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Medium-term Fluctuations and the “Great Ratios” of Economic Growth

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Abstract. Evidence for the OECD countries show that the “great ratios”, such as the unemployment rate, factor shares, Tobin’s q and the investment-capital ratio, fluctuate significantly on medium-term frequencies of 10-40 years duration. To explain these medium-term fluctuations, we establish a macro-dynamic model where the q -theory of investment is combined with sluggish real-wage adjustment in the labour market. In this framework, responses to shocks show persistence and amplification. A high degree of real-wage rigidity combined with a low elasticity of factor substitution leads to damped internal oscillations and hump-shaped impulse-response functions.

JEL: E3, G1, O4.

Key words: Medium-term cycles; Tobin’s q ; real-wage Phillips curve; elasticity of factor substitution; endogenous oscillations.

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1 Introduction

The economic literature has long distinguished between long-run growth and fluctuations around this long-run trend on business cycle frequencies. However, Schumpeter (1939), Rostow (1980), Blanchard (1997), Aghion and Howitt (1998) and Comin and Gertler (2006) argue that this distinction may be misleading since economies display growth and unemployment cycles on medium term frequencies of, say, 10-40 years duration. Considering the OECD countries over the past two centuries medium-term upturns free of any significant downturns have been experienced in the mid-19th century, the decade leading up to WWI, in the period 1950-1973 and the long upturn before the Global Financial Crisis (Madsen, 2010). Conversely, the OECD countries underwent medium-term downturns in the periods 1860-1896, 1913-1946, and 1974-1993. These events suggest that major shocks, such as shocks initiated in the labour market (labour supply shocks) and technology shocks (labour demand shocks), may be propagated through wage rigidities and capital adjustment costs.

This paper seeks to explain medium-term fluctuations in the “great ratios” such as unemployment rates, factor shares, Tobin’s q and the investment-capital ratio in response to shocks to labour demand and labour supply, where labour supply shocks are predominantly of the wage-push type while the labour demand shocks are predominantly driven by technology shocks. To comply with these features, we introduce convex capital adjustment costs and sluggish wage adjustment. Allowing for convex capital adjustment costs establishes an explicit link between asset markets and investment. Furthermore, firms’ investment decisions and households’ saving decisions become separate but mutually compatible through the endogenous adjustment of current and expected future interest rates. As shown by Abel and Blanchard (1983), this allows for appealing medium-term dynamics; however, the dynamics of the Abel-Blanchard model is confined by the assumption of perfect competition in the labour market and therefore full employment.

The Abel-Blanchard framework is here extended to allow for medium-term fluctuations through wage rigidity while maintaining long-run constancy of the “great ratios”. We replace the assumption of perfect competition in the labour market with sluggish real-wage adjustment in the form of a real-wage Phillips curve. This changes the model from a two-dimensional system to a three-dimensional system with two state variables, thus opening up for rich dynamics, including internal (endogenous) damped oscillations. For alternative degrees of substitutability of capital for labour we examine how the system reacts to infrequent sizeable technology and wage-push shocks.

Consider a shock that drives productivity-corrected wages above their long-run equilibrium. This induces substitution toward less labour intensive production methods and a reduction in Tobin's q . This pushes unemployment above its steady state equilibrium and lowers capital investment. The excess unemployment reduces wage growth, and unemployment and Tobin's q , consequently, move back towards their steady state values. However, the adjustment towards equilibrium is cyclical in that the re-instatement of profitability pushes the rate of unemployment *below* its equilibrium level; thereby generating the basis for a new profits squeeze. Depending on the parameter values, internally generated (damped) cycles may arise, entailing correlated hump-shaped responses of the "great ratios" to various shocks. This possibility of internal oscillations challenges the claim by Kydland and Prescott (1990, p. 5) that "cyclical laws of motion do not arise ... for economies with reasonable statements of people's ability and willingness to substitute."

This paper is not the first to introduce wage stickiness into a growth model. Goodwin (1967) was the first to formally bring in the real-wage Phillips curve as a driving force in business cycles. Assuming a Leontief production function (hence no capital-labour substitution at all) and with all wages consumed and all profits saved, the real-wage Phillips curve leads to self-sustained oscillations (closed cycles) in labour's income share and unemployment. Such recurrent, regular cycles, however, do not correspond well to the irregularity of the period length and the amplitude that characterise the data. Akerlof and Stiglitz (1969) introduced a neoclassical production function into the Goodwin model and showed that the presence of factor substitution implied either a dampening or a complete elimination of the cycles, depending on parameter values (other contributions, along this line, include Ito, 1980; Marrewijk, 1993, and Flaschel *et al.*, 1997).

A limitation of the Goodwin as well as the Akerlof and Stiglitz model is the assumption of fixed saving rates for wage and profits income and the treatment of investment as essentially passive. Instead, by combining sluggish real-wage adjustment with convex capital adjustment costs we obtain a more realistic perspective where the investment decisions are taken by forward-looking firms and the interest rate is determined by separate investment and saving decisions, the latter taken by utility maximising households.

We use a continuous-time non-stochastic formulation and focus on analytical results rather than a precise empirical application. The focus is on how real-wage rigidity, substitutability of capital and labour and convexity of adjustment costs affect the amplification and persistence characteristics of the impulse-propagation mechanism of the model economy. We conjecture that the medium- and long-term perspective allows putting monetary aspects and nominal

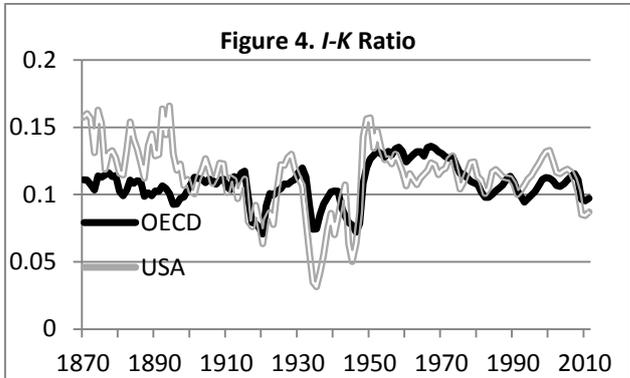
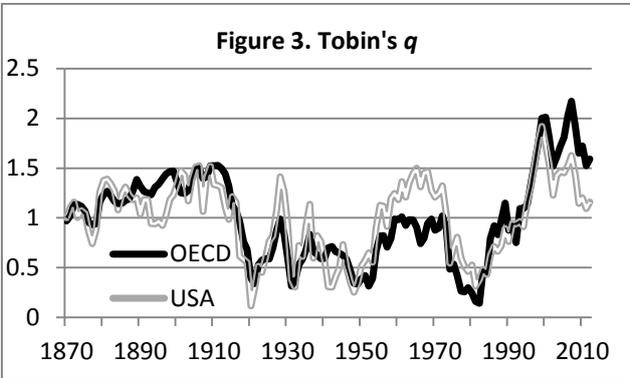
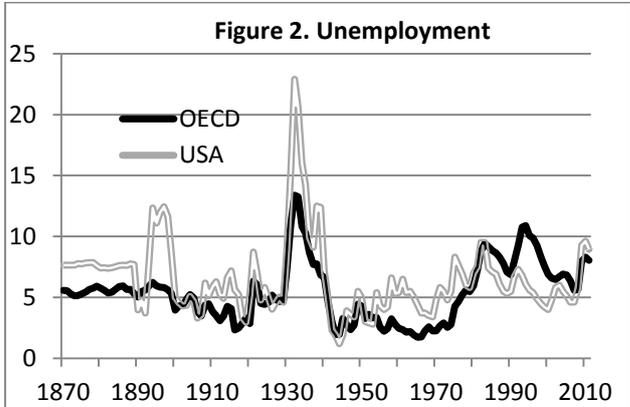
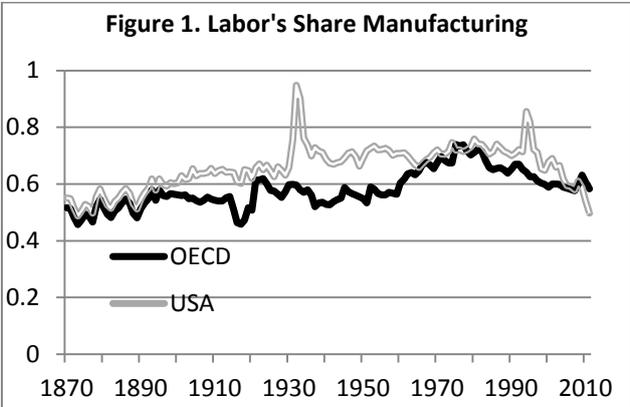
rigidities aside as a first approximation. This supply-side approach is related to the real business cycle (RBC) literature, which has also drawn attention to Tobin's q (Baxter and Crucini, 1993; Christiano and Fisher, 2003). In contrast to these studies and the general RBC approach, however, we do not model fluctuations in employment as reflecting fluctuations in labour supply. Instead, we model them as reflecting fluctuations in the rate of unemployment, thus enhancing internal propagation of shocks.

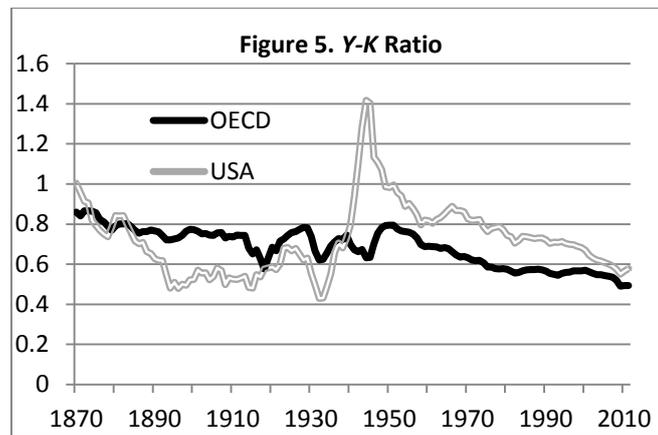
We address response persistence that goes beyond the time scale of ordinary business cycles of 4-8 years' duration. The medium-term perspective is complementary to recent contributions emphasizing a link between medium-term fluctuations and endogenous innovation and technology adoption. In the "New Schumpeterian" theory lumpy advances in technology lead to long swings in economic activity (see Aghion and Howitt, 1998, Chapter 8). In the Comin and Gertler (2006) approach pro-cyclical R&D and endogenous technology adoption provide an avenue through which shocks generate medium-term movements in productivity. Blanchard (1997) develops a model of employment and capital accumulation under the assumption of monopolistic competition while allowing for adjustment costs of capital. In his model adverse shifts in labour demand induced by shifts in the distribution of rents or labour-saving technological progress lead to medium-term movements in unemployment and the capital income share. The present paper complements these contributions by emphasising the role of sluggish real wage and capital adjustment for generation of medium-term swings in response to shocks.

The paper is organised as follows. The next section uses long historical data for 13 OECD countries to give a rough indication that the great ratios fluctuate substantially on medium-term frequencies. Our extended Abel-Blanchard model with sluggish real-wage adjustment is set up in Section 3 and steady state and stability properties are characterized in Section 4. Specific functional forms and the conditions leading to internal oscillations are considered in Section 5, where also a comparison with related models in the literature is drawn. The model is calibrated in Section 6 and Section 7 explores how the presence of internal fluctuations and persistence of the effects of shocks depend on the slope of the real-wage Phillips curve, the elasticity of factor substitution and the convexity of adjustment costs. Section 8 simulates the impulse-response functions for specific shock types and the mechanisms behind the oscillations are spelled out. Section 9 concludes.

2 The great ratios over the past 142 years

This section gives graphical evidence indicating that medium term fluctuations are distinct features of the movement in the great ratios. Data are shown over the period 1870-2011 for the US and an unweighted average of the following 13 OECD countries: Canada, the US, Australia, Belgium, Denmark, Finland, France, Germany, the Netherlands, Norway, Spain, Sweden and the UK. The data sources are listed in Groth and Madsen (2013). Separate figures are presented for the US because the labour market has behaved quite differently from that of continental Europe in the post-WWII period (Blanchard, 1997). Continental Europe was much slower to react to the adverse supply shocks in the 1970s and the early 1980s than the US; thus creating a wedge between these two groups in terms of unemployment and labour's share. The medium-term is defined as fluctuations of 10 to 40 years of duration, following Abramovitz (1993). We shall only give a descriptive account and not attempt any formal analysis of the data.





Labour's income share in manufacturing is displayed in Figure 1. Manufacturing data are used because it has a substantially lower fraction of self-employed and family workers than economy-wide data and is, therefore, not much affected by the decline in the fraction of self-employment in total employment over the past 142 years. The graph indicates that the labour income share fluctuates around a roughly constant level of about two thirds over the time-span covered by the graphs. The peaks are approximately 15 percent above the troughs, which, potentially, can generate considerable fluctuations in unemployment and in Tobin's q .

Three medium-term cycles of 25-40 years of duration seem discernible in the OECD data. The increasing labour share from 1870 was reversed in the beginning of the 20th century as the growth in nominal wages did not keep up with inflation and rising productivity. The increasing labour share immediately after WWI may reflect pushy labour unions, inspired by the October revolution in Russia, and the desire to re-establish the purchasing power of wages that was eroded by the great inflation during WWI. The most marked increase in labour's share during the period 1870-2011 occurs in the 1960s and the first half of the 1970s and was probably triggered by adverse supply shocks, such as increasing commodity and food prices and strong labour unions, resulting in wages being pushed above their full employment equilibrium (Bruno and Sachs, 1985; Blanchard, 1997). The subsequent decline occurs in the wake of persistently high unemployment rates and weakening union power.

Unemployment also displays pronounced medium-term fluctuations as witnessed by Figure 2 and these fluctuations seem positively related to labour's income shares. Unemployment is above its trend in the 1890s, the interwar period and in the 1980s and 1990s. The movement of Tobin's q , the ratio of the market value of firms to the replacement cost of their capital, is shown in Figure 3. Long data on Tobin's q are only available for the US. However, since for the US there is a strong relationship between Tobin's q and the deviation of real share prices from their trend, the estimates for the non-US OECD countries are based on real stock prices. More

precisely, the log of consumer-price-deflated stock prices is regressed on a time trend and a constant and Tobin's q is presented by the residuals from these regressions.¹ For the US Figure 3 shows that Tobin's q tends towards constancy in the long run, as predicted by the model in this paper.

Tobin's q moves counter to labour's income share, which is consistent with our model. With elasticity of factor substitution less than one (in line with recent econometric studies, cf. Antràs 2004), increased labour income share, brought about by wage pressure, lowers the rate of profit and Tobin's q . This may explain the relatively low q in the interwar period and the 1970s. Conversely, the high q during the decades up to the Great Recession in 2008- and most of the 1950s and 1960s may reflect weakness of unions during these periods.

The I/K ratio in Figure 4 also indicates pronounced swings on medium-term frequencies and co-variation with Tobin's q , unemployment and labour's income share. The relatively low I/K in the late 19th century, the interwar period as well as the 1980s and 1990s coincides with a relatively low Tobin's q , high unemployment and high labour share in the same periods. Finally, as shown in Figure 5 the output-capital ratio in the OECD has been decreasing somewhat over the period considered, likely reflecting capital deepening. For the US the $Y-K$ ratio has been below its trend in the late 19th century, the interwar period and in the 1980s and 1990s.

Overall, the figures indicate distinct medium-term movements in the 'great ratios' and these movements often overshadow the movements at business cycle frequencies. This is particularly true for Tobin's q , the investment ratio and unemployment.

3 A Ramsey model with sluggish capital and wage adjustment

This section sets up the model to shed light on possible mechanisms behind medium-term fluctuations in the great ratios. The essential ingredients are the income struggle in the labour market, represented by the real-wage Phillips curve, and firms' investment decisions.

