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Optimal hedging with the cointegrated vector autoregressive model

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Abstract

We derive the optimal hedging ratios for a portfolio of assets driven by a Cointegrated Vector Autoregressive model (CVAR) with general cointegration rank. Our hedge is optimal in the sense of minimum variance portfolio.

We consider a model that allows for the hedges to be cointegrated with the hedged asset and among themselves. We find that the minimum variance hedge for assets driven by the CVAR, depends strongly on the portfolio holding period. The hedge is defined as a function of correlation and cointegration parameters. For short holding periods the correlation impact is predominant. For long horizons, the hedge ratio should overweight the cointegration parameters rather then short-run correlation information. In the infinite horizon, the hedge ratios shall be equal to the cointegrating vector. The hedge ratios for any intermediate portfolio holding period should be based on the weighted average of correlation and cointegration parameters.

The results are general and can be applied for any portfolio of assets that can be modeled by the CVAR of any rank and order.

Keywords: hedging, cointegration, minimum variance portfolio

JEL Classification: C22, C58, G11

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1 Introduction

The idea of minimum variance portfolio dates back to [Markowitz, 1952]. It is defined as a portfolio of individually risky assets that, when taken together, result in the lowest possible risk level for return. Such a portfolio hedges each investment with an offsetting investment; the individual investor's choice on how much to offset investments depends on the level of risk and expected return he/she is willing to accept. The investments in a minimum variance portfolio are individually riskier than the portfolio as a whole. The name of the term comes from how it is mathematically expressed in Markowitz Portfolio Theory, in which volatility is used as a replacement for risk, and in which less variation in volatility correlates to less risk in an investment. Since the seminal paper of [Markowitz, 1952], the notion of minimum variance portfolio and minimum variance hedging has been explored and extended heavily in both financial and econometric literature, see [Grinold and Kahn, 1999]. However, the common denominator of those methods remain the same. They either aim at minimizing volatility of a portfolio itself or volatility of some function of a portfolio. This function often represents the evolution of the portfolio over time. This is also the purpose of the hedging problem we define.

In general, the hedging methods can be divided in two classes: static and dynamic methods. The static hedging techniques assume that the hedged portfolio is selected given information available in period t, and remains unchanged during the entire holding period $t, \dots, t+h$. On the contrary, the dynamic hedging methods allow for rebalancing of the portfolio during the holding period.

Our method is static. We find the optimal hedging ratios for a portfolio of assets driven by a Cointegrated Vector Autoregressive model (CVAR). We start with an example of a simple process, which relates the hedged asset to hedges via a cointegrating relation. The hedges are exogenous and are modeled by random walks.

The general results that we find, define the optimal hedging ratios as a function of correlation and cointegration parameters in the model. We find that a minimum variance portfolio held for one period should be based on the hedge ratios driven only by correlation. At the infinite horizon, the hedge ratios will be equal to a cointegrating vector. The hedge ratios for any intermediate portfolio holding period should be based on the weighted average of the correlation and cointegration parameters. Our result are general and can be applied to a CVAR model of any rank and order.

2 A simple example of hedging cointegrated variables

2.1 The model

We consider a bivariate cointegration regression model with an endogenous variable y_{1t} cointegrated with an exogenous variable y_{2t}

$$\begin{aligned} y_{1t} &= \beta y_{2t} + u_{1t}, \\ y_{2t} &= y_{2,t-1} + u_{2t}, \end{aligned} \tag{1}$$

where $u_t = (u_{1t}, u_{2t})'$ are independent identically distributed (i.i.d.) random errors with mean zero and variance

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}.$$

An alternative formulation of this model is the error correction form which we find by substituting $y_{2t} = y_{2,t-1} + u_{2t}$ into the first equation of (1) to obtain

$$\begin{aligned} \Delta y_{1t} &= -(y_{1t-1} - \beta y_{2,t-1}) + v_{1t}, \\ \Delta y_{2t} &= v_{2t}, \end{aligned} \tag{2}$$

where $v_{1t} = \beta u_{2t} + u_{1t}$ and $v_{2t} = u_{2t}$, so that v_t are i.i.d. with mean zero and variance

$$\Xi = \begin{pmatrix} \phi_{11} + \beta^2\phi_{22} + 2\beta\phi_{12} & \phi_{12} + \beta\phi_{22} \\ \phi_{21} + \beta\phi_{22} & \phi_{22} \end{pmatrix} = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}.$$

