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Abstract

What explains the persistence of unemployment? The literature on hysteresis, which is based on unit root testing in autoregressive models, consists of a vast number of univariate studies, i.e. that analyze unemployment series in isolation, but few multivariate analyses that focus on the *sources* of hysteresis. As a result, this question remains largely unanswered. This paper presents a multivariate econometric framework for analyzing hysteresis, which allows one to test different hypotheses about non-stationarity of unemployment against one another. For example, whether this is due to a persistently changing equilibrium, slow adjustment towards the equilibrium (persistent fluctuations), or perhaps even a combination of the two. Different hypotheses of slow adjustment, as implied by theories of hysteresis, nominal rigidities or labor hoarding can also be compared. A small illustrative application to UK quarterly data on prices, wages, output, unemployment and crude oil prices, suggests that, for the period 1988 up to the onset of the financial crisis, the non-stationarity of UK unemployment cannot be explained as a result of slow adjustment, including sluggish wage formation as emphasized by the hysteresis theories. Instead, it is the equilibrium that has evolved persistently as a consequence of exogenous oil prices shifting the price setting relation (in the unemployment-real wage space) in a non-stationary manner.

JEL: C1, C32, E00, E24.

Keywords: Hysteresis, Unemployment Hysteresis, Persistence, Cointegration, Structural VAR, Equilibrium unemployment, Multivariate Time series analysis, Price- and Wage Setting, Wage formation, Crude oil prices, UK unemployment.

1 Introduction

In the wake of the European unemployment experience from the 1970s to the mid-80s, economists began emphasizing that high unemployment tends to persist - a phenomenon they dubbed *hysteresis* (Blanchard and Summers 1986).[1,2] Although the interest in unemployment hysteresis probably peaked in the late 1980s/early 90s (Røed 1997) the recent economic crisis has brought this important topic into focus again. A few examples of this are Ball (2009), Andersen (2010), Amable and Mayhew (2011), O'Shaughnessy (2011), and Delong and Summers (2012). However, some of these economists stress that, in spite of a vast empirical literature, our knowledge about the mechanisms underlying hysteresis remains limited. To quote Ball (2009), "..hysteresis is an important phenomenon, but one that is not well understood. This means more research is needed." Other examples are Andersen (2010) and Delong and Summers (2012).

So what may explain our ignorance when it comes to hysteresis? At least part of this can be attributed to the fact that the empirical literature has, by far, been dominated by *univariate* econometric studies, i.e. which analyze time series of unemployment in isolation.[3] In particular, following Blanchard and Summers (1986) the vast majority of empirical papers have been based on univariate Dickey-Fuller type tests for unit roots in autoregressive models (see e.g. the surveys in Røed (1997), and more recently, Gustavsson and Österholm (2009) and Arestis and Sawyer (2009).[4] In this literature evidence of hysteresis in the form of unit root non-stationarity in unemployment is typically being associated with mechanisms that imply sluggishness in wage formation such as those suggested by Blanchard and Summers (1986) and Lindbeck and Snower (1986). In contrast, when unit root tests reject in favor of stationarity this is taken as evidence of a constant equilibrium (i.e. *Natural*) rate of unemployment (often conditional on a few level breaks).[5]

Such a reasoning is, however, too coarse, and indeed, by modeling unemployment in

[1] I would like to thank Christian Groth, Geraldine Henningsen, Søren Johansen and Diana Framroze Møller.

[2] This paper is about "unit root hysteresis" to be distinguished from "genuine hysteresis" (see Amable, Henry, Lordon, and Topol 1995 and Göcke 2002). Here, hysteresis (or persistence) implies that the characteristic roots in linear (V)AR models are close to but not necessarily equal to one (with moduli always outside the unit circle). However, in the empirical model exact unit roots are imposed and thus regarded as approximations. This interpretation is thus similar to that in Blanchard and Summers (1986) (see their footnote 1). Note that, some later studies define (unit root) hysteresis more precisely by the condition that *transitory* impulses have *permanent* effects (see e.g. Røed 1997 and Jacobson, Vredin, and Warne 1997). This will be made clear below. For excellent surveys on hysteresis, see Røed (1997), and more recently, O'Shaughnessy (2011).

[3] Below, I discuss the multivariate studies.

[4] Over the years, many different approaches have been developed, but most of them are still rooted in the conventional Dickey-Fuller type tests (Dickey and Fuller 1981), with the more recent studies focusing on increasing the power of such tests, using e.g. panel data models (De Lee, Lee, and Chang (2009), p. 326).

[5] An example of this view is found in Camarero and Tamarit (2004).

isolation, univariate methods provide no foundation on which to draw such conclusions. More generally, they offer no way of distinguishing between different causes of unemployment hysteresis (unit root non-stationarity). And there are many distinctions to be made: Clearly, hysteresis may be explained by many other mechanisms than those suggested by any existing theory, but even within the realm of mainstream macroeconomic reasoning, there are several possibilities. Unemployment may display hysteresis because it *adjusts slowly* (jointly with other endogenous variables) to changes in its underlying exogenous determinants. In contrast, unemployment may also be hysteretic due to a *slowly moving exogenous determinant of equilibrium unemployment*, even though it may in fact be quick to adjust to changes in such a determinant. Moreover, whereas sluggish wage formation, i.e. wages responding slowly to unemployment, provides an explanation why unemployment adjusts slowly towards the equilibrium, there can be several other reasons for this. In general, slow adjustment is the eventual result of a slow interaction between goods-, financial- and labor markets, an interaction that is also influenced by other *rigidities.* Well-known examples of the latter could be related to menu costs, sticky wages and labor hoarding, but costs of adjusting production to accommodate demand (e.g. Danziger 2008) could also play a role. Finally, the matter is even more subtle, since when hysteresis is not supported empirically, that is when a unit root test rejects, this is not necessarily inconsistent with theoretical hypotheses of unemployment hysteresis: For example, in open economies, other stabilizing mechanisms, such as real wage resistance, may be present and may dominate the destabilization caused by hysteresis generating mechanisms (see e.g. Carlin and Soskice 2006). Hence, in practice, finding unit root non-stationarity of unemployment is not necessarily evidence of ill-functioning labor markets with sluggish wage formation, and stationarity is not sufficient to reject the mechanisms suggested by hysteresis theories in favor of the Natural Rate theories.

Taking such considerations and distinctions into account is of utmost importance, since different causes of hysteresis point to different policies.[6] To a large extent macroeconomists are aware of this and the inadequacy of univariate methods in this respect has indeed been emphasized previously: An example is Andersen (2010), when referring to univariate measures of unemployment persistence in general, stating that, "They do not clearly separate the role of shocks and their persistence from the persistence generating mechanisms arising from sluggish adjustment of various forms".

To make progress in terms of understanding the *sources* of unemployment hysteresis we thus need to take its systemic nature seriously in applied empirical work, and, as a result, the adoption of multivariate econometric methods is inevitable. This is not a novel insight, and indeed, the system aspect of unemployment hysteresis has been emphasized before (see e.g. Amable, Henry, Lordon, and Topol 1995 and Göcke 2002). Moreover,

[6] For discussions of policy implications of hysteresis, see e.g. Røed (1997), Andersen (2010), O'Shaughnessy (2011) and Delong and Summers (2012).

concrete multivariate econometric analyses do exist (see Jacobson, Vredin, and Warne 1997 and Dolado and Jimeno 1997).[7] However, while the latter two studies are clearly superior relative to univariate-based studies, when it comes to gaining insights into the sources of hysteresis, their focus is different from that of the present paper.[8] Here, the overall purpose is to present a multivariate econometric framework for understanding the causes and nature of unit root non-stationarity of unemployment. This framework allows us to systematically incorporate distinctions such as the abovementioned into the econometric analysis. In particular, it allows one to test whether unemployment is hysteretic because it adjusts slowly towards the equilibrium, whether this is because the equilibrium itself moves in a persistent manner, or even if a combination of the two can be supported. The studies of Jacobson et al. (1997) and Dolado and Jimeno (1997) are mainly concerned with testing different hypotheses of what I here denote *slow adjustment*, i.e. hysteresis which has the property that even *transitory* shocks have *permanent* effects (see Footnote 2 and Section 5).[9]

To make my framework relevant for applications based on macroeconomic theory, it builds directly on a representative *structural model,* i.e. a linear Structural VAR (SVAR) model.[10] The SVAR is sufficiently general for formulating different hypotheses about hysteresis, in order to address the abovementioned distinctions, within the same model. This implies a statistical analysis that allows one to evaluate the empirical performance of these hypotheses on an equal footing, something that greatly facilitates learning about hysteresis from the data.[11]

The framework is unfolded in three steps. In the first step I present an overall conceptual framework for characterizing unemployment hysteresis by its source in a multivariate setting. This builds directly on a general SVAR with exogenous variables. The endogenous-exogenous dichotomy implies that one can analyze hysteresis in the full system of variables, and thus unemployment, as coming from two overall sources: It can come from the sub-system of exogenous variables, for example from a persistently evolving exogenous determinant of equilibrium unemployment, or it can come from sluggish (slow) adjustment in the sub-system of endogenous variables for fixed values of the exogenous variables. The theories by Blanchard and Summers (1986) and Lindbeck and Snower (1986), let alone the abovementioned rigidities, are examples of the latter

[7] See also Hansen and Warne (2001) which build on Jacobson et al. (1997).

[8] Their econometric approach is also different from that adopted here. I discuss the differences of the present study relative to Jacobson, Vredin, and Warne 1997 in Section 5.

[9] Section 2 places these analyses into the context of the present framework. In Section 5, I further compare these studies to my analysis.

[10] In this paper a *linear Structural VAR model* refers to a log-linear model to be interpreted as a log-linear approximation of an underlying stable system of non-log-linear difference equations. Note that, Jacobson et al. (1997) and Dolado and Jimeno (1997), also use SVARs although they adopt a common trends approach (see Section 5).

