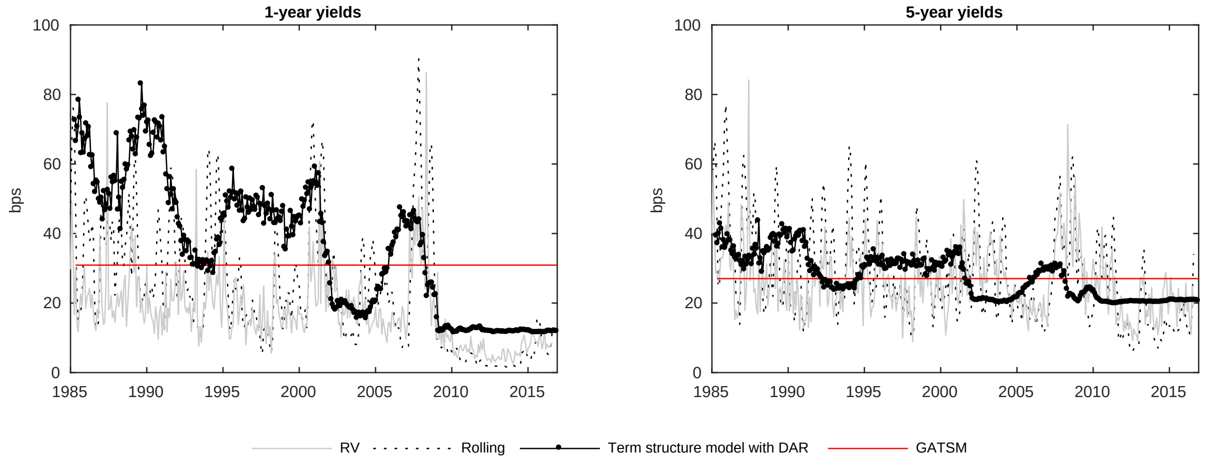

Figure 7: Conditional volatilities of 1- and 5-year yields implied by the DAR term structure model and the GATSM. The former is computer by a local linearization. Proxies for conditional volatilities of the data given by respectively realized variances (RV) based on daily data and rolling-sample variance with a 6-month look-back are plotted for reference. Reported in bps.

These results are consistent with Filipović et al. (2017), who find evidence that yield volatilities are level-dependent near the zero-lower bound but unspanned at higher levels. In fact, an expanding literature emphasizes that separate factors drive respectively first- and second-order moments of the yield curve (Cieslak and Povala, 2016, Haubrich et al., 2012). This includes unspanned stohcastic volatility models studied in Andersen and Benzoni (2010), Collin-Dufresne and Goldstein (2002), Creal and Wu (2015), Joslin et al. (2014).

7 Conclusion

This paper presented a macro-finance term structure model based on the error-correcting double autoregression. The model accommodates the extreme persistence observed in interest rates and inflation without implying non-stationarity by reconciling unit roots and stationarity. This property is driven by a level-dependent conditional volatility, hence the model is volatility-induced stationary.

We demonstrated empirically that US Treasury bond yields exhibit volatility-induced stationarity especially in the short end of the yield curve. Moreover, volatility-induced stationarity alters the error-correcting mechanism and improves term structure modeling compared to results obtained with the cointegrated VAR. Specifically, while the DAR and VAR models agree on the relations that determine the yield curve, the models carry different predictions about the error-correcting mechanisms. With respect to term structure modeling, volatility-induced stationarity improves out-of-sample forecasting and term premia estimation.

Although the DAR process is heteroskedastic, the resulting term structure model does not match conditional yield volatilities well. Fulfilling that objective may require restrictions that separate factors of the conditional variance from those of the conditional mean; that second-order moments are targeted in estimation as in Cieslak and Povala (2016); or inclusion of derivatives in the information set as in Almeida et al. (2011), Bikbov and Chernov (2011), and Jagannathan et al. (2003). We leave this topic for future research.

A Specification and Estimation of the DAR

We specify the model in (2) by use of conventional methods. In particular, the rank is determined by the likelihood-ratio test in Johansen (1991), which is based on the Gaussian VAR, with critical values obtained using the wild bootstrap procedure in Cavaliere et al. (2014). The lag structure of the test is specified by general-to-specific LR tests, information criteria, and misspecification tests. For the choice of three months lags, the residuals are not autocorrelated according to univariate Ljung-Box tests, see the right panel of Table 8. The likelihood-ratio test, for which results are reported in Table 7, suggests a reduced rank of $r = 2$. We interpret these findings as indications that the DAR model in (2) may be well-specified by $r = 2$ and lag length $K = 3$ as well. The left panel of Table 8 confirms this presumption.