3.1 Firms

Production is carried out by many identical competitive firms with the technology

$$Y_t = F(K_t, A_t L_t), \quad (3.1)$$

¹Regressing the log of Tobin's q on the deviation of the log of real share prices from a time-trend, for the US, using Cochrane-Orcutt correction for first-order serial correlation yields:

$$\ln q_t = 0.92 q_t^c - 0.20, \quad \bar{R}^2 = 0.98, \quad DW = 2.35,$$

(29.7) (2.23)

where q^c is the residual from regressing the log of consumer-price-deflated share prices on a time-trend, and the numbers in parentheses are absolute t -statistics. Estimation period is from 1871 to 2006.

where F is a neoclassical production function with constant returns to scale; Y is output gross of adjustment costs and physical capital depreciation; K is capital; L is labour and A is the economy-wide efficiency of labour. Technological progress is Harrod-neutral with a constant exogenous rate, γ :

$$A_t = A_0 e^{\gamma t}, \quad \gamma > 0. \quad (3.2)$$

For notational convenience, the number of firms is set equal to one (the representative firm).

There are convex capital adjustment costs (installation costs), J_t , and these are internal to the firm. We assume that $J_t = G(K_t, I_t)$, where G is strictly convex in gross investment, I_t , and that the firm has perfect foresight.² Given the expected time path for wages and interest rates, $(w_t, r_t)_{t=0}^{\infty}$, the firm chooses $(L_t, I_t)_{t=0}^{\infty}$ to maximise the present value of expected cash flows,

$$V_0 = \int_0^{\infty} [F(K_t, A_t L_t) - w_t L_t - I_t - G(K_t, I_t)] e^{-\int_0^t r_s ds} dt, \quad (3.3)$$

subject to

$$\dot{K}_t = I_t - \delta K_t, \quad \delta \geq 0, \quad (3.4)$$

and the requirement that $K_t \geq 0$ for all $t \geq 0$. Here δ is a constant capital depreciation rate.

Taxes and subsidies are ignored. We assume that G is homogeneous of degree one, thereby allowing us to write $G(K, I) \equiv g(I/K)K$, where

$$g(0) = 0 = g'(0) \text{ and } g''(\bullet) > 0. \quad (3.5)$$

The following necessary and sufficient conditions for an interior solution apply:

$$F_2(K_t, A_t L_t) A_t = w_t, \quad (3.6)$$

$$G_2(K_t, I_t) = g' \left(\frac{I_t}{K_t} \right) = q_t - 1, \quad (3.7)$$

$$\dot{q}_t = (r_t + \delta) q_t - F_1(K_t, A_t L_t) + G_1(K_t, I_t), \quad (3.8)$$

$$\lim_{t \rightarrow \infty} q_t K_t e^{-\int_0^t r_s ds} = 0, \quad (3.9)$$

where q_t is the shadow price of installed capital and $G_1(K, I) = g(I/K) - g'(I/K)I/K$. The first-order condition (3.7) defines the investment-capital ratio as an implicit function of q ,

$$\frac{I_t}{K_t} = m(q_t), \text{ where } m(1) = 0 \text{ and } m'(q) = \frac{1}{g''(m(q))} > 0. \quad (3.10)$$

The differential equation (3.8) may be written as a no-arbitrage condition,

² Although there are additional and partly alternative explanations for sluggish investment (irreversibility, bankruptcy costs, "time to plan"), we stick to the convex adjustment cost approach for tractability reasons.

$$\frac{F_1(K_t, L_t) - G_1(K_t, I_t) - \delta q_t + \dot{q}_t}{q_t} = r_t, \quad (3.11)$$

saying that along the optimal path the marginal rate of return must equal the interest rate. Solving (3.8), using the transversality condition (3.9), gives q_t as the present value of the extra cash-flows brought about (along the optimal path) by the marginal unit of installed capital at time t , i.e.,

$$q_t = \int_t^\infty (F_1(K_\tau, A_\tau L_\tau) - G_1(K_\tau, I_\tau)) e^{-\int_t^\tau (r_s + \delta) ds} d\tau. \quad (3.12)$$

3.2 Households

As in the standard Ramsey setup, there is a fixed number of infinitely-lived households, all alike. For notational convenience, the number of households is one. The representative household has N members and, in the simplest interpretation of the model, each member supplies one unit of labour inelastically.³ The size of the household grows at a constant exogenous rate n , i.e., $N_t = N_0 e^{nt}$. The household consumes and saves, and saving may go into bonds or equity of firms or take the form of capital gains on already acquired equity. Capital income per capita at time t is $r_t a_t$, where a_t is per capita financial wealth. Labour income at time t is $w_t v_t N_t$, where $v_t \in [0, 1]$ is actual labour per household member.

The household's preferences, as seen from time 0, are given by

$$U_0 = \int_0^\infty \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-(\rho-n)t} dt, \quad \theta > 0, \quad (3.13)$$

where θ is the (absolute) elasticity of marginal utility of consumption and ρ is the pure rate of time preference.⁴ The household chooses a path $(c_t)_{t=0}^\infty$ to maximise U_0 subject to $c_t \geq 0$ and

$$\dot{a}_t = (r_t - n)a_t + w_t v_t - c_t, \quad a_0 \text{ given}, \quad (3.14)$$

$$\lim_{t \rightarrow \infty} a_t e^{-\int_0^t (r_s - n) ds} \geq 0, \quad (3.15)$$

where (3.15) is the No-Ponzi-Game condition. Taxes and unemployment benefits are ignored. As is well-known, the necessary and sufficient conditions for an interior solution to this problem are the Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta} (r_t - \rho), \quad (3.16)$$

and the transversality condition that (3.15) holds with equality.

³ An alternative interpretation, which we come back to below, allows elastic labour supply.

⁴ In case $\theta = 1$, the expression $(c^{1-\theta} - 1)/(1-\theta)$ should be interpreted as $\ln c$.

3.3 The labour market

The above elements, together with the assumption of perfect competition in the labour market, essentially constitute the model by Abel and Blanchard (1983). In view of the large fluctuations in unemployment displayed in Figure 2 above, we introduce real wage rigidity in the short run due to, say, efficiency wages or trade unions. Hence, employment is determined by labour demand, L_t , and is generally below labour supply, N_t . We have

$$v_t N_t = L_t, \quad (3.17)$$

where, as a rule, the employment ratio, v_t , is less than 1.

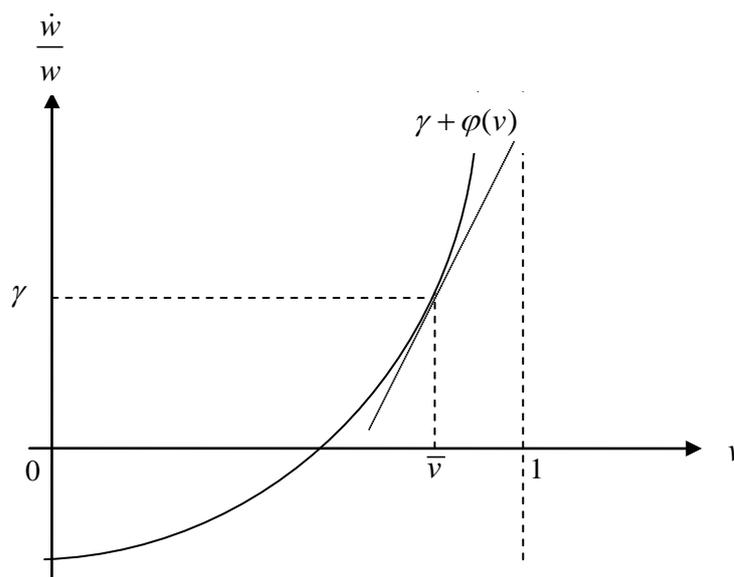


Figure 6. The real-wage Phillips curve.

The *real-wage Phillips curve* is given by:

$$\frac{\dot{w}_t}{w_t} = \gamma + \varphi(v_t), \quad \varphi' > 0, \varphi'' \geq 0, \varphi(\bar{v}) = 0 \text{ for some } \bar{v} \in (0,1), \quad (3.18)$$

where at any given t the real wage, w_t , is predetermined. The employment ratio at which w_t grows at the same rate as technology is denoted \bar{v} and named the *natural rate of employment*.⁵ To ensure that the employment ratio never exceeds one, one might impose the condition $\lim_{v \rightarrow 1} \varphi(v) = \infty$, as illustrated in Figure 6. Our focus will be on dynamics near \bar{v} , however. In the

⁵ Blanchard and Katz (1999) summarise the empirical evidence and provide a theoretical explanation supporting such a description of wage movements in the US. The conditions needed for a theoretical “wage curve” (delivered by an efficiency-wage or bargaining model) to lead to a Phillips curve like (3.18) apply well to the US. Our abstraction from divergence between actual and expected inflation is likely to be an acceptable approximation in a medium-run perspective. Atkins et al. (1993) find that Canadian data are well described by a real-wage Phillips curve.

numerical simulations of Section 8 we apply a linear approximation of (3.18) and consider shocks of modest size so that the system never hits the $v = 1$ ceiling.

The model allows an alternative interpretation of the household labour supply preferences. What is essential is the rationing in the labour market, not the simplifying assumption that labour, ℓ_t , is supplied inelastically. In the alternative interpretation, leisure, $1 - \ell_t$, enters the instantaneous utility function as in a standard RBC model: $u(c_t, 1 - \ell_t) = \ln c_t + \xi \ln(1 - \ell_t)$, $\xi > 0$. Then, in accordance with the theory of rationing,⁶ the condition $0 \leq \ell_t \leq \min(v_t, 1)$ would be an additional constraint in the household's optimisation problem and imply the additional first-order condition $\ell_t = \min(v_t, 1 - \xi c_t / w_t)$. As in the specification without utility of leisure, in steady state c_t and w_t would grow at the same rate, γ , thus implying constancy of c_t / w_t . By choosing ξ small enough, the employment offered by the firms at the going real wage will be binding (i.e., smaller than the labour supply) not only in steady state, but also during the transition towards steady state. Thus, the Phillips curve as described by (3.18) remains in force.

3.4 Dynamic general equilibrium

In view of the linearly homogenous production and adjustment cost functions, we know from Hayashi's theorem (Hayashi, 1982) that our q , often called "marginal q ", in competitive equilibrium equals "average q ", often called *Tobin's q* . If V denotes the market value of the aggregate capital stock, we thus have

$$q = \frac{V}{K} = \frac{Na}{K} \text{ for all } t, \quad (3.19)$$

where the second equality follows from the fact that the total financial wealth of the households equals V ; thus linking per capita financial wealth, a , to the evolution of q and K .⁷ From an empirical point of view the importance of (3.19) is that it links our theoretical q to something measurable, V/K .

Letting k denote firms' capital intensity, $k \equiv K/(AL)$, and f be the production function on intensive form we can write $Y/(AL) \equiv y = f(k) \equiv F(k, 1)$, $f(0) \geq 0$, $f' > 0$, $f'' < 0$. Denoting the efficiency-corrected real wage \tilde{w} , we have $\tilde{w} \equiv w/A$. Thus, the firms' first-order condition, (3.6), can be written

$$F_2(K, AL) = f(k) - kf'(k) = \tilde{w}. \quad (3.20)$$

This, implicitly, defines k as a function of \tilde{w} ,

⁶ See e.g. Malinvaud (1977).

⁷ For simplicity, the time-subscripts will be omitted from now on when not needed.

$$k = k(\tilde{w}), \quad k'(\tilde{w}) = -\frac{1}{k(\tilde{w})f''(k(\tilde{w}))} > 0. \quad (3.21)$$

Since w and A are predetermined it follows that \tilde{w} and k will also be predetermined at any given point in time. Inserting (3.7), (3.10) and (3.21) into (3.8) yields

$$\dot{q} = (r + \delta)q - f'(k(\tilde{w})) + g(m(q)) - m(q)(q - 1). \quad (3.22)$$

The economy-wide capital intensity, x , is the ratio of the capital stock to the efficiency-corrected labour force:

$$x \equiv \frac{K}{AN}.$$

Log-differentiating x with respect to t , using (3.4) and (3.10), gives:

$$\dot{x} = [m(q) - (\delta + \gamma + n)]x. \quad (3.23)$$

Owing to unemployment, x differs from the firms' capital intensity, k . The employment ratio is the ratio of the two:

$$v = \frac{L}{N} = \frac{x}{k(\tilde{w})}. \quad (3.24)$$

In effect the Phillips curve, (3.18), can be written as:

$$\dot{\tilde{w}} = \varphi \left(\frac{x}{k(\tilde{w})} \right) \tilde{w}. \quad (3.25)$$

Finally, equilibrium in the goods market requires that $F(K, AL) - g(I/K)K = cN + I$.

Substituting (3.10), the consumption-capital ratio can be written as:

$$\frac{cN}{K} = \frac{f(k(\tilde{w}))}{k(\tilde{w})} - g(m(q)) - m(q) \equiv \hat{c}(q, \tilde{w}). \quad (3.26)$$

Note that $\hat{c}_q = -qm'(q) < 0$ (for $q > 0$) and $\hat{c}_{\tilde{w}} = -\tilde{w}k'(\tilde{w})/k(\tilde{w})^2 < 0$ following from (3.7) and (3.20), respectively. From the Keynes-Ramsey rule, (3.16), we have:

$$r = \theta \dot{\tilde{c}} / \tilde{c} + \rho + \theta \gamma, \quad (3.27)$$

where

$$\tilde{c} \equiv \frac{c}{A} \equiv \frac{cN}{K} \frac{K}{AN} = \hat{c}(q, \tilde{w})x. \quad (3.28)$$

Log-differentiating the right-hand side with respect to t , using (3.26), yields:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{-m'(q)q\dot{q} - \frac{\tilde{w}k'(\tilde{w})}{k(\tilde{w})^2}\dot{\tilde{w}}}{\hat{c}(q, \tilde{w})} + \frac{\dot{x}}{x}. \quad (3.29)$$

Substituting this together with (3.27) into (3.22) gives:

$$\dot{q} = \frac{\theta \left[\hat{c}(q, \tilde{w}) \frac{\dot{x}}{x} - \frac{\tilde{w} k'(\tilde{w})}{k(\tilde{w})^2} \dot{\tilde{w}} \right] q}{\hat{c}(q, \tilde{w}) + \theta m'(q) q^2} + \frac{\hat{c}(q, \tilde{w}) [(\rho + \gamma \theta + \delta) q - f'(k(\tilde{w})) + g(m(q)) - m(q)(q-1)]}{\hat{c}(q, \tilde{w}) + \theta m'(q) q^2}. \quad (3.30)$$

In view of (3.23) and (3.25), this can be written as

$$\dot{q} = Q(q, \tilde{w}, x).$$

Our extension of the two-dimensional Abel-Blanchard model thus results in a three-dimensional dynamic system, consisting of (3.30), (3.25) and (3.23), in terms of q , \tilde{w} and x . The variables \tilde{w} and x are state variables (predetermined variables), whereas q is a jump variable. The equilibrium path of the economy, $(q, \tilde{w}, x)_{t=0}^{\infty}$, is determined by these three differential equations and the firm's transversality condition, (3.9).⁸ Despite its seemingly complexity, the system (3.30)-(3.25)-(3.23) turns out to be analytically tractable.

4 Steady state and stability

In a steady state we have $\dot{q} = \dot{\tilde{w}} = \dot{x} = 0$ and steady-state values of the variables are marked by an asterisk. The steady-state values are determined recursively in the following way. In view of (3.28), \tilde{c} is constant in steady state. So (3.27) reduces to

$$r^* = \rho + \theta \gamma, \quad (4.1)$$

as in the standard neoclassical model without capital adjustment costs. The value of q in steady state is given by the condition $\dot{x} = 0 < x$ in (3.23), i.e.

$$q^* = m^{-1}(\delta + \gamma + n) > 1, \quad (4.2)$$

where the inequality follows from (3.10) and $\delta + \gamma + n > 0$. The condition $\dot{q} = 0$ in (3.22), together with (4.1), gives $(\rho + \theta \gamma + \delta) q^* - f'(k(\tilde{w})) + g(\delta + \gamma + n) - (\delta + \gamma + n)(q^* - 1) = 0$.