[11] In other words, all hypotheses about hysteresis translate into parameter restrictions i.e. sub-models which are all nested in the same statistical model (the Unrestricted VAR). See Section 4.2.

source. As in the univariate tradition, hysteresis corresponds to characteristic roots close to unity, and thus, in the empirical implementation a given hysteresis hypothesis will be tested statistically as parameter restrictions implying *exact* unity roots.[12] This amounts to testing a specific restricted Cointegrated VAR (CVAR). The division into the two abovementioned sub-systems therefore leads to a two-dimensional *taxonomy* of multivariate tests, i.e. classes of CVARs, corresponding to different hysteresis hypotheses. The taxonomy is refined with respect to the "degree of hysteresis" which can be more or less pronounced. Here, the focus is on hysteresis that can be described as I(1) - or I(2) - non-stationary processes, which together cover most of the empirically relevant cases (see Johansen 1996). This taxonomy systematically categorizes all potential (classes of) reasons for unit root non-stationarity of unemployment, thereby placing both the theoretical models of hysteresis and the analyses of Jacobson et al. (1997) and Dolado and Jimeno (1997) into a well-defined context. This taxonomy structures the rest of the analysis, but, hopefully, it will also be useful for macroeconomists by bridging the theory side, i.e. various hysteresis models, theories of real and nominal rigidities etc., with the empirical side, i.e. unit root non-stationarity of unemployment in a general (systemic) setting.

In the second step I elaborate by deriving the exact form of the CVARs from the taxonomy. I focus on the cases for which the two abovementioned sub-systems involve at most I(1) processes. These restricted CVARs are relatively general, in that they are represented in terms of a set of block matrices (based on the endogenous-exogenous dichotomy), but can nevertheless still be implemented directly in empirical analyses. Hence, some of them correspond to sluggish mutual adjustment of the endogenous variables (given the exogenous variables), some correspond to a slowly moving exogenous determinants of equilibrium unemployment, while others correspond to a combination of these two possibilities. An advantage of this general block matrix formulation is that it can be applied without knowing the exact form of the structural model: It is sufficient to know which variables are exogenous (according to the theory) and which are endogenous. This approach could, for example, be used to conclude that hysteresis comes from the exogenous variables and hence has nothing to do with rigidities and/or wage formation-based explanations (i.e. slow adjustment). In this part I also analyze the corresponding Moving Average (MA) representations of the CVARs in order to derive, what may be termed, the *hysteresis equation.* This is simply an expression for unemployment that shows from which variables in the system unemployment gets its non-stationarity or hysteresis from and how.

Finally, in the third step, I elaborate further by exemplifying some of the central ideas. I analyze quarterly data (1988-2006) for the UK economy based on a SVAR with

[12] See Footnote 2.

four endogenous variables: prices, wages, output and unemployment, and one exogenous variable, the price of crude oil. The latter is supposed to influence equilibrium unemployment via price setting (see Section 4.1). This SVAR is meant primarily as a simple illustration, but is still general enough to illustrate many of the abovementioned distinctions by providing a set of different parameter restrictions, or CVARs, each corresponding to fundamentally different types of hysteresis. In particular, it allows a statistical test of whether the non-stationarity of UK unemployment for this period is due to slow adjustment or rather a slowly moving exogenous determinant of equilibrium unemployment (i.e. oil prices). Interestingly, I find evidence in favor of the latter, whereas for these data, hysteresis does not seem to result from slow adjustment, and in particular, not the kind related to sluggish wage formation.

Based on a general SVAR, which represents a wide range of applied macroeconomic models, Section 2 takes the birds-eye-view on the sources of unemployment hysteresis and ends with outlining the two-dimensional taxonomy of multivariate hysteresis tests. In Section 3, I derive the classes of the block matrix CVARs corresponding to the taxonomy, while Section 4 considers the five-variable SVAR for the UK economy. Section 5 compares the analysis to Jacobson et al. (1997) and Mosconi and Giannini (1992), and finally, concluding remarks and perspectives are given in Section 6.

2 A systemic taxonomy for characterizing hysteresis

This section presents a taxonomy that characterizes hysteresis in two dimensions - i.e. with respect to its source and its degree. In a multivariate setup there can be two *pure* sources leading to unemployment hysteresis: It may arise from sluggishness in the mutual adjustment in the endogenous variables or it may arise from slowly evolving exogenous variables that influence equilibrium unemployment. The degree can be more or less pronounced and here the focus is on hysteresis that can be described in terms of I(1) or I(2) non-stationary stochastic processes (see Johansen 1996 and Section 3).

The taxonomy is based on a Structural VAR (SVAR) with two lags.[13,14] A SVAR is sufficiently general to represent a wide range of generic economic models. It is based on a *theoretical model* that determines which variables are endogenous, x_{1t} $(p_1 \times 1)$, and which are exogenous, x_{2t} $(p_2 \times 1)$, and may be written in its Structural Error-Correction-

[13] All results generalize straigthforwardly with more lags.

[14] This and the next section build on the econometric analyses of Davidson and Hall (1991) and Mosconi and Giannini (1992).

Mechanism (ECM) form as,

$$
\begin{aligned}
A_{11}\Delta x_{1t} &= -A_{12}\Delta x_{2t} + m_{1t} + F_{11}x_{1t-1} + F_{12}x_{2t-1} - C_{11}\Delta x_{1t-1} - C_{12}\Delta x_{2t-1} + \varepsilon_{1t}, \\
A_{22}\Delta x_{2t} &= m_{2t} + F_{22}x_{2t-1} - C_{22}\Delta x_{2t-1} + \varepsilon_{2t},
\end{aligned}
\tag{1}
$$

with A_{11} $(p_1 \times p_1)$ and A_{22} $(p_2 \times p_2)$ with ones on the diagonal, m_{it} are deterministic terms and $(\varepsilon_{1t}', \varepsilon_{1t}')' \sim N(0, \Sigma)$, Σ being positive definite and diagonal. The theoretical model specifies the equations for x_{1t}, in particular, the parameter matrices, F_{11} and F_{12}. In addition to unemployment, x_{1t} could for example include prices, wages, output, long-term unemployment etc., whereas the exogenous variables may comprise unemployment benefits, interest rates, exchange rates, foreign prices /output etc.. The corresponding reduced form is,

$$
\begin{aligned}
\Delta x_{1t} &= \mu_{1t} + \Pi_{11}x_{1t-1} + \Pi_{12}x_{2t-1} + \Gamma_{11}\Delta x_{1t-1} + \Gamma_{12}\Delta x_{2t-1} + \upsilon_{1t}, \\
\Delta x_{2t} &= \mu_{2t} + \Pi_{22}x_{2t-1} + \Gamma_{22}\Delta x_{2t-1} + \upsilon_{2t},
\end{aligned}
\tag{2}
$$

where the parameters as functions of the structural parameters in (1) are given in Appendix A.1.

The theoretical model can in most cases be thought of as a model for the endogenous variables *given* the exogenous variables, and it is assumed that this *conditional* model has a stable steady state. Although a theoretical model by construction does not explain how the exogenous variables have been generated, it is assumed here that these variables are determined in another larger system that also has a stable steady state. The result of these equilibrium assumptions is that the SVAR has a stable equilibrium, and thus that *all* roots, $z \in \mathbb{C}$, of the characteristic equation corresponding to (2) have modulus, $|z| > 1$. However, in the present context hysteresis means that some of these roots are *close* to $z = 1$.[15]

Due to the endogenous-exogenous dichotomy the characteristic equation splits into a product of two polynomial,

$$
\det(A_1(z)) \det(A_2(z)) = 0, \tag{3}
$$

where, $A_1(z) \equiv I_{p_1}(1-z) - \Pi_{11}z - \Gamma_{11}(1-z)z$ and $A_2(z) \equiv I_{p_2}(1-z) - \Pi_{22}z - \Gamma_{22}(1-z)z$, $z \in \mathbb{C}$. Hence, for the given structural model, *all* roots corresponding to the *full* system, and hence, *all* potential sources of unemployment hysteresis, can be analyzed based on the union of the roots, respectively of $\det(A_1(z)) = 0$ and $\det(A_2(z)) = 0$.

The dichotomous partitioning in (3) reflects that one can think of the full process x_t, in particular the endogenous variables, as being related to two processes: The first may

[15] Note that, this interpretation of the concept, hysteresis, corresponds to that in Blanchard and Summers (1986). In practice, "close" should be interpreted as "not significantly different from unity".

be termed the *counterfactual* x_1-process (denoted x_{1t}^c henceforth). This is the process that would result if, counterfactually, *the x_2-process, were to be fixed at some level.* The second process is simply the exogenous x_2-process given by the second block line in (2).

The counterfactual process can be written as,

$$\Delta x_{1t}^c = \Pi_{11} x_{1t-1} + \Gamma_{11} \Delta x_{1t-1} + \mathcal{D}_{1t} + \upsilon_{1t}, \tag{4}$$

where $\mathcal{D}_{1t}$ is constant (see 2), and from which it appears that $A_1(z)$ in (3) is the corresponding characteristic polynomial (Mosconi and Giannini 1992). The other polynomial, $A_2(z)$, is the characteristic polynomial corresponding to the exogenous x_2-process.

These two processes or subsystems thus correspond to the two pure sources of hysteresis in the system of the endogenous variables and thus in unemployment: That is, roots of $A_1(z)$ close to 1 reflect slow mutual adjustment (towards steady state) between the endogenous variables. If no other roots of (3) are close to unity, this can be referred to as the case of (pure) *slow adjustment.* In contrast, if the only roots of (3) that are close to 1 are roots of $A_2(z)$, this mirrors sluggishly evolving exogenous variables. In the context of unemployment the latter type of hysteresis may be denoted (pure) *equilibrium hysteresis*, to signify that the persistence of the endogenous variables originates from slowly evolving exogenous determinants of equilibrium unemployment.[16]

Thus, this simple representation shows that, for a given structural model, hysteresis in the system can arise for three reasons: Slow adjustment, equilibrium hysteresis, or a combination.[17] Hysteresis in the system is necessary for hysteresis in unemployment but whether it translates into this depends on the parameters of the model (see below).

