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# Optimal Sales Schemes for Network Goods

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## Abstract

This paper examines the optimal sequencing of sales in the presence of network externalities. A firm sells a good to a group of consumers whose payoff from buying is increasing in total quantity sold. The firm selects the order to serve consumers so as to maximize expected sales. It can serve all consumers simultaneously, serve them all sequentially, or employ any intermediate scheme. We show that the optimal sales scheme is purely sequential, where each consumer observes all previous sales before choosing whether to buy himself. A sequential scheme maximizes the amount of information available to consumers, allowing success to breed success. Failure can also breed failure, but this is made less likely by consumers' desire to influence one another's behavior. We show that when consumers differ in the weight they place on the network externality, the firm would like to serve consumers with lower weights first. Our results suggests that a firm launching a new product should first target independent-minded consumers who can serve as opinion leaders for those who follow.

**JEL-codes:** M31, D42, D82, L12

**Key Words:** Product launch, Network externality, Sequencing of sales

## 1 Introduction

In many situations, firms must decide on the sequence in which consumers are served, and in particular whether to serve consumers simultaneously or sequentially. Consider a nightclub that puts in place a visible queue outside its entrance. This queue essentially makes decisions sequential, as it allows customers to observe whether earlier arrivals will attend the club before they decide whether to attend themselves. Apple used a similar approach when launching the iPhone 5, publicizing pre-order sales figures prior to the official

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release, so that consumers looking to buy would know how many others had bought as well.<sup>1</sup> Firms must also decide on the sequencing of sales when launching new products across different markets. They may follow a sprinkler strategy, where they release the new product in all markets simultaneously, or a waterfall strategy, where they release it in one market after another. These decisions can be relevant for products as diverse as books, movies, cars, and video games (see Aoyagi (2010), Bhalla (2013) and the examples therein). Sony’s European Marketing director recently touched on this dilemma regarding the launch of its new gaming console:

“We will launch [the PlayStation 4] this year. Exactly what regions, what timing, is being worked through. Which regions in 2013 - is it all of them, is it some of them? Is there some degree of phasing? We’ll reveal that in more detail later, but we can’t yet.”<sup>2</sup>

A firm’s decision on how to sequence sales, and in particular whether to use a simultaneous or sequential scheme, can have a crucial impact when consumption decisions are interdependent, so when each consumer’s willingness to buy depends on how many others buy as well. In particular, the choice of sales scheme will matter for the numerous products that exhibit positive network externalities. These are products where a consumer’s payoff from buying is increasing in aggregate sales (see, e.g., Katz and Shapiro (1985), Farrell and Saloner (1986)). In the case of items such as the iPhone or the PlayStation 4, these externalities can arise through the availability of complementary goods. High sales then increase the range of applications or game titles that are subsequently produced. Externalities can also arise because of consumer social concerns, particularly in settings where consumption is a social experience (Becker, 1991). For example, customers may prefer attending nightclubs that others attend, or watching movies alongside other viewers. Purchasing popular books or music may also generate an additional payoff by facilitating social interactions or simply by associating the owner with the ‘in thing’.

Intuitively, an advantage of sequential sales is that success can breed success. Consumers who observe high initial product sales then become more likely to buy the product themselves. A disadvantage of sequential sales is that failure can also breed failure, where low initial sales dissuade other consumers from buying. For this reason, it is not a priori clear what advice marketers should provide about how to sequence sales in the presence of network externalities. To our knowledge, our paper is the first to address this issue in a setting where sales dynamics arise from consumers’ optimal purchase decisions. Each consumer’s decision will depend both on the actual behavior of consumers he observes and on rational expectations about consumers he does not.

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<sup>1</sup>See “iPhone 5 Pre-Orders Top Two Million in First 24 Hours”, Apple.com, September 17, 2012.

<sup>2</sup>See “PlayStation 4 launch in ‘at least’ one country in 2013”, digitalspy.co.uk, February 22, 2013.

Specifically, we consider a firm that sells a homogeneous good to a group of consumers, where each consumer's payoff from buying the good is the sum of two terms: an intrinsic payoff, and a network payoff that is increasing in the total number of consumers who buy. The baseline analysis assumes that consumers differ only in their intrinsic payoff from buying. This payoff is private information. At the start of the game, the firm partitions consumers into different cohorts, where consumers in each cohort buy simultaneously, but consumers in different cohorts buy sequentially, having observed sales from all previous cohorts. When deciding whether to buy, consumers do not observe sales from their own cohort or from any later cohort. They instead based their purchase decision on rational expectations about what these sales will be. The firm chooses the number and the size of each cohort. Each consumer's decision whether or not to buy is one off and irreversible.

Our main result is that the optimal sales scheme is purely sequential, with a single consumer per cohort. A sequential scheme maximizes the amount of information available to consumers, who make their purchase decisions one after another and observe all previous sales. The crucial point with this scheme is not just that each consumer can observe others. Rather, it is that each consumer realizes that others can observe him. Consumers then take into account how their purchase can influence those who follow, which increases their own incentive to buy. In this sense, consumers may be willing to wait outside a nightclub precisely because their presence encourages others to join the queue. We show that a sequential scheme remains optimal regardless of whether the firm can commit to maintain it after consumers start making their purchases. It also remains optimal when consumers differ not just in their intrinsic payoff, but also in the weight they play on the network payoff. In this case, we show that the firm would like to serve consumers in increasing order of these dependence weights, so that more independent-minded consumers act first and can serve as opinion leaders. We also explore how these results depend on consumers' understanding of each others' incentives and on the possibility of consumer communication.

We now discuss how this paper relates to previous work. An extensive literature has looked at a variety of issues related to network externalities.<sup>3</sup> A strand of this literature has looked at the timing of sales (or the timing of entry, or the timing of technological adoption, depending on the context). However, unlike in our paper, this literature assume that the timing is determined by consumers, not by the firm. Each consumer then makes an optimal decision about when to buy, given the observed actions of others and expectations about their future behavior (see, e.g., Farrell and Saloner (1985) for technological adoption;

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<sup>3</sup>We do not attempt to survey this literature here. Contributions include Padmanabhan et al. (1997) on quality upgrades, Basu et al. (2003) on complementary goods and product attributes, and Sun et al. (2004) on product strategies for innovators, among many others.

Cabral et al. (1999) in relation to pricing; Gale (2001), Ochs and Park (2010) for network games; and Farrell and Klemperer (2007) for a review related to competition). While this approach has been instructive in many ways, it offers little advice to firms that have some direct control over the sequencing of sales.

Conceptually, this paper also has elements in common with diffusion models in the spirit of Bass (1969), which analyze how internal and external influence affect product take-up in a population.<sup>4</sup> Typically, these models assume that internal influence depends on the actions of other consumers, whereas external influence can depend on marketing variable such as price and advertising (Kalish (1985)), Dhebar and Oren (1985), Dockner and Jorgensen (1988), Xie and Sirbu (1995)). In our framework, internal influence also stems from the actions of other consumers, but this influence is based on individual optimizing behavior. External influence arises not through price and advertising but through the firm's sequencing of sales. It is this sequencing by the firm, rather than information diffusion between consumers, that determines equilibrium sales dynamics.

A number of recent papers have looked at the optimal sequencing of sales when consumers engage in social learning. These papers do not consider network externalities, but instead assume consumers have private information about product quality, which they may learn from observing each others' purchases. The optimal sales scheme then depends on the firm's beliefs about quality in a way that varies with the precise specification. SgROI (2002) shows that when prior beliefs about quality are high, simultaneously serving a group of "guinea pigs" can help avoid an information cascade where all subsequent consumers refrain from buying. Liu and Schiraldi (2012) show that the optimal scheme is often fully simultaneous when prior beliefs about quality are low. Bhalla (2013) suggests instead that a firm should use simultaneous sales when its updated beliefs about quality are high, as long as it can adjust its price over time.<sup>5</sup> The closest paper from a technical point of view is Aoyagi (2010) where a seller also divides consumers into different cohorts. Aoyagi shows that the optimal scheme is fully sequential, but this result depends crucially on his assumption of dynamic pricing.

We view our contribution as complementary to those from the literature on social learning. Our results apply to a different set of economic situations, where consumers may not have private information about product quality, but where they care directly about each others' behavior. The mechanism driving our results is also very different. It relies on consumers' desire to influence one another, which arises naturally with network externalities, but which plays no role under social learning.