Assuming this equation in $k(\tilde{w})$ has a positive solution, k^* , the equation can be rewritten as:

$$f'(k^*) = (\rho + \theta \gamma + \delta) q^* + g(\delta + \gamma + n) - (\delta + \gamma + n)(q^* - 1) \equiv p. \quad (4.3)$$

From (3.20) we find:

$$\tilde{w}^* = f(k^*) - k^* f'(k^*) \equiv \tilde{w}(k^*). \quad (4.4)$$

Since $\dot{\tilde{w}} = 0$, (3.18) implies $v = \bar{v}$. Hence, from (3.24):

$$x^* = \bar{v} k^* > 0. \quad (4.5)$$

⁸ The transversality condition of the household is equivalent to this condition.

In steady state, K grows at the rate $\gamma + n$ and the transversality condition (3.9) is satisfied if and only if $r^* > \gamma + n$, i.e.,

$$\rho - n > (1 - \theta)\gamma, \quad (4.6)$$

which we assume holds. By (4.6), p in (4.3) satisfies $p > g(\delta + \gamma + n) + \delta + \gamma + n > 0$. To ensure existence of a steady state the following condition has to hold:

$$\lim_{k \rightarrow 0} f'(k) > p > \lim_{k \rightarrow \infty} f'(k), \quad (4.7)$$

where p is given in (4.3).

As to the steady-state values of the remaining great ratios, the investment-capital ratio is given by $(I/K)^* = \delta + \gamma + n$. Due to the adjustment costs, there is a difference between gross production Y and GDP . Indeed, $GDP \equiv Y - J = Y - g(I/K)K$. The output-capital ratio (the ratio of value added to capital) is given by:

$$\left(\frac{GDP}{K}\right)^* = \frac{f(k^*)}{k^*} - g(\delta + \gamma + n). \quad (4.8)$$

The consumption-capital ratio is:

$$\left(\frac{cN}{K}\right)^* = \frac{f(k^*)}{k^*} - g(\delta + \gamma + n) - (\delta + \gamma + n) \equiv \hat{c}(q^*, \tilde{w}^*). \quad (4.9)$$

The share of wages in income is:

$$\left(\frac{wL}{GDP}\right)^* = \left(\frac{wL/Y}{GDP/Y}\right)^* = \frac{1 - \varepsilon(k^*)}{1 - \frac{g(\delta + \gamma + n)k^*}{f(k^*)}}, \quad (4.10)$$

where $\varepsilon(k)$ is the elasticity of gross production with respect to capital, i.e.,

$$\varepsilon(k) \equiv kf'(k) / f(k) \in (0, 1). \quad (4.11)$$

Without further specification of the functions f , g and φ , we are able to demonstrate that the steady state, when it exists, is always (locally) saddle-point stable:

Proposition 1. Assume that the parameter restriction (4.6) and the technology condition (4.7) hold. Then:

- (i) A steady state, (q^*, \tilde{w}^*, x^*) , exists and is unique.
- (ii) Let the initial values $\tilde{w}_0 > 0$ and $x_0 > 0$ be given. Then there exists a neighbourhood of (\tilde{w}^*, x^*) such that with (\tilde{w}_0, x_0) belonging to this neighbourhood, there exists a unique equilibrium path, $(q, \tilde{w}, x)_{t=0}^{\infty}$, and it converges towards the unique steady state.

Proof. (i) By (4.6), p in (4.3) is positive. Then, using (4.7), there exists a $k^* > 0$ satisfying (4.3); k^* is unique since $f'' < 0$. By (4.6), the steady state is not only technically feasible but also consistent with general equilibrium. Moreover, non-negativity and strict concavity of $f(k)$ implies $f(k^*)/k^* > f'(k^*)$. Thereby (4.6) combined with (4.3) ensures that the right-hand side of (4.8) as well as the denominator in (4.10) are positive. Finally, positivity of the right-hand side of (4.9) is ensured by Lemma A2 of Appendix A. (ii) See Appendix A. Q.E.D.

The presence of, at least, local stability makes the model consistent with Kaldor's "stylised facts". The fact that stability is present in a system with *two* predetermined variables opens up for rich dynamics, including internal damped oscillations. The remaining part of the paper explores the theoretical and quantitative implications of this.

5 Parameterisation and internal oscillations

To proceed we introduce the following functional forms. The φ function in the Phillips curve is assumed to be linear in the deviation of the employment ratio from its natural level, i.e.,

$$\varphi(v) = \lambda(v - \bar{v}), \quad \lambda > 0, \quad (5.1)$$

where λ signifies the degree of real wage flexibility. We shall call the inverse of λ the degree of real wage stickiness. The capital adjustment costs are assumed to be quadratic:

$$g\left(\frac{I}{K}\right) = \frac{1}{2\beta}\left(\frac{I}{K}\right)^2, \quad \beta > 0, \quad (5.2)$$

implying that $I/K = m(q) = \beta(q-1)$. Here β signifies the degree of investment flexibility and the inverse of β is the convexity of adjustment costs, thus measuring the degree of investment sluggishness.⁹ With this specification of adjustment costs, Tobin's q in steady state takes the value

$$q^* = 1 + \frac{\delta + \gamma + n}{\beta},$$

and (4.3) can be written

$$f'(k^*) = \rho + \theta\gamma + \delta + \frac{\delta + \gamma + n}{\beta} \left[\rho - n - (1 - \theta)\gamma + \frac{1}{2}(\delta + \gamma + n) \right] \equiv p(\beta, \xi) > 0, \quad (5.3)$$

where $\xi \equiv (\delta, \gamma, n, \rho, \theta)$ and the inequality is implied by (4.6). Note that in view of (4.6), $\partial p / \partial \beta < 0$.

Finally, the production function is assumed to be of CES form:

⁹ The degree of convexity is $g''/g' = (I/K)^{-1} = [\beta(q-1)]^{-1}$.

$$f(k) = B(\alpha k^\psi + 1 - \alpha)^{1/\psi}, \quad \psi < 1, B > 0, 0 < \alpha < 1, \quad (5.4)$$

where ψ is the “substitution parameter”, α the “distribution parameter” and B the “efficiency parameter”. The elasticity of gross production with respect to capital then is $\varepsilon(k) = \alpha / (\alpha + (1 - \alpha)k^{-\psi})$. The elasticity of factor substitution is $1 / (1 - \psi) \equiv \sigma$, a positive constant. For $\sigma = 1$ (i.e. $\psi = 0$), (5.4) should be interpreted as the Cobb-Douglas function $f(k) = Bk^\alpha$ in which case the Inada conditions are satisfied so that the technology condition (4.7) and thereby also (i) and (ii) of Proposition 1 automatically hold. Outside the Cobb-Douglas case, the Inada conditions are not satisfied and a further precondition is needed to ensure that (4.7) holds:

Lemma 1. Given the functional specifications (5.1), (5.2) and (5.4), assume the parameter restriction (4.6). Let $p(\beta, \xi)$ be given as in (5.3) and suppose that either $\sigma = 1$ or the composite inequality

$$p(\beta, \xi) \begin{cases} < \alpha^{\sigma/(\sigma-1)} B & \text{if } 0 < \sigma < 1, \\ > \alpha^{\sigma/(\sigma-1)} B & \text{if } \sigma > 1, \end{cases} \quad (5.5)$$

holds. Then the technology condition (4.7) holds.

Proof. See Appendix B. Q.E.D.

It follows that outside the Cobb-Douglas case, imposing the composite inequality (5.5) ensures existence and stability of a unique steady state. In case $\sigma < 1$, (5.5) implies a lower bound on the investment flexibility for the steady state to exist. If $\sigma > 1$, (5.5) implies an upper bound on the investment flexibility for the steady state to exist.

To be able to study how alternative values of the elasticity of substitution affects the transitional dynamics *for fixed steady-state values*, k^* , $f(k^*)$ and $f'(k^*)$, we “normalise” the CES function in the following way. For a given time unit (one year, say), let the values of the parameters in (5.3) be fixed. This fixes a required value of $f'(k^*)$. Then, starting from a baseline value of σ (which is a dimensionless parameter), the fixed $f'(k^*)$ together with a baseline value of the output-capital ratio, $f(k^*)/k^*$, fixes the parameters α and B in (5.4) (see Appendix C). When we consider an alternative value of σ , we adjust α and B so that k^* , $f(k^*)$ and $f'(k^*)$ remain unchanged as long as $p(\beta, \xi)$ in (5.3) is unchanged. In accordance with recent empirical evidence (see below) we choose the baseline value for σ , denoted $\bar{\sigma}$, to be less than one.

Besides allowing meaningful comparative analysis, this “normalisation” permits a simpler expression than in (5.5) for the domain of β allowing a steady state. Indeed, given $\bar{\sigma} < 1$ and given associated baseline values of α and B , denoted $\bar{\alpha}$ and \bar{B} , respectively, the domain for β allowing a steady state becomes simply $\beta > \underline{\beta}$, where $\underline{\beta}$ is the value of β required for the equation $p(\beta, \xi) = \bar{\alpha}^{\bar{\sigma}/(\bar{\sigma}-1)} \bar{B}$ to hold, cf. (5.5). By (5.3), this value is

$$\underline{\beta} = \frac{(\delta + \gamma + n)[\rho - n - (1 - \theta)\gamma + \frac{1}{2}(\delta + \gamma + n)]}{\bar{\alpha}^{\bar{\sigma}/(\bar{\sigma}-1)} \bar{B} - (\rho + \theta\gamma + \delta)} \equiv \underline{\beta}(\bar{\sigma}, \bar{\alpha}, \bar{B}, \xi). \quad (5.6)$$

Important for the ability of a model to generate correlated *hump-shaped* responses to a variety of disturbances is whether internal (endogenous) oscillations occur. They do so if and only if there is a pair of conjugate complex eigenvalues. A necessary and sufficient condition for complex eigenvalues to emerge in our three-dimensional system is that the discriminant, Δ , of the Jacobian matrix associated with the system, evaluated in the steady state, is positive. The next proposition provides analytical results about the conditions for this as well as its converse to hold.

Proposition 2. Given the functional specifications (5.1), (5.2) and (5.4), assume the parameter restriction (4.6). Let $\underline{\beta}(\bar{\sigma}, \bar{\alpha}, \bar{B}, \xi)$ be defined as in (5.6). Then for any $\beta > \underline{\beta}(\bar{\sigma}, \bar{\alpha}, \bar{B}, \xi)$:

- (i) For any given $\sigma > 0$ there exists $\lambda_0 > 0$ such that $\lambda < \lambda_0$ implies $\Delta > 0$.
- (ii) For any given $\lambda > 0$ there exists $\sigma_0 > 0$ such that $\sigma < \sigma_0$ implies $\Delta > 0$.
- (iii) There exists $\sigma_1 > 0$ such that $\sigma > \sigma_1$ implies $\Delta < 0$ at least for some $\lambda > 0$.

Proof. See Appendix D. Q.E.D.

Thus, by (i) and (ii) of the proposition, given the two other key parameters, both a high enough degree of wage sluggishness (low λ) and a low enough degree of factor substitutability (low σ) are alone sufficient for internal oscillations to arise. Claim (iii) of the proposition tells us that there exist parameter combinations such that internal oscillations do not occur and that a large σ helps in this direction.

The economic interpretation of the occurrence of internal oscillations is that an increase in profitability carries the seeds of its own destruction by pushing the level of unemployment

below its long-run equilibrium level.¹⁰ Before discussing the empirical issue whether the parameter values that generate internal oscillations are within an empirically plausible range, it may be illuminating to briefly consider the major limiting cases of our model. Some of these cases have similarities with known models in the literature.

Limiting cases and related literature

The limiting case in which $\lambda = \infty$ is the Abel-Blanchard (1983) model with instantaneous adjustment of the real wage so that full employment obtains at all times. The real wage thus ceases to be a state variable and the dynamics become two-dimensional with a saddle-point stable steady state. Hence, there is no possibility of internal oscillations whatever the size of the elasticity of substitution between capital and labour.¹¹

The limiting case $\sigma = 0$ may be called a Ramsey-Goodwin-Tobin model since it combines two key assumptions in Goodwin (1967) (fixed technical coefficients and a real-wage Phillips curve) with a representative Ramsey household and Tobin's q . This gives a three-dimensional model which is saddle-point stable and has always internal oscillations whatever the size of λ and β (the proof is similar to that for Propositions 1 and 2). Thus, replacing the original Goodwin model's fixed saving rates out of profits and wages, respectively, with a representative Ramsey household turns Goodwin's self-sustained oscillations (closed cycles) into damped oscillations, whatever the slope of the real-wage Phillips curve and the degree of convexity of the adjustment costs. Instead of an aggregate saving rate depending negatively on the wage income share as with Goodwin, it is now the investment-income ratio, $I/Y = (I/K)(K/Y)$ where K/Y is constant, that, via Tobin's q , depends negatively on labour's income share.¹²

The limiting case $\beta = \infty$ (absence of convex capital adjustment costs) yields a model we may call a Ramsey-Akerlof-Stiglitz model. It replaces the fixed saving rates in the two-dimensional Akerlof and Stiglitz (1969) model by a utility-maximising representative Ramsey household, thereby changing the dynamics to a three-dimensional system in \tilde{c} , \tilde{w} and x . A rise in real wages induces a rise in the capital intensity and thereby, in the absence of convex capital adjustment costs, a fall in the interest rate. The higher real wages therefore decrease current saving of the representative household. Thus, the real-wage Phillips curve mechanism essentially makes it appear *as if* there is a lower fixed rate of saving out of wage income than out of profits like in

¹⁰ We explore the details of the oscillation mechanism in Section 8 below.

¹¹ The "opposite" limiting case, $\lambda = 0$, implies degenerate dynamics in the sense that any path of the form

$(q^*, \tilde{w}_0, \bar{v}k(\tilde{w}_0))_{t=0}^{\infty}$ is an equilibrium path.

¹² The "opposite" limiting case, $\sigma = \infty$, is degenerate and so is the case $\beta = 0$.

the original Akerlof-Stiglitz model. The combination with a Ramsey household preserves the saddle-point stability of the simple neoclassical Ramsey model. Owing to the sluggish real wage adjustment, however, internal oscillations may - but need not - arise.¹³

6 Calibrating the model

To assess quantitatively the dynamic properties of the model, including the likelihood of internal oscillations, we need to assign values to the parameters. There are two categories of parameters, the *background parameters* and the *key parameters*, β , λ and σ . The parameters belonging to the first category are shown in Table 1. The time unit is one year. To evaluate the sensitivity of the results, alternative values are considered for the parameters. The shaded row in the table contain “intermediate” values, called *baseline values*, whereas the first and last row show the minimum and maximum values, respectively, considered in sensitivity analyses.

Table 1. Values for the background parameters

	δ	γ	n	θ	ρ	\bar{v}
min	0.05	0.015	0.000	0.7	0.02	0.92
baseline	0.07	0.020	0.010	1.0	0.05	0.95
max	0.10	0.025	0.015	5.0	0.07	0.97

The baseline values shown for the growth rates of technology, γ , and labour supply, n , follow Barro and Sala-i-Martin (2004). Regarding the capital depreciation rate δ , the minimum value used here is the baseline value in Barro and Sala-i-Martin (2004), whereas a higher value is commonly used in the business cycle literature. There is no agreement about the magnitude of θ , the elasticity of marginal utility. Following the standard RBC literature, we choose a value of 1 as baseline. Since there are microeconomic studies suggesting a considerably higher value (cf. Attanasio and Weber, 1995), we also consider this possibility. Our baseline value for the rate of time preference is 0.05, thereby implying a long-run real interest rate of $r^* = 0.07$, given the baseline values for θ and γ .¹⁴ Numerous post-war estimates suggest that the equilibrium unemployment rate has been in the region of 4-8 percent (see for instance Coe, 1985, and Staiger et al., 1997). In line with data in Figure 2 above, as baseline value we have chosen 5 percent, hence $\bar{v} = 0.95$.