Analogously to the univariate Dickey-Fuller approach, the empirical tests for hysteresis are based on *approximating* the roots that are close to unity by exact unit roots, i.e. $z = 1$.[18] This means that the variables are integrated, i.e. I(d), d being the order of integration, and in the multivariate setting cointegration may arise. For a vast majority of macroeconomic applications it suffices to consider processes for which d is at most 2. Thus, for both processes, x_{1t}^c and x_{2t}, corresponding to an increasing degree of persistence/hysteresis, I(0), i.e. no or little persistence, I(1) and I(2) are considered, respectively.

This leads us to the overall *taxonomy* shown in Table 1 for classifying statistical hypotheses of hysteresis in the full system and thus potentially in unemployment. As the table shows, even when restricting the focus to at most I(2) processes, there are

[16] Another term could be *exogenous hysteresis*, which would be more precise since the equilibrium may depend on endogenous variables and the related hysteresis mechanism would in that case belong under "slow adjustment".

[17] This has previously been noted for example in Davidson and Hall (1991).

[18] For another example for which the imposition of cointegration/exact unit roots is also explicitly regarded as an approximation of a highly persistent process, see Møller and Sharp (2013).

already eight *cases* of hysteresis, each corresponding to a class of testable parameter restrictions, that is, restricted CVARs.

Table 1: A systemic taxonomy for hysteresis hypotheses (for the full system). Each case represents a class of restricted CVARs corresponding to specific hypotheses of hysteresis.

	Counterfactual sub-system, x_1^c		
Exo. sub-system, x_2	**I(0)**	**I(1)**	**I(2)**
I(0)	No hysteresis (stationarity)	**I** Pure slow adjustment, I(1)	**VIII** Pure slow adjustment, I(2)
I(1)	**II** Pure equil. hyst., I(1)	**III**	**VII**
I(2)	**IV** Pure equil. hyst., I(2)	**V**	**VI**

The taxonomy should clarify the relation between non-stationarity or unit roots on the one hand and hysteresis as implied by the theoretical models on the other. For example, the Cases II and IV will generally imply a unit root in unemployment but do not concern the restrictions implied by hysteresis theories, or any other hypothesis of slow adjustment. In contrast, all other cases except these two, involve slow adjustment, and in Cases I and VIII, this is the only source for hysteresis in unemployment.[19] In other words these classes of CVARs are compatible with the theories suggested by Blanchard and Summers (1986) and Lindbeck and Snower (1986), but, as mentioned, also models with various sorts of rigidities such as, menu costs, labor hoarding, costs of adjusting production to accommodate demand (e.g. Danziger 2008) and real wage resistance etc. (Layard, Nickell, and Jackman 2005 and Carlin and Soskice 2006), are consistent with CVARs belonging in this group. In contrast, an example of equilibrium hysteresis, whose pure form corresponds to Cases II and IV, could be a persistently evolving replacement ratio, which is often assumed to be an exogenous determinant of the natural rate of unemployment. In general, a key distinguishing property between the CVAR classes corresponding to the pure cases of equilibrium hysteresis and slow adjustment, respectively, is that, only in the latter case, even temporary changes in the exogenous variables and shocks to the equations will have permanent influence, i.e. in practice "long after they have disappeared". This is often what economists mean by hysteresis (see e.g. Røed 1997, Papell, Murray, and Ghiblawi 2000 and Göcke 2002 and references therein). This is also what the studies, Jacobson et al. (1997) and Dolado and Jimeno (1997), are about. In particular, the restrictions they analyze can be related to the second column in the table (see also Footnote 2 and Section 5).

When evidence of unit roots is found based on univariate autoregressive models

[19] In the terminology of Davidson and Hall (1991), all cases except II and IV are denoted as *unstable models.*

of unemployment, there is nothing in the empirical analysis to suggest which of all these cases applies: Neither, is it possible to find out whether the evidence favors pure equilibrium hysteresis or pure slow adjustment: Nor is it possible to discriminate between the various sub-cases of the latter type of persistence.[20] This is a major drawback as these different cases and sub-cases are fundamentally different and thus have radically different policy implications. For example, if the evidence is in favor of slow adjustment related to wage formation this points towards labor market policies, whereas the finding of a persistently evolving exogenous foreign price level (e.g. oil prices) affecting equilibrium unemployment has no immediate policy implications. Moreover, hysteresis effects may also play an important role in improving the efficacy and sustainability of fiscal policy (see Delong and Summers 2012). In contrast, the advantage of a multivariate approach is that for each possible explanation of hysteresis it gives a particular set of parameter restrictions in the form of a restricted CVAR (see below). Moreover, each of all these CVARs are sub-models (i.e. nested) in the same overall statistical model, i.e. the unrestricted VAR. This makes it straightforward to compare the empirical performance of the various explanations, something which is necessary in learning about the causes and nature of unemployment hysteresis.

3 Statistical hypotheses of unemployment hysteresis

This section elaborates on the taxonomy in Table 1 by deriving the general parameter restrictions, i.e. the classes of CVARs (in block-matrix notation), that correspond to the various *cases*. As mentioned although fairly general these can still be used directly in empirical applications. I focus on Cases I-III for which the counterfactual and the exogenous processes are at most I(1). That is, I use the assumptions about x_{1t}^c and x_{2t} that effectively define each of these three cases to derive the general cointegration restrictions for the full process (i.e. the forms of the CVAR parameter matrices, α and β as shown below). From this, the corresponding Structural MA representation can be derived in order to determine why and which variables make unemployment become hysteretic. The details of the computations are found in Appendices A.1, A.2 and A.3 and for the more general technical details the reader is referred to Johansen (1996) and Abadir and Magnus (2005).

[20] Finally, as is often the most realistic, both persistently evolving exogenous variables and sluggish adjustment in the endogenous variables contribute simultaneously to hysteresis making the univariate approach even more inadequate.

3.1 The preliminaries

Define the full process by $x_t \equiv (x'_{1t}, x'_{2t})'$. The block matrices of the SVAR in (1) can then be collected accordingly to yield,

$$A\Delta x_t = m_t + Fx_{t-1} - C\Delta x_{t-1} + \varepsilon_t, \tag{5}$$

with the reduced form corresponding to (2),

$$\Delta x_t = \mu_t + \Pi x_{t-1} + \Gamma_1 \Delta x_{t-1} + \upsilon_t, \tag{6}$$

where $\mu_t \equiv A^{-1}m_t, \Pi \equiv A^{-1}F$, $\Gamma_1 \equiv -A^{-1}C$ and $\upsilon_t \equiv A^{-1}\varepsilon_t$ (see Appendix A.1). The characteristic polynomial is $A(z) \equiv I(1-z) - \Pi z - \Gamma_1(1-z)z$, $z \in \mathbb{C}$. When the *exact* unit roots (i.e. cointegration) are imposed for x^c_{1t} and/or x_{2t}, i.e. when parameter restrictions are imposed such that, $\det(A_1(1)) = 0$ and/or $\det(A_2(1)) = 0$, it follows from (3) that $\det(A(1)) = 0$, implying that Π in (6) has reduced rank, $r < p$, where $p \equiv p_1 + p_2$. This is parameterized as,

$$\Pi = \alpha\beta', \tag{7}$$

where the matrices α and β are $p \times r$ of rank r, with $0 \leq r < p$. Since it is assumed, for both x^c_{1t} and x_{2t}, that the remaining roots are all outside the unit disc (i.e. $|z| > 1$), it follows that (6) with (7) imposed is a Cointegrated VAR generating I(1) variables if and only if $\det(\alpha'_\perp(I - \Gamma_1)\beta_\perp) \neq 0$, where $\alpha_\perp$ and $\beta_\perp$ (both $p \times p - r$) are the respective orthogonal complements of α and β.

To elucidate the sources of hysteresis consider the Structural Moving Average (SMA) representation. This is given by,

$$x_t = \mathcal{C}\sum_{i=1}^{t} \varepsilon_i + \mathcal{X}_t, \tag{8}$$

where $\mathcal{C} \equiv \beta_\perp(\alpha'_\perp(I - \Gamma_1)\beta_\perp)^{-1}\alpha'_\perp A^{-1}$, $\mathcal{X}_t \equiv \mathcal{C}\sum_{i=1}^{t} m_i + \mathcal{C}(L)(\varepsilon_t + m_t) + Z_0$, with $\mathcal{C}(L)$, a convergent lag-polynomial, capturing the transitory influence from the stochastic ε-shocks and Z_0 depending on the initial values (see Johansen 1996). Premultiplying (8) with e_u, the unit vector picking out unemployment, gives,

$$u_t = \mathcal{U}_t + \gamma' ST_t, \tag{9}$$

where,

$$\mathcal{U}_t \equiv e'_u \mathcal{X}_t, \;\; \gamma' \equiv e'_u \mathcal{C} \tag{10}$$

and $ST_t \equiv (ST_{t,1}, ST_{t,2}, .., ST_{t,p})'$ with $ST_{t,j} \equiv \sum_{i=1}^{t} \varepsilon_{ij}$ for $j = 1, .., p$, being the *Stochastic Trend* arising from the cumulation of the shocks to variable j in the system.