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<sup>4</sup>For a survey of related types of diffusion models, see Mahajan et al. (1990).

<sup>5</sup>These papers relate to a broader literature on how firms can influence social learning, through means such as pricing or product testing (see, e.g., Ottaviani and Prat (2001), Bar-Isaac (2003), Bose et al. (2006), Bose et al. (2008), Gill and SgROI (2008), Gill and SgROI (2012)).

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 contains the main analysis and derives the optimal sales scheme. Section 4 discusses issues of robustness, and Section 5 then concludes. Proofs of all Propositions and of technical lemmas can be found in the appendix.

## 2 Model

A seller of a network good faces a market of  $n$  consumers who each have unit demand. Consumers differ in their type  $\theta$ , where each  $\theta_i$  is an independent draw from a uniform distribution  $U \sim [\underline{\theta}, \bar{\theta}]$ . Type is private information.

If consumer  $i$  buys a unit of the good, then he obtains payoff

$$\theta_i + \frac{\lambda_i}{n-1} \sum_{j \neq i} x_j, \quad (1)$$

where  $x_j = 1$  if consumer  $j$  buys and  $x_j = 0$  if he does not. If consumer  $i$  does not buy, then he obtains payoff  $u_0$ .

A consumer's payoff from buying is increasing in his type. It is also increasing in the number of other consumers who buy, giving a positive network externality. The weight consumer  $i$  places on this network externality is  $\lambda_i \in (0, 1]$ , which is public information.

In this setting, the seller effectively chooses the order in which consumer can buy. At  $t = 0$ , the seller chooses a number of cohorts  $m \leq n$ , and how to partition the  $n$  consumers between the  $m$  different cohorts,  $I = \{I_1, \dots, I_m\}$ . The seller commits to this choice of partition. We will refer to  $I$  as the seller's choice of sales scheme. Although we do not explicitly modeling pricing,  $u_0$  can be interpreted as the value of consumers' outside option plus a price fixed at  $t = 0$ .

At  $t = 1$ , all consumers in cohort  $I_1$  simultaneously decide whether to buy a unit of the good. Similarly, for any period  $t$  with  $2 \leq t \leq m$ , all consumers in cohort  $I_t$  simultaneously decide whether to buy, having observed the choice of consumers in all previous cohorts  $I_{t'}$  for  $t' \leq t - 1$ . A consumer who chooses not to buy cannot change his mind and buy in a later period. After consumers in cohort  $I_m$  make their choice, the game ends.

For consumer  $i \in I_t$ , the relevant history is the number of consumers in cohorts  $I_1, \dots, I_{t-1}$  who chose to buy. Denote this number by  $K_t$ . For a given  $I$  such that  $i \in I_t$ , the strategy of consumer  $i$  is a decision rule that, for any  $K_t$ , specifies whether or not to buy,  $x_i = 0$  or  $x_i = 1$ .

We look for a subgame perfect equilibrium where the strategy of consumer  $i$  maximizes his expected payoff, for any history  $K_t$ . All expectations follow from other consumers' equilibrium strategies. We are

interested in the sale  $I$  that maximizes equilibrium expected sales,  $\sum_{1 \leq t \leq m} \mathbb{E}(N_t)$ . We assume  $\underline{\theta} + 1 < u_0 < \bar{\theta}$  to guarantee interior solutions.

### 3 Analysis

To begin with, consider the incentives of some consumer  $i$ , who must decide whether or not to buy after observing sales from previous cohorts. Suppose that consumer  $i$  is in cohort  $I_t$ , and that he observes  $K_t$  in previous sales. Let  $N_t$  denote the number of consumers who will buy in his own cohort  $I_t$ , and let  $N_{t'}$  denote the number of consumers who will buy in a later cohort  $I_{t'}$ . Neither  $N_t$  nor  $N_{t'}$  are known when consumer  $i$  makes his purchase decision, so this decision must rely in part on expectations. Consumer  $i$  will find it optimal to buy himself if

$$\theta_i + \frac{\lambda_i}{n-1} \left( K_t + \mathbb{E}(N_t - x_i | K_t) + \sum_{t+1 \leq t' \leq m} \mathbb{E}(N_{t'} | K_t, x_i = 1) \right) \geq u_0. \quad (2)$$

The left-hand-side of (2) gives consumer  $i$ 's expected payoff from buying, which follows from (1). It consists of the intrinsic payoff from buying,  $\theta_i$ , plus the expected network payoff, which depends on three components: how many consumers have already bought,  $K_t$ , how many are expected to buy from the current cohort, and how many are expected to buy from later cohorts. All consumers in later cohorts will observe that consumer  $i$  chose to buy before making their own choice, which means that consumer  $i$ 's action can influence their behavior. This is why the final expectation in (2) is conditional on consumer  $i$ 's decision to buy,  $x_i = 1$ . Consumer  $i$  will find it optimal to buy if the left-hand-side of (2) exceeds  $u_0$ , which is the payoff from not buying.

Looking at (2), the incentive for any consumer  $i$  to buy is increasing in his intrinsic utility from buying, given by his type  $\theta_i$ . It follows that consumers will use cut-off strategies according to their type. Rearranging (2) shows the best response of consumer  $i \in I_t$  after history  $K_t$  is to buy if and only if  $\theta \in [\theta_i^*, \bar{\theta}]$ , where

$$\theta_i^* = u_0 - \frac{\lambda_i}{n-1} \left( K_t + \sum_{j \in I_t \setminus \{i\}} \mathbb{E}(x_j | K_t) + \sum_{t' \geq t+1} \sum_{j \in I_{t'}} \mathbb{E}(x_j | K_t, x_i = 1) \right). \quad (3)$$

Consumer  $i$  buys if his type exceeds a threshold value, given by the right-hand-side of (3), which depends on the particular history he observes. The consumer of type  $\theta_i^*$  earns exactly  $u_0$  from buying and is therefore indifferent with his outside option.

The optimal decision rule given by (3) is deterministic. However, from the perspective of consumer  $i$ , the decisions of other consumers in his own cohort and those in later cohorts appear to be probabilistic. The

reason is that type is private information. Consumer  $i$  is unsure whether these consumers' types will exceed the relevant threshold value for the particular histories they end up facing. He is also uncertain about what these particular histories will look like, since histories depend on the actions (and hence the types) of yet other consumers. From his own perspective, the probabilities that other consumers will buy, given that he buys himself, appear as expectations on the right-hand-side of (3). Consumer  $i$  can use his knowledge of the distribution of types, the size of each cohort, and the fact that others also use decision rule (3) as an equilibrium strategy, to determine these probabilities and compute his own cutoff.

By this same reasoning, the probability that consumer  $i$  will buy after history  $K_t$ , from the perspective of those who observe the history but are uncertain about his type, is

$$\mathbb{E}(x_i|K_t) = \frac{\bar{\theta} - \theta_i^*}{\bar{\theta} - \underline{\theta}}, \quad (4)$$

with  $\theta_i^*$  given by (3). Parameter assumptions  $\underline{\theta} + 1 < u_0 < \bar{\theta}$  and  $\lambda_i \leq 1$ , combined with (3), ensure that  $\mathbb{E}(x_i|K_t)$  is always strictly greater than zero and strictly less than one.

From an ex-ante perspective, the overall probability that consumer  $i$  will buy depends on his probability of buying after a particular history  $K_t$  and on the ex-ante probability distribution over all possible histories. The set of all histories is very large, since each  $K_t$  can take on any value from zero to  $\sum_{i=1}^{t-1} n_i$ , where  $n_i$  denotes the number of consumers in cohort  $i$ . Our assumption of a uniform distribution of types reduces the problem from analysing the whole distribution of relevant histories to just the expected value of the number of consumers who will buy,  $\mathbb{E}(K_t)$ . Formally, a uniform distribution means that the right-hand-side of (4) is linear in type, which ensure that the right-hand-side of (3) is linear in type as well. This assumption makes the analysis tractable and allows us to establish equilibrium existence and uniqueness.