Let $\mathcal{I}_t$ denote the information in the process y_s for $s \leq t$, then the best linear predictor of $y_{1,t+1}$ from $y_{2,t+1}$ and $\mathcal{I}_t$ solves the problem

$$\min_{\delta} Var(y_{1,t+1} - \delta y_{2,t+1} | \mathcal{I}_t, y_{2,t+1}).$$

The solution is

$$\delta^* = \frac{Cov(y_{1,t+1}, y_{2,t+1}|\mathcal{I}_t)}{Var(y_{2,t+1}|\mathcal{I}_t)} = \xi_{22}^{-1}\xi_{21} = \beta + \phi_{22}^{-1}\phi_{21} = \beta^{Corr}, \tag{3}$$

and the minimal variance is

$$\xi_{11} - \xi_{12}\xi_{22}^{-1}\xi_{21} = \phi_{11} - \phi_{12}\phi_{22}^{-1}\phi_{21}.$$

we call β^{Corr} the β-correction parameter.

2.2 The hedging problem and its solution in (1)

In financial market one can consider a long or a short position in a given security. The long position means the holder of the position owns the security and will profit if the price of the security goes up. The short position is defined as the sale of a borrowed security, with the expectation that the asset will fall in value. Then, the investor must eventually return the borrowed security by buying it back from the market. Because it can be purchased cheaper than at the time of borrowing, the difference in price results in profit for the investor.

With two assets with prices y_{1t} and y_{2t}, we want to hedge one unit of the first asset using β_h of the second asset, and create a portfolio with value

$$s_t = y_{1t} - \beta_h y_{2t}. \tag{4}$$

Her h indicates that we want to keep the portfolio for h periods and determine β_h to minimizes the conditional variance of the value at time $t + h$ given the information up to time t, $\mathcal{I}_t = \sigma(y_s, s \leq t)$, that is

$$\min_{\beta_h} Var(s_{t+h}|\mathcal{I}_t) = \min_{\beta_h} Var(y_{1,t+h} - \beta_h y_{2,t+h}|\mathcal{I}_t). \tag{5}$$

In portfolio hedging, traditionally a long position in asset y_{1t} is hedged with a short position in another asset y_{2t}. Thus the sign in front of the weight, or hedging parameter, β_h symbolizes the market convention regarding hedging practice.

The optimal portfolio is selected in period t and it is held up to period $t+h$. It is a static hedge, as the portfolio is not rebalanced during periods $t, \ldots, t + h$.

We present an optimal hedge which explores both the long term cointegration parameter β, but also the correlation between the random errors in y_{1t} and y_{2t}, and formulate it as

Theorem 1 *Let $y_t, t = 1, \ldots, T$, be bivariate and given by*

$$y_{1t} = \beta y_{2t} + u_{1t},$$
$$y_{2t} = y_{2,t-1} + u_{2t},$$

where u_i are i.i.d. $(0, \Phi)$. Then the optimal hedge coefficient for the portfolio $s_t = y_{1t} - \beta_h y_{2t}$ at time horizon h is given by

$$\beta_h^* = \beta + h^{-1}\phi_{22}^{-1}\phi_{21} = h^{-1}(\beta(h-1) + \beta^{Corr}), \tag{6}$$

and the minimal variance is

$$Var(y_{1,t+h} - \beta_h^* y_{2,t+h}|\mathcal{I}_t) = \phi_{11} - h^{-1}\phi_{12}\phi_{22}^{-1}\phi_{21}.$$

Proof. We find from (1) that

$$y_{1,t+h} - \beta_h y_{2,t+h} = y_{1,t+h} - \beta y_{2,t+h} - (\beta_h - \beta)y_{2,t+h} = u_{1,t+h} - (\beta_h - \beta)y_{2,t+h}$$

and

$$y_{2,t+h} = y_{2t} + u_{2,t+1} + \cdots + u_{2,t+h}.$$

This implies that

$$Var(s_{t+h}|\mathcal{I}_t) = \phi_{11} + (\beta_h - \beta)^2 h\phi_{22} - 2\phi_{12}(\beta_h - \beta)$$

which is minimized for $\beta_h = \beta + h^{-1}\phi_{22}^{-1}\phi_{21}$, which proves the result. ■