¹³ The Ramsey-Akerlof-Stiglitz model shares the standard weakness of simple Ramsey models that the framework allows no steady state except in a knife-edge case when applied to a small open economy with an exogenous real interest rate given from the world capital market. This is because of the absence of convex capital adjustment costs.

¹⁴ King and Rebelo (1999, p. 493) suggest that the average real rate of return to capital in the US has been 0.065 over the period 1948-86. Madsen (2003) finds the ex post real rate of return to shares to be on average 8.2 per year for 19 OECD countries over the period from 1871 (or later) to 2002.

Our focus will be on the roles of λ , σ and β , which measure three “flexibilities”; that of real wages, factor substitution and investment, respectively. To expose in detail the qualitative and quantitative impact of these *key parameters* on the dynamics, we consider a sizeable range of values around an “intermediate” level which we label “baseline”, as displayed in Table 2.

Table 2. Range for the key parameters

	λ	σ	β
min	0.15	0.4	0.03
baseline	0.25	0.5	1.00
max	0.50	1.0	10.00

As to the slope of the Phillips curve, λ , an estimate in the neighbourhood of 0.3 is suggested by Coe (1985) and Fuhrer (1999) and in the neighbourhood of 0.2 by Madsen (1998). We choose 0.25 as baseline value for λ but also allow for considerably higher values. Regarding the elasticity of factor substitution, it is common in RBC theory to assume a value equal to one (the Cobb-Douglas case). Taking the absence of Hicks-neutrality of technical progress into account, however, Antràs (2004) finds estimates of the elasticity of substitution in the range 0.4 to 1.0, with fifteen of eighteen estimates significantly below one (at 5% level). The survey by Chirinko (2008) concludes that the interval (0.4, 0.6) is the plausible range. We take the intermediate value, 0.5, as our baseline value, $\bar{\sigma}$, for the elasticity of factor substitution, implying a baseline $\bar{\psi}$ equal to $\bar{\psi} \equiv 1 - 1/\bar{\sigma} = -1.0$. In regard to the value of β , the study by Summers (1981) suggests a value in the vicinity of 0.03. In our setup this implies an implausibly slow adjustment process in the economy and an implausibly high q^* (recall that q^* should indicate the cost in steady state of an extra unit of installed capital). To come closer to the estimates of q^* reported in Blanchard et al. (1993) and the level indicated by our Figure 3, we choose our baseline investment flexibility, $\bar{\beta}$, so as to get $q^* = 1.10$ when $\beta = \bar{\beta}$ and δ , γ and n take their baseline values. This requires $\bar{\beta} = 1.0$.

Owing to the “normalisation” of the CES production function the elasticity of factor substitution, σ , affects only the transitional dynamics, not the steady-state values. Similarly, the wage flexibility parameter, λ , influences only transitional dynamics since we take \bar{v} (the natural rate of employment) as exogenous (determined at a deeper level).¹⁵ Hence in Table 3, where the sensitivity of the steady-state values of important ratios is displayed, the only key parameter to

¹⁵ This is in contrast to Goodwin (1967). His Marxian-flavoured model has \bar{v} increasing in γ and decreasing in λ .

be varied is the investment flexibility, β . The range for β displayed in Table 3 is such that existence of a steady state is never in danger. Indeed, the lower bound for β defined in (5.6) turns out to be *very* low in our calibration (close to 0.005 for all value combinations for the background parameters allowed according to Table 1).

Table 3. How central steady-state values vary with the investment flexibility.

	$\beta = 0.03$	$\beta = 0.15$	$\beta = 0.40$	$\beta = 1.00$	$\beta = 2.00$	$\beta = 10.00$
q^*	4.33	1.67	1.25	1.10	1.05	1.01
$f'(k^*)$	0.44	0.20	0.16	0.15	0.14	0.14
k^*	0.42	0.81	0.94	1.00	1.02	1.04
$1 - \varepsilon(k^*)$	0.52	0.68	0.71	0.72	0.72	0.73
$(wL / GDP)^*$	0.63	0.71	0.72	0.73	0.73	0.73
$(J / K)^*$	0.167	0.033	0.013	0.005	0.003	0.001
$(GDP / K)^*$	0.75	0.58	0.54	0.53	0.52	0.52
$(I / GDP)^* = \frac{(I / K)^*}{(GDP / K)^*}$	0.13	0.17	0.18	0.19	0.19	0.19

Note. The calculations are based on baseline values of $\delta, \gamma, n, \theta$ and ρ . Hence $r^* = 0.07$.

The first row in Table 3 shows a large range for $q^* = 1 + (\delta + \gamma + n) / \beta$, which of course is decreasing as β rises.¹⁶ Via (5.3) this shifts $f'(k^*)$ downward (second row) which in turn reflects upward shifts in k^* (third row) since β does not affect the technology parameters and thereby not the production function f . The fourth row shows the associated upward shifts in the elasticity of gross production with respect to effective labour input (recall that $\bar{\sigma} < 1$). This is reflected in the level and shifts in the labour income share, $(wL/GDP)^*$, in the fifth row although in a somewhat “distorted” way when the capital adjustment costs per unit of existing capital (sixth row) are high, that is, when β is low. It is also only for low β that the output-capital ratio and the investment-income ratio are markedly affected by the adjustment costs (seventh and eighth row, respectively).

The shaded column where $\beta = \bar{\beta} = 1$ represents our baseline case in which, by construction, the gross investment-GDP ratio is 0.19 (which is the US trend level over more than a century, cf. Barro and Sala-i-Martin, 2004, p. 15). Apparently “too high” labour income shares, about

¹⁶ Allowing for corporate taxation and an investment tax credit would lower the general level of the theoretical q^* . The same would happen if not gross investment, but net investment were the source of adjustment costs.

0.72, arise unless β is very small. This reflects the general “problem” of reconciling a labour income share at the conventional 2/3 level with a realistic investment share of GDP and capital growth rate under competitive conditions in an aggregate model (see Appendix E).

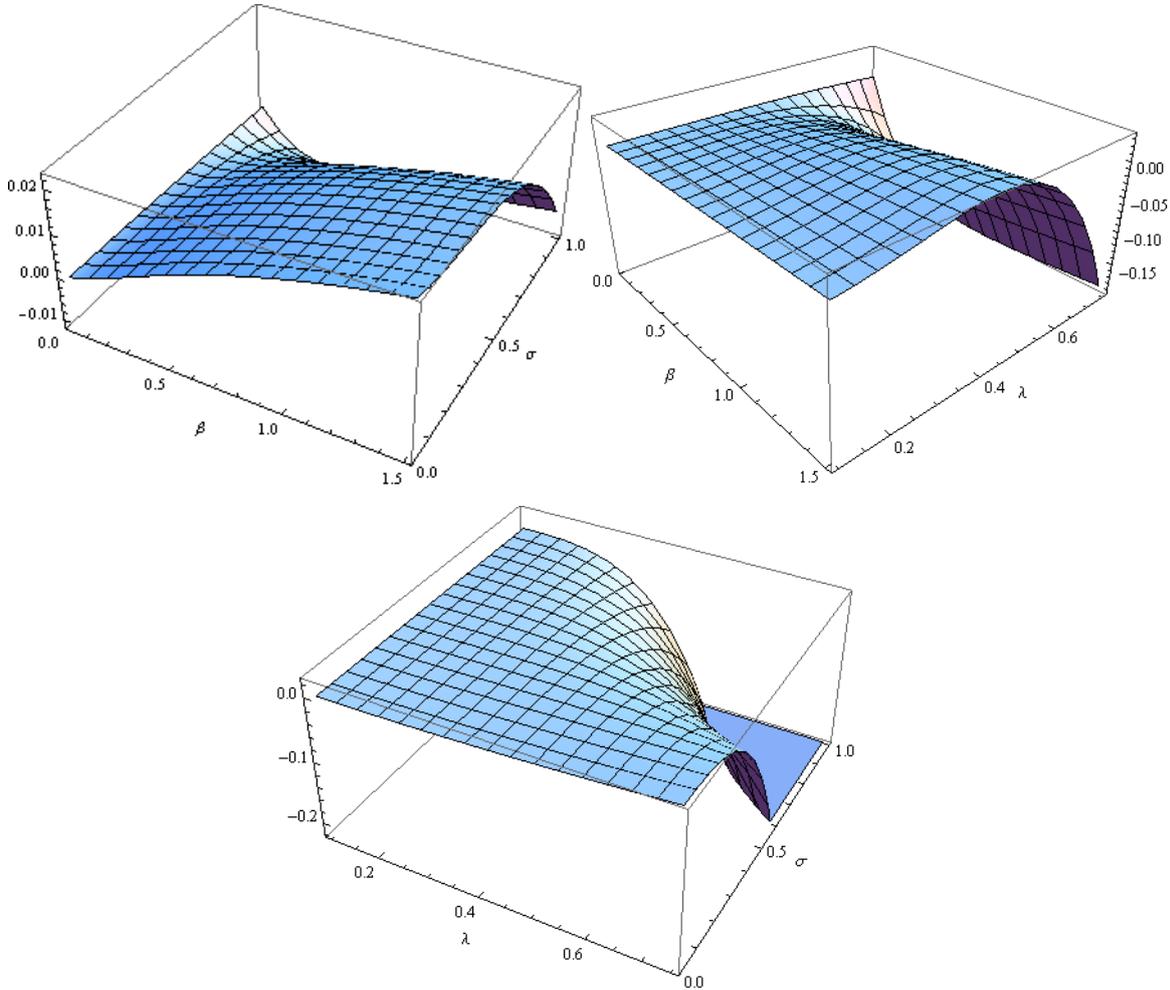


Figure 7. The discriminant of the Jacobian matrix as a function of (β, σ) (left panel), (β, λ) (right panel) and (λ, σ) (lower panel), given baseline values of the remaining parameters in each case.

7 Fluctuations and persistence

We are now ready to quantify the transitional dynamics of the model. It was shown in Section 5 that existence of internal oscillations is a theoretical possibility. Indeed, the discriminant of the Jacobian matrix was shown to be positive, reflecting existence of complex eigenvalues, for a “sufficiently low” wage flexibility, λ , or a “sufficiently low” elasticity of factor substitution, σ .

The three panels in Figure 7, graphing the size of the discriminant as a function of (β, σ) , (β, λ) and (λ, σ) , respectively, for given baseline values of the remaining parameters, quantify this claim. What's more, the figure demonstrates that our baseline case, $(\beta, \lambda, \sigma) = (1.0, 0.25, 0.5)$, clearly belongs to the region with internal oscillations. It takes an elasticity of factor substitution not much below one combined with a quite high slope of the real-wage Phillips curve for the economy to end up in the no-oscillations region. A low level of investment flexibility helps in this direction.

These features are also visible in Table 4, where cells with numbers in the “period-length” columns signify the existence of internal oscillations and cells with just a dot signify non-existence. In addition, for alternative parameter values within the oscillatory region the table shows the magnitude of the (asymptotic) period length (wave length); that is, the time distance from peak-to-peak of the oscillations.¹⁷ A *higher elasticity of factor substitution* is seen to increase the period length and a *higher investment flexibility* (lower degree of convexity) is, as expected, seen to lower it. As to the wage flexibility, its influence on the period length is generally modest and non-monotonic. Anyhow, the displayed period lengths are sizeable. In the two shaded cells, representing our baseline case, appears a period length of 28 years.

Table 4 also reports *persistence* of the effects of disturbances. Persistence of the effect on a convergent variable is measured by the (asymptotic) half-life of the deviation from the steady-state value generated by a once-for-all disturbance of the steady state.¹⁸ It is noticeable that the half-life in both the oscillatory and the non-oscillatory region tends to be longer than the 2-3 years in standard RBC models. In the case of oscillatory adjustment, longer half-life is associated with lower wage flexibility and lower elasticity of factor substitution. The intuition is as follows. A low wage flexibility means that the corrective feedbacks from real-wage responses to employment changes are delayed, while a low elasticity of factor substitution implies that the corrective feedbacks from employment responses to real-wage changes are delayed. As for the investment flexibility, its influence on half-life is non-monotonic. The absence of complete monotonicity is in consonance with the non-linearity of the model. Finally, by and large, the numbers in Table 4, in particular those for half-life, are not very sensitive to the choice of values of the background parameters within the range shown in Table 1.

¹⁷ Owing to the non-linearity of the model, the period length is only ultimately, i.e., for $t \rightarrow \infty$, a constant. Hence the adjective “asymptotic”.

¹⁸ The formal definitions of asymptotic half-life in the non-oscillatory and the oscillatory region, respectively, are given in Appendix F, where also period length is formally defined.

Table 4. Half-life and, in case of complex eigenvalues, period length for alternative values of λ , β and σ (otherwise baseline case).

λ	β	$\sigma = 0.4$		$\sigma = 0.5$		$\sigma = 0.6$		$\sigma = 1.0$	
		Half-life	Period length						
0.15	0.03	10.2	85.8	8.7	97.8	7.5	125.9	12.7	.
-	0.40	6.8	36.7	5.8	37.4	5.1	38.9	3.2	71.1
-	1.00	7.1	31.7	6.1	31.8	5.3	32.2	3.3	40.7
-	10.00	7.8	28.3	6.7	28.0	5.8	27.9	3.6	30.3
0.20	0.03	7.8	86.3	6.6	117.3	6.6	.	13.7	.
-	0.40	5.1	33.8	4.3	35.6	3.7	38.9	3.9	.
-	1.00	5.3	28.7	4.5	29.3	3.8	30.6	2.4	71.9
-	10.00	5.7	25.2	4.8	25.2	4.1	25.5	2.5	33.3
0.25	0.03	6.4	94.5	5.3	325.8	8.4	.	14.2	.
-	0.40	4.1	32.4	3.5	35.7	3.0	43.1	4.4	.
-	1.00	4.2	26.9	3.5	28.2	3.0	30.8	3.1	.
-	10.00	4.5	23.2	3.7	23.6	3.2	24.4	1.9	53.5
0.30	0.03	5.4	119.8	7.1	.	9.1	.	14.5	.
-	0.40	3.5	32.0	2.9	37.9	2.5	57.8	4.7	.
-	1.00	3.5	25.9	3.0	28.2	2.5	33.1	3.5	.
-	10.00	3.7	21.9	3.1	22.7	2.6	24.4	2.4	.
0.35	0.03	4.7	374.2	7.8	.	9.6	.	14.7	.
-	0.40	3.0	32.5	2.5	43.7	2.5	.	4.9	.
-	1.00	3.0	25.5	2.5	29.2	2.1	39.6	3.7	.
-	10.00	3.1	21.1	2.6	22.5	2.2	25.4	2.7	.
0.50	0.03	8.9	.	10.2	.	11.4	.	15.8	.
-	0.40	3.5	.	3.9	.	4.3	.	5.8	.
-	1.00	2.8	.	3.2	.	3.5	.	4.7	.
-	10.00	2.3	.	2.6	.	2.9	.	3.8	.

Note. The time unit is one year; “.” signifies that no eigenvalues are complex.

The finding that the parameter combinations leading to internal oscillations are within a reasonable range deserves attention. It suggests that the model is capable of generating correlated *hump-shaped* responses to a variety of shocks. From an empirical point of view this is a desirable feature for a business cycle model also when the focus is on the medium term rather than the short term (Cogley and Nason, 1995). Since, by Proposition 1, the internal oscillations are always *damped*, the situation resembles the wooden rocking horse analogy referred to in Ragnar Frisch’s classic “impulse-propagation” paper (Frisch, 1933).