**Proposition 1.** *For any sales scheme  $I$ , the game has a unique subgame perfect equilibrium. That is, for any consumer  $i \in I_t$  and history  $K_t$ , the cut-off  $\theta_i^*$  is uniquely defined.*

Proposition 1 does not suggest that the seller can predict equilibrium sales with certainty. Sales depend on willingness to pay, which depend in part on consumer types, and type is not something the seller observes. What the seller can predict is exactly how many sales to expect on average. This certainty follows from the sellers' knowledge of the distribution of types and of the unique set of cut-offs that follow from consumer equilibrium strategies. Formally, it is our assumption  $\lambda_i \leq 1$  that yields uniqueness. This assumption also means that our qualitative results, while relying on the presence of network externalities, do not require these externalities to be particularly strong.

The fact that consumer equilibrium strategies are unique allows us to be precise in referring to the seller's



optimal sales scheme. The seller can compare the unique value of expected sales under all possible schemes, each of which partitions consumers into a different set of cohorts, and then select the scheme that yields the largest expected value. Multiplicity of equilibria would make this procedure less clear. Multiple values of expected sales would then be consistent with each different scheme, where the realized value of sales would depend on the state of consumer sentiment. Equilibrium expected sales could be high if consumers expect them to be high but also low if consumers expect them to be low. The seller would then need to know when a change in sales scheme would trigger a change in sentiment by coordinating consumers on different expectations, something for which formal theory provides little guidance.

Given uniqueness, we now proceed with our main results regarding the seller's optimal scheme. We first consider the baseline situation where all consumers place the same weight on the network payoff.

**Proposition 2.** *Suppose that  $\lambda_i = \lambda \leq 1$  for all  $i = 1, \dots, n$ . Then the sales scheme I that maximizes expected sales  $\sum_{1 \leq i \leq n} \mathbb{E}(x_i)$  has a single consumer per cohort.*

Proposition 2 provides support for the sequential product launch strategies followed by Apple, Sony and others mentioned in the Introduction, which give some consumers the opportunity to buy before others. A purely sequential scheme provides consumers with the maximum amount of information about each others' purchases, essentially making their decisions visible to one another. This visibility can allow success to breed success. High sales from consumer who are served first can then encourage increased sales from consumers who are served later. The Proposition shows that a sequential scheme is optimal despite the fact that failure can also breed failure, where low early sales can depress sales from those who follow.

The intuition for the result is as follows. Under a sequential scheme, consumers not only observe others' purchases, but they also realize their own purchases will be observed. The very fact of being observed makes buying more attractive, since consumers who are served early understand that those who see them buy will become more likely to buy themselves. The key formal point is that the expectations in (3) for consumers in cohorts  $t' \geq t + 1$  all condition on the purchase of consumer  $i$  in cohort  $t$ . It follows that a sequential scheme will tend to yield high initial sales, starting a virtuous cycle where early success is then compounded. The situation is very different under simultaneous sales, where consumers make their choices based solely on expectations, rather than on observed sales.

Proposition 2 shows that the form of the optimal sales scheme is very stark. Taken literally, a purely sequential scheme may be difficult to implement in practice, since firms often have imperfect control over the order in which consumer are served. For example, a firm can reasonably control the timing of sales between different markets by deciding where to launch its product, but may have less control over the timing of sales within a particular market, where consumers themselves decide when to visit the store. Our result

implies that such a constrained firm should then use a sales scheme that is as close to sequential as possible. Formally, the proof of Proposition 2 shows that for any scheme that places multiple consumers in a cohort, the seller can always increase expected sales by serving one consumer in this cohort before the others.

Our results so far show that the exact value of  $\lambda$  does not have a qualitative impact on the optimal scheme. However, the value of  $\lambda$  does have a quantitative impact on how the optimal scheme affects expected sales. A sequential sales scheme helps the seller by exploiting network externalities between consumers. The stronger these externalities, the larger the impact we would tend to expect from a sequential scheme. We now show simulation results that provide support for this idea. Specifically, we set  $\lambda_i = \lambda$  for all  $i$ , and calculate expected sales under both a simultaneous and a sequential scheme, for different values of  $\lambda$ .

Figure 1

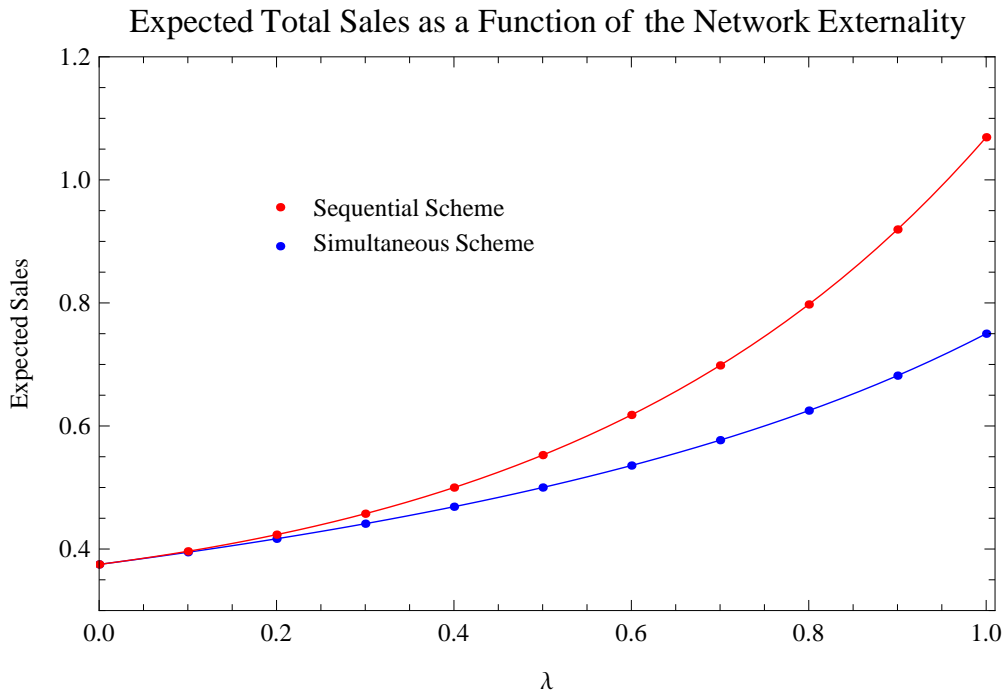


Figure 1 plots expected sales as a function of  $\lambda$  under the two schemes, with the simultaneous scheme represented in blue and the sequential scheme in red. As required by Proposition 2, expected sales are higher under a sequential scheme for all values of  $\lambda > 0$ . Figure 1 also shows that the difference in expected sales between the two schemes is increasing in  $\lambda$ . In particular, when  $\lambda = 1$ , a sequential scheme allows the seller to increase expected sales by approximately 40%.<sup>6</sup>

<sup>6</sup>The simulation uses parameter values  $n = 5$ ,  $\bar{\theta} = 2$ ,  $\underline{\theta} = 0$ , and  $u_0 = 1.85$

Proposition 2 shows that a sequential sales scheme will always maximize expected sales, which is the seller’s goal in our setting with risk neutrality. However, it can also be instructive to compare the overall distribution of sales under a sequential scheme and a simultaneous scheme, rather than simply the distributions’ means. The fact that low early sales under a sequential scheme can be self reinforcing (i.e. failure can breed failure) suggests such a scheme might carry more downside risk. We now show by simulation that this is not necessarily the case. In fact, Figure 2 shows that for certain parameter values, the distribution of sales under a sequential scheme is unambiguously better for the seller than the distribution under a simultaneous scheme, in the sense of first order stochastic dominance.