$r \leq 0$	$r \leq 1$	$r \leq 2$	$r \leq 3$
90.89	32.62	15.99	5.06
[0.00]	[0.03]	[0.07]	[0.11]

Table 7: Likelihood-ratio test of the null, $\mathcal{H}_0$: $r \leq r^\star < p$, against $r = p$. P-values obtained with the wild bootstrap in brackets.

	DAR				VAR			
	r_t	R_t	π_t	g_t	r_t	R_t	π_t	g_t
Ljung-Box test	7.71	1.67	0.61	4.96	3.72	1.48	1.05	5.75
	[0.10]	[0.80]	[0.96]	[0.29]	[0.44]	[0.83]	[0.90]	[0.22]
Engle's ARCH test	23.81	4.37	2.53	1.58	36.15	9.92	20.82	0.60
	[0.00]	[0.04]	[0.11]	[0.21]	[0.00]	[0.00]	[0.00]	[0.44]
Kolmogorov-Smirnov test	0.07	0.06	0.06	0.06	0.09	0.06	0.10	0.05
	[0.05]	[0.09]	[0.12]	[0.16]	[0.00]	[0.12]	[0.00]	[0.12]

Table 8: Residual specification tests: Ljung-Box test of no autocorrelation. Engle's test of no ARCH effects. Kolmogorov-Smirnov test of standard normal distribution. P-values in brackets.

Then, we estimate (2) with $r = 2$ and $K = 3$ by maximum likelihood under just-identifying restrictions. Nielsen and Rahbek (2014) prove that the maximum likelihood estimator in a particular bi-variate specification of the model in (1) is consistent and asymptotically normal. Since there are no results available for the general multivariate specification, we confirm by simulations that the maximum likelihood estimator exhibit reasonable properties in the four-dimensional model. The simulations are available upon request. Then, we impose further restrictions to obtain models with economically sensible interpretations. The restrictions are imposed sequentially starting with setting the most insignificant estimates to zero first in the relations $\beta' X_t$ and then in the adjustment matrix. At each step, the restrictions are tested by LR tests and the short-run coefficient estimates are compared.

B Bond Yield Approximation

Define $\varepsilon_t^{\mathbb{Q}} = \Omega_t^{1/2} z_t^{\mathbb{Q}}$. From the factor dynamics under the $\mathbb{Q}$-measure in (5), write

$$\begin{aligned} X_t X_t' =& \mu^{\mathbb{Q}} \mu^{\mathbb{Q}'} + \mu^{\mathbb{Q}} X_{t-1}' \Phi^{\mathbb{Q}'} + \Phi^{\mathbb{Q}} X_{t-1} \mu^{\mathbb{Q}'} + \Phi^{\mathbb{Q}} X_{t-1} X_{t-1}' \Phi^{\mathbb{Q}'} + \left(\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_{t-1} \right) \varepsilon_t^{\mathbb{Q}'} \\ &+ \varepsilon_t^{\mathbb{Q}} \left(\mu^{\mathbb{Q}'} + X_{t-1}' \Phi^{\mathbb{Q}'} \right) + \varepsilon_t^{\mathbb{Q}} \varepsilon_t^{\mathbb{Q}'}, \end{aligned}$$

or by using the vectorization operator, $\text{vec}\,(A)$, that stacks the columns of the matrix A into a vector and its relation with the Kronecker product denoted $\otimes$,

$$\begin{aligned} \text{vec}\,(X_t X_t') =& \text{vec} \left(\mu^{\mathbb{Q}} \mu^{\mathbb{Q}'} \right) + \left(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}} + \mu^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}} \right) X_{t-1} + \left(\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}} \right) \text{vec} \left(X_{t-1} X_{t-1}' \right) \\ &+ \left(I_4 \otimes \left(\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_{t-1} \right) + \left(\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_{t-1} \right) \otimes I_4 \right) \varepsilon_t^{\mathbb{Q}} + \text{vec} \left(\varepsilon_t^{\mathbb{Q}} \varepsilon_t^{\mathbb{Q}'} \right). \end{aligned}$$