The hedge defined in (6) is a weighted average of the correlation correction, β^{Corr}, and the cointegration parameter with weights: $1/h$ and $(h-1)/h$. The resulting formula complies with the stylized facts about the short- and long-term hedging in the sense that for a short period, $h = 1$, we hedge only based on the correlation

$$\beta_h^\star = \beta^{Corr},$$

whereas for a long period, when $h \to \infty$, we hedge only based on cointegration

$$\beta_h^\star = \beta.$$

2.3 Hedging in a simple multivariate model

This result generalizes immediately to the case of many possibly correlated hedges as given by the model with n assets, from which $n-1$ are exogenous, and are used as hedges. Let $y_t = (y_{1t}, y_{2t}')' \in \mathbb{R}^{1+(n-1)}, t = 1, \ldots, T$ be given by

$$\begin{aligned} y_{1t} &= \beta' y_{2t} + u_{1t} \\ y_{2t} &= y_{2,t-1} + u_{2t}, \end{aligned} \tag{7}$$

where u_t are i.i.d. with mean zero and variance

$$\Phi = \begin{pmatrix} \phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}.$$

The main difference between the univariate case (with one hedge modeled as a random walk y_{2t}) and the multivariate case, is the possibility of correlation between the innovations of the random walks $y_{2t}, y_{3t}, \ldots, y_{nt}$. Two hedges modeled by correlated random walks are substitutes. In the extreme case that two potential hedges are fully correlated, having only one of them to hedge y_{1t} is enough for an optimal portfolio. The optimal hedges that we derive for the portfolio based on assets modeled according to (7) takes into account not only the cointegration parameters, but also the correlation in order to account for optimal amount of hedges in a portfolio. By comparing the variance of the optimal portfolio, for the multivariate case with the result for the univariate case we can see what is gained by including more hedges.

Theorem 2 *Let $y_t = (y_{1t}, y_{2t}')' \in \mathbb{R}^{1+(n-1)}, t = 1, \ldots, T$ be given by*

$$\begin{aligned} y_{1t} &= \beta' y_{2t} + u_{1t}, \\ y_{2t} &= y_{2,t-1} + u_{2t}, \end{aligned}$$

where u_t are i.i.d. $(0, \Phi)$. Then the optimal hedge coefficient for the portfolio $s_t = y_{1t} - \beta_h' y_{2t}$ at time horizon h is given by

$$\beta_h^* = \beta + h^{-1}\Phi_{22}^{-1}\Phi_{21} = h^{-1}(\beta(h-1) + \beta^{Corr}),$$

and the minimal variance is

$$Var(y_{1,t+h} - \beta_h^{*\prime} y_{2,t+h} | \mathcal{I}_t) = \phi_{11} - h^{-1}\Phi_{12}\Phi_{22}^{-1}\Phi_{21}.$$

Proof. The proof is the same as for Theorem 1. ∎

It is seen that the variance of the optimal portfolio increases with the horizon h, from the conditional variance of u_{1t} given u_{2t}, $\phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{21}$ for $h = 1$, to the variance of the cointegrating relation ϕ_{11} for $h \to \infty$. It is also seen that including more correlated hedges, the variance of u_{1t} given u_{2t} will decrease, because $\phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{21}$ can be interpreted as a conditional variance of u_{1t} given u_{2t}, and conditioning on more variables decreases the conditional variance. The limit for $h \to \infty$ is always given by ϕ_{11}.

3 Hedging in the general cointegration model

The analysis of exogenous hedges is now generalized to the general cointegration model, see [Johansen, 1996], where we use the error correction formulation. For notational convenience we give the result for the model with one lag, but the result for more lags can be derived from the companion form. Thus, we consider a model that allows the hedges to be cointegrated with y_{1t} and among themselves such that the number of cointegrating relations could be more than one. The assumption of full mean reversion of models (1) and (7) is dropped and general adjustment coefficients are allowed.

Theorem 3 *Let $y_t \in \mathbb{R}^n, t = 1, \ldots, T$ be given by*

$$\Delta y_t = \alpha\gamma' y_{t-1} + v_t,$$

where v_t are i.i.d.$(0, \Xi)$ and α and γ are $r \times n$ matrices, and the eigenvalues of $\rho = I_r + \gamma'\alpha$ have absolute value less than 1. Then for $C = \gamma_\perp(\alpha'_\perp\gamma_\perp)^{-1}\alpha'_\perp$ we find

$$\Sigma_h = Var(y_{t+h}|\mathcal{I}_t) = hC\Xi C' + \alpha(\gamma'\alpha)^{-1}[\sum_{i=0}^{h-1}\rho^i\gamma'\Xi\gamma\rho'^i](\alpha'\gamma)^{-1}\alpha' \\ - \alpha(\gamma'\alpha)^{-2}(I_r - \rho^h)\gamma'\Xi C' - C\Xi\gamma(I_r - \rho'^h)(\alpha'\gamma)^{-2}\alpha'. \tag{8}$$