Figure 2

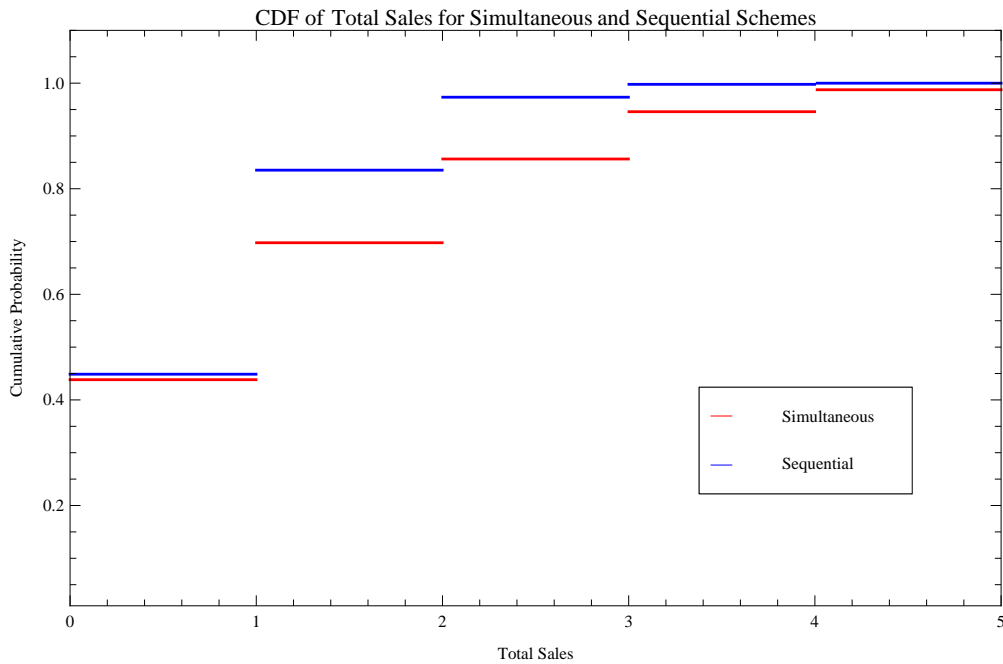


Figure 2 plots the cumulative distribution function of total sales under two different schemes, with the simultaneous scheme represented in blue and the sequential scheme in red. The former CDF lies entirely above the latter, showing the relation of first order stochastic dominance.<sup>7</sup> A sequential scheme here serves the dual purpose of increasing expected sales while decreasing the probability of a poor outcome where sales are very low.

We now show that our results on sequential sales generalize to the situation where consumers may place different weights on the network payoff.

<sup>7</sup>The simulation uses parameter values  $n = 5$ ,  $\theta = 2$ ,  $\underline{\theta} = 0$ ,  $u_0 = 1.85$ , and  $\lambda = 1$ .

**Proposition 3.** *The sales scheme I that maximizes expected sales  $\sum_{1 \leq i \leq n} \mathbb{E}(x_i)$  has a single consumer per cohort, increasingly ordered in  $\lambda$ , i.e.  $\lambda_1 \leq \dots \leq \lambda_n$ .*

The optimal sales scheme with ex-ante heterogeneous consumers remains purely sequential. The intuition for this result echoes that from Proposition 2. There, we argued that the qualitative advantage of a sequential scheme does not depend on the exact weight consumers place on the network payoff,  $\lambda$ . In a similar way, this qualitative advantage does not depend on the exact difference between these weights,  $\lambda_i$  and  $\lambda_j$ , for  $i \neq j$ . As long as these weights are strictly positive, a sequential scheme will increase expected sales by increasing visibility, allowing success to breed success.

Proposition 3 goes further by showing what the optimal ordering should look like: consumers should be served in increasing order of the weight they place on the network payoff. This result echoes the notion that a firm launching a new product should target independent-minded consumers first, who can serve as opinion leaders for those who follow. These innovators (low  $\lambda$ ) will decide whether or not to buy the product based on their own private information or their own personal tastes. Their decision to buy can then encourage imitators (high  $\lambda$ ) who care about their actions to jump on the bandwagon. Of course, in practice it may be difficult for a firm to serve literally every innovator before serving a single imitator. A more reasonable interpretation of Proposition 3 is that a firm should focus its marketing activities surrounding product launch on one group of consumers before the others.

Choosing the optimal ordering presents the seller with a new trade off. Serving consumers in increasing order of weights means that later consumers (high  $\lambda$ ) have a strong incentive to follow those who buy. In principle, doing so reinforces the benefits when success breeds success. However, these benefits are limited by the fact that early consumers (low  $\lambda$ ) do not become particularly likely to buy just because others will follow. Another way to understand the trade off is that consumers with high weights are likely to set a good example, but they are also more likely to follow a good example once it has been set. Proposition 3 shows that the second effect outweighs the first so the optimal order is increasing in these weights. Even though the key feature of sequential sales is that consumers behave differently when their actions are visible, the optimal scheme grants the largest visibility to consumers who care little about being observed.

One surprising feature of the optimal scheme is that consumers with lower expected willingness to pay are actually served first. Another is that the quantitative benefits of the optimal ordering will not always increase in the difference between consumers' weights. To take a simple case, suppose there are two consumers, with  $\lambda_1 = 0$  and  $\lambda_2 \geq 0$ . The first consumer's action then influences the second consumer. However, because this influence will not affect the first consumer's behavior, expected sales are independent of the precise value of  $\lambda_2$ . Expected sales are in fact the same as without network externalities and are independent of the seller's

scheme.

We conclude this section by turning to the issue of consumer communication. Typically, models of sequential decision making with private information assume that consumers cannot directly speak with one another. All information transmission instead takes place indirectly when consumers observe each others' choices. We also make this assumption, which is reasonable in many market situations.

The question remains whether direct consumer communication could even make a difference in our setting. If consumers could directly speak with one another before buying, and also chose to truthfully reveal their private information, then there would certainly be no scope for further learning. The seller's choice of sales scheme would then be immaterial as consumers would already know each others' types. However, it is not clear why consumers here would reveal their types truthfully. To the contrary, each consumer might prefer to exaggerate his willingness to pay, in the hopes of convincing others to buy, which can only make him better off.

We explore this issue by considering a cheap talk game in the period before the first consumer decides whether to buy. Perhaps surprisingly, an equilibrium with full information revelation in fact exists, even though each consumer has a weakly dominant strategy to claim he is the highest type.

**Proposition 4.** *Suppose that at  $t = 0$ , consumers can simultaneously send a public message about their type,  $m \subseteq [\underline{\theta}, \bar{\theta}]$ . Then an equilibrium exists with full revelation,  $m_i = \theta_i$  for every  $i = 1, \dots, n$ , where all consumers truthfully reveal their type.*

The intuition for this result is that all consumers have a weak incentive to exaggerate in an equilibrium with full revelation, but this incentive is never strict. A consumer who exaggerates his type may well convince others to buy by convincing them he will buy himself. However, this consumer will only benefit from convincing others if he actually does end up buying. And if he truly has an incentive to buy, then there is no reason for him to exaggerate his type in the first place.

Proposition 4 does not guarantee that consumers will truthfully reveal their private information if they have the chance to speak. After all, a babbling equilibrium also exists in the cheap talk game where no information is revealed. The result instead demonstrates that truth telling is possible, so that the seller's choice of sales scheme is most likely to matter in two types of settings: where either direct communication between consumers is difficult, or where consumers are unlikely to coordinate on telling the truth.

## 4 Discussion and Robustness

Our analysis has assumed that the seller commits to his choice of sales scheme, solutions are interior, any pricing is static, and types are uniformly distributed. We now briefly comment on how each of these assumptions relates to our result that sequential sales are optimal.

The fact that the seller can commit to a sales scheme is unimportant for the results. The analysis shows that for any cohort  $I_t$  with at least two consumers, given any history  $K_t$ , the seller always benefits by having one consumer act before the other. This means that a seller who chooses a sequential scheme at  $t = 0$  has no incentive to change his mind after observing the actions of any number of consumers. If the first consumers do not buy, then the seller may well regret ex-post having a fully sequential scheme. Nonetheless, he will still prefer the remaining consumers to act sequentially.

Our assumption on parameter values ensures that after any history, the probability a consumer will buy lies strictly between zero and one, so that the equilibrium of the consumer game is interior and unique. Relaxing this assumption would mean that multiple values of expected sales could be consistent with each scheme as discussed following Proposition 1. If network effects were sufficiently strong, then any sales scheme could generate both a good equilibrium outcome where all consumers buy and a bad equilibrium outcome where nobody buys. A sequential scheme might then still be useful in helping with equilibrium selection, if observing an early purchase can coordinate the remaining consumers on the Pareto dominant outcome.