Next, we compute the conditional expectation given $\mathcal{F}_{t-1} = \{X_{t-1}, \ldots, X_1\}$ under $\mathbb{Q}$-measure probabilities, $\mathbb{E}_{t-1}^{\mathbb{Q}}(\cdot)$, of this expression. It follows that

$$\begin{aligned} \mathbb{E}_{t-1}^{\mathbb{Q}} \left(\text{vec}\,(X_t X_t') \right) =& \text{vec} \left(\mu^{\mathbb{Q}} \mu^{\mathbb{Q}'} \right) + \left(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}} + \mu^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}} \right) X_{t-1} + \left(\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}} \right) \text{vec} \left(X_{t-1} X_{t-1}' \right) \\ &+ \text{vec}\,(\Omega_t). \end{aligned}$$

To derive a bond yield expression in closed-form, we introduce the following approximation:

$$\text{vec}\,(X_t X_t') \approx \text{vec} \left(\mu^{\mathbb{Q}} \mu^{\mathbb{Q}'} \right) + \left(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}} + \mu^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}} \right) X_{t-1} + \left(\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}} \right) \text{vec} \left(X_{t-1} X_{t-1}' \right) + \text{vec}\,(\Omega_t),$$

where $\approx$ denotes an equality that is valid only approximately. Given this equation, the zero-coupon bond price takes the form

$$P_{t,n+1} = \exp \left(A_{n+1} + B_{n+1}' X_t + C_{n+1}' \text{vec}\,(X_t X_t') \right).$$

It is straightforward to prove this claim and derive recursive formulas for the loadings:

$$\begin{aligned}\log P_{t,n+1} = & - r_t + A_n + B_n'(\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t) + C_n'\text{vec}\left(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}'}\right) + C_n'\left(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}} + \mu^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}}\right) X_t \\ & + C_n'\left(\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}}\right) \text{vec}\left(X_t X_t'\right) + C_n'\text{vec}\left(\Omega_t\right) + \log \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(B_n'\varepsilon_{t+1}^{\mathbb{Q}}\right)\right] \\ = & - r_t + A_n + B_n'(\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t) + C_n'\text{vec}\left(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}'}\right) + C_n'\left(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}} + \mu^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}}\right) X_t \\ & + C_n'\left(\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}}\right) \text{vec}\left(X_t X_t'\right) + C_n'\text{vec}\left(\Omega_{t+1}\right) + \frac{1}{2} B_n'\Omega_{t+1} B_n.\end{aligned}$$

Gathering terms result in factor loadings recursions given by

$$\begin{aligned}A_{n+1} &= A_n + B_n'\mu^{\mathbb{Q}} + C_n'\left(\text{vec}\left(\mu^{\mathbb{Q}}\mu^{\mathbb{Q}'}\right) + \text{vec}\left(\Sigma_0\Sigma_0'\right)\right) + \frac{1}{2} B_n'\Sigma_0\Sigma_0' B_n \\ B_{n+1}' &= -\iota_1 + B_n'\Phi^{\mathbb{Q}} + C_n'\left(\Phi^{\mathbb{Q}} \otimes \mu^{\mathbb{Q}} + \mu^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}}\right) \\ C_{n+1}' &= C_n'\left(\Phi^{\mathbb{Q}} \otimes \Phi^{\mathbb{Q}} + \Sigma_1 \otimes \Sigma_1\right) + \frac{1}{2}\left(\left[B_n'\Sigma_1\right] \otimes \left[B_n'\Sigma_1\right]\right).\end{aligned}$$

To be consistent with $r_t = \iota_1' X_t$, the recursions are initiated at $n = 0$ with $A_0 = 0$, $B_0 = \mathbf{0}_{p\times 1}$, $C_0 = \mathbf{0}_{p^2\times 1}$.

C Additional Tables and Figures

	$\mathcal{H}_0$	r_t	R_t	π_t	g_t
ADF test	unit root	2.16	3.19	-16.98	-14.83
		[0.44]	[0.17]	[0.00]	[0.01]
KPSS test	stationarity	1.49	1.89	0.39	0.43
		[0.00]	[0.00]	[0.08]	[0.06]

Table 9: Augmented Dickey-Fuller and KPSS tests for respectively unit roots and stationarity. P-values in brackets.