Equation (9) is key to understanding the nature and origin of unemployment hysteresis, and will be referred to as the *hysteresis equation.* In particular, whereas the term, $\mathcal{U}_t$, captures the initial values, deterministic components and *transitory* stochastic influences, it is the stochastic trend term, $\gamma' ST_t$, that is the focus here since this is what describes the persistent stochastic movements in unemployment, i.e. hysteresis. From (10) it is clear that, for a given structural model, the multivariate approach fully maps out from which variables and how unemployment gets its stochastic trend or hysteretic behavior. Specifically, the γ vector of weights is fully specified by the SVAR parameters via Γ_1, A, $\alpha_\perp$ and $\beta_\perp$, where the latter two matrices can be derived from α and β (Johansen (1996), p. 48) that contain the structural parameters in Π which remain after the imposition of unit roots.

In contrast to this detailed characterization of unemployment hysteresis is the univariate approach. For the illustration it suffices to consider an AR(1) process, in which case (9) reduces to,

$$u_t = u_0 + ST_t, \tag{11}$$

where u_0 is the initial value and $ST_t \equiv \sum_{i=1}^{t} \varepsilon_{ui}$ is the stochastic trend. Clearly, in contrast to (9), although the stochastic trend gives a relevant statistical description of hysteresis, (11) cannot tell which other macroeconomic variables this comes from.

The I(1) analysis above illustrates the essentials. If $\det(\alpha'_\perp(I - \Gamma_1)\beta_\perp) = 0$, and a further full rank condition holds, I(2) arises (see Johansen (1996), p. 58). In this case the hysteresis equation now involves stochastic trends of the second order, i.e. twice cumulated shocks, $\sum_{s=1}^{t}\sum_{i=1}^{s}\varepsilon_i$, as well as the first order trends as in the I(1) case, $\sum_{i=1}^{t}\varepsilon_i$ (see below). As mentioned these second order trends resemble the I(1) trends but simply describe an even more persistent and smooth (stochastic) component in the series. As shown below, I(2) in the full system will arise in Case III$_b$.

Now, based on the CVARs for x_{1t}^c and x_{2t}, by using block-matrix notation, the purpose is now to derive α and β in (7) for each of the cases I-III. This will provide our testable hysteresis restrictions and the implied hysteresis equation corresponding to (9) and (10), for the full process. For all cases it holds that for the CVARs for x_{1t}^c and x_{2t}, there are no roots inside the complex unit disc. Second, recalling (2) it follows that for all cases the full system Π matrix is,

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ 0 & \Pi_{22} \end{pmatrix}, \tag{12}$$

where the blocks are defined in Appendix A.1. Using similar notation as for the full system, defining $r_i \equiv r(\Pi_{ii})$, then whenever $r_i \leq p_i$, this is parameterized as $\Pi_{ii} = \alpha_{ii}\beta'_{ii}$ with α_{ii} and β_{ii} $(p_i \times r_i)$ of rank, r_i, for $i = 1, 2$. For $r_i = p_i$, $\alpha_{ii} = \Pi_{ii}$ and $\beta_{ii} = I_{p_i}$. The respective $p_i \times (p_i - r_i)$ orthogonal complements are denoted $\alpha_{ii\perp}$ and $\beta_{ii\perp}$. Since $\Pi_{11} = A_{11}^{-1}F_{11}$ and A_{11}^{-1} is non-singular, a reduced rank restriction on Π_{11} is a reduced rank restriction on the structural parameter F_{11}, which therefore can be decomposed as

$F_{11} = \delta_{11}\lambda'_{11}$, where δ_{11} and λ_{11} are $p_1 \times r_1$ of rank, r_1. The latter two matrices will contain the structural parameters of F_{11} that remain after the imposition of the reduced rank (unit root) restriction. In particular, as opposed to using the reduced form notation, $\Pi_{11} = \alpha_{11}\beta'_{11}$, I will use this notation when relevant in order to state the estimable cointegration parameters in terms of the structural parameters only. The orthogonal complements of δ_{11} and λ_{11} (β_{11}) can be denoted as $\delta_{11\perp}$ and $\lambda_{11\perp}$, respectively, and these will also be functions of the structural parameters only. Since the "structural" parameters for the exogenous process are not important (second block in (1)) I will use a similar notation (i.e. $F_{22} = \delta_{22}\lambda'_{22}$ etc.) only as long as it simplifies the exposition.

In all of the considered cases at least one of the processes, x^c_{1t} and x_{2t}, generates hysteresis implying that either $r_1 < p_1$ or $r_2 < p_2$ or both. For the case of pure slow adjustment (Case I), by construction it holds that $r_2 = p_2$ and $r_1 < p_1$, while the opposite holds for pure equilibrium hysteresis (Case II), i.e. $r_1 = p_1$ and $r_2 < p_2$. In either of the pure cases it thus follows that the rank of Π (and α and β), r, equals $r_1 + r_2$.[21] When combining slow adjustment with equilibrium hysteresis (i.e. when $r_1 < p_1$ and $r_2 < p_2$) there is no non-singular block on the diagonal of Π, and the rank can in this case be computed as shown in Appendix A.2 which implies,

$$r = r_1 + r_2 + s, \tag{13}$$

where, $s = r(\delta'_{11\perp}F_{12}\beta_{22\perp})$. Hence, this applies for Case III.[22]

Now, as for the full system, the assumptions about the rank of $\alpha'_{ii\perp}(I - \Gamma_{ii})\beta_{ii\perp}$ and, in the event this is reduced a further rank condition (see below) will determine exactly which case applies.

3.2 Slow Adjustment (Case I)

In this case the roots of $A_1(z)$ that are close to one are approximated by $z = 1$. There are no roots of $A_2(z)$ that are close to 1, i.e. $r_2 = p_2$, and hence, $r = r_1 + p_2$, with $0 \leq r_1 < p_1$. The I(1) condition holds for x^c_1, that is, $r(\alpha'_{11\perp}(I - \Gamma_{11})\beta_{11\perp}) = p_1 - r_1$.

For $0 < r_1 < p_1$, α and β in (7) are given by,

$$\alpha = \begin{pmatrix} A_{11}^{-1}\delta_{11} & \Pi_{12} \\ 0 & \Pi_{22} \end{pmatrix}, \beta = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & I_{p_2} \end{pmatrix}, \tag{14}$$

where Π_{12} and Π_{22} are defined in Appendix A.1, and where, as mentioned above, Π_{11} (in 2) has been decomposed as $\Pi_{11} = \alpha_{11}\beta'_{11}$, with $\alpha_{11} \equiv A_{11}^{-1}\delta_{11}$, $\beta_{11} \equiv \lambda_{11}$, where δ_{11} and

[21] This is because in these cases there is a non-singular block on the diagonal in (12) so that the rank of Π is simply the sum of the ranks of the diagonal matrices.

[22] More generally, the latter condition applies also to Cases V, VI and VII, whereas, similar to Cases I and II, for Cases IV and VIII it holds that $r = r_1 + r_2$.

λ_{11} (both $p_1 \times r_1$ of rank, r_1) depend exclusively on the structural long-run parameter matrix, F_{11} (see Appendix A.3.1).

Partitioning $ST_t = \sum_{i=1}^{t} \varepsilon_i$, as $(ST'_{1t}, ST'_{2t})' \equiv ((\sum_{i=1}^{t} \varepsilon_{1i})', (\sum_{i=1}^{t} \varepsilon_{2i})')'$ and e_u as $e_u = (e'_{1u}, 0')'$, e_{1u} being $p_1 \times 1$, it follows that the hysteresis equation will have the form,

$$u_t = \mathcal{U}_t + \gamma'_1 ST_{1t} + \gamma'_2 ST_{2t}, \tag{15}$$

where $\gamma'_1 \equiv e'_{1u}\mathcal{C}_1, \gamma'_2 \equiv -e'_{1u}\mathcal{C}_1 F_{12}F_{22}^{-1}$ and $\mathcal{C}_1 \equiv \lambda_{11\perp}(\delta'_{11\perp}A_{11}(I_{p_1} - \Gamma_{11})\lambda_{11\perp})^{-1}\delta'_{11\perp}$ is the structural long-run impact matrix corresponding to the counterfactual x_1 process (See Appendix A.3.1).

For $r_1 = 0$, α and β will have the simple forms, $\alpha = (\Pi'_{12}, \Pi'_{22})'$ and $\beta = (0, I_{p_2})$ and the hysteresis equation is simply (15) but where the $\mathcal{C}_1$ matrix in γ_1 now becomes $\mathcal{C}_1 \equiv (I_{p_1} - \Gamma_{11})^{-1}A_{11}^{-1}$.

Hence, as (15) shows, when hysteresis in the system stems from slow mutual adjustment between the endogenous variables, the stochastic trend, or hysteresis, in unemployment will in general come from both the cumulation of shock to the exogenous variables *and* from the cumulation of the shocks in the equations for the endogenous variables.[23] As mentioned, special cases of this class of CVARs could be derived from the hysteresis theories as suggested by, e.g. Blanchard and Summers (1986), Lindbeck and Snower (1986) and Layard, Nickell, and Jackman (2005), when the exogenous variables are best described as I(0). So instead of testing for a unit root in a single reduced form equation for unemployment, the restrictions, (14), need to be tested in a VAR.

3.3 Equilibrium Hysteresis (Case II)

In this case the roots of $A_2(z)$ that are close to one are approximated by $z = 1$. There are no roots of $A_1(z)$ that are close to 1, i.e. $r_1 = p_1$, and hence, $r = r_2 + p_1$, with $0 \leq r_2 < p_2$.

Using the definitions in Appendix A.1 it follows that for $0 < r_2 < p_2$ (and $r_1 = p_1$),

$$\alpha = \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1}\delta_{22} \\ 0 & A_{22}^{-1}\delta_{22} \end{pmatrix} \text{ and } \beta = \begin{pmatrix} F'_{11} & 0 \\ F'_{12} & \lambda_{22} \end{pmatrix}, \tag{16}$$

which both have rank, $p_1 + r_2 = r$. This shows that in this case the parameters of interest, F_{11} and F_{12}, can be analyzed directly as cointegration coefficients. Furthermore, (16) shows how one can partition the β vectors into a subset containing the parameters of interest only, and another containing the non-theoretical parameters only.