Frisch envisioned the presence of recurrent small shocks such that the resulting series of superimposed damped oscillations generated sustained waves.¹⁹ Without taking a stand on this issue, we shall emphasise that there is an alternative view of business cycle fluctuations, namely that they are dominated by the propagation of large specific events and are not at all alike (Blanchard and Watson, 1986). Another view, taken by leading “real business cycle” theorists, adheres to Slutsky’s “sum of small random causes” perspective instead of large and infrequent shocks, but rejects the notion of internal oscillations. Dissociating themselves from older business cycle approaches, Kydland and Prescott (1990, p. 5) claim that “cyclical laws of motion do not arise ... for economies with reasonable statements of people’s ability and willingness to substitute.” What Kydland and Prescott here have in mind is a representative agent model with market clearing under *flexible* wages and prices, so that fluctuations in employment are fluctuations in labour supply. The present model suggests, however, that if sluggish wage adjustment is taken into account, internal oscillations are quite likely, although not guaranteed. This allows for appealing impulse-response patterns without relying on postulated autocorrelation structures of recurrent shocks.

8 Responses to specific shocks

By numerical simulation we now examine how the system propagates specific shocks. We look at responses to isolated and unanticipated one-time shocks at time $t = 0$, starting from a steady state equilibrium.²⁰ Three kinds of shocks are considered: a positive permanent technology shock, a transitory wage push shock and a permanent wage push shock in the form of a permanent rise in the natural rate of unemployment, $1 - \bar{v}$. The simulations we show use baseline values for the background parameters. Numerous additional simulations have been performed, indicating that the impulse-response functions are not very sensitive to the values of the background parameters within the ranges shown in Table 1.

8.1 A positive permanent technology shock

Suppose that at time $t = 0$, an unanticipated two percent increase in labour efficiency, A_0 , occurs.

¹⁹ Though apparently irregular, according to Frisch (1933) such waves tend under certain conditions to have approximately the same *average* period as the original internal cycle.

²⁰ The solution method is based on the relaxation algorithm described in Trimborn *et al.* (2008).

Amplification of the employment response

Real wage sluggishness *amplifies* the effect of the shock on employment. At the outset the positive productivity shock lowers the efficiency-corrected real wage, \tilde{w} . Provided that the elasticity of factor substitution is not too small, this induces firms to combine their existing capital equipment with more labour.²¹ Capital intensity drops and the employment rate goes up. The response in labour hours to the productivity shock is larger than it would be in a standard RBC model with the same elasticity of factor substitution. In the RBC model the response in labour hours is a manifestation of intertemporal substitution in labour supply, which requires an immediate rise in the real wage. This rise in labour costs dampens the employment and output responses. In contrast, with sluggish real wages and involuntary unemployment, employment adjusts to the higher labour demand at *unchanged* wages.

Letting h denote the proportionate size of the shock, the *instantaneous amplification* can be measured as $(v_0(h) - \bar{v}) / h$, where $v_0(h)$ is the employment rate immediately after the shock. For alternative values of the elasticity of factor substitution, σ , the first row in Table 5 shows the instantaneous amplification when $h = 0.02$ (the instantaneous amplification is independent of λ). Amplification is seen to increase significantly with σ . The amplification in the Cobb-Douglas case ($\sigma = 1$) is 2.47, which is a profound response. In the standard RBC model with Cobb-Douglas production function the instantaneous amplification is only 0.50 (King and Rebelo, 1999, p. 970).

Table 5. Amplification in the employment rate in response to a positive productivity shock of 0.02.

		$\sigma = 0.4$	$\sigma = 0.5$	$\sigma = 0.6$	$\sigma = 1.0$
$\frac{v_0(0.02) - \bar{v}}{0.02}$		0.39	0.72	1.07	2.47
$\frac{v_{\max}(0.02) - \bar{v}}{0.02}$	$\lambda = 0.15$	0.89	0.96	1.12	2.47
	$\lambda = 0.25$	0.63	0.75	1.07	2.47
	$\lambda = 0.35$	0.51	0.72	1.07	2.47
	$\lambda = 0.50$	0.41	0.72	1.07	2.47

Note. Baseline values of β and background parameters; $h = 0.02$.

²¹ We have $\partial(K/L) / \partial A = f(k) (\varepsilon(k) - \sigma) / f'(k)$. Hence, given K , the requirement is that $\sigma > \varepsilon(k)$, which at least holds close to the steady state when $\sigma \geq 0.4$ and $\beta \geq 0.15$, cf. Table 3.

The remaining four rows in Table 5 display the *peak amplification* measured as $(v_{\max}(h) - \bar{v}) / h$, where $v_{\max}(h)$ is the peak employment rate after the shock. As the table shows, for low elasticity of factor substitution, σ , and low wage flexibility, λ , the peak employment is considerably greater than the employment rate immediately after the shock. This is due to the propagation generated from the internal cycle mechanism. In the standard RBC model there are no internal oscillations and instantaneous and peak amplification of employment are the same.

Impulse-response functions in the baseline case compared with the Cobb-Douglas case

Figure 8 depicts impulse-response functions for the variables of interest in two scenarios where the wage and investment flexibilities, λ and β , respectively, are at their baseline values. The two scenarios differ with respect to the value of the elasticity of factor substitution, σ . The solid curves in Figure 8 represent our baseline case, $\sigma = 0.5$, which has internal oscillations. The dotted curves represent the Cobb-Douglas case, $\sigma = 1.0$, which is non-oscillatory. Among the variables represented are four of the “great ratios”, q , v , GDP/K and wL/GDP . The fifth great ratio, I/K , equals $\beta(q - 1)$ and its movement is therefore indirectly visible from the q -diagram. The six upper-left and two lower panels show time paths of the main trendless variables, with the dashed horizontal lines representing their steady state levels; for trending variables the upper-right panels show percentage deviations from the value attained in the absence of the shock. This format is applied also in the subsequent figures.

Owing to the higher level of productivity, the efficiency-corrected real wage, \tilde{w} , and the unemployment rate, $1 - v$, are in both scenarios for several years below “normal” (i.e. steady state level). In the q panel we see a modest immediate upward jump, reflecting the raised profitability of the marginal unit of installed capital. In view of the efficiency-corrected capital-labour ratio being low after the shock (the k panel), the above-normal output-capital ratio shown in the GDP/K panel is as expected. The wL/GDP panel in Fig. 8 shows the responses of the labour income share in the two scenarios. The initial drop in wL/GDP in the $\sigma = 0.5$ scenario is as expected in view of the initial drop in k . In the Cobb-Douglas scenario one might have expected no change in wL/GDP but we see an initial lift. The explanation lies in the relatively high initial investment level (the I/N panel) which implies a high initial adjustment cost and thereby a perceptible drag on value added.

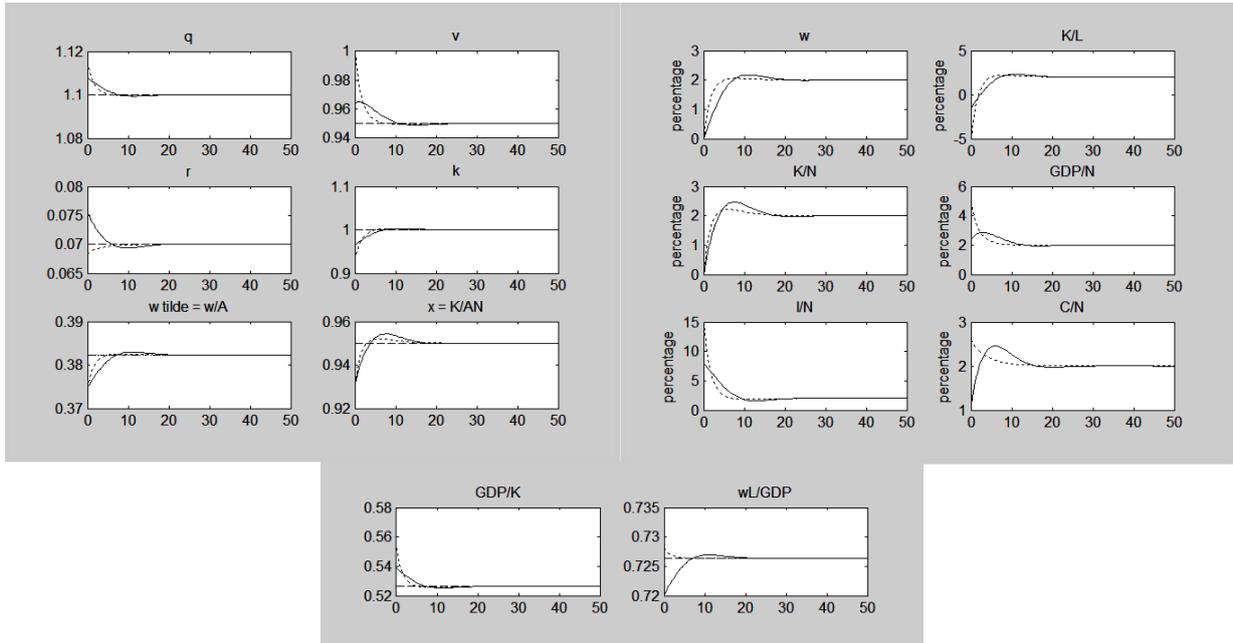


Figure 8. A positive permanent technology shock (2%). Impulse-response functions in the case $\lambda = 0.25$ and $\beta = 1.0$ combined with $\sigma = 0.5$ (solid curves) and $\sigma = 1.0$ (dotted curves), respectively.

The upper-right panel for w in Figure 8 exhibits the gradual upward adjustment of the real wage per hour due to the improved bargaining power of the employees vis-à-vis the employers. The positive, permanent 2 percent technology shock makes room for a 2 percent higher real wage in the long run. As indicated by the K/L panel, the immediate boost to employment elicits a downward jump in the K/L ratio followed by a gradual rise, with slight “overshooting” in the $\sigma = 0.5$ scenario. This rise comes about because L gradually decreases to normal. There is above-normal growth in K (the K/N panel) due to the high level of investment (the I/N panel) induced by the high marginal q . The build-up of K postpones the return of the employment rate to normal. The higher productivity and temporarily higher employment raise value added *per capita* considerably (GDP/N panel), most in the Cobb-Douglas scenario; in the long run value added settles down at 2 percentage points above the level without the shock. In accordance with a stylised business cycle fact, the amplification effect on investment is significantly larger than on consumption (compare the I/N and C/N panels).

Apart from generating a wealth effect on consumption, the productivity shock induces an intertemporal substitution effect through the response of the interest rate. In the $\sigma = 0.5$ scenario the interest rate is for some time after the shock above its “normal” (the solid curve in the r panel), thus causing the time path of consumption to be initially steeper than in the steady state (reflected by the solid curve in the C/N panel). In the Cobb-Douglas scenario, by contrast, the interest rate displays an initial dip instead of a rise immediately after the shock (the dotted curve

in the r panel), thus causing the time path of consumption to be initially less steep than in the steady state (reflected by the dotted curve in the C/N panel).

What is the explanation of this qualitatively different behaviour of r ? By the assumed quadratic adjustment costs, (3.11) takes the form:

$$r = \frac{f'(k) + \frac{1}{2}\beta(q-1)^2 + \dot{q}}{q} - \delta. \quad (8.1)$$

The initial upward jump in q as well as the subsequent capital losses, $|\dot{q}|$, are greater in the Cobb-Douglas case than in the $\sigma = 0.5$ case (see the q panel) because higher factor substitutability allows faster adjustment (shorter half-life, cf. Table 4). This leads to an initial dip in r in the former case.

A case with more pronounced oscillatory adjustment

Here we still consider a positive permanent productivity shock (2%), but this time the wage flexibility and the elasticity of factor substitution are at the low end of the “allowed” interval in Table 2, whereas the investment flexibility is still at baseline level. Pronounced oscillations arise, as displayed in Figure 9.

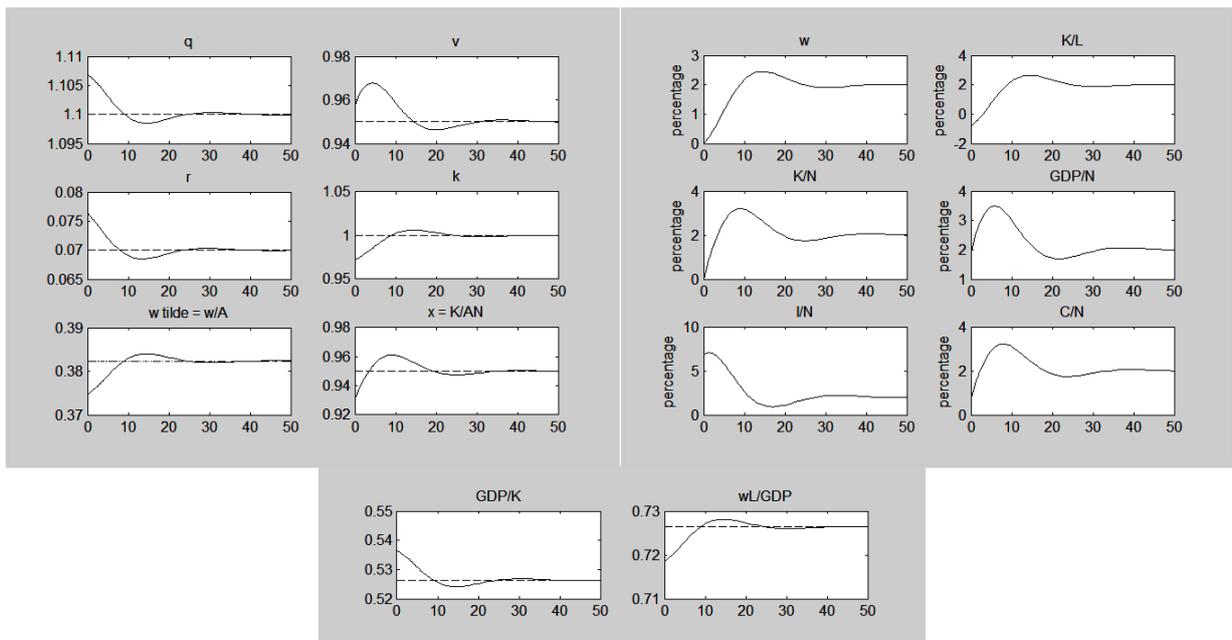


Figure 9. A positive permanent technology shock (2%). Impulse-response functions when $\lambda = 0.15$, $\sigma = 0.4$ and $\beta = 1.0$.

The timing of the movements, displayed in the six top left-hand panels in the figure, are basic for understanding the mechanism underlying the oscillations. To begin with the positive two percent shock to A_0 has qualitatively the same effects as in Figure 8: (a) the efficiency-corrected real wage, \tilde{w}_0 , and the efficiency-corrected per-capita capital stock, x_0 , are lowered; (b) the low \tilde{w}_0 induces high employment and low capital intensity, k , via factor substitution; and (c) the resulting high marginal productivity of capital²² immediately lifts q above its mean level, q^* . The available baseline investment flexibility, $\beta = 1.0$, triggers capital accumulation fast enough to bring x back close to its mean level, x^* , before q has come down to q^* again. Therefore x continues its upward movement and the system enters a phase where $x > x^*$. This “overreaction” receives a check, however, by q subsequently overreacting on its way back towards q^* . As q passes the mean level, q^* , x peaks and begins descending, since $q < q^*$ implies $I / K < \delta + \gamma + n$.

The gradual fall of q from its high level during the boom is brought about by the diminishing returns to capital, reflecting that firms’ capital intensity, k , increases through high investment and factor substitution due to above-normal real wage increases. These real wage increases are a result of the high employment rate, v . In view of the *low* slope, λ , of the Phillips curve, it takes time for w to catch-up with labour productivity, so that when the system enters the phase where $x > x^*$, \tilde{w} is not yet back at normal. Thus, when \tilde{w} and k reach normal, x is high, thereby prolonging the phase where $v \equiv x / k$ is above normal; this causes \tilde{w} , hence also k , to increase further. In this way the high level of employment carries the seed to its own destruction. When the rising k has brought v back to its mean level, so that \tilde{w} stops rising, k is at its maximum. Since the marginal product of capital and q are then low, capital accumulation is low and x thereby falling. This brings about a further decrease in v and a contraction sets off.