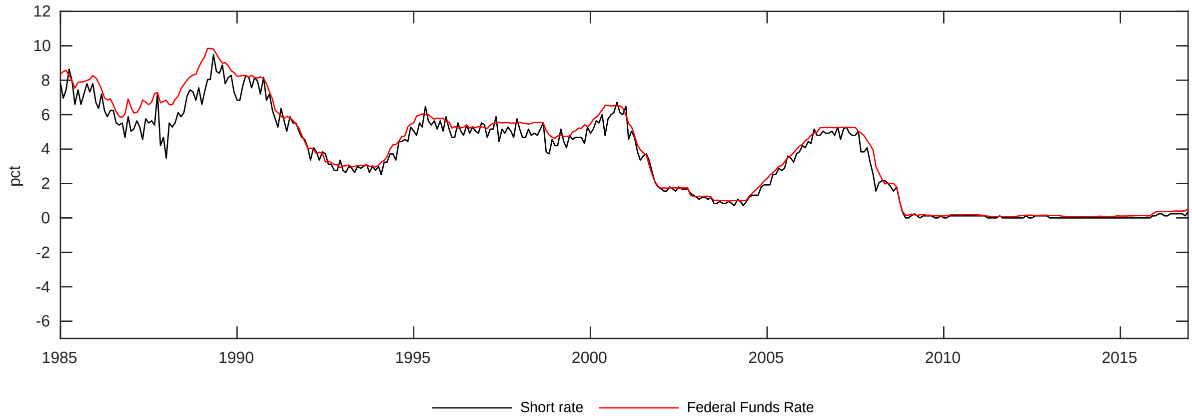

Figure 8: The short rate and the effective Federal Funds Rate from Federal Reserve Economic Data between January 1985 and December 2016 with monthly frequency.

	DAR				VAR			
Σ_0 $(\times 10^{-3})$	0.045				0.369			
	(0.010)				(0.022)			
	0.049	0.217			0.029	0.228		
	(0.037)	(0.012)			(0.015)	(0.010)		
	-0.016	0.022	0.247		0.020	-0.007	0.275	
	(0.040)	(0.016)	(0.017)		(0.013)	(0.016)	(0.016)	
	0.012	0.014	0.002	0.113	0.013	0.017	-0.001	0.114
	(0.015)	(0.007)	(0.009)	(9,995)	(0.005)	(0.005)	(0.008)	(0.005)
Σ_1	-0.096	-0.019	-0.015	-0.008				
	(0.013)	(0.006)	(0.006)	(0.005)				
	-0.015	0.007	0.012	-0.028				
	(0.014)	(0.010)	(0.011)	(0.012)				
	-0.006	-0.007	-0.012	0.076				
	(0.018)	(0.011)	(0.018)	(0.016)				
	-0.002	-0.000	-0.008	-0.0074				
	(0.006)	(0.004)	(0.007)	(0.007)				
μ' $(\times 10^{-4})$	-0.041	0.288	-0.273	-0.244	-0.637	-0.215	-0.122	-0.241
	(0.060)	(0.217)	(0.263)	(0.106)	(0.448)	(0.119)	(0.425)	(0.095)
Γ_1	-0.175	0.059	0.027	0.036	-0.427	0.183	0.014	-0.064
	(0.064)	(0.034)	(0.022)	(0.039)	(0.076)	(0.088)	(0.060)	(0.161)
	-0.026	0.027	0.150	0.140	-0.028	0.034	0.148	0.069
	(0.034)	(0.055)	(0.050)	(0.102)	(0.032)	(0.059)	(0.054)	(0.100)
	0.039	0.032	0.444	-0.087	0.020	0.024	0.449	0.029
	(0.032)	(0.056)	(0.059)	(0.113)	(0.030)	(0.056)	(0.077)	(0.140)
	0.027	0.010	0.068	0.244	0.020	0.021	0.065	0.236
	(0.016)	(0.027)	(0.022)	(0.048)	(0.015)	(0.027)	(0.023)	(0.049)
Γ_2	-0.028	-0.032	0.011	0.014	-0.125	0.164	0.035	0.527
	(0.072)	(0.023)	(0.019)	(0.045)	(0.082)	(0.080)	(0.065)	(0.165)
	-0.035	-0.119	0.064	0.048	-0.025	-0.117	0.057	0.005
	(0.040)	(0.055)	(0.044)	(0.106)	(0.039)	(0.053)	(0.048)	(0.114)
	0.036	-0.063	-0.162	0.292	0.015	-0.035	-0.124	0.480
	(0.032)	(0.049)	(0.066)	(0.118)	(0.033)	(0.052)	(0.079)	(0.137)
	0.008	-0.015	0.025	0.187	0.010	-0.009	0.017	0.193
	(0.017)	(0.025)	(0.029)	(0.053)	(0.016)	(0.025)	(0.033)	(0.049)
Γ_3	-0.061	-0.022	-0.0359	0.061	0.030	-0.064	-0.190	-0.070
	(0.056)	(0.022)	(0.0220)	(0.038)	(0.074)	(0.083)	(0.062)	(0.141)
	-0.045	0.065	-0.090	0.001	-0.026	0.056	-0.078	-0.063
	(0.034)	(0.058)	(0.043)	(0.096)	(0.034)	(0.061)	(0.044)	(0.099)
	0.007	0.064	0.122	-0.084	0.009	0.066	0.063	0.050
	(0.029)	(0.051)	(0.056)	(0.113)	(0.030)	(0.056)	(0.070)	(0.111)
	-0.039	-0.015	0.034	0.288	-0.035	-0.017	0.032	0.278
	(0.015)	(0.025)	(0.025)	(0.052)	(0.014)	(0.024)	(0.028)	(0.053)