[23] Note the proportionality between the coefficients, γ_1 and γ_2, i.e. $\gamma'_2 = -\gamma'_1 F_{12}F_{22}^{-1}$. To understand this look at (4): changes in exogenous represented by change in $\mathcal{D}_{1t}$ enter the equation the same way as v_{1t}.

In Appendix A.3.2 the orthogonal complements and the structural long-run impact matrix are computed. Based on this and the above partitioning for the $p \times 1$ vectors, ST_t and e_u, it follows the hysteresis equation will have the form,

$$u_t = \mathcal{U}_t + \gamma' ST_{2t}, \tag{17}$$

where $\gamma' \equiv -e'_{1u} F_{11}^{-1} F_{12} \mathcal{C}_2$.

For $r_2 = 0$, α and β are simply given as, $\alpha = ((A_{11}^{-1})', 0)'$ and $\beta = (F_{11}, F_{12})'$ and the hysteresis equation is (17) with $\mathcal{C}_2$ in γ being $\mathcal{C}_2 = (I - \Gamma_{22})^{-1} A_{22}^{-1}$.

The hysteresis equation, (17), shows that in this case of pure equilibrium hysteresis, the only sources of non-stationarity in unemployment are the stochastic trends arising from the exogenous variables. In particular, although unemployment will in general be hysteretic (non-stationary) in this case, this has nothing to do with the theoretical implications derived from Blanchard and Summers (1986) and Lindbeck and Snower (1986).

3.4 The combinations: Slow adjustment towards a slowly moving equilibrium (Case III)

Compared to the pure cases in the two previous sections there are now two important differences. First, as noted above, $r \neq r_1 + r_2$ and instead (13) applies.[24] Second, in contrast to the pure cases it now turns out that, despite the fact that each of the processes, x_1^c and x_2, are at most I(1), this is not sufficient to ensure that the full process (and hence potentially u_t) is I(1). That is, as analyzed in e.g. Mosconi and Giannini (1992) the full process may be I(2).[25]

In this case there are two sub-cases which I denote Cases III$_a$ and III$_b$. In both cases it holds that

$$\det(\alpha'_{ii\perp}(I - \Gamma_{ii})\beta_{ii\perp}) \neq 0, \text{ for } i = 1, 2. \tag{18}$$

That is, x_{1t}^c and x_{2t} are each I(1). In terms of (13), Case III$_a$ is defined by $s = 0$ and implies that x_t is I(1), while Case III$_b$ - corresponding to the "high instability" case in Mosconi and Giannini (1992) - is defined for, $0 < s \leq \min(p_1 - r_1, p_2 - r_2)$ and implies that x_t is I(2).

[24] It can be shown that the pure cases (I and II) each implies that $\overline{\alpha}'_{11\perp} A_{11}^{-1} F_{12} \overline{\beta}_{22\perp} = 0$, which means that the formulae for the rank (13) is still valid, and in these cases implies that $r = r_1 + r_2$.

[25] Although these cases are not considered here, similar implications would hold for the other cases of combinations, i.e. Cases V-VII in Table 1. For example, Cases V and VII involve I(3) variables.

3.4.1 Case III$_a$ - The I(1) case

In this case $\delta'_{11\perp}F_{12}\lambda_{22\perp} = 0$.[26] As shown in Appendix A.3.3.1, for $0 < r_i < p_i$, $i = 1, 2$, the cointegration parameters now have the form,

$$\alpha = \begin{pmatrix} A_{11}^{-1}\delta_{11} & \alpha_{12} \\ 0 & A_{22}^{-1}\delta_{22} \end{pmatrix} \text{ and } \beta' = \begin{pmatrix} \lambda'_{11} & \beta'_{21} \\ 0 & \lambda'_{22} \end{pmatrix}, \tag{19}$$

where $\alpha_{12} \equiv A'_{11}\delta_{11\perp}(\delta'_{11\perp}A_{11}A'_{11}\delta_{11\perp})^{-1}\delta'_{11\perp}(F_{12}\overline{\lambda}_{22} - A_{12}A_{22}^{-1}\delta_{22})$ and $\beta'_{21} \equiv (\delta'_{11}(A_{11}A'_{11})^{-1}\delta_{11})^{-1}\delta'_{11}(A_{11}A'_{11})^{-1}(F_{12} - A_{12}A_{22}^{-1}F_{22})$. The proof that $s = 0$ is necessary and sufficient for $x_t \sim \mathrm{I}(1)$ is given in Appendix A.3.3.1, where it is also shown that the hysteresis equation is given by,

$$u_t = \mathcal{U}_t + \gamma'_1 ST_{1t} + \gamma'_2 ST_{2t}, \tag{20}$$

where $\gamma'_1 \equiv e'_{1u}\mathcal{C}_1$ and $\gamma'_2 \equiv e'_{1u}\breve{\mathcal{C}}_{12}$ where $\mathcal{C}_1$ is the structural long-run impact matrix (see equation 8) for the x_1^c process, $\breve{\mathcal{C}}_{12} \equiv \mathcal{C}_1 A_{11} W \mathcal{C}_2 -$
$\mathcal{C}_1 F_{12}\overline{\lambda}_{22}(\delta'_{22}(A_{22}A'_{22})^{-1}\delta_{22})^{-1}\delta'_{22}(A_{22}A'_{22})^{-1} - \overline{\lambda}_{11}(\delta'_{11}(A_{11}A'_{11})^{-1}\delta_{11})^{-1}\delta'_{11}(A_{11}A'_{11})^{-1}F_{12}\mathcal{C}_2$, with $\mathcal{C}_2$ is defined accordingly for x_{2t}, and W is defined in Appendix A.3.3.1.[27]

The cases for which $r_1 = 0$ and/or $r_2 = 0$ are straightforward to derive from the above. However, since $r_1 = 0$ and, in particular, $r_1 = 0$ together with $r_2 = 0$ do not seem to be particularly relevant for empirical applications, I focus on the case $r_2 = 0$ while still $0 < r_1 < p_1$. In this case α simply reduces to the first (block) column in (19), while β' is given by the first (block) row of β in (19) with $\beta'_{21} \equiv (\delta'_{11}(A_{11}A'_{11})^{-1}\delta_{11})^{-1}\delta'_{11}(A_{11}A'_{11})^{-1}F_{12}$. The hysteresis equation still has the form as in (20) with $\gamma'_1 \equiv e'_{1u}\mathcal{C}_1$ and $\gamma'_2 \equiv e'_{1u}\breve{\mathcal{C}}_{12}$ but $\breve{\mathcal{C}}_{12}$ reduces to, $\breve{\mathcal{C}}_{12} \equiv \mathcal{C}_1 A_{11} W \mathcal{C}_2 - \overline{\lambda}_{11}(\delta'_{11}(A_{11}A'_{11})^{-1}\delta_{11})^{-1}\delta'_{11}(A_{11}A'_{11})^{-1}F_{12}\mathcal{C}_2$, with $\mathcal{C}_2 \equiv (I_{p_2} - \Gamma_{22})^{-1}A_{22}^{-1}$. In such a case there are no cointegration relations between the exogenous variables and, hence each of these corresponds to a driving forces in the system.

3.4.2 Case III$_b$ - The I(2) case

The remaining sub-cases of Case III are defined for, $0 < s \leq \min(p_1 - r_1, p_2 - r_2)$. In Appendix A.3.3.2 it is shown that now x_t is no longer I(1), but instead I(2). Compared to the previous cases the computations now become more involved. I therefore focus on the case for which $0 < s < \min(p_1 - r_1, p_2 - r_2)$ and $0 < r_i < p_i$, $i = 1, 2$. The

[26] This is equivalent to the condition in equation (8) in Mosconi and Rahbek (1997).

[27] It is shown in the appendix that the Cases I and II, i.e. respectively, (15) and (17), can be obtained as (simpler) special cases of this more complicated expression. Note also that, as in Case I, unemployment is influenced by both stochastic trends, ST_{1t} and ST_{2t}. However, there is a difference in that in the former case there is a proportionality between the coefficients, γ_1 and γ_2 (i.e. $\gamma'_1 = e'_{1u}\mathcal{C}_1$ and $\gamma'_2 = -\gamma'_1 F_{12}F_{22}^{-1}$). This homogenous influence is not present here and in general.

details are found in Appendix A.3.3.2. But overall, the $(p_1 - r_1) \times (p_2 - r_2)$ matrix product, $\overline{\alpha}'_{11\perp} A_{11}^{-1} F_{12} \overline{\beta}_{22\perp}$, which was zero in Case III$_a$, has now rank $s > 0$ and can be decomposed as $\overline{\alpha}'_{11\perp} A_{11}^{-1} F_{12} \overline{\beta}_{22\perp} = HG'$ where H is $(p_1 - r_1) \times s$ and G is $(p_2 - r_2) \times s$, both with rank, s, and contain structural parameters only.