The ensuing below-normal employment triggers a downward adjustment of \tilde{w} and k , thereby restoring the marginal product of capital and q . The efficiency-corrected per-capita capital stock, x , is back at its mean level, x^* , before q has fully recovered. Hence, x has to fall below x^* . When q reaches q^* , x is at its minimum. An upward adjustment in x takes off by q exceeding q^* . This overreaction of q is brought about by the low employment rate, which implies that wages increase less than labour productivity, so that firms’ capital intensity falls and the marginal productivity of capital rises. In this way a new boom is underway – and so the cyclical process

²² The changes in $f'(k)$ are not shown but they are one-to-one reflections of the changes in k although in the opposite direction, as $f''(k) < 0$.

goes on forever. But due to strong damping, already the second boom is barely visible in Figure 9.

In accordance with business cycle empirics (Zarnowitz, 1992) the model thus predicts that Tobin's q is *leading* employment and GDP, whereas the efficiency-corrected per-capita capital stock, x , is a *lagging* variable. The model envisages the efficiency-corrected real wage, \tilde{w} , and firms' capital intensity, k , to be even more lagging, at least when the "impulse" is a productivity shock. The six *right-hand* panels in Figure 9 display the hump-shaped percentage deviations of the six trending variables. That investment *leads* consumption and GDP is clearly visible. The period length of the oscillation is sizeable and the damping of the cycle is strong.

What is the intuition behind that internal oscillations occur when λ and σ are low rather than high? In the boom caused by the positive productivity shock, when the wage flexibility, λ , is low, the initial below-normal \tilde{w} is slow to adjust and this is reflected as *slow* increases in firms' below-normal capital intensity, $k(\tilde{w})$. Consequently, the above-normal employment rate, v , is brought down relatively slowly so that it boosts \tilde{w} up to normal before it has itself come down. When it finally comes down, it has thus in the meantime heightened \tilde{w} *above* normal. Through factor substitution, this takes $k(\tilde{w})$ *above* normal and at the same time, due to a reduction in Tobin's q , there is reduced capital accumulation. The combined effect of these circumstances is that employment is forced down below normal. So, now a reverse adjustment takes place and so on.

This story also suggests why a low σ tends to bring about oscillations. A low σ implies *little* change in firms' capital intensity when \tilde{w} moves. There is thus only modest feedback on v implying a slow "error-correction" similar to that caused by a low λ . Finally, Figure 7 and Table 4 brought to light that a "not too small" investment flexibility, β , increases the scope for oscillations. The reason is that a "not too small" β implies relatively fast increases in the capital stock, when q is above normal. Hence, before q is back at normal, $x - x^*$ has already changed sign and a reverse adjustment is needed.

8.2 A transitory wage push shock

Suppose that for some reason, outside the model, a change in the relative strength of the parties in the labour market generates a "wage push shock". We first consider the case of a *transitory* wage push shock.

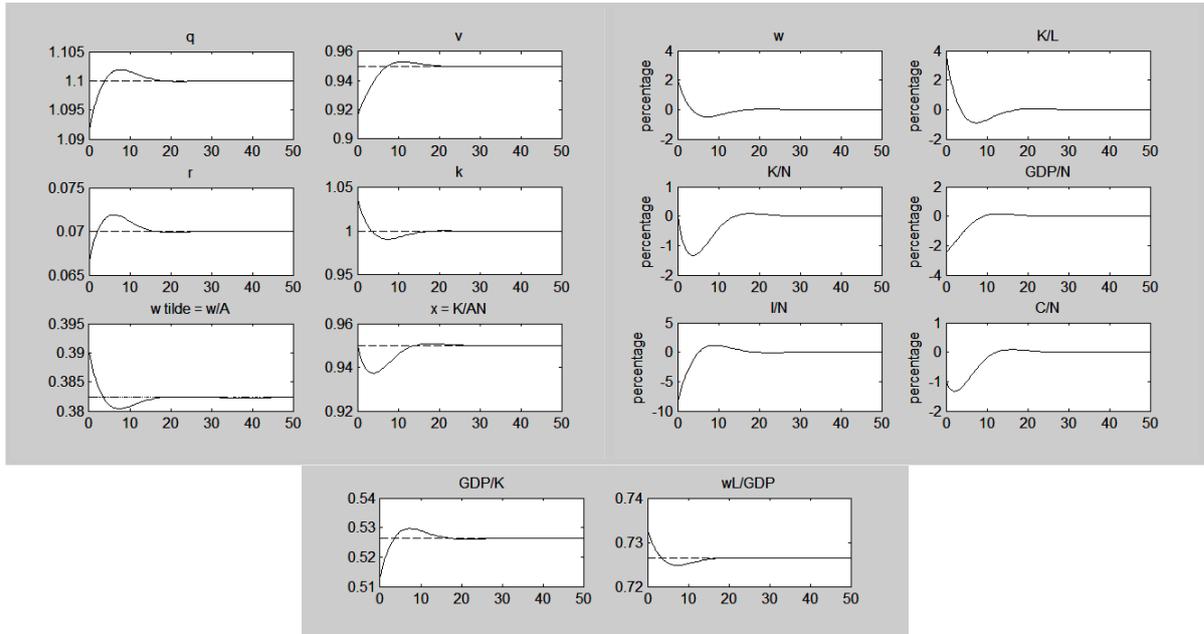


Figure 10. A transitory wage push shock (2%). Impulse-response functions, when $\lambda = 0.25$, $\sigma = 0.5$ and $\beta = 1.0$.

Starting from steady at time $t = 0$, Figure 10 displays impulse-response functions for a 2 percent transitory wage push shock at time $t = 0$, i.e. $\tilde{w}_0 = 1.02 \times \tilde{w}^*$ (given baseline values of all parameters). Through a profits squeeze the initially high wage level carries the seed to its own destruction. Indeed, immediately after the shock, the high labour costs trigger a fall in employment and a rise in firms' capital intensity. The resulting low marginal product of capital implies a drop in q and so capital accumulation will be low for a while, implying a falling x . As employment is below normal, the efficiency-corrected wage, \tilde{w} , is falling and employment improves. Before employment is back at \bar{v} , \tilde{w} is back at its mean level, \tilde{w}^* , and so \tilde{w} continues its downward movement and the system enters a phase where $\tilde{w} < \tilde{w}^*$. After a while this "overreaction" (in a negative direction) becomes corrected, however, because in the meantime q has risen above normal and this induces capital accumulation above normal, which pushes employment above normal. The ensuing boom not only restores the wage level, but does it so fast that a new profits squeeze begins – and so on. Ups and downs will continue, but due to the damping, the ups and downs are hardly visible in Figure 10. The amplitudes of the output-capital ratio and the wage income share are fairly small.

8.3 A permanent fall in \bar{v}

Figure 11 displays impulse-response functions, given a fall in the natural rate of employment by 2 percent (rise in the natural rate of unemployment); with respect to the key parameters, the baseline case ($\lambda = 0.25$, $\sigma = 0.5$ and $\beta = 1.0$) is maintained. The interpretation of the fall in \bar{v} may

be that (outside the model) a permanent shift in trade union preferences has taken place at time 0, attaching more weight to wage increases and less to employment. The long-term effect will indeed be a fall in the time average of the employment rate. Yet, the endeavour to obtain higher wages will end up being frustrated, as the \tilde{w} panel in Figure 11 shows. The reason is that, whatever the equilibrium rate of employment, the rate of capital accumulation net of depreciation must in steady state equal the unchanged natural growth rate, $\gamma + n$; this requires the same investment stimulus, q^* , as before the shock occurred. To maintain the same value of q^* requires the same marginal productivity of capital in steady state as before the shock, cf. (4.3). An unchanged capital intensity, k^* , and thereby unchanged efficiency-corrected real wage, \tilde{w}^* , is therefore required, which amounts to an unchanged wage level since the technology path is by assumption unaffected.

The mechanism driving the natural rate of employment, \bar{v} , down is the following. Immediately after the shock, the unchanged employment rate is above the level consistent with steady state under the new trade union preferences. As envisaged by the \tilde{w} panel, the situation results in wages rising faster than productivity, a state of affairs leading to its own destruction. Indeed the high efficiency-corrected real wages induce firms to choose more capital intensive techniques. Firms also slow down capital accumulation due to the low q caused by the low marginal productivity of capital that prevails as long as \tilde{w} and k are above normal. The employment rate has to overreact in its downward movement. This is because, when v has come down to its new long-run mean and \tilde{w} and k therefore peak, capital accumulation is still below the natural growth rate due to the low q , which causes x to continue its fall, thus drawing v down below its new long-run mean. Then the high \tilde{w} and k begin to fall, thus gradually restoring q and capital accumulation. After a while, this stops the drop in v and brings about an upturn in v . Damped oscillations continue but with v and x fluctuating around their new lower steady state levels. In the new steady state wages are back at the same equilibrium path as before the shock. But employment is 2% lower on the new steady state path along which, therefore, also the per capita capital stock, output and consumption are 2% lower (the upper-right panels in Figure 11). The amplitudes of the output-capital ratio and the wage income share are again fairly small (the two lower panels in Figure 11).

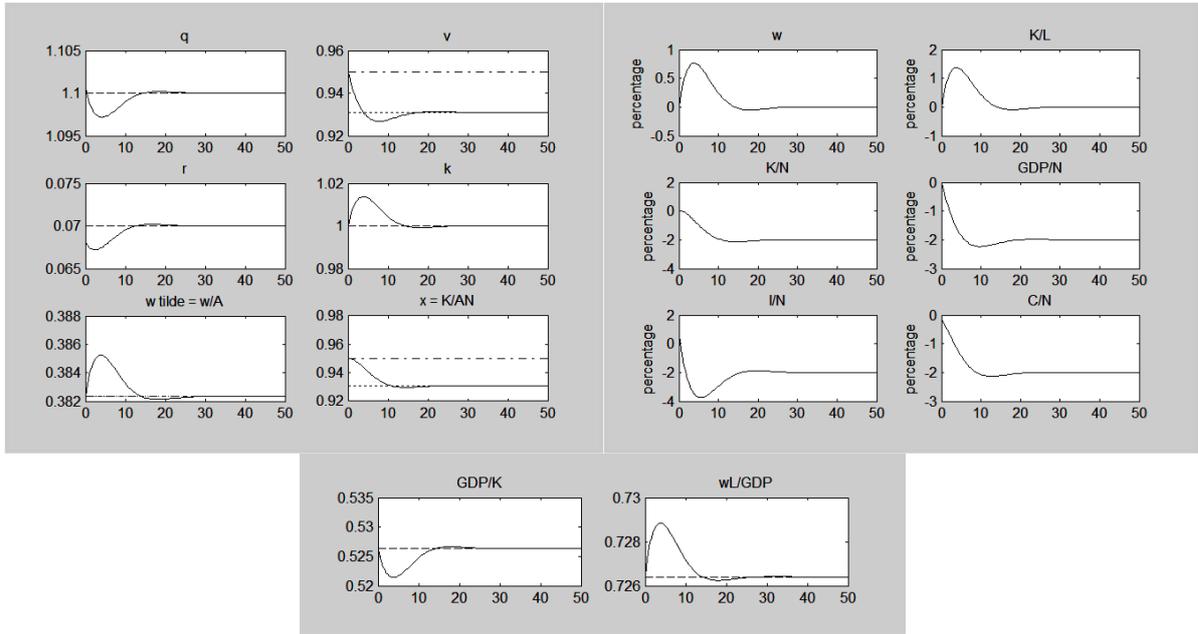


Figure 11. A permanent wage push shock (fall in \bar{v} by 2%). Impulse-response functions, when $\lambda = 0.25$, $\sigma = 0.5$ and $\beta = 1.0$.

9 Conclusion

This paper has argued that business cycle fluctuations may be more persistent than suggested by standard real business cycle theory because of sluggish adjustment of real wages and investment to wage push and technology shocks. Wages have been extraordinary slow to adjust to the high unemployment that prevailed in the late 19th century, the interwar period and the 1970s and 1980s. Similarly, investment has been slow to react to the high values of Tobin's q in the decades leading up to WWI and in the *post*-1992 period.

Extending the model of Abel and Blanchard (1983) this paper has shown that wage push and technology shocks can lead to cycles in employment, investment, the Y - K ratio and labour's income share at medium term frequencies due to sluggish adjustment in investment and wages towards their steady state. For plausible values of the elasticity of factor substitution real wage sluggishness creates *internal* oscillations in the endogenous variables. Thereby correlated hump-shaped responses in the endogenous variables to a variety of shocks arise without relying on assumed particular autocorrelation structures of recurrent shocks. The scope for internal oscillations to arise is larger the smaller is the elasticity of factor substitution.

Calibration of the model showed an asymptotic period length of the cycle in response to shocks of 25-35 years, thus giving rise to a medium-term perspective. A higher degree of convexity in capital adjustment costs increases the period length while sluggish real-wage adjustment amplifies employment fluctuations. In response to aggregate productivity shocks

Tobin's q and investment lead fluctuations in employment and output, while real wages are lagging. The simultaneous correlation between real wages and output will thus appear weak in accordance with a stylized business cycle fact. The oscillations are heavily damped which is a feature consistent with the irregularity of empirical business cycle fluctuations.

Appendix

A. Proof of (ii) of Proposition 1 (Section 4)

As a preliminary step, by straightforward calculation, we find the Jacobian matrix of our three-dimensional dynamic system, (3.30), (3.25) and (3.23), evaluated in the unique steady state, to be $J =$

$$\begin{bmatrix} \frac{\hat{c}(q^*, \tilde{w}^*)[\theta q^* m'(q^*) + \rho - n - (1 - \theta)\gamma]}{h(q^*, \tilde{w}^*)} & \frac{\varepsilon(k^*)^2 \hat{c}(q^*, \tilde{w}^*) + \theta q^* \sigma(k^*)^2 \varphi'(\bar{v})\bar{v}}{\varepsilon(k^*)^2 k^* h(q^*, \tilde{w}^*)} & \frac{-\theta q^* \frac{\tilde{w}^*}{k^*} \sigma(k^*) \varphi'(\bar{v})}{\varepsilon(k^*) k^* h(q^*, \tilde{w}^*)} \\ 0 & -\frac{\sigma(k^*)}{\varepsilon(k^*)} \varphi'(\bar{v})\bar{v} & \frac{\tilde{w}^*}{k^*} \varphi'(\bar{v}) \\ m'(q^*)x^* & 0 & 0 \end{bmatrix},$$

where $h(q^*, \tilde{w}^*) \equiv \hat{c}(q^*, \tilde{w}^*) + \theta q^{*2} m'(q^*) > 0$ and $\sigma(k)$ is the elasticity of substitution between capital and labour in gross production:

$$\sigma(k) = -\frac{f'(k)(f(k) - kf'(k))}{kf(k)f''(k)} > 0.$$

The derivation of the elements, where $\sigma(k)$ occurs, relies on the elasticity formula in Lemma A1 below.