Proof. See Appendix. ■

In the following we assume $y_t = (y_{1t}, y'_{2t})' \in \mathbb{R} \times \mathbb{R}^{n-1}$, and that there is a cointegrating relation of the form $y_{1t} + \beta'_1 y_{2t}$, and that the cointegrating vectors have been written, without loss of generality, as

$$\gamma = (\gamma_1, \gamma_2) = \begin{pmatrix} 1 & 0 \\ \beta_1 & \beta_2 \end{pmatrix} \tag{9}$$

for $\gamma_1 \in \mathbb{R}^n$ and $\gamma_2 \in \mathbb{R}^{n\times(r-1)}$. We define the variance of the stationary variables $\gamma' y_t$ as

$$\Gamma = Var(\gamma' y_t) = Var\begin{pmatrix} y_{1t} + \beta'_1 y_{2t} \\ \beta'_2 y_{2t} \end{pmatrix} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}. \tag{10}$$

If the portfolio is chosen as a cointegrating relation, we find the optimal portfolio in the next Theorem.

Theorem 4 *Let $y_t = (y_{1t}, y'_{2t})' \in \mathbb{R} \times \mathbb{R}^{n-1}$. If the cointegrating relations are normalized as in (9), then the variance of the stationary relation $y_{1t} - \beta' y_{2t}$ is minimized for $\beta \in \mathbb{R}^{n-1}$ by*

$$\beta^*_{coint} = -\beta_1 + \beta_2\Gamma_{22}^{-1}\Gamma_{21}. \tag{11}$$

Proof. The cointegrating vector $(1, -\beta')'$ is a linear combination of the vectors in γ, and therefore there exists a vector $\xi_{(r-1)\times 1}$ such that

$$\begin{pmatrix} 1 \\ -\beta \end{pmatrix} = \begin{pmatrix} 1 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta_2 \end{pmatrix}\xi = \begin{pmatrix} 1 \\ \beta_1 + \beta_2\xi \end{pmatrix},$$

so that $y_{1t} - \beta' y_{2t} = y_{1t} + \beta'_1 y_{2t} + \xi'\beta'_2 y_{2t}$, which has variance $\Gamma_{11} + \xi'\Gamma_{21} + \Gamma_{12}\xi + \xi'\Gamma_{22}\xi$, which is minimized for

$$\xi^* = -\Gamma_{22}^{-1}\Gamma_{21}$$

see (10), giving the optimal cointegrating portfolio

$$y_{1t} - \beta^{*\prime}_{coint} y_{2t} = y_{1t} + \beta'_1 y_{2t} - \Gamma_{12}\Gamma_{22}^{-1}\beta'_2 y_{2t}.$$

■

Note that in the result in Theorem 4 the parameter β_1 is not identified, because

$$\gamma_\xi = \begin{pmatrix} 1 & 0 \\ \beta_1 + \beta_2\xi & \beta_2 \end{pmatrix}$$

spans the same space as γ. The result in (11), however, is invariant to identification because if γ_ξ were the cointegrating relations, then using the expression in (11) we find

$$\begin{aligned}\beta^*_{coint,\xi} &= -(\beta_1 + \beta_2\xi) + \beta_2 Var(\beta_2 y_{2t})^{-1} Cov(\beta_2' y_t, y_{1t} + (\beta_1 + \beta_2\xi) y_{2t}) \\ &= -(\beta_1 + \beta_2\xi) + \beta_2\Gamma_{22}^{-1}(\Gamma_{21} + \Gamma_{22}\xi) = -\beta_1 + \beta_2\Gamma_{22}^{-1}\Gamma_{21} = \beta_{coint}.\end{aligned}$$

Thus the result does not depend on the identification of β_1.

In the case of $n-1$ exogenous random walks, the optimizing portfolio approaches the cointegrating vector. In general case, the cointegration rank r might deviate from 1, but we show that in general the optimal portfolio converges to the optimal cointegrated portfolio, see (11).