The cointegrating parameters become,

$$\alpha = \begin{pmatrix} A_{11}^{-1}\delta_{11} & \alpha_{12} & A'_{11}\delta_{11\perp}H \\ 0 & A_{22}^{-1}\delta_{22} & 0 \end{pmatrix}, \text{ and } \beta' = \begin{pmatrix} \lambda'_{11} & \beta'_{21} \\ 0 & \lambda'_{22} \\ 0 & G'\lambda'_{22\perp} \end{pmatrix}, \tag{21}$$

where α_{12} and β'_{21} are defined as in Case III$_a$ (see equation 19). It appears that compared to Case III$_a$ there are now s additional cointegration relations, $G'\lambda'_{22\perp}$, which involve exogenous variables but to which only the endogenous variables adjust through the coefficients in $A'_{11}\delta_{11\perp}H$.[28]

It is shown in Appendix A.3.3.2 that the full process is I(2). Hence, in terms of the hysteresis equation one now needs to distinguish between the integration order (i.e. 1 and 2) of a stochastic trend at time t. Therefore, denote the first order stochastic trend by $ST_{j1t} \equiv \sum_{i=1}^{t} \varepsilon_{ji}$, and the second order stochastic trend by, $ST_{j2t} \equiv \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_{ji}$, for $j = 1, 2$, which is the variable index. The γ coefficients will resemble this notation, i.e. γ_{j1} and γ_{j2} correspond to ST_{j1t} and ST_{j2t}, respectively. Using this and (63) and (64) in Appendix A.3.3.2 the hysteresis equation can be written as,

$$u_t = \mathcal{U}_t + \gamma'_{22} ST_{22t} + \gamma'_{11} ST_{11t} + \gamma'_{21} ST_{21t}, \tag{22}$$

where $\gamma'_{22} \equiv e'_{1u}\mathcal{C}_1 F_{12} \mathcal{C}_2$, $\gamma'_{11} \equiv e'_{1u}\mathcal{C}_1$ and $\gamma'_{21} \equiv e'_{1u}\Phi_{12}$.

4 A small illustration: UK unemployment 1988-2006

The above analysis has illustrated the main ideas in terms of a general structural model (5). In this section I consider a specific example of the latter, i.e. a small macroeconomic model consisting of four endogenous variables: prices, wages, output and unemployment, and one exogenous variable, the price of crude oil, which is supposed to influence equilibrium unemployment via price setting (see below). The exogeneity captures that UK oil supply and demand is not big enough to influence the world price of crude oil. I confront the model with quarterly time series data for the UK economy 1988 up to the beginning of the financial crisis. Although this example is admittedly simple and is meant primarily as an illustration, it is still general enough to provide a set of different

[28] This case is thoroughly analyzed in Mosconi and Giannini (1992), and relative to that paper, my contribution here (as mentioned) consists in deriving the expression in terms of the structural parameters only as well as the MA representation.

parameter restrictions consistent with the different types of hysteresis cf. Cases I, II and III in Table 1. Note that, compared to the more general block-matrix formulation above this more detailed example allows one to test different hypotheses of slow adjustment against each other.

The overall structure of the model is akin to the framework presented in the well-known book Layard, Nickell and Jackman (2005, first ed. in 1991). This framework (henceforth *the LNJ framework*) has had a great influence on the work of applied macro-economists and policy makers and is often regarded as a key reference representing the consensus view on European unemployment (e.g. Blanchard 2006 and Holden and Nymoen 2002).[29]

4.1 A Layard-Nickell-Jackman-based model

The equations (24) through (27) below are consistent with a general equilibrium model of unemployment building on aggregate supply and demand, imperfect competition and adaptive expectations. On the supply side goods markets are characterized by monopolistic competition, and unions and firms bargain over wages in the labor markets. The demand side can be interpreted as some reduced form IS-Taylor Rule system. The short-run equilibrium is the goods market equilibrium, and the long-run equilibrium (steady state), in addition, involves labor market equilibrium.[30]

At any given point in time, capital, technology, and expectations are given. Based on the expected aggregate price and demand levels monopolistically competitive firms choose a price and plan production in order to maximize expected short-run profits. The chosen price is realized, and stays fixed within the period due to positive costs associated with changing it. Given a realization of an aggregate demand shock, a level of aggregate demand will result at this price level. It is assumed that actual production accommodates this level fully. Given output, actual employment can then be determined from the production function. This determines actual unemployment since the labor force by assumption is given. Finally, unemployment and prices impact on nominal wage setting.

A few remarks are in order here: First, I state the model in the aggregate variables directly since the exogenous variables and the parameters in the disaggregate model (and hence, its solution) are independent of the "individual index" i. Second, the model is stated in terms of log-linear relations which should be interpreted as log-linear approx-

[29] This approach provided the theoretical underpinnings of the policy recommendations in the influential OECD Jobs Study from 1994 (see Mitchell and Muysken 2008, p. 72).

[30] For this type of model, the latter is not unique and is often referred to as a medium-run and not a long-run equilibrium. A "true" unique long-run equilibrium (steady state) would additionally entail a constant net-foreign asset-to-income ratio to ensure *external balance* (see Carlin and Soskice 1990, Layard et al. 2005, Carlin and Soskice 2006, Groth 2009). However, to keep this example simple I have chosen to abstract from this.

imations, which are made around an underlying stable and unique steady state similar to that described in footnote 30. Hence, in the following all variables are stated in logarithms.

To elaborate, firms set prices as a markup over *expected* marginal costs. The markup is allowed to depend on activity and crude oil prices. The latter assumption is a shortcut of modeling the role of oil prices. I assume that the marginal product of labor is constant, which implies the aggregate production function,

$$y_t = a_t + n_t + \varepsilon_{yt}, \tag{23}$$

where y_t is output, n_t is employment and where the (logarithmic) technological state is described by an unsystematic component, ε_{yt}, and a deterministic labor augmenting part, a_t, evolving as $a_t = g_A t$ with $g_A > 0$.[31] The assumption of constant marginal returns captures that "under normal circumstances" the *average* utilization of capital is sufficiently below the capacity limit, so that for a realistic range of variation in employment the marginal product does not diminish. Excess capacity *on average* is a very reasonable assumption for many reasons. For example, it could represent an equilibrium state because firms use it to deter entrance (Fudenberg and Tirole 1983). Under this assumption the capital stock does not influence marginal costs and thus price setting. This is an advantage, both because it simplifies dynamics and since, as is well-known, capital stock data are in general of dubious quality. The aggregate pricing relation can therefore be written as,

$$\begin{aligned} p_t \;=\; & \rho_0 - g_A t + z_t^p + \rho_1 y_{t-1} + \rho_{21} u_{t-1} + \rho_{22} u_{t-2} + \rho_3 p_{t-1}^o \\ & + \rho_4 p_{t-2}^o + \rho_5 w_{t-1} + \rho_6 w_{t-2} + \kappa_1 p_{t-1} + \kappa_2 p_{t-2} + \varepsilon_{pt}, \end{aligned} \tag{24}$$

where p is aggregate output price, w denotes wages, u is unemployment and p^o, the price of crude oil (all in logs). The autoregressive terms (i.e. $\kappa_1 p_{t-1} + \kappa_2 p_{t-2}$) in this equation and similar terms in the subsequent equations have been added to make the dynamics of adjustment more flexible (realistic). It is assumed that $\kappa_1 + \kappa_2 \neq 1$ (similar conditions hold for the equations below), which implies that these parameters do not affect the unit roots of the model. The LNJ framework is not specific about how activity will influence the price markup and I have therefore allowed for both output and unemployment, to enter. Although p^o could enter the other equations below, for example aggregate demand and wages setting, I have chosen to abstract from this to keep the example as simple as possible. The rationale behind the lagged effect of p^o on p in relation (24) could be adaptive expectations, which is how the nominal wage rate enters. However, since my

[31] The latter follows from an underlying technological state, A_t evolving as $A_t = (1+g_A)A_{t-1}$ assuming that g_A is moderate.

exposition is merely a short-cut of including oil prices, I have no specific prior with respect to the coefficients ρ_3 and ρ_4, other than $\frac{\rho_3+\rho_4}{1-\kappa_1+\kappa_2} > 0$, and following Layard et al. (2005), there are no prior restrictions on the signs of ρ_1 and $\rho_{21}+\rho_{22}$. The term, $z_t^p+\varepsilon_{pt}$, captures the (combined) influence from various exogenous unmodelled variables, where ε_{pt} is an unsystematic stochastic error term, while z_t^p accounts for the more systematic changes in such variables. In the empirical implementation the latter will be approximated by some expression of deterministic components (e.g. impulse dummies, trends and level shift dummies etc.). Similar terms are added to the equations for the other observable variables below.

The relation in (24) determines p_t and can be referred to as the *short-run Price Setting relation* (PS), to be distinguished from the *long-run PS* relation obtained when expectations are correct, as implied by a long-run equilibrium. It is the latter relation that is usually referred to as the *Price Setting* relation or *feasible real wage.*

In the next step output, y_t, is determined according to,

$$y_t = \phi_1 + z_t^d + \phi_2 p_t + \phi_3 p_{t-1} + \phi_4 p_{t-2} + \kappa_3 y_{t-1} + \kappa_4 y_{t-2} + \varepsilon_{dt}, \tag{25}$$

This is the Aggregate Demand (AD) relation with dynamics (lags) added. It is a short-cut to be interpreted roughly as the reduced form that results from an IS relation and a simple Taylor Rule, the latter of which is based on trend-adjusted output and inflation relative to some target. It allows for an effect from both the price *level*, through the real exchange rate (IS), and the *inflation rate,* working through the Taylor Rule and the real interest rate. The long-run level effect is thus expected to be negative, whereas the sign of the latter inflation rate (short-run) effect is indeterminate. The term, z_t^d, may include trends and dummy variables proxying the unobserved (or unmodelled) exogenous variables that shift the Taylor Rule and the IS relation (see Layard et al. 2005).

When production is determined from (25), the production function, (23), gives the required amount of labor input, which in turn will be actual employment since firms have the right to manage. Using the definition of unemployment, assuming that the labor force is exogenous, and using (23) to eliminate n_t, it follows that,

$$u_t = \lambda_0 + z_t^u + \lambda_1 y_t + \lambda_2 y_{t-1} + \lambda_3 y_{t-2} + \kappa_5 u_{t-1} + \kappa_6 u_{t-2} + \varepsilon_{yt}, \tag{26}$$

where some further adjustment has been added and where changes (trend-like and discrete shifts) in the size of the labor force, in aggregate capital and hours of work, as well as a_t (and shifts in a_t), are potentially captured by z_t^u. It is assumed that $\frac{\lambda_1+\lambda_2+\lambda_3}{1-\kappa_5-\kappa_6} < 0$.