Despite not explicitly modeling pricing, our framework is consistent with an initial price that figures implicitly in each consumer's outside option. A question is then whether sequential sales would remain optimal if the seller could adjust its price over time. The answer is yes in a situation with commitment, where the seller initially fixes a pricing policy, specifying the relevant price for each particular history, and then implements the policy as different histories are observed. A fully sequential scheme then offers an additional benefit: it maximizes the set of histories the seller can observe, and hence the set of pricing policies from which he can choose. The seller can always ignore as much of this information as he likes by setting a policy with the same price after different histories.<sup>8</sup>

Working with a uniform distribution of types yields equilibrium existence and uniqueness, as discussed prior to Proposition 1. It also has an effect that relates to the variance of early sales. Intuitively, variance can be quite high under a sequential scheme, since consumers can observe and imitate one another. This

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<sup>8</sup>The situation without commitment is more subtle, since the optimal price after a particular history may differ from the price specified under the optimal policy. Specifically, the seller may want to increase the price after a consumer buys, but a consumer is less likely to buy if he expects the price to then increase. Our mechanism supporting sequential sales will continue to hold as long as price changes are not too severe.

reasoning suggests that a sequential scheme may tend to generate more extreme histories.

The variance of early sales plays no role with a uniform distribution of types. All that matters about early sales is their expected value, which is maximized under a sequential scheme. However, variance could potentially matter if consumer type followed a different distribution. For example, if many consumers had low type, and only a very good history would persuade them to buy, then high variance could help by increasing the probability of such a history. If instead many consumers had high type, so only a very bad history would dissuade them from buying, then high variance could hurt by the same reasoning. Our analysis would then underestimate the benefit of sequential sales in the first case but overestimate it in the second case.

## 5 Conclusion

In settings with positive network externalities, a firm's choice of sales scheme can matter a great deal. Consumers may be more willing to buy mobile phones or video games, movies tickets or books, if they believe that others will likely buy them as well. Put another way, many consumers are more willing to buy a new product if they believe it will be a 'hit'. We derive the optimal sequencing of sales in such a setting, including the choice between simultaneous sales, sequential sales, and any number of intermediate schemes.

The sequencing of sales matters because it determines whether consumers can observe each others' purchase decisions before they make their own. The advantage of sequential sales is that success can breed success, where high early sales encourage later consumers to buy as well. The disadvantage is that failure can also breed failure, where later consumers do not buy precisely because they observe low early sales.

We show that despite this trade-off, the optimal sales scheme is purely sequential. The intuition is that a sequential scheme has two effects. It provides the maximum amount of information to later consumers about whether the good is a hit before they decide whether to buy. Crucially, it also increases early consumer confidence that others will follow them if they buy themselves. This second effect makes it more likely that later consumers will observe a hit, which is what drives our results. These results are consistent with firms' use of the waterfall strategy for product release, first releasing a new product in specific markets before releasing it in others.

We also explore various issues of robustness such as commitment and consumer communication. We show that if consumers differ in the weight they place on the network externality, it is still optimal to serve them sequentially, but to serve consumers with lower weights first. Serving independent-minded consumers first is consistent with firms' targeting of opinion leaders, whose initial take-up of a product can help generate later

success.

## 6 Appendix\*

Throughout all proofs, we let  $\lambda_i$  denote the weight placed on the network payoff by consumer  $i = 1, \dots, n$ . Doing so covers the case where all consumers place the same weight on the network payoff ( $\lambda_i = \lambda$  for all  $i = 1, \dots, n$ ) and the case where they do not ( $\lambda_i \neq \lambda_j$  for some  $i \neq j$ ).

### 6.1 Technical Lemmas\*

**Lemma A.1.** *Suppose  $K_t$  consumers buy up until cohort  $I_t$ , and consider consumer  $j \in I_{t'}$  with  $t' \geq t + 1$ . Suppose a set of consumers  $M \subseteq \cup_{l=t}^{t'-1} I_l$  choose to buy. Then*

$$\mathbb{E}(x_j | K_t, M) = \frac{\bar{\theta} - \mathbb{E}(\theta_j^* | K_t, M)}{\bar{\theta} - \underline{\theta}},$$

where

$$\mathbb{E}(\theta_j^* | K_t, M) = u_0 - \frac{\lambda_j}{n-1} \left( K_t + \sum_{t \leq l \leq t'} \sum_{i \in I_l \setminus \{j\}} \mathbb{E}(x_i | K_t, M) + \sum_{l \geq t'+1} \sum_{i \in I_l} \mathbb{E}(x_i | K_t, M, x_j = 1) \right).$$

*Proof.* By (3), for any  $K_{t'}$ , consumer  $j \in I_{t'}$  uses cut-off

$$\theta_j^* = u_0 - \frac{\lambda_j}{n-1} \left( K_{t'} + \sum_{i \in I_{t'} \setminus \{j\}} \mathbb{E}(x_i | K_{t'}) + \sum_{l \geq t'+1} \sum_{i \in I_l} \mathbb{E}(x_i | K_{t'}, x_j = 1) \right), \quad (5)$$

so that

$$\mathbb{E}(x_j | K_{t'}) = \frac{\bar{\theta} - \theta_j^*}{\bar{\theta} - \underline{\theta}}. \quad (6)$$

We now work with (5) and (6) to obtain  $\mathbb{E}(x_j | K_t, M)$ . Let  $\mathcal{K}$  be the set of all  $K_{t'}$  consistent with  $(K_t, M)$ . For each  $K_{t'}$  we multiply (5) with  $p(K_{t'} | K_t, M)$  and sum up over all  $K_{t'} \in \mathcal{K}$ . Since  $\sum_{\mathcal{K}} p(K_{t'} | K_t, M) = 1$ , we have that  $\mathbb{E}(\theta_j^* | K_t, M)$  is equal to

$$u_0 - \frac{\lambda_j}{n-1} \left( \sum_{\mathcal{K}} K_{t'} p(K_{t'} | K_t, M) + \sum_{i \in I_{t'} \setminus \{j\}} \sum_{\mathcal{K}} p(K_{t'} | K_t, M) \mathbb{E}(x_i | K_{t'}) + \sum_{l \geq t'+1} \sum_{i \in I_l} \sum_{\mathcal{K}} p(K_{t'} | K_t, M) \mathbb{E}(x_i | K_{t'}, x_j = 1) \right),$$

Note that



$$\mathbb{E}(x_j|K_t, M) = \sum_{\mathcal{K}} p(K_{t'}|K_t, M) \mathbb{E}(x_j|K_{t'}),$$

$$\mathbb{E}(x_i|K_t, M, x_j = 1) = \sum_{\mathcal{K}} p(K_{t'}|K_t, M) \mathbb{E}(x_j|K_{t'}, x_j = 1),$$

and  $\sum_{\mathcal{K}} K_{t'} p(K_{t'}|K_t, M) = K_t + \sum_{t \leq l \leq t'} \sum_{i \in I_l \setminus \{j\}} \mathbb{E}(x_i|K_t, M)$ . Therefore,

$$\mathbb{E}(\theta_j^*|K_t, M) = u_0 - \frac{\lambda_j}{n-1} \left( K_t + \sum_{t \leq l \leq t'} \sum_{i \in I_l \setminus \{j\}} \mathbb{E}(x_i|K_t, M) + \sum_{l \geq t'+1} \sum_{i \in I_l} \mathbb{E}(x_i|K_t, M, x_j = 1) \right).$$

□

**Lemma A.2.** For any consumer  $i$  in cohort  $I_t$  with history  $K_t$ ,

$$\frac{d\mathbb{E}(x_i|K_t)}{dK_t} > 0.$$

*Proof.* Write out  $\mathbb{E}(x_i|K_t) = \frac{\bar{\theta} - \theta_i^*}{\bar{\theta} - \underline{\theta}}$  with

$$\theta_i^* = u_0 - \frac{\lambda_i}{n-1} \left( K_t + \sum_{j \in I_t \setminus \{i\}} \mathbb{E}(x_j|K_t) + \sum_{t' \geq t+1} \sum_{j \in I_{t'}} \mathbb{E}(x_j|K_t, x_i = 1) \right).$$

Write out each term in the second summation as  $\mathbb{E}(x_j|K_t, x_i = 1) = \frac{\bar{\theta} - \mathbb{E}(\theta_j^*|K_t, x_i = 1)}{\bar{\theta} - \underline{\theta}}$  with

$$\begin{aligned} \mathbb{E}(\theta_j^*|K_t, x_i = 1) &= u_0 - \frac{\lambda_j}{n-1} \left( K_t + 1 + \sum_{j' \in I_t \setminus \{i\}} \mathbb{E}(x_{j'}|K_t) \right. \\ &\quad \left. + \sum_{t+1 \leq l \leq t'} \sum_{j' \in I_l \setminus \{j\}} \mathbb{E}(x_{j'}|K_t, x_i = 1) + \sum_{l \geq t'+1} \sum_{j' \in I_l} \mathbb{E}(x_{j'}|K_t, x_i = 1, x_j = 1) \right), \end{aligned}$$