Theorem 5 *Let $y_t = (y_{1t}, y_{2t}')' \in \mathbb{R} \times \mathbb{R}^{n-1}$ and decompose Σ_h, see (8), as*

$$\Sigma_h = \begin{pmatrix} \Sigma_{h11} & \Sigma_{h12} \\ \Sigma_{h21} & \Sigma_{h22} \end{pmatrix}.$$

Then the optimal hedge for the portfolio $s_{t+h} = y_{1,t+h} - \beta_h' y_{2,t+h} = (1, -\beta_h')' y_{t+h}$ at time horizon h is given by

$$\beta_h^* = \Sigma_{h22}^{-1}\Sigma_{h21}, \tag{12}$$

and the minimal variance is

$$Var(y_{1,t+h} - \beta_h^* y_{2,t+h} | \mathcal{I}_t) = \Sigma_{h11} - \Sigma_{h12}\Sigma_{h22}^{-1}\Sigma_{h21}.$$

Moreover it holds that, with β_{coint} defined by (11), we have

$$\beta_h^* \to \beta_{coint}^* \text{ for } h \to \infty. \tag{13}$$

Proof: See Appendix.

3.1 A special case

The results in Theorem 5 cover the cases considered so far, and we give here a simple special case with one cointegrating relation and $n-1$ exogenous hedges, but we do not assume full mean reversion. Thus, the equations can be written in error correction form as

$$\begin{aligned}\Delta y_{1t} &= \alpha_1(y_{1,t-1} - \beta' y_{2,t-1}) + \beta' u_{2t} + u_{1t} \\ \Delta y_{2t} &= u_{2t}\end{aligned}$$

where u_t are i.i.d.$(0, \Phi)$. For $\alpha_1 = -1$ we find model (7). We find the covariance matrix

$$\Xi = Var\begin{pmatrix} \beta' u_{2t} + u_{1t} \\ u_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} + \beta'\Phi_{21} + \Phi_{12}\beta + \beta'\Phi_{22}\beta & \beta'\Phi_{22} + \Phi_{12} \\ \Phi_{21} + \Phi_{22}\beta & \Phi_{22} \end{pmatrix},$$

and the parameters $\alpha = (\alpha_1, 0, \ldots, 0)' = \alpha_1 e_1'$, $\gamma = (1, -\beta')'$, $\gamma_\perp' = (\beta, I_{n-1})$ and $\alpha_\perp = e_{1\perp}$, where $e_1 = (1, 0, \ldots, 0)'$, but also

$$C = I_n - \alpha(\gamma'\alpha)^{-1}\gamma' = \begin{pmatrix} 0 & \beta' \\ 0 & I_{n-1} \end{pmatrix}.$$

Inserting into the general expression (8), we first find that because $\alpha_\perp = e_{1\perp}$ we get

$$\Sigma_{h22} = he'_{1\perp}\Xi e_{1\perp} = h\Phi_{22}.$$

Next we see that $e'_{1\perp}C = e'_{1\perp}$, $\rho = 1 + \gamma'\alpha = 1 + \alpha_1$, $C'e_1 = (0, \beta')$ and we find

$$\begin{aligned}\Sigma_{h21} &= e'_{1\perp}\Sigma_h e_1 = he'_{1\perp}\Xi(0,\beta') - e'_{1\perp}\Xi\gamma(1-(1+\alpha_1)^h)\alpha_1^{-1} \\ &= h\Phi_{22} - (\Phi_{21} + \Phi_{22}\beta - \Phi_{22}\beta)(1-(1+\alpha_1)^h)\alpha_1^{-1} \\ &= h\Phi_{22} - \Phi_{21}(1-(1+\alpha_1)^h)\alpha_1^{-1}\end{aligned}$$

and therefore

$$\beta_h^* = \beta + h^{-1}\Phi_{22}^{-1}\Phi_{21}((1+\alpha_1)^h - 1)\alpha_1^{-1}. \tag{14}$$

In particular we can take $\alpha_1 = -1$, see (1), and find the full mean reversion case which gives

$$\beta_h^* = \beta + h^{-1}\Phi_{22}^{-1}\Phi_{21},$$

see Theorem 2, but in general the result for β_h^* depends in a complicated way on the parameters of the model.

4 Summary

We derive the optimal hedging ratios for a portfolio of assets driven by a Cointegrated Vector Autoregressive model with general cointegration rank. Our hedge is optimal in the sense of minimum variance portfolio. For illustration we start with the exogenous case, in which the hedged asset depends on hedges via a cointegration relation, and the hedges are exogenous, modeled by random walks. Then we consider the CVAR, that allows for the hedges to be cointegrated with the hedged asset and among themselves. We find that the minimum variance hedge for assets driven by the CVAR, depends strongly on the portfolio holding period. The hedge is defined as a function of correlation and cointegration parameters. For short holding periods the correlation impact is predominant. For long horizons, the hedge ratio should emphasize the cointegration parameters rather then short-run correlation information. At the infinite horizon, the hedge ratios shall be equal to a cointegrating vector, which is the optimal cointegrated portfolio. The hedge ratios for any intermediate portfolio holding period should be based on the weighted average of correlation and cointegration parameters.