Finally, the nominal wage, w_t, is determined. Wage formation is a central part of the LNJ framework and is where the hypotheses of unemployment hysteresis appear. I assume that wages are determined according to the short-run Wage-Setting (WS) relation,

$$w_t = \omega_0 + z_t^w + \omega_1 w_{t-1} + \omega_2 w_{t-2} + \omega_3 u_t + \omega_4 u_{t-1} + \omega_5 p_{t-1} + \omega_6 p_{t-2} + \varepsilon_{wt}, \tag{27}$$

where $\omega_1 + \omega_2 \neq 1$ and where z_t^w may include changes in various "exogenous wage pressure variables" e.g. union power, and exogenous movements in the (expected) replacement ratio (Layard et al. (2005) eq. 21, p. 202). It is assumed that, $\frac{\omega_3+\omega_4}{1-\omega_1+\omega_2} < 0$, for unemployment to have a stabilizing effect on the system. However, this term may be close to zero, which is exactly what the typical hypotheses of unemployment hysteresis would imply. The LNJ framework emphasizes two dominating of such hypotheses: The first is the insider-outsider theory, claiming that when workers are fired, the remaining employed workers increase their wage targets, and that due to e.g. collective agreement contracts the unemployed cannot underbid to get their jobs back (Blanchard and Summers 1986, Lindbeck and Snower 1986). The other hypothesis asserts that when total unemployment increases, so does its share of long-term unemployed, and this attenuates the downward pressure on wage inflation from higher unemployment needed to bring down unemployment again (Layard and Nickell (1987)). In the present context, both of these hypotheses imply that, $\omega_3 + \omega_4 = 0$, as an approximation.

Based on the empirical evidence below it is assumed that oil prices can be described by the following autoregressive model,

$$p_t^o = \pi_1 + z_t^o + \pi_2 p_{t-1}^o + \pi_3 p_{t-2}^o + \varepsilon_{ot}, \tag{28}$$

where z_t^o reflects the major exogenous *oil shocks* (see e.g. Blanchard and Galí 2007).

This completes the description of the structural model corresponding to (5), whose matrices corresponding to the equations (24) through (28) are given in Appendix A.4.

To analyze the different cases of hysteresis (unit-root non-stationarity) that this macroeconomic model generates, consider the dichotomous partitioning corresponding to (3). As discussed in the previous sections, hysteresis in the system, and hence potentially in unemployment, imply characteristic roots, z, close to 1 which may be tested statistically as the parameter restrictions that imply exact unit roots (cointegration). In other words, restrictions that imply $\det(A(1)) = 0$ or $\det(A_1(1))\det(A_2(1)) = 0$. However, calculating this condition based on the matrices in Appendix A.4 leads to a cumbersome expression. I therefore introduce the accumulated dynamic multipliers defined in Table 5 in Appendix A.4, where $M_{x,y}$ denotes the accumulated (multiplier) effect on variable x from variable y. Based on this, the unit root restriction, $\det(A_1(1))\det(A_2(1)) = 0$, for this model becomes,

$$\underbrace{([M_{p,y} + (M_{p,u} + M_{p,w}M_{w,u})M_{u,y}]M_{y,p} + M_{p,w}M_{w,p} - 1)}_{\det(A_1(1))}\underbrace{(\pi_2 + \pi_3 - 1)}_{\det(A_2(1))} = 0. \tag{29}$$

First of all, in contrast to its univariate counterpart, $\rho - 1 = 0$, where ρ is the sum of the autoregressive parameters in an AR model, this equation is a complicated expression depending on many parameters. This reflects the fact that in a dynamic system of variables, slow adjustment may come from many sources. In particular, note that the hypotheses of unemployment hysteresis concern the parameter, $M_{w,u}$, which constitute only a small part of this overall restriction. Equation (29) also shows how slow adjustment in the system is related to nominal wage rigidity (sticky wages), $M_{w,p} = 0$, rigidity in prices, for example, $M_{p,w} = 0$, excessive labor hoarding effects, i.e. that unemployment is very slow to adjust to output ($M_{u,y} = 0$). Hence, even within the framework of the simple structural model, (24) through (28) the relationship between unit roots, on the one hand, and, hysteresis hypotheses and persistence generating rigidities on the other, is quite complicated.

However, given price-wage homogeneity, which is assumed in most applications, the expression does have a more clear interpretation. Homogeneity, is defined by the restrictions $M_{p,w} = 1$ and $M_{w,p} = 1$ and results from the assumption that in the long run i.e. in the hypothetical absence of any other shocks, expectational errors vanish. Under this restriction, the first term in (29), $\det(A_1(1))$, reduces to $[M_{p,y} + (M_{p,u} + M_{p,w}M_{w,u})M_{u,y}]M_{y,p}$. This is a price-quantity interaction term consisting of a product of two terms: $M_{y,p}$ the accumulated multiplier effect from prices on output, and $[M_{p,y} + (M_{p,u} + M_{p,w}M_{w,u})M_{u,y}]$ which is the total effect of output on prices consisting of the sum of the direct effect, $M_{p,y}$, and the indirect effect which propagates through the system, starting by affecting unemployment ($M_{u,y}$) which, in turn, affects prices directly ($M_{p,u}$) and indirectly effect via wages ($M_{p,w}M_{w,u}$). Under homogeneity it is thus relatively clear why slow adjustment may occur: Either prices have no effect on output and/or vice versa. This expression also shows that even under homogeneity the hysteresis hypotheses, implying $M_{w,u} = 0$, are in general *not* sufficient for unit roots in unemployment. It appears that what is needed for this to be the case is "a flat PS curve", i.e. that price setting is independent of activity ($M_{p,y} = -M_{p,u}M_{u,y}$).

In this example *equilibrium hysteresis* is particularly simple and results when $\pi_2 + \pi_3 - 1 = 0$, corresponding to $r_2 = 0$. When this is the case oil prices are I(1) and unemployment will inherit this I(1) stochastic trend (non-stationarity) via equilibrium unemployment. The interpretation is that non-stationary oil prices will make the (trend adjusted) PS curve shift up and down (in unemployment real wage space) in a non-stationary manner, which will translate into non-stationarity of equilibrium unemployment. How strongly this non-stationarity will transmit into unemployment also depend on the slope of the WS relation, as well as on the extent of slow adjustment, and all this is fully described by the hysteresis equation.

4.2 Confronting the data

Based on the model above I now test a few hypotheses about hysteresis against the quarterly UK data. As the empirical analysis below will show, oil prices are both exogenous and I(1), in particular, these series can be described as an independent random walk - the simplest I(1) process. Moreover, this I(1) stochastic trend behavior seems to be a relevant description of the non-stationarity and it is reasonable for this example to disregard I(2) for these data. This means that we can focus on Cases **II** and $\mathbf{III}_a$, and in order to illustrate some of the points emphasized previously, I will focus on two overall questions:

- Most importantly, I will investigate whether hysteresis of actual UK unemployment for this period, is the result of *slow adjustment*, or whether it is due to *equilibrium hysteresis*, i.e. a persistently changing underlying equilibrium level of unemployment.

- I will also test whether slow adjustment of the form based on sluggish wage formation (as emphasized by theoretical models for unemployment hysteresis), can account for the non-stationarity of unemployment alone.

The econometric test procedure for the respective cases is as follows.[32] To ensure a statistically adequate analysis, a well-specified unrestricted VAR was specified first and then the cointegration rank was determined (Johansen 2006). The evidence quite clearly supported a CVAR with either $r = 3$ or $r = 4$. A given hysteresis case is then tested as a sub-model of one of these CVARs (which impose no other restrictions than the general reduced rank restriction, $r = 3$ or $r = 4$). For all cases, I report the likelihood ratio test statistic and p-value corresponding to this test, where, in addition, I also removed insignificant coefficients subsequently. The estimates of α and β corresponding to the restricted CVARs, i.e. the sub-models, are also given, and in light of these the identification of the structural parameters is considered. The data are described in Table 6 in Appendix A.4.

Table 2 reports the p-value and the estimates of α and β' (with rows, β'_1 to β'_3) corresponding to the general restriction of slow adjustment, i.e. $\det(A_1(1)) = 0$ in (29). Note that, β has been augmented with the trend term, which is merely a matter of

[32] All estimation is based on PcGive, OxMetrics 6.10 and CATS in RATS (see respectively, Doornik and Hendry 1998, Doornik 2010 and Dennis, Hansen, and Juselius 2006).

notation, and that price-wage homogeneity ($M_{p,w} = 1$ and $M_{w,p} = 1$) is *not* imposed. This hypothesis is a special case of Case III$_a$. The I(1) condition ($s = 0$) amounts to the restriction that oil prices do not enter the cointegrating relations, cf. the zero column in β' corresponding to p^o_t. These restrictions are strongly rejected with a p-value of less than 0.1%.

Table 2: Testing slow adjustment of the general from, $\det(A_1(1) = 0$, in equation 29 (a special case of Case III$_a$): The likelihood ratio test, with p-value and estimates of α and β, of the corresponding CVAR against an unrestricted CVAR with $r = 3$.