Consider a player in a cohort  $k > t + 1$ . Let  $M_k$  be the subset so players such that (i) each player  $i \in M_k$  decided to buy, (ii) for  $i, j \in M_k$ ,  $i \in I_{n_i}, j \in I_{n_j}$   $n_i \neq n_j$  and  $n_i > t$ . Let for  $l \leq k$   $M_k^l \subseteq M_k : \forall i \in M_k^l, i \in I_{n_i} \Rightarrow n_i < l$ . Then

Denoting the number of distinct equations for  $\mathbb{E}(x_j|K_t, x_i = 1, M_k)$  by  $A$ , including terms with zero coefficient on the right-hand side of each equation gives a system of  $A$  equations in  $A$  unknowns. By  $\lambda_j \leq 1$  and  $\underline{\theta} + 1 < u_0 < \bar{\theta}$ , consumers with type sufficiently close to  $\bar{\theta}$  have a dominant strategy to buy. This means the unique solution gives  $\mathbb{E}(x_i|K_t) > 0$ , for any  $K_t$ .

Differentiating each equation in the system with respect to  $K_t$  gives  $\frac{d\mathbb{E}(x_i|K_t)}{dK_t} = \frac{\bar{\theta} - \frac{d\theta_i^*}{dK_t}}{\bar{\theta} - \underline{\theta}}$  with

$$\begin{aligned} \frac{d\mathbb{E}(\theta_j^*|K_t, x_i = 1, M_k)}{dK_t} &= u_0 - \frac{\lambda_j}{n-1} \left( 1 + \sum_{j' \in I_t \setminus \{i\}} \frac{d\mathbb{E}(x_{j'}|K_t)}{dK_t} \right) \\ &+ \sum_{t+1 \leq l \leq k} \sum_{j' \in I_l \setminus M_k} d\mathbb{E}(x_{j'}|K_t, x_i = 1, M_k^l) dK_t + \sum_{l \geq k+1} \sum_{j' \in I_l} d\mathbb{E}(x_{j'}|K_t, x_i = 1, M_k, x_j = 1) dK_t. \end{aligned}$$

This system of  $A$  linear equations in  $A$  unknowns is identical to the first one, except that each conditional expectation is replaced by its derivative, and  $K_t$  has been set equal to 1. The associated matrix for this system has diagonal entries of 1 and off-diagonal entries of either 0 or  $\frac{-1}{\theta - \underline{\theta}} \frac{\lambda_i}{n-1} < 0$ , where the number of non-zero off-diagonal entries in each row cannot exceed  $\sum_{t' \geq t} \sum_{i \in I_{t'}} n_i - 1 \leq n - 1$ . By  $\lambda_i \leq 1$  and  $\underline{\theta} + 1 < \bar{\theta}$ , the sum of the absolute values of off-diagonal entries in each row is therefore strictly less than one. Hence, this matrix is strictly diagonally dominant. By the Gershgorin theorem (1931), the system then has a unique solution, with  $\frac{d\mathbb{E}(x_i|K_t)}{dK_t} > 0$ . □

**Lemma A.3.** For any consumer  $j \in I_{t'}$ , with  $t' \geq t + 1$ ,  $\mathbb{E}(x_j|K_t)$  is strictly increasing in  $\sum_{i \in I_t} \mathbb{E}(x_i|K_t)$ .

*Proof.* We proceed by induction. First, let  $t' = t + 1$ . By Lemma 1, for any  $x_j \in I_{t+1}$ , write out  $\mathbb{E}(x_j = 1|K_t) = \frac{\bar{\theta} - \mathbb{E}(\theta_j^*|K_t)}{\bar{\theta} - \underline{\theta}}$  with

$$\mathbb{E}(\theta_j^*|K_t) = u_0 - \frac{\lambda_j}{n-1} \left( \left[ K_t + \sum_{i \in I_t} \mathbb{E}(x_i|K_t) \right] + \sum_{i \in I_{t+1} \setminus \{j\}} \mathbb{E}(x_i|K_t) + \sum_{l \geq t+2} \sum_{i \in I_l} \mathbb{E}(x_i|K_t, x_j = 1) \right).$$

Again by Lemma A.1, write out each expectation  $\mathbb{E}(x_i|K_t, x_j = 1)$  in the last summation, and so on to generate a system of equations. Each of these equations will include the same expression in square brackets.

We can identify the expression in square brackets with a constant  $K_{t+1}$ . A strict increase in  $\sum_{i \in I_t} \mathbb{E}(x_i|K_t)$  is then equivalent to a strict increase in  $K_{t+1}$ . Hence by Lemma A.2,  $\mathbb{E}(x_j|K_t)$  must strictly increase.

Now suppose the result holds for all  $t + 1, \dots, t' - 1$ . We show that the result also holds for  $t'$ . For a consumer  $j \in I_{t'}$ , write

$$\mathbb{E}(\theta_j^*|K_t) = u_0 - \frac{\lambda_j}{n-1} \left( \left[ K_t + \sum_{t \leq l \leq t'-1} \sum_{i \in I_l} \mathbb{E}(x_i|K_t) \right] + \sum_{i \in I_{t'} \setminus \{j\}} \mathbb{E}(x_i|K_t) + \sum_{l \geq t'+1} \sum_{i \in I_l} \mathbb{E}(x_i|K_t, x_j = 1) \right).$$

Once again using Lemma A.1, write out each expectation  $\mathbb{E}(x_i|K_t, x_j = 1)$  in the last summation, and so on to generate a system of equations which all include the same term in square brackets. By the induction

hypothesis, the term in square brackets strictly increases, which is again equivalent to an increase in  $K_{t'}$ . By Lemma A.2,  $\mathbb{E}(x_j|K_t)$  must strictly increase. □

**Lemma A.4.** *Suppose all the cohorts  $I_j$ ,  $j \leq t-1$  are singletons. Let  $t' \leq t-1$ . Consider a history  $K_{t'}$  and any consumer  $i \in I_l$  with  $l \geq t'$ . Let  $\mathcal{K}_{t'}$  be a set of histories  $K_t$  consistent with  $K_{t'}$ , and let  $a \in \mathbb{R}$  be some parameter of arbitrary nature. Then if  $\frac{d \sum_{j \in I_t} \mathbb{E}(x_j|K_t)}{da} > 0$  for all  $K_t \in \mathcal{K}_{t'}$ , then  $\frac{\mathbb{E}(x_i|K_{t'})}{da} > 0$ .*

*Proof.* First consider consumer  $t-1 \in I_{t-1}$ , given history  $K_{t-1}$ . By (4) and (3), write  $\mathbb{E}(x_{t-1}|K_{t-1}) = \frac{\bar{\theta} - \theta_{t-1}^*}{\bar{\theta} - \theta}$  with

$$\theta_{t-1}^* = u_0 + \frac{\lambda_{t-1}}{n-1} \left( K_{t-1} + \sum_{l \geq t} \sum_{j \in I_l} \mathbb{E}(x_j|K_{t-1}, x_{t-1} = 1) \right).$$

Since  $I_{t-1}$  is a singleton,  $(K_{t-1}, x_{t-1} = 1)$  is just some history  $K_t$ . If the expectation of the summation for cohort  $I_t$  strictly increases, then, by Lemma A.3, each expectation in the summation for cohorts  $I_l$  with  $l \geq t+1$  then increases as well. Thus  $\mathbb{E}(x_{t-1}|K_{t-1})$  must strictly increase, for any history  $K_{t-1}$ .