Our results are general and can be applied for any portfolio of assets that can be modeled by the CVAR of any rank and order. The further research aims at a dynamic version of the developed methodology. In that case the static hedge kept for the entire portfolio holding horizon shall be replaced by a hedge that is dynamically rebalanced during this period.

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5 Appendix

Proof of Theorem 3. From the equations we find that the cointegrating relation $\gamma' y_t$ is an $r-$dimensional $AR(1)$ process with autoregressive parameter $\rho = I_r + \gamma'\alpha$, given by

$$\gamma' y_t = \rho\gamma' y_{t-1} + \gamma' u_t.$$

By forward recursion from $i = t+1, \dots, t+h$ we find

$$\gamma' y_{t+h} = \rho^h \gamma' y_t + \rho^{h-1}\gamma' u_{t+1} + \cdots + \rho\gamma' u_{t+h-1} + \gamma' u_{t+h}.$$

Similarly we find that $\alpha'_\perp y_t$ is a random walk

$$\alpha'_\perp y_{t+h} = \alpha'_\perp y_t + \alpha'_\perp u_{t+1} + \cdots + \alpha_\perp u_{t+h}.$$

We combine these results using the identity

$$I_n = \gamma_\perp(\alpha'_\perp\gamma_\perp)^{-1}\alpha'_\perp + \alpha(\gamma'\alpha)^{-1}\gamma' = C + \alpha(\gamma'\alpha)^{-1}\gamma',$$

where $\alpha_\perp$ is an $n \times (n-r)$ matrix of rank $n-r$ for which $\alpha'\alpha_\perp = 0$. We find

$$\begin{aligned} y_{t+h} &= Cy_{t+h} + \alpha(\gamma'\alpha)^{-1}\gamma' y_{t+h} \\ &= \sum_{i=t+1}^{t+h}(Cu_i + \alpha(\gamma'\alpha)^{-1}\rho^{t+h-i}\gamma' u_i) + (C + \alpha(\gamma'\alpha)^{-1}\rho^h\gamma')y_t = z_{1t} + z_{2t}, \end{aligned}$$

where z_{1t} is independent of z_{2t} so that $\Sigma_h = Var(y_{t+h}|\mathcal{I}_t) = Var(z_{1t})$. Using $\sum_{i=0}^{h-1}\rho^i = (I_r-\rho)^{-1}(I_r-\rho^h) = -(\gamma'\alpha)^{-1}(I_r-\rho^h)$, we therefore find that

$$\Sigma_h = Var(z_{1t}) = \sum_{i=0}^{h-1}[C + \alpha(\gamma'\alpha)^{-1}\rho^i\gamma']\Xi[C' + \gamma\rho'^i(\alpha'\gamma)^{-1}\alpha'], \tag{15}$$

which reduces to the expression in (8). ■

Proof of Theorem 5. The optimal portfolio β_h^* is the best linear predictor of y_{1t+h} given y_{2t+h} and $\mathcal{I}_t$ and is given by the coefficient

$$\beta_h^* = \Sigma_{h22}^{-1}\Sigma_{h21},$$

and the minimal (conditional) variance is

$$Var(y_{1,t+h} - \beta_h^* y_{2,t+h}|\mathcal{I}_t) = \Sigma_{h11} - \Sigma_{h12}\Sigma_{h22}^{-1}\Sigma_{h21}.$$

We therefore have to discuss the limits of the matrices Σ_{h22} and Σ_{h21}.We note that $h^{-1}\Sigma_h \to C\Xi C'$, which is singular, and that may also hold for the limit of $h^{-1}\Sigma_{n22}$. Thus, we have to analyse the matrix in more detail.

We first note the expression for γ and $\gamma_\perp$ given by

$$\gamma = \begin{pmatrix} 1 & 0 \\ \beta_1 & \beta_2 \end{pmatrix} \text{ and } \gamma_\perp = \begin{pmatrix} -\beta_1'\beta_{2\perp} \\ \beta_{2\perp} \end{pmatrix},$$

so that for $e_1 = (1, 0, \ldots, 0)' \in \mathbb{R}^n$ we find the first row of C is

$$C_1' = e_1'C = -\beta_1'\beta_{2\perp}(\alpha_\perp'\gamma_\perp)^{-1}\alpha_\perp',$$

and we define the remaining rows as

$$C_2' = e_{1\perp}'C = \beta_{2\perp}(\alpha_\perp'\gamma_\perp)^{-1}\alpha_\perp'.$$

Similarly we define the first row of $\alpha(\gamma'\alpha)^{-1}$ by $\alpha_1' = e_1'\alpha(\gamma'\alpha)^{-1}$, and the remaining rows are $\alpha_2' = e_{1\perp}'\alpha(\gamma'\alpha)^{-1}$.