Table 10: Short-run factor dynamics in the DAR and VAR. Standard errors in parentheses.

References

Abbritti, M., Erba, S.D., Moreno, A., Sola, S., 2013. Global Factors in the Term Structure of Interest Rates. IMF working paper .

Ahn, D.H., Dittmar, R.F., Gallant, A.R., 2002. Quadratic term structure models: Theory and evidence. The Review of Financial Studies 15, 243–288.

Albin, J.M.P., Jensen, B.A., Muszta, A., Richter, M., 2006. On Volatility Induced Stationarity for Stochastic Differential Equations. D-CAF Working Paper Series .

Almeida, C., Graveline, J.J., Joslin, S., 2011. Do interest rate options contain information about excess returns? Journal of Econometrics 164, 35–44.

Andersen, T.G., Benzoni, L., 2010. Do Bonds Span Volatility Risk in the U.S. Treasury Market? A Specification Test for Affine Term Structure Models. The Journal of Finance 65, 603–653.

Ang, A., Piazzesi, M., 2003. A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. Journal of Monetary Economics 50, 745–787.

Ang, A., Piazzesi, M., Wei, M., 2006. What Does the Yield Curve Tell us about GDP Growth? Journal of Econometrics 131, 359–403.

Barnichon, R., 2010. Building a composite Help-Wanted Index. Economics Letters 109, 175–178.

Bauer, M.D., Rudebusch, G.D., Wu, J.C., 2014. Term Premia and Inflation Uncertainty: Empirical Evidence from an International Panel Dataset: Comment. American Economic Reivew 101, 1514–1534.

Beechey, M., Hjalmarsson, E., Österholm, P., 2009. Testing the expectations hypothesis when interest rates are near integrated. Journal of Banking and Finance 33, 934–943.

Bernanke, B.S., 2006. Reflections on the Yield Curve and Monetary Policy. URL: `https://www.federalreserve.gov/newsevents/speech/bernanke20060320a.htm`.

Bernanke, B.S., 2015. The Taylor Rule: A benchmark for monetary policy? URL: `https://www.brookings.edu/blog/ben-bernanke/2015/04/28/`

`the-taylor-rule-a-benchmark-for-monetary-policy/`.

Bikbov, R., Chernov, M., 2011. Yield Curve and Volatility: Lessons from Eurodollar Futures and Options. Journal of Financial Econometrics 9, 66–105.

Campbell, J.Y., Shiller, R.J., 1987. Cointegration and Tests of Present Value Models. Journal of Political Economy 95, 1062–1088.

Cavaliere, G., Rahbek, A., Taylor, A.M.R., 2014. Bootstrap Determination of the Co-Integration Rank in Heteroskedastic VAR Models. Econometric Reviews 33, 606–650.

Chan, K.C., Karolyi, G.A., Longstaff, F.A., Sanders, A.B., 1992. An Empirical Comparison of Alternative Models of the Short-Term Interest Rate. Journal of Finance XLVII.

Cieslak, A., Povala, P., 2016. Information in the term structure of yield curve volatility. The Journal of Finance 71, 1393–1436.

Collin-Dufresne, P., Goldstein, R.S., 2002. Do bonds span the fixed income markets? theory and evidence for unspanned stochastic volatility. The Journal of Finance 57, 1685–1730.