Lemma A1. $\frac{\tilde{w}}{k(\tilde{w})} k'(\tilde{w}) = \frac{\sigma(k)}{\varepsilon(k)}.$

Proof. $\frac{\tilde{w}}{k} k' = -\frac{f(k) - kf'(k)}{k} \frac{1}{kf''(k)} = -\frac{f'(k)(f(k) - kf'(k))}{kf(k)f''(k)} \frac{f(k)}{kf'(k)} = \sigma(k) \frac{1}{\varepsilon(k)}. \text{ Q.E.D.}$

We may write the characteristic polynomial of J as $P(\mu) = \mu^3 + b_1\mu^2 + b_2\mu + b_3$, where

$$b_1 = -\text{tr } J, \quad b_2 = \sum_{k>i} \begin{vmatrix} j_{ii} & j_{ik} \\ j_{ki} & j_{kk} \end{vmatrix}, \quad b_3 = -|J| \equiv -\det J. \quad (10.1)$$

and j_{ik} denotes the element in the i 'th row and k 'th column of J , $i, j = 1, 2, 3$. The determinant of J is

$$\begin{aligned} \det J &= \frac{\varepsilon(k^*)^2 \hat{c}(q^*, \tilde{w}^*) + \theta q^* \sigma(k^*)^2 \varphi'(\bar{v}) \bar{v}}{\varepsilon(k^*)^2 k^* h(q^*, \tilde{w}^*)} \frac{\tilde{w}^*}{k^*} \varphi'(\bar{v}) m'(q^*) x^* \\ &\quad - \frac{\theta q^* \frac{\tilde{w}^*}{k^*} \sigma(k^*) \varphi'(\bar{v})}{\varepsilon(k^*) k^* h(q^*, \tilde{w}^*)} \frac{\sigma(k^*)}{\varepsilon(k^*)} \varphi'(\bar{v}) \bar{v} m'(q^*) x^*, \\ &= \frac{\hat{c}(q^*, \tilde{w}^*)}{h(q^*, \tilde{w}^*)} m'(q^*) \frac{\tilde{w}^*}{k^*} \varphi'(\bar{v}) \bar{v} > 0 \quad (\text{in view of } x^* = \bar{v} k^*). \end{aligned} \quad (10.2)$$

Hence, $b_3 < 0$. The trace of J is

$$\text{tr } J = \frac{\hat{c}(q^*, \tilde{w}^*) [\theta q^* m'(q^*) + \rho - n - (1 - \theta)\gamma]}{h(q^*, \tilde{w}^*)} - \frac{\sigma(k^*)}{\varepsilon(k^*)} \varphi'(\bar{v}) \bar{v}. \quad (10.3)$$

We see that the determinant is always positive. In view of the parameter restriction (4.6), the first term on the right hand side of (10.3) is positive. Thus, the trace is the difference between two positive terms. The slope of the Phillips curve, $\varphi'(\bar{v})$, can in principle take any positive value and this value does not affect the steady-state values q^* , k^* and \tilde{w}^* . Hence, there exists a number $\bar{\lambda} > 0$ such that

$$\text{tr } J \underset{<}{\geq} 0 \text{ for } \varphi'(\bar{v}) \underset{>}{\leq} \bar{\lambda},$$

respectively. It follows that the trace, and therefore b_1 , can not be signed *a priori*.

As to b_2 , applying (10.1) gives

$$\begin{aligned} b_2 &= - \frac{\hat{c}(q^*, \tilde{w}^*) [\theta q^* m'(q^*) + \rho - n - (1 - \theta)\gamma] \sigma(k^*) \varphi'(\bar{v}) \bar{v}}{[\hat{c}(q^*, \tilde{w}^*) + \theta m'(q^*) q^{*2}] \varepsilon(k^*)} + \frac{\theta q^* \frac{\tilde{w}^*}{k^*} \sigma(k^*) \varphi'(\bar{v}) m'(q^*) x^*}{[\hat{c}(q^*, \tilde{w}^*) + \theta m'(q^*) q^{*2}] \varepsilon(k^*) k^*} \\ &= \frac{\theta q^* m'(q^*) \left[\frac{\tilde{w}^*}{k^*} - \hat{c}(q^*, \tilde{w}^*) \right] - \hat{c}(q^*, \tilde{w}^*) [\rho - n - (1 - \theta)\gamma]}{[\hat{c}(q^*, \tilde{w}^*) + \theta m'(q^*) q^{*2}] \varepsilon(k^*)} \sigma(k^*) \varphi'(\bar{v}) \bar{v} \\ &= - \frac{\rho - n - (1 - \theta)\gamma}{\varepsilon(k^*)} \sigma(k^*) \varphi'(\bar{v}) \bar{v} < 0, \end{aligned}$$

where the second equality relies on $x^*/k^* = \bar{v}$; the third equality as well as the inequality rely on Lemma A2 below.

Lemma A2. $\frac{\tilde{w}^*}{k^*} - \hat{c}(q^*, \tilde{w}^*) = -[\rho - n - (1 - \theta)\gamma] q^* < 0$.

Proof. $\frac{\tilde{w}^*}{k^*} - \hat{c}(q^*, \tilde{w}^*) = \frac{f(k^*) - k^* f'(k^*)}{k^*} - \left[\frac{f(k^*)}{k^*} - (\delta + \gamma + n) - g(\delta + \gamma + n) \right]$
 $= -f'(k^*) + \delta + \gamma + n + g(\delta + \gamma + n) = -[\rho - n - (1 - \theta)\gamma] q^* < 0,$

where the last equality follows from (4.3), and the inequality relies on the parameter restriction (4.6). Q.E.D.

Remark. Lemma A2 reflects that $\tilde{w}/k - \hat{c} \equiv (Y - cN - P)/K \equiv (S - P)/K$, where P is gross capital income and S is gross saving. It is a general principle that dynamic efficiency, ensured by (4.6), requires $S < P$ in steady state, which is equivalent to the growth-corrected rate of return being positive.

Let the eigenvalues of J (i.e., the roots of $P(\mu)$) be denoted μ_1 , μ_2 and μ_3 , respectively. Since our dynamic system has one jump variable, q , and two predetermined variables, \tilde{w} and x , (ii) of Proposition 1 requires that the eigenvalues of J are signed as in Lemma A3 below.

Lemma A3. Given (4.6), J has one positive eigenvalue and two eigenvalues which are either real and negative or complex with negative real part.

Proof. The Hurwitz matrix associated with $P(\mu)$ is:

$$H = \begin{pmatrix} b_1 & b_3 & 0 \\ 1 & b_2 & 0 \\ 0 & b_1 & b_3 \end{pmatrix}.$$

The leading principal minors are

$$\begin{aligned} D_1 &= b_1 = -\text{tr } J, \\ D_2 &= b_1 b_2 - b_3, \\ D_3 &= (b_1 b_2 - b_3) b_3 = D_2 b_3. \end{aligned}$$

There are two cases to consider. Case 1: $\text{tr } J > 0$. Here, $D_1 < 0, D_2 > 0$ and $D_3 < 0$, so that, by the (general) Routh-Hurwitz Theorem (Lancaster and Tismenetsky, 1985, p. 480), J has no pure imaginary eigenvalues and the number of eigenvalues with positive real part equals the number of alterations of sign in the sequence $\{1, D_1, D_2/D_1, D_3/D_2\}$. This number is one, since the three last elements in the sequence are negative.

Case 2: $\text{tr } J \leq 0$. Since

$$\mu_1 \mu_2 \mu_3 = \det J > 0, \tag{10.4}$$

J has at least one positive (real) eigenvalue, say $\mu_3 > 0$. Moreover, in view of $\mu_1 + \mu_2 + \mu_3 = \text{tr } J \leq 0$, we have

$$\mu_1 + \mu_2 = \text{tr } J - \mu_3 < 0. \tag{10.5}$$

By (10.4), if μ_1 and μ_2 are real, they are either both positive or both negative. But in view of (10.5), they can not both be positive. Hence, μ_1 and μ_2 are both negative. If instead μ_1 and μ_2 are complex, they have the same real part, say a . Then $\mu_1 + \mu_2 = 2a < 0$, by (10.5), showing that $a < 0$. Q.E.D.

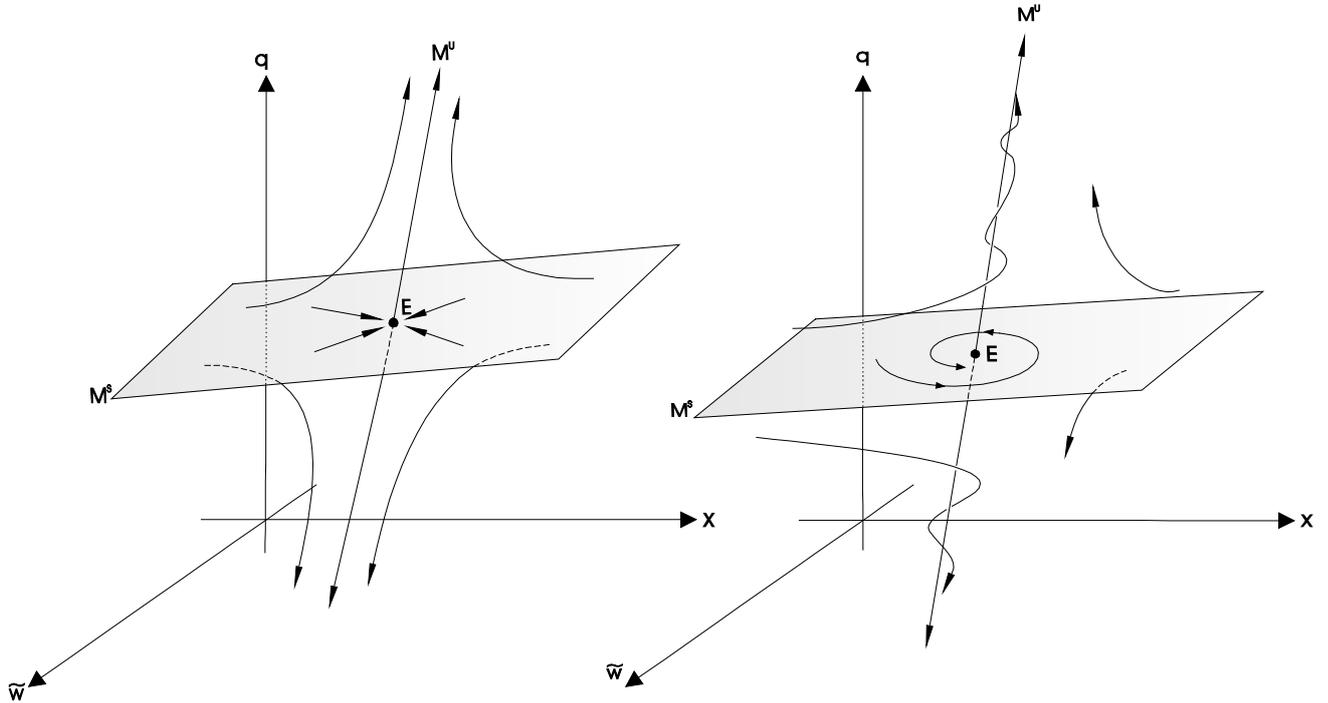


Figure A1. Phase portrait of the dynamics in case of all eigenvalues real (left panel) and in case of complex eigenvalues (right panel).

From Lemma A3 follows that the steady state has a two-dimensional stable manifold, which is tangent to the linear subspace spanned by two (generalised) eigenvectors corresponding to the two eigenvalues with negative real part. We call this linear subspace M^s . Figure A1 shows the qualitative features of the phase portrait of the dynamic system near the steady state in the case of no complex eigenvalues (left panel) and in the case of two conjugate complex eigenvalues (right panel). The structure of J is such that this linear subspace is never parallel to the q axis.²³ Therefore, given the initial values \tilde{w}_0 and x_0 in a small neighbourhood of \tilde{w}^* and x^* , respectively, there exists a unique q_0 such that the initial point (q_0, \tilde{w}_0, x_0) belongs to M^s , hence entailing a unique solution path contained in the stable manifold. This solution path satisfies all criteria of an equilibrium. That the transversality condition (3.9) is satisfied, follows from the

²³ A proof, using that $j_{21} = 0$, $j_{23} \neq 0$ and $j_{31} \neq 0$, is provided in Supplementary Material (Groth and Madsen, 2013).

parameter restriction (4.6). All other solutions to the dynamic system diverge from the steady-state point E and approach, for $t \rightarrow \infty$, one of the branches of the one-dimensional unstable manifold, which is tangent to the line M^u parallel to an eigenvector corresponding to the positive eigenvalue of J . These solutions violate either the transversality condition (3.9) or the feasibility constraint that $K_t \geq 0$ for all $t \geq 0$. This completes the proof of (ii) Proposition 1.

B. The parameterised case and proof of Lemma 1 (Section 5)

Let the Phillips curve, the adjustment cost function and the production function be specified as in the equations (5.1), (5.2) and (5.4), respectively; by (5.4), $\sigma \equiv 1/(1-\psi)$ is the now constant value of $\sigma(k)$.

By (5.4), the solution for $k^* > 0$ in (5.3) (when it exists) is

$$k^* = (1-\alpha)^{\frac{1}{\psi}} \left((\alpha B)^{\frac{\psi}{\psi-1}} p(\beta, \xi)^{\frac{\psi}{1-\psi}} - \alpha \right)^{\frac{-1}{\psi}}, \quad (10.6)$$

where $p(\beta, \xi) \equiv \rho + \theta\gamma + \delta + \frac{\delta + \gamma + n}{\beta} \left(\rho - n - (1-\theta)\gamma + \frac{1}{2}(\delta + \gamma + n) \right)$ and $\xi \equiv (\delta, \gamma, n, \rho, \theta)$.

Proof of Lemma 1. Assume the parameter restriction (4.6). Then $p(\beta) > 0$. Suppose $\psi = 0$, i.e. $\sigma = 1$. Then the Inada conditions are satisfied and so the technology condition (4.7) holds automatically, thereby ensuring existence of a unique steady-state value of k , whatever the size of the investment flexibility, β . Suppose $\psi \neq 0$ ($\psi < 1$), i.e. $\sigma \neq 1$ ($\sigma > 0$). Then the right-hand side of (10.6) is well-defined if and only if

$$(\alpha B)^{\frac{\psi}{\psi-1}} p(\beta, \xi)^{\frac{\psi}{1-\psi}} > \alpha. \quad (10.7)$$

This inequality is thus necessary and sufficient for existence of a steady-state value $k^* > 0$. At the same time, given the definition $\psi \equiv (\sigma - 1)/\sigma$ and $\psi \in (-\infty, 1)$, the inequality (10.7) is seen to be equivalent to (5.5). Q.E.D.

C. Normalisation of the CES function (Section 5)

Above we have expressed the CES function in the conventional way introduced by Arrow et al. (1961). As underlined by e.g. Klump and Saam (2008), essentially following La Grandville (1989), it is expedient to consider the CES function in “family” form, also called “normalised” form. That is, we look for a formula encompassing CES production functions that are distinguished by the substitution parameter, ψ , but at some arbitrary baseline point $k = \bar{k} > 0$

have the same output per unit of effective labour and output elasticity with respect to capital, namely equal to $\bar{y} = f(\bar{k})$ and $\bar{\varepsilon} = \varepsilon(\bar{k})$, respectively. It turns out that the simplest formula satisfying this is

$$y = \bar{y} \left(\bar{\varepsilon} \left(\frac{k}{\bar{k}} \right)^\psi + 1 - \bar{\varepsilon} \right)^{1/\psi}, \quad \psi < 1, 0 < \bar{\varepsilon} < 1, \bar{y} > 0, \bar{k} > 0. \quad (10.8)$$

Dividing through by \bar{y} , the resulting formula shows a relationship between output and capital input measured in a dimensionless way as index numbers.²⁴

Considering alternative values of ψ ($\equiv 1 - \sigma^{-1}$) and at the same time adjusting the two other technology parameters, α and B , in (5.4), so that at $k = \bar{k}$ the same values of $f(k)$ and $f'(k)$ obtain, corresponds to applying the “normalised” form (10.8) with alternative values of ψ (recall that $\varepsilon(k) \equiv kf'(k)/f(k)$). In our context, as reference point we take a baseline steady state, $(\bar{k}, f(\bar{k}), f'(\bar{k}))$, defined in the following way. We let the background parameters δ, γ, n, ρ and θ take their baseline values, cf. Table 1, i.e. $\xi = \bar{\xi}$, and β its baseline value $\bar{\beta}$ (the value at which $q^* = 1.10$, i.e. $\bar{\beta} = 1$). Next, in accordance with (5.3) we let the value of $f'(\bar{k})$ equal $p(\bar{\beta}, \bar{\xi})$ and the value of $f(\bar{k})/\bar{k}$ be consistent with an investment-GDP ratio of 0.19, cf. Table 3. Since the value of \bar{k} in our baseline steady state is a matter of measurement units, we set \bar{k} equal to 1.