From (8) we find, multiplying by $e_{1\perp}$ and its transposed, that

$$\begin{aligned}\Sigma_{h22} = {} & hC_2'\Xi C_2 + \alpha_2'\sum_{i=0}^{h-1}\rho^i\gamma'\Xi\gamma\rho'^i\alpha_2 \\ & - \alpha_2'(\gamma'\alpha)^{-1}(I_r - \rho^h)\gamma'\Xi C_2 - C_2'\Xi\gamma(I_r - \rho'^h)(\alpha'\gamma)^{-1}\alpha_2,\end{aligned}$$

with a similar expression for Σ_{h21}. We introduce the coefficients, see (10),

$$\begin{aligned}\Gamma_h &= \sum_{i=0}^{h-1}\rho^i\gamma'\Xi\gamma\rho'^i \to \sum_{i=0}^{\infty}\rho^i\gamma'\Xi\gamma\rho'^i = Var(\gamma' y_t) = \Gamma, \\ \Theta_h &= (I_r - \rho^h)\gamma'\Xi\alpha_\perp \to \gamma'\Xi\alpha_\perp = \Theta, \\ \Omega &= (\alpha_\perp'\gamma_\perp)^{-1}\alpha_\perp'\Xi\alpha_\perp(\gamma_\perp'\alpha_\perp)^{-1},\end{aligned}$$

to simplify the reductions and find the expressions

$$\begin{aligned}\Sigma_{h22} &= h\beta_{2\perp}\Omega\beta_{2\perp}' + \alpha_2'\Gamma_h\alpha_2 - \alpha_2'\Theta_h\beta_{2\perp}' - \beta_{2\perp}\Theta_h'\alpha_2 \\ \Sigma_{h21} &= -h\beta_{2\perp}\Omega\beta_{2\perp}'\beta_1 + \alpha_2'\Gamma_h\alpha_1 + \alpha_2'\Theta_h\beta_{2\perp}'\beta_1 - \beta_{\perp 2}\Theta_h'\alpha_1.\end{aligned}$$

In order to study the limits of Σ_{h22} and Σ_{h21} we introduce the non-singular $(n-1)\times(n-1)$ matrices

$$A = (\beta_2, \bar\beta_{2\perp}) \text{ and } A_h = (\beta_2, h^{-1}\bar\beta_{2\perp}),$$

and the rows of α

$$\alpha_1 = (\alpha'\gamma)^{-1}\alpha' e_1 \text{ and } \alpha_2 = (\alpha'\gamma)^{-1}\alpha' e_{1\perp}.$$

Then, it holds that

$$I_r = \gamma'\alpha(\gamma'\alpha)^{-1} = \begin{pmatrix} 1 & \beta_1' \\ 0 & \beta_2' \end{pmatrix}\begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix} = \begin{pmatrix} \alpha_1' + \beta_1'\alpha_2' \\ \beta_2'\alpha_2' \end{pmatrix} = \begin{pmatrix} e_{1r} \\ e_{1r\perp} \end{pmatrix} \tag{16}$$

and hence

$$\begin{aligned}\beta_2'\alpha_2'\Gamma\alpha_2\beta_2 &= e_{1\perp}'\Gamma e_{1\perp} = \Gamma_{22} \\ \beta_2'\alpha_2'\Gamma(\alpha_1 + \alpha_2\beta_1) &= e_{1\perp}'\Gamma e_1 = \Gamma_{21}.\end{aligned}$$