Conley, T.G., Hansen, L.P., Luttmer, E.G.J., Scheinkman, J.A., 1997. Short-Term Interest Rates as Subordinated Diffusions. The Review of Financial Studies 10, 525–577.

Creal, D.D., Wu, J.C., 2015. Estimation of Affine Term Structure Models with Spanned or Unspanned Stochastic Volatility. Journal of Econometrics 185, 60–81.

Duffee, G.R., 2011. Information in (and not in) the term structure. Review of Financial Studies 24, 2895–2934.

Engle, R.F., Granger, C.W.J., 1987. Co-integration and Error Correction: Representation, Estimation, and Testing. Econometrica 55, 251–76.

Engle, R.F., Kroner, K.F., 1995. Multivariate Simultaneous Generalized ARCH. Econometric Theory 11, 122–150.

Filipović, D., Larsson, M., Trolle, A.B., 2017. Linear-rational term structure models. The Journal of Finance 72, 655–704.

Goliński, A., Zaffaroni, P., 2016. Long memory affine term structure models. Journal of Econometrics 191, 33–56.

Gürkaynak, R.S., Sack, B., Wright, J.H., 2007. The U.S. Treasury yield curve: 1961 to the present. Journal of Monetary Economics 54, 2291–2304.

Hall, A., Anderson, H., Granger, C., 1992. A Cointegration Analyisis of Treasury Bill Yields. The Review of Economics and Statistics 74, 116–126.

Haubrich, J., Pennacchi, G., Ritchken, P., 2012. Inflation Expectations, Real Rates, and Risk Premia: Evidence from Inflation Swaps. The Review of Financial Studies 25, 1588–1629.

Jagannathan, R., Kaplin, A., Sun, S., 2003. An evaluation of multi-factor CIR models using LIBOR, swap rates, and cap and swaption prices. Journal of Econometrics 116, 113–146.

Jardet, C., Monfort, A., Pegoraro, F., 2013. No-arbitrage Near-Cointegrated VAR(p) term structure models, term premia and GDP growth. Journal of Banking & Finance 37, 389–402.

Johansen, S., 1991. Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models. Econometrica 59, 1551–1580.

Joslin, S., Konchitchki, Y., 2018. Interest rate volatility, the yield curve, and the macroeconomy. Journal of Financial Economics 128, 344–362.

Joslin, S., Priebsch, M., Singleton, K.J., 2014. Risk Premiums in Dynamic Term Structure Models with Unspanned Macro Risks. Journal of Finance 69, 1197–1233.

Kim, D.H., Orphanides, A., 2007. The Bond Market Term Premium: What is It, and How Can we Measure It? BIS Quarterly Review , 27–40.

Kim, D.H., Orphanides, A., 2012. Term Structure Estimation with Survey Data on Interest Rate Forecasts. Journal of Financial and Quantitative Analysis 47, 241–272.

Leippold, M., Wu, L., 2002. Class Asset Pricing under the Quadratic. The Journal of Financial and Quantitative Analysis 37, 271–295.

Ling, S., 2004. Estimation and testing of stationarity for double-autoregressive models. Journal of Statistics 66, 63–78.

Litterman, R., Scheinkman, J., 1991. Common Factors Affecting Bond Returns . Journal of Fixed Income 1, 54–61.

Ludvigson, S.C., Ng, S., 2009. Macro factors in bond risk premia. The Review of Financial Studies 22, 5027–5067.

Nicolau, J., 2005. Processes with volatility-induced stationarity: an application for interest rates. Statistica Neerlandica 59, 376–396.

Nielsen, H.B., Rahbek, A., 2014. Unit root vector autoregression with volatility induced stationarity. Journal of Empirical Finance 29, 144–167.

Osterrieder, D., 2013. Interest Rates with Long Memory : A Generalized Affine Term-Structure Model.

Realdon, M., 2006. Quadratic Term Structure Models in Discrete Time. Finance Research Letters 3, 277–289.

Shea, G.S., 1992. Benchmarking the Expectations Hypothesis of the Interest-Rate Term Structure: An Analysis of Cointegration Vectors. Journal of Business & Economic Statistics 10, 347–366.

Shiller, R., 1979. The Volatility of Long-Term Interest Rates and Expectations Models of the Term Structure. The Journal of Political Economy 87, 1190–1219.