For any consumer  $j$  in cohort  $l \geq t$ , write  $\mathbb{E}(x_j|K_{t-1}) = \frac{\bar{\theta} - \mathbb{E}(\theta_j^*|K_{t-1})}{\bar{\theta} - \theta}$  with

$$\mathbb{E}(\theta_j^*|K_{t-1}) = u_0 + \frac{\lambda_j}{n-1} \left( \left[ K_{t-1} + \mathbb{E}(x_{t-1}|K_{t-1}) \right] + \sum_{t \leq m \leq l} \sum_{i \in I_m \setminus \{j\}} \mathbb{E}(x_i|K_{t-1}) + \sum_{m \geq t+1} \sum_{i \in I_m} \mathbb{E}(x_i|K_{t-1}, x_i = 1) \right)$$

Looking at the term in square brackets, an increase in  $\mathbb{E}(x_{t-1}|K_{t-1})$  is equivalent to an increase in  $K_{t'}$ . Hence  $\mathbb{E}(x_j|K_{t-1})$  must strictly increase as well, for any history  $K_{t-1}$ .

Now consider consumer  $t-2$ . By the same logic, a strict increase in  $\mathbb{E}(x_{t-1}|K_{t-1})$  for any history  $K_{t-1}$  will give a strict increase in  $\mathbb{E}(x_i|K_{t-2})$  for any history  $K_{t-2}$ , for any consumer  $i \in I_l$  with  $l \geq t-2$ . Since all cohorts  $I_1, \dots, I_{t-2}$  are singletons, continuing in this way gives the result. □

## 6.2 Proofs of the Propositions\*

**Proposition 1.** *For any sales scheme  $I$ , the game has a unique subgame perfect equilibrium. That is, for any consumer  $i \in I_t$  and history  $K_t$ , the cut-off  $\theta_i^*$  is uniquely defined.*

*Proof.* Consider a subgame starting with cohort  $I_t$  to act after a history summarized by  $K_t$ . By (4) and (3), for each consumer  $i \in I_t$ , write out  $\mathbb{E}(x_i|K_t) = \frac{\bar{\theta} - \theta_i^*}{\bar{\theta} - \theta}$ , with

$$\theta_i^* = u_0 - \frac{\lambda_i}{n-1} \left( K_t + \sum_{j \in I_t \setminus \{i\}} \mathbb{E}(x_j | K_t) + \sum_{t' \geq t+1} \sum_{j \in I_{t'}} \mathbb{E}(x_j | K_t, x_i = 1) \right).$$

By Lemma A.1, write out each term in the second summation as  $\mathbb{E}(x_j | K_t, x_i = 1) = \frac{\bar{\theta} - \mathbb{E}(\theta_j^* | K_t, x_i = 1)}{\bar{\theta} - \underline{\theta}}$ , with

$$\begin{aligned} \mathbb{E}(\theta_j^* | K_t, x_i = 1) &= u_0 - \frac{\lambda_j}{n-1} \left( K_t + 1 + \sum_{j' \in I_t \setminus \{i\}} \mathbb{E}(x_{j'} | K_t) \right. \\ &\quad \left. + \sum_{t+1 \leq l \leq t'} \sum_{j' \in I_l \setminus \{j\}} \mathbb{E}(x_{j'} | K_t, x_i = 1) + \sum_{l \geq t'+1} \sum_{j' \in I_l} \mathbb{E}(x_{j'} | K_t, x_i = 1, x_j = 1) \right). \end{aligned}$$

Again by Lemma A.1, write out each term  $\mathbb{E}(x_{j'} | K_t, x_i = 1, x_j = 1)$  in the last summation as  $\mathbb{E}(x_{j'} | K_t, x_i = 1, x_j = 1) = \frac{\bar{\theta} - \mathbb{E}(\theta_{j'}^* | K_t, x_i = 1, x_j = 1)}{\bar{\theta} - \underline{\theta}}$ , and so on. Consider a player in a cohort  $k > t + 1$ . Let  $M_k$  be the subset so players such that (i) each player  $i \in M_k$  decided to buy, (ii) for  $i, j \in M_k$ ,  $i \in I_{n_i}, j \in I_{n_j}$ ,  $n_i \neq n_j$  and  $n_i > t$ . Let for  $l \leq k$   $M_k^l \subseteq M_k : \forall i \in M_k^l, i \in I_{n_i} \Rightarrow n_i < l$ . Then

$$\begin{aligned} \mathbb{E}(\theta_j^* | K_t, x_i = 1, M_k) &= u_0 - \frac{\lambda_j}{n-1} \left( K_t + 1 + \#M_k + \sum_{j' \in I_t \setminus \{i\}} \mathbb{E}(x_{j'} | K_t) \right. \\ &\quad \left. + \sum_{t+1 \leq l \leq k} \sum_{j' \in I_l \setminus M_k} \mathbb{E}(x_{j'} | K_t, x_i = 1, M_k^l) + \sum_{l \geq k+1} \sum_{j' \in I_l} \mathbb{E}(x_{j'} | K_t, x_i = 1, M_k, x_j = 1) \right). \end{aligned}$$

Denoting the number of distinct equations for  $\mathbb{E}(x_j | K_t, x_i = 1, M_k)$  by  $A$ , including terms with zero coefficient on the right-hand side of each equation gives a system of  $A$  equations in  $A$  unknowns.

The associated matrix for this system has diagonal entries of 1 and off-diagonal entries of either 0 or  $\frac{-1}{\bar{\theta} - \underline{\theta}} \frac{\lambda_i}{n-1} < 0$ , where the number of non-zero off-diagonal entries in each row cannot exceed  $\sum_{t' \geq t} \sum_{i \in I_{t'}} n_i - 1 \leq n - 1$ . By  $\lambda_i \leq 1$  and  $\underline{\theta} + 1 < \bar{\theta}$ , the sum of the absolute values of off-diagonal entries in each row is therefore strictly less than one. Hence, this matrix is strictly diagonally dominant. By the Gershgorin theorem (1931), the system then has a unique solution, so  $\mathbb{E}(x_i | K_t)$  and  $\theta_i^*$  are uniquely defined.  $\square$

**Proposition 2.** *Suppose that  $\lambda_i = \lambda \leq 1$  for all  $i = 1, \dots, n$ . Then the sales scheme I that maximizes expected sales  $\sum_{1 \leq i \leq n} \mathbb{E}(x_i)$  has a single consumer per cohort.*

*Proof.* For a given  $I$ , let  $I_t$  be the first cohort that is not a singleton. Consider any history  $K_t$ . For consumer  $i \in I_t$ , write  $\mathbb{E}(x_i | K_t) = \frac{\bar{\theta} - \theta_i^*}{\bar{\theta} - \underline{\theta}}$  with

$$\theta_i^* = u_0 - \frac{\lambda_i}{n-1} \left( K_t + \sum_{j \in I_t \setminus \{i\}} \mathbb{E}(x_j | K_t) + \sum_{t' \geq t+1} \sum_{j \in I_{t'}} \mathbb{E}(x_j | K_t, x_i = 1) \right).$$

If consumer  $i$  instead acts before all others in cohort  $I_t$ , we have

$$\theta_i^* = u_0 - \frac{\lambda_i}{n-1} \left( K_t + \sum_{j \in I_t \setminus \{i\}} \mathbb{E}(x_j | K_t, x_i = 1) + \sum_{t' \geq t+1} \sum_{j \in I_{t'}} \mathbb{E}(x_j | K_t, x_i = 1) \right).$$

For any other consumer  $j$  remaining in cohort  $I_t$ , or in a cohort  $I_{t'}$  with  $t' \geq t+1$ , write  $\mathbb{E}(x_j | K_t, x_i = 1)$  in turn as

$$\mathbb{E}(\theta_j^* | K_t, x_i = 1) = u_0 - \frac{\lambda_j}{n-1} \left( \left[ K_t + 1 \right] + \sum_{t \leq l \leq t'} \sum_{j' \in I_l \setminus \{j\}} \mathbb{E}(x_{j'} | K_t, x_i = 1) + \sum_{l \geq t'} \sum_{j' \in I_l} \mathbb{E}(x_{j'} | K_t, x_i = 1, x_j = 1) \right).$$

Looking at the terms in square brackets, allowing consumer  $i$  to act before others in cohort  $I_t$  is equivalent to replacing  $\mathbb{E}(x_i | K_t) < 1$  with 1, which is equivalent to replacing history  $K_t$  by  $K_t + 1$ . Hence by Lemma A.2 and Lemma A.3,  $\mathbb{E}(x_j | K_t, x_i = 1) > \mathbb{E}(x_j | K_t)$  for all consumers in cohorts  $t' \geq t$ . Moreover, now consumer  $i$  and all cohorts prior to him are singletons. Thus Lemma A.4 with  $t' = 1$  implies that  $\mathbb{E}(x_j)$  strictly increases for all consumers. Continuing in this way with every consumer  $i' \in I_t$ , then with every consumer in the next non-singleton cohort  $I_{t'}$  with  $t' \geq t$ , and so on gives the result.  $\square$