Test: $\chi^2(10) = 30.645$, p-value; 0.0007											
	$\widehat{\alpha}$			$\widehat{\beta}'$							
					p_t	y_t	u_t	w_t	p^o_t		$Trend$
Δp_t	0.198 [3.536]	1.282 [5.518]	0.083 [5.134]	$\widehat{\beta}'_1$	1	0	0	2.886 [12.539]	0		−0.035 [−14.484]
Δy_t	−0.265 [−6.302]	−1.344 [−7.682]	−0.079 [−6.473]	$\widehat{\beta}'_2$	0	1	0	−0.789 [−35.124]	0		0
Δu_t	0.307 [3.212]	0	−0.065 [−2.047]	$\widehat{\beta}'_3$	0	0	1	6.44 [7.643]	0		−0.052 [−5.849]
Δw_t	0	0	−0.012 [−6.561]								
Δp^o_t	0	0	0								

Note: The brackets contain t-ratios.
The coefficients, $\alpha_{3,2}, \alpha_{4,1}, \alpha_{4,2}$ and trend coefficient in β'_2 were insignificant and thus removed

Under this restriction the cointegrating coefficients on wages in β'_1, β'_2 and β'_3 are given as, $\frac{M_{p,w}}{(M_{u,y}M_{p,u}+M_{p,y})M_{y,p}-1}$, $\frac{M_{p,w}M_{y,p}}{(M_{u,y}M_{p,u}+M_{p,y})M_{y,p}-1}$, and $\frac{M_{p,w}M_{y,p}M_{u,y}}{(M_{u,y}M_{p,u}+M_{p,y})M_{y,p}-1}$, respectively. Although the overall test rejects the signs of the estimated α and β coefficients are in line with Table 5, in the Appendix A.4.

Next, consider the special case of Case III$_a$, that imposes $M_{w,u} = 0$, i.e. the main restriction consistent with the hysteresis theories à la Blanchard and Summers (1986), Lindbeck and Snower (1986) and Layard and Nickell (1987). As discussed above, for this restriction to imply a unit root in unemployment in this model it is necessary to impose, in addition, homogeneity ($M_{p,w} = M_{w,p} = 1$) and a flat price setting relation ($M_{p,y} = -M_{p,u}M_{u,y}$). It seems plausible for the latter restriction to hold as a result of $M_{p,y} = 0$ and $M_{p,u} = 0$, which is thus imposed. Table 3 reports the estimation results under this restriction.

Table 3: Testing Hysteresis Theories (a special case of Case III$_a$): The likelihood ratio test, with p-value and estimates of α and β, of the corresponding CVAR against an unrestricted CVAR with $r = 3$.

Test: $\chi^2(15) = 72.017$, p-value; < 0.0001										
		$\widehat{\alpha}$					$\widehat{\beta}'$			
					p_t	y_t	u_t	w_t	p_t^o	$Trend$
Δp_t	-0.178 [-6.329]	0	0	$\widehat{\beta}_1'$	1	0	0	0.355 [2.269]	0	-0.009 [-5.247]
Δy_t	0	-0.185 [-4.130]	0	$\widehat{\beta}_2'$	0	1	0	0	0	-0.008 [-58.430]
Δu_t	0	0	-0.072 [-4.098]	$\widehat{\beta}_3'$	0	0	1	-11.513 [-4.380]	0	0.132 [4.653]
Δw_t	0	0.158 [5.390]	0							
Δp_t^o	0	0	0							

Note: The brackets contain t-ratios.
The coefficients, $\alpha_{2,1}, \alpha_{3,1}, \alpha_{3,2}, \alpha_{4,1}, \alpha_{4,3}$, and the wage coefficient in β_2' were insignificant and thus removed.

Since this hypothesis imposes more restrictions on the parameter space than the more general one in Table 2 it is not surprising that it is also strongly rejected.

Given the lack of support to slow adjustment let us now turn to testing whether, alternatively, the unit root evidence can be explained as a result of pure equilibrium hysteresis, cf. Case II. As Table 4 suggests, this does indeed seem to be the case. The restriction defining this case, i.e. $\pi_2 + \pi_3 - 1 = 0$, while $M_{y,p}(M_{u,y}\rho_2 + \rho_1) \neq 0$, and the additional zeros suggested by the data were accepted with a p-value as high as 0.92.

Table 4: Testing Equilibrium Hysteresis (a special case of Case II): The likelihood ratio test, with p-value and estimates of α and β, of the corresponding CVAR against an unrestricted CVAR with $r = 4$.

Test: $\chi^2(9) = 3.8266$, p-value; 0.9224											
		$\widehat{\alpha}$						$\widehat{\beta}'$			
						p_t	y_t	u_t	w_t	p_t^o	$Trend$
Δp_t	0	1.120 [6.167]	0.078 [5.408]	0.204 [2.367]	$\widehat{\beta}_1'$	1	0	0	0	-0.053 [-1.873]	-0.005 [-8.361]
Δy_t	-0.160 [-3.575]	-1.267 [-8.373]	-0.077 [-6.870]	-0.275 [-4.080]	$\widehat{\beta}_2'$	0	1	0	0	0.058 [3.747]	-0.009 [-30.384]
Δu_t	0	0	-0.090 [-4.673]	1.113 [5.679]	$\widehat{\beta}_3'$	0	0	1	0	-0.862 [-4.712]	0.027 [8.010]
Δw_t	0.097 [2.835]	0	-0.016 [-4.777]	-0.140 [-3.969]	$\widehat{\beta}_4'$	0	0	0	1	0	-0.010 [-61.485]
Δp_t^o	0	0	0	0							

Note: The brackets contain t-ratios.
The coefficients, $\alpha_{1,1}, \alpha_{3,1}, \alpha_{3,2}, \alpha_{4,2}$ and the oil price coefficient in β_4' were insignificant and thus removed.

Under these restrictions, β and, to some extent, α, have relatively clear interpretations, and identify some of the key parameters. In particular,

$$\alpha = \begin{pmatrix} 0 & \rho_1 & \rho_2 & \rho_5+\rho_6 \\ * & * & * & * \\ 0 & 0 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \beta' = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\rho_3+\rho_4}{M_{y,p}(M_{u,y}\rho_2+\rho_1)} \\ 0 & 1 & 0 & 0 & \frac{\rho_3+\rho_4}{M_{u,y}\rho_2+\rho_1} \\ 0 & 0 & 1 & 0 & \frac{M_{u,y}(\rho_3+\rho_4)}{M_{u,y}\rho_2+\rho_1} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{30}$$

where $*$ denotes a complicated short-run adjustment coefficient and $\rho_2 \equiv \rho_{21} + \rho_{22}$. By computing the structural long-run impact matrix one can show that the coefficients in β' are the long-run effects from crude oil prices on the respective variable. That is, in the first row of β', $\frac{\rho_3+\rho_4}{M_{y,p}(M_{u,y}\rho_2+\rho_1)}$ is (minus) the long-run effect of a long-run change of one unit in crude oil prices on the domestic price level etc.[33] As mentioned no prior with respect to sign of ρ_1 and ρ_2, but the estimates from Table 4 suggest that both are positive. The estimate of $\frac{\rho_3+\rho_4}{M_{u,y}\rho_2+\rho_1}$ in the second row of β' is positive, which implies that the remaining estimates in $\widehat{\beta}'$ are consistent with the expected signs, i.e. $M_{y,p} < 0$ and $M_{u,y} < 0$. The hysteresis equation, (17), becomes,

$$u_t = \mathcal{U}_t - \frac{M_{u,y}(\rho_3+\rho_4)}{M_{u,y}\rho_2+\rho_1}\frac{1}{1+\pi_3}\sum_{i=1}^{t}\varepsilon_{oi} \tag{31}$$

where $(1+\pi_3)^{-1}$ is positive which follows from the fact that all roots except unit roots are strictly outside the complex unit disc. The latter is empirically supported. Equation 31 shows that oil prices influence unemployment permanently via influencing the Pricing Setting relation and thus equilibrium unemployment. Specifically, consider an isolated positive shock to crude oil prices of magnitude $1+\pi_3$, i.e. simply normalized to produce a long-run unit change in the oil prices (that is the term $(1+\pi_3)^{-1}\sum_{i=1}^{t}\varepsilon_{oi}$ changes by one unit). This will have a positive long-run effect on prices, of magnitude, $-\frac{\rho_3+\rho_4}{M_{y,p}(M_{u,y}\rho_2+\rho_1)} > 0$, which will then lower output by, $-M_{y,p}\left(-\frac{\rho_3+\rho_4}{M_{y,p}(M_{u,y}\rho_2+\rho_1)}\right) = \frac{\rho_3+\rho_4}{(M_{u,y}\rho_2+\rho_1)}$, which eventually will raise unemployment, by $-M_{u,y}\frac{\rho_3+\rho_4}{(M_{u,y}\rho_2+\rho_1)} = -\frac{M_{u,y}(\rho_3+\rho_4)}{(M_{u,y}\rho_2+\rho_1)}$. Hence, as claimed in Section 3 the hysteresis equation fully uncovers the the origin of unemployment persistence, i.e. by being expressed exclusively in terms of the structural parameters.

In relation to the analysis in Section 2, it is essential to note here that the hypothetical change in oil prices is permanent. If, on the other hand, this were temporary the effect on unemployment would die out relatively fast. As mentioned, this is in contrast to what is often meant by hysteresis, namely that transitory influences have permanent or long-lasting effects, and in its pure form this concerns the pure cases of slow adjustment, i.e. Cases I and VIII in Table 1.

[33] It should be noted that the zero restrictions which were exclusively motivated by the data and not the theoretical hypothesis were simplifying the estimated β matrix considerably in this case.

Note also that, the restrictions imposed in Table 3, $M_{p,y} = 0$ and $M_{p,u} = 0$ are equivalent to $\rho_1 = 0$ and $\rho_2 = 0$, respectively. The rejection of that hypothesis is thus consistent with the significant estimates of the latter two coefficients in Table 4.

5 Comparison to some previous studies

In terms of its focus – i.e. the sources of unemployment hysteresis - this study is closely related to the multivariate studies, Jacobson et al. (1997) and Dolado and Jimeno (1997), and in terms of the method, it builds mainly on the econometric analysis in Mosconi and Giannini (1992). In this section I therefore compare my analysis to Jacobson et al. (1997) and Mosconi and Giannini (1