Inserting these results above we get

$$\begin{aligned}
A_h'\Sigma_{h22}A &= \begin{pmatrix} \beta_2'\alpha_2'\Gamma_h\alpha_2\beta_2 & \beta_2'\alpha_2'(\Gamma_h\alpha_2\bar{\beta}_{2\perp}-\Theta_h) \\ h^{-1}(\bar{\beta}_{2\perp}'\alpha_2'\Gamma_h-\Theta_h')\alpha_2\beta_2 & \Omega+h^{-1}(\bar{\beta}_{2\perp}'\alpha_2'\Gamma_h\alpha_2\bar{\beta}_{2\perp}-\bar{\beta}_{2\perp}'\alpha_2'\Theta_h-\Theta_h'\alpha_2\bar{\beta}_{2\perp}') \end{pmatrix} \\
&\to \begin{pmatrix} \Gamma_{22} & e_{1\perp}'(\Gamma\alpha_2\bar{\beta}_{2\perp}-\Theta) \\ 0 & \Omega \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
A_h'\Sigma_{h21} &= \begin{pmatrix} \beta_2'\alpha_2'(\Gamma_h\alpha_1+\Theta_h\beta_{2\perp}'\beta_1) \\ -\Omega\beta_{2\perp}'\beta_1+h^{-1}(\bar{\beta}_{2\perp}'\alpha_2'\Gamma_h\alpha_1+\bar{\beta}_{2\perp}'\alpha_2'\Theta_h\beta_{2\perp}'\beta_1-\beta_{2\perp}\Theta_h'\alpha_1) \end{pmatrix} \\
&\to \begin{pmatrix} e_{1\perp}'(\Gamma\alpha_1+\Theta\beta_{2\perp}'\beta_1) \\ -\Omega\beta_{2\perp}'\beta_1 \end{pmatrix}.
\end{aligned}$$

This shows that

$$\begin{aligned}
\beta_h^* &\to (\beta_2,\bar{\beta}_{2\perp})\begin{pmatrix} \Gamma_{22} & e_{1r\perp}'(\Gamma\alpha_2\bar{\beta}_{2\perp}-\Theta) \\ 0 & \Omega \end{pmatrix}^{-1}\begin{pmatrix} e_{1r\perp}'(\Gamma\alpha_1+\Theta\beta_{2\perp}'\beta_1) \\ -\Omega\beta_{2\perp}'\beta_1 \end{pmatrix} \\
&= (\beta_2,\bar{\beta}_{2\perp})\begin{pmatrix} \Gamma_{22}^{-1} & -\Gamma_{22}^{-1}e_{1r\perp}'(\Gamma\alpha_2\bar{\beta}_{2\perp}-\Theta)\Omega^{-1} \\ 0 & \Omega^{-1} \end{pmatrix}\begin{pmatrix} e_{1r\perp}'(\Gamma\alpha_1+\Theta\beta_{2\perp}'\beta_1) \\ -\Omega\beta_{2\perp}'\beta_1 \end{pmatrix} \\
&= (\beta_2,\bar{\beta}_{2\perp})\begin{pmatrix} \Gamma_{22}^{-1}e_{1r\perp}'\Gamma(\alpha_1+\alpha_2\bar{\beta}_{2\perp}\beta_{2\perp}'\beta_1) \\ -\beta_{2\perp}'\beta_1 \end{pmatrix} = \beta_2\Gamma_{22}^{-1}e_{1r\perp}'\Gamma(\alpha_1+\alpha_2\bar{\beta}_{2\perp}\beta_{2\perp}'\beta_1)-\bar{\beta}_{2\perp}\beta_{2\perp}'\beta_1.
\end{aligned}$$

We now substitute $\bar{\beta}_{2\perp}\beta_{2\perp}' = I_n - \beta_2\bar{\beta}_2'$ and find the expression

$$\begin{aligned}
&\beta_2\Gamma_{22}^{-1}e_{1r\perp}'\Gamma(\alpha_1+\alpha_2(I_{n-1}-\beta_2\bar{\beta}_2')\beta_1)-(I_{n-1}-\beta_2\bar{\beta}_2')\beta_1 \\
&= (\beta_2\Gamma_{22}^{-1}e_{1r\perp}'\Gamma(\alpha_1+\alpha_2\beta_1)-\beta_1)-\beta_2\Gamma_{22}^{-1}e_{1r\perp}'\Gamma\alpha_2\beta_2\bar{\beta}_2'\beta_1+\beta_2\bar{\beta}_2'\beta_1 \\
&= (\beta_2\Gamma_{22}^{-1}\Gamma_{21}-\beta_1)-\beta_2\Gamma_{22}^{-1}\Gamma_{22}\bar{\beta}_2'\beta_1+\beta_2\bar{\beta}_2'\beta_1 \\
&= (-\beta_1+\beta_2\Gamma_{22}^{-1}\Gamma_{21})-(\beta_2\Gamma_{22}^{-1}\Gamma_{22}-\beta_2)\bar{\beta}_2'\beta_1 = -\beta_1+\beta_2\Gamma_{22}^{-1}\Gamma_{21},
\end{aligned}$$

see (16). ∎