As long as $\xi = \bar{\xi}$ and $\beta = \bar{\beta}$, we want the steady state of the model to remain $(\bar{k}, f(\bar{k}), f'(\bar{k}))$ whatever the value of ψ ($\equiv 1 - \sigma^{-1}$). That is, we simply apply the “normalised” form (10.8) for alternative values of ψ . On the other hand, when considering the role of deviations of β from $\bar{\beta}$ or of ξ from $\bar{\xi}$, we do this for a fixed technology, namely the “normalised” form (10.8) with a fixed $\psi = \bar{\psi} \equiv 1 - \bar{\sigma}^{-1}$ (where $\bar{\sigma} = 0.50$). Provided a steady state still exists (i.e. provided $\beta > \underline{\beta}(\bar{\sigma}, \bar{\alpha}, \bar{B}, \xi)$, cf. (5.6)), in view of (5.3) and $p(\beta, \xi) \neq p(\bar{\beta}, \bar{\xi})$, this fixed technology now implies a steady-state value $f'(k^*) \neq f'(\bar{k})$, hence $k^* \neq \bar{k}$ and $y^* \neq \bar{y}$.

D. Proof of Proposition 2 (Section 5)

The Jacobian matrix, J , defined in Appendix A, simplifies to:

²⁴ For details, see Supplementary Material (Groth and Madsen, 2013). The original Arrow et al. (1961) form, (5.4), appears to have only three parameters, while (10.8) has four. The reason is that (5.4) has $\bar{k} = 1$ as implicit baseline point.

$$\begin{bmatrix} \frac{\hat{c}^* [\theta \beta q^* + \rho - n - (1 - \theta) \gamma]}{\hat{c}^* + \theta \beta q^{*2}} & \frac{\varepsilon(k^*)^2 \hat{c}^* + \theta q^* \sigma^2 \lambda \bar{v}}{\varepsilon(k^*)^2 k^* (\hat{c}^* + \theta \beta q^{*2})} & \frac{-\theta q^* \frac{\tilde{w}^*}{k^*} \sigma \lambda}{\varepsilon(k^*) k^* (\hat{c}^* + \theta \beta q^{*2})} \\ 0 & -\frac{\sigma}{\varepsilon(k^*)} \lambda \bar{v} & \frac{\tilde{w}^*}{k^*} \lambda \\ \beta x^* & 0 & 0 \end{bmatrix},$$

where $\sigma \equiv 1 / (1 - \psi)$; moreover

$$\begin{aligned} q^* &= 1 + \frac{\delta + \gamma + n}{\beta}, \\ \varepsilon(k) &\equiv \frac{k f'(k)}{f(k)}, \text{ where } f(k) = B(\alpha k^\psi + 1 - \alpha)^{1/\psi}, \\ \tilde{w}^* &= \tilde{w}(k^*) \equiv f(k^*) - k^* f'(k^*) = (1 - \alpha) B(\alpha k^{*\psi} + 1 - \alpha)^{1/\psi - 1}, \\ \hat{c}^* &= \hat{c}(q^*, \tilde{w}^*) = \frac{f(k^*)}{k^*} - (\delta + \gamma + n) - \frac{1}{2\beta} (\delta + \gamma + n)^2. \end{aligned}$$

The coefficients of the characteristic polynomial $P(\mu) = \mu^3 + b_1 \mu^2 + b_2 \mu + b_3$ are now

$$\begin{aligned} b_1 &= -\frac{\beta \hat{c}^* [\theta(\beta + \delta + \gamma + n) + \rho - n - (1 - \theta) \gamma]}{\beta \hat{c}^* + \theta(\beta + \delta + \gamma + n)^2} + \frac{1}{\varepsilon(k^*)} \sigma \lambda \bar{v}, \\ b_2 &= -\frac{\rho - n - (1 - \theta) \gamma}{\varepsilon(k^*)} \sigma \lambda \bar{v} < 0, \\ b_3 &= -\frac{\beta^2 \hat{c}^* \tilde{w}(k^*)}{\beta \hat{c}^* + \theta(\beta + \delta + \gamma + n)^2} \lambda \bar{v} \left(= -\frac{\beta^2 \hat{c}^* (1 - \alpha) B(\alpha k^{*\psi} + 1 - \alpha)^{1/\psi - 1}}{\beta \hat{c}^* + \theta(\beta + \delta + \gamma + n)^2} \lambda \bar{v} \right) < 0. \end{aligned}$$

Regarding b_3 , because of our normalization of the CES function it is the first equality sign which is applied.

Defining $s = (3b_2 - b_1^2) / 3$ and $z = (2b_1^3 - 9b_1 b_2 + 27b_3) / 27$, the discriminant of J is

$$\Delta = 4s^3 + 27z^2 \equiv \Delta(\lambda, \sigma, \beta),$$

considering Δ as a function of λ , σ and β , defined for $\lambda \geq 0$, $\sigma \geq 0$ and $\beta > \underline{\beta}(\bar{\sigma}, \bar{\alpha}, \bar{B}, \xi)$.

There are two (conjugate) complex eigenvalues if and only if $\Delta > 0$. To prove part (i) of Proposition 2, notice that varying λ does not affect the steady-state values k^* , $\varepsilon(k^*)$, $\tilde{w}(k^*)$, and \hat{c}^* . As $\lambda \rightarrow 0$, implies $b_2 \rightarrow 0$ and $b_3 \rightarrow 0$, we have, for arbitrary $\sigma > 0$ and $\beta > \underline{\beta}(\bar{\sigma}, \bar{\alpha}, \bar{B}, \xi)$,

$$\lim_{\lambda \rightarrow 0} \Delta = 4\left(\frac{-b_1^2}{3}\right)^3 + 27\left(\frac{2b_1^3}{27}\right)^2 = 0 = \Delta(0, \sigma, \beta).$$

At the same time, at any $\lambda > 0$,

$$\begin{aligned} \frac{\partial \Delta}{\partial \lambda} &= 4s^2\left(3\frac{\partial b_2}{\partial \lambda} - 2b_1\frac{\partial b_1}{\partial \lambda}\right) + 2z(6b_1^2\frac{\partial b_1}{\partial \lambda} - 9b_1\frac{\partial b_2}{\partial \lambda} - 9b_2\frac{\partial b_1}{\partial \lambda} + 27\frac{\partial b_3}{\partial \lambda}) \\ &= (12s^2 - 18zb_1)\frac{\partial b_2}{\partial \lambda} - (8s^2b_1 - 12zb_1^2 + 18zb_2)\frac{\partial b_1}{\partial \lambda} + 2 \cdot 27z\frac{\partial b_3}{\partial \lambda}. \end{aligned}$$

In view of $\lambda = 0$ implying $b_2 = 0$, $s = -b_1^2/3$ and $z = 2b_1^3/27$, at $\lambda = 0$ we get

$$\begin{aligned} \frac{\partial \Delta}{\partial \lambda} &= (12\frac{b_1^4}{9} - 18 \cdot 2\frac{b_1^3}{27}b_1)\frac{\partial b_2}{\partial \lambda} - (8b_1\frac{b_1^4}{9} - 12 \cdot 2\frac{b_1^3}{27}b_1^2)\frac{\partial b_1}{\partial \lambda} + 2 \cdot 27\frac{2b_1^3}{27}\frac{\partial b_3}{\partial \lambda} \\ &= 4b_1^3\frac{\partial b_3}{\partial \lambda} > 0, \end{aligned}$$

where the positivity follows from $b_1 < 0$ at $\lambda = 0$ and $\partial b_3 / \partial \lambda < 0$ always. Both Δ and the partial derivative $\partial \Delta / \partial \lambda$ are continuous. Hence, for any $\sigma > 0$ and $\beta > \underline{\beta}(\bar{\sigma}, \bar{\alpha}, \bar{B}, \xi)$, there exists $\lambda_0 > 0$ such that $\lambda < \lambda_0$ implies $\Delta > 0$. This proves (i) of Proposition 2.

Similarly, in view of our normalization of the CES production function, varying σ does not affect the steady-state values k^* , $\varepsilon(k^*)$, $\tilde{w}(k^*)$ and \hat{c}^* . For $\sigma = 0$ we have $b_1 < 0$ (in view of (4.6)) and $b_2 = 0$; $b_3 (< 0)$ is independent of σ . Hence, for arbitrary $\lambda > 0$ and $\beta > \underline{\beta}$,

$$\Delta(\lambda, 0, \beta) = 4\left(\frac{-b_1^2}{3}\right)^3 + \frac{1}{27}(2b_1^3 + 27b_3)^2 = 27b_3^2 + 4b_1^3b_3 > 0.$$

Moreover, $\lim_{\sigma \rightarrow 0} \Delta(\lambda, \sigma, \beta) = \Delta(\lambda, 0, \beta)$. By continuity it follows that, for any $\lambda > 0$ and $\beta > \underline{\beta}(\bar{\sigma}, \bar{\alpha}, \bar{B}, \xi)$, there exists $\sigma_0 > 0$ such that $\Delta > 0$ if $\sigma < \sigma_0$. This proves (ii) of Proposition 2.

Finally, for any $\beta > \underline{\beta}(\bar{\sigma}, \bar{\alpha}, \bar{B}, \xi)$ there exists positive constants C_0, C_1, C_2 and C_3 such that $b_1 = -C_0 + C_1\sigma\lambda$, $b_2 = -C_2\sigma\lambda$ and $b_3 = -C_3\lambda$. Now, choose the product of σ and λ to satisfy

$$\sigma\lambda = C_0 / C_1. \quad (10.9)$$

Then $b_1 = 0$, $s = b_2 = -C_2\sigma\lambda$ and $z = b_3 = -C_3\lambda$, implying

$$\Delta = -4(C_2\sigma\lambda)^3 + 27(-C_3\lambda)^2 = \lambda^2 \left[-4C_2^3(C_0 / C_1)\sigma^2 + 27C_3^2 \right], \quad (10.10)$$

by (10.9). Define,

$$\sigma_1 = \left(\frac{27C_3^2}{4C_2^3(C_0 / C_1)} \right)^{1/2}.$$

Then (10.10) shows that $\Delta < 0$ for any $\sigma > \sigma_1$ combined with a $\lambda = C_0 / (C_1\sigma)$ (so that (10.9) remains valid). This proves (iii) of Proposition 2.

E. The problem of reconciling a labour income share of 2/3 with a realistic investment share (Section 6)

In Section 6 we referred to the general problem of reconciling a labour income share of 2/3 with a realistic investment share and rate of capital accumulation under competitive conditions in a one-sector model. To illustrate, we consider the US investment share of GDP which on average over more than a century has been almost 19% (Barro and Sala-i-Martin, 2004, p. 15). Ignoring capital adjustment costs, $I/Y = (I/K)/(Y/K) \leq 0.19$ implies $f(k)/k = Y/K \geq (I/K)/0.19 = (\dot{K}/K + \delta)/0.19$. Consequently, under competitive conditions and absence of adjustment costs we have

$$\frac{wL}{Y} = 1 - \frac{f'(k)}{f(k)/k} = 1 - \frac{r + \delta}{f(k)/k} \geq 1 - \frac{r + \delta}{(\dot{K}/K + \delta)/0.19} \equiv \underline{\omega}.$$

With $\dot{K}/K + \delta$ no smaller than 0.10 (= $\delta + \gamma + n$ in baseline case) and $r + \delta$ no smaller than 0.14 (= $r + \delta$ in baseline case), the lower bound, $\underline{\omega}$, for the competitive labour income share will be no smaller than 0.734, which is well above the conventional benchmark of 0.667. Although the introduction of convex adjustment costs may help, at least under the chosen specification in (5.2) it can not solve the problem unless the adjustment costs are unrealistically high (β unrealistically low).

F. Half-life and period length (Section 7)

The asymptotic half-life and, in the case of complex eigenvalues, the asymptotic period length are calculated in the following way. Let $\mathbf{x} = (x_1, x_2, x_3) = (q, \tilde{w}, x)$ and $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) = (q^*, \tilde{w}^*, x^*)$. As shown in Appendix A, exactly two eigenvalues of J have negative real part. We write these two eigenvalues as $\mu_1 = a_1 + ib$ and $\mu_2 = a_2 - ib$, where $a_1 < 0$ and $a_2 < 0$. In case μ_1 and μ_2 are complex, $b \neq 0$ and $a_1 = a_2 = a$. Otherwise, $b = 0$ and, generically, $a_1 \neq a_2$.

First, consider the case where μ_1 and μ_2 are real. Suppose $a_2 < a_1 < 0$. In a neighbourhood of the steady state, the unique convergent solution for x_{it} , $i = 1, 2, 3$, can be approximated by $x_{it} = C_{i1}e^{a_1 t} + C_{i2}e^{a_2 t} + x_i^*$, where C_{i1} and C_{i2} are constants determined by the given initial values of \tilde{w} and x . Let $\Delta x_i \equiv x_i - x_i^*$. Then $|\Delta x_i|$ is the distance of x_i from its steady-state value. For $\Delta x_i \neq 0$ and $C_{i1} \neq 0$, the instantaneous rate of decline of this distance is

$$-\frac{d|\Delta x_i|/dt}{|\Delta x_i|} = -\frac{C_{i1}e^{a_1 t} a_1 + C_{i2}e^{a_2 t} a_2}{C_{i1}e^{a_1 t} + C_{i2}e^{a_2 t}} = -\frac{a_1 + \frac{C_{i2}}{C_{i1}} e^{(a_2 - a_1)t} a_2}{1 + \frac{C_{i2}}{C_{i1}} e^{(a_2 - a_1)t}}.$$

In view of $a_2 < a_1 < 0$, there exists $t_1 > 0$ such that for all $t > t_1$, $|C_{i2}/C_{i1}| e^{(a_2 - a_1)t} < 1$ and

$$-\frac{d|\Delta x_i|/dt}{|\Delta x_i|} \rightarrow -a_1$$

for $t \rightarrow \infty$. We refer to $-a_1$ as the *asymptotic speed of adjustment*; the *asymptotic half-life* is $\ln 2 / (-a_1)$.

In case $a_2 = a_1 = a < 0$, the unique convergent solution for x_{it} , $i = 1, 2, 3$, can be approximated by $x_{it} = (C_{i1} + C_{i2}t)e^{at} + x_i^*$, and by similar reasoning we find $-a$ as the asymptotic speed of adjustment. The asymptotic half-life is in this case $\ln 2 / (-a)$.

When μ_1 and μ_2 are conjugate complex, then, in a neighbourhood of the steady state, the unique convergent solution for x_{it} , $i = 1, 2, 3$, can be approximated by $x_{it} = D_i e^{at} \cos(bt + \omega_i) + x_i^*$, where $a (< 0)$, D_i and ω_i are constants determined by the given initial values of \tilde{w} and x . Right away, the rate of damping, $-a$, suggests itself as the relevant measure of the asymptotic speed of adjustment. The corresponding asymptotic half-life of the amplitude, $D_i e^{at}$, is again $\ln 2 / (-a)$. The time path of x_{it} features damped oscillations with *asymptotic period length* equal to $2\pi / b$.

Half-life as well as period length are the same for q , \tilde{w} and x . For variables with a trend, for instance the real wage, w_t , it is not the level but the ratio w_t / w_t^* that converges, where w_t^* is the trend level, $w_t^* \equiv \tilde{w}^* A_t$. We think of the asymptotic half-life in the adjustment process in this case as the asymptotic half-life of the relative deviation $(w_t - w_t^*) / w_t^*$, which by division with A_t is seen to equal $(\tilde{w}_t - \tilde{w}^*) / \tilde{w}^*$. This asymptotic half-life is the same as that for q , \tilde{w} and x .

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