**Proposition 3.** *The sales scheme I that maximizes expected sales  $\sum_{1 \leq i \leq n} \mathbb{E}(x_i)$  has a single consumer per cohort, increasingly ordered in  $\lambda$ , i.e.  $\lambda_1 \leq \dots \leq \lambda_n$ .*

*Proof.* We prove the result directly. Consider two subsequent consumers:  $i$  and  $i+1$ . Suppose there were  $K$  consumers who bought before consumer  $i$ . Then

$$\mathbb{E}(x_{i+1} | K, x_i = 1) = \frac{\bar{\theta} - u_0 + \frac{\lambda_{i+1}}{n-1} \left( K + 1 + \sum_{j=i+2}^n \mathbb{E}(x_j | K, x_i = x_{i+1} = 1) \right)}{\bar{\theta} - \underline{\theta}},$$

$$\mathbb{E}(x_{i+1} | K, x_i = 0) = \frac{\bar{\theta} - u_0 + \frac{\lambda_{i+1}}{n-1} \left( K + \sum_{j=i+2}^n \mathbb{E}(x_j | K, x_i = 0, x_{i+1} = 1) \right)}{\bar{\theta} - \underline{\theta}}.$$

Now we look at consumer  $i$ , where

$$\mathbb{E}(x_i | K) = \frac{1}{\bar{\theta} - \underline{\theta}} \left( \bar{\theta} - u_0 + \frac{\lambda_i}{n-1} \left( K + \mathbb{P}(x_{i+1} = 1 | K, x_i = 1) \left( 1 + \sum_{j=i+2}^n \mathbb{E}(x_j | K, x_i = x_{i+1} = 1) \right) + \mathbb{P}(x_{i+1} = 0 | K, x_i = 1) \sum_{j=i+2}^n \mathbb{E}(x_j | K, x_i = 1, x_{i+1} = 0) \right) \right).$$

Clearly in our setting  $\mathbb{P}(x_{i+1} = 1 | h) = \mathbb{E}(x_{i+1} | h)$ , for any history  $h$ .

Now define:

$$\begin{aligned}
S(\lambda_i, \lambda_{i+1}) &= \sum_{j=i}^n \mathbb{E}(x_j | K) = \\
&\mathbb{P}(x_i = 1 | K) \left( \mathbb{P}(x_{i+1} = 1 | K, x_i = 1) \left( 2 + \sum_{j=i+2}^n \mathbb{E}(x_j | K, x_i = x_{i+1} = 1) \right) + \right. \\
&\quad \left. \mathbb{P}(x_{i+1} = 0 | K, x_i = 1) \left( 1 + \sum_{j=i+2}^n \mathbb{E}(x_j | K, x_i = 1, x_{i+1} = 0) \right) \right) + \\
&\mathbb{P}(x_i = 0 | K) \left( \mathbb{P}(x_{i+1} = 1 | K, x_i = 0) \left( 1 + \sum_{j=i+2}^n \mathbb{E}(x_j | K, x_i = 0, x_{i+1} = 1) \right) + \right. \\
&\quad \left. \mathbb{P}(x_{i+1} = 0 | K, x_i = 0) \sum_{j=i+2}^n \mathbb{E}(x_j | K, x_i = 0, x_{i+1} = 0) \right).
\end{aligned}$$

We now use the fact that all expectations are linear in prior sales:  $\mathbb{E}(x_j | K, x_i = 0, x_{i+1} = 1) = \mathbb{E}(x_j | K, x_i = 1, x_{i+1} = 0)$  and  $2\mathbb{E}(x_j | K, x_i = 1, x_{i+1} = 0) = \mathbb{E}(x_j | K, x_i = 0, x_{i+1} = 0) + \mathbb{E}(x_j | K, x_i = x_{i+1} = 1)$ . Thus

$$S(\lambda_i, \lambda_{i+1}) - S(\lambda_{i+1}, \lambda_i) = -\frac{(2 + Q_2 - Q_0)^3 (Q_2 + K + 1)(\lambda_i - \lambda_{i+1})\lambda_i\lambda_{i+1}}{8(n-1)^3(\bar{\theta} - \underline{\theta})^3}, \quad (7)$$

where

$$\begin{aligned}
Q_2 &= \sum_{j=i+2}^n \mathbb{E}(x_j | K, x_i = x_{i+1} = 1), \\
Q_0 &= \sum_{j=i+2}^n \mathbb{E}(x_j | K, x_i = x_{i+1} = 0).
\end{aligned}$$

Lemma A.2 implies that  $Q_2 > Q_0$ . Hence by (7), we have  $S(\lambda_i, \lambda_{i+1}) - S(\lambda_{i+1}, \lambda_i) > 0$  if and only if  $\lambda_i < \lambda_{i+1}$ . If  $\lambda_i > \lambda_{i+1}$ , then allowing consumer  $i + 1$  to act before consumer  $i$  will strictly increase  $\sum_{j=i}^n \mathbb{E}(x_j | K)$ , for any history  $K$ . Moreover, all  $i' < i$  are singletons. So by Lemma A.4, allowing consumer  $i + 1$  to act before consumer  $i$  will also strictly increase  $\sum_{1 \leq i \leq n} \mathbb{E}(x_i)$ . □

**Proposition 4.** *Suppose that at  $t = 0$ , consumers can simultaneously send a public message about their type,  $m \subseteq [\underline{\theta}, \bar{\theta}]$ . Then an equilibrium exists with full revelation,  $m_i = \theta_i$  for every  $i = 1, \dots, n$ , where all consumers truthfully reveal their type.*

*Proof.* For consumer  $i$ , define  $N_i$  as the smallest value of  $N$  for which

$$\theta_i + \frac{\lambda_i}{n-1}N \geq u_0.$$

$N_i$  is the minimum number of other consumers who would have to buy for consumer  $i$  to want to buy himself.

Each consumer's incentives depend only on the number of other consumers he expects to buy. Hence, without loss of generality, we can assume each consumer  $i$  sends a message about  $N_i$  rather than about  $\theta_i$ :  $m_i \subseteq \{0, \dots, n\}$ .

Consider a candidate equilibrium with full revelation:  $m_i = N_i$  for every  $i = 1, \dots, n$ . Define  $X_l$  as the number of messages  $m = l$  in this candidate equilibrium, for each  $l = 0, \dots, n$ . Define  $N_{max}$  as the maximum value of  $j$  such that  $\sum_{l=0}^j X_l \geq j + 1$ . We consider a candidate equilibrium where each consumer  $i$  buys if and only if  $(N_{max} - \mathbb{1}_{m_i \leq N_{max}}) \geq N_i$ .

By  $m_i = N_i$  for all  $i = 1, \dots, n$  and the definition of  $N_{max}$ , each consumer who buys receives payoff of at least  $u_0$ . Each consumer who does not want to buy receives payoff strictly less than  $u_0$  if he did. Hence any profitable deviation must include a change of message.

Suppose consumer  $i$  deviates by announcing  $m'_i = N_k \neq N_i$ . Let  $X'_l$  be the number of messages  $m = l$  following this deviation, for each  $l = 0, \dots, n$ . We have  $X'_{N_i} = X_{N_i} - 1$ ,  $X'_{N_k} = X_{N_k} + 1$ , and  $X'_l = X_l$  for all  $l \neq N_i, N_k$ . Define  $N'_{max}$  as the maximum value of  $j$  such that  $\sum_{l=0}^j X'_l \geq j + 1$ .

We know that  $\sum_{l=0}^j X'_l \leq \sum_{l=0}^j X_l$  for all  $j \geq N_i$ . Hence  $N'_{max} > N_{max}$  can only hold if  $N'_{max} < N_i$ . The condition  $N'_{max} > N_{max}$  is necessary for the deviation to be profitable, since the number of consumers other than  $i$  who buy must increase. But the condition  $N'_{max} \geq N_i$  is also necessary, since  $(N'_{max} - \mathbb{1}_{m'_i \leq N'_{max}}) \geq N_i$  must hold for consumer  $i$  to buy following the deviation. Hence the deviation cannot be profitable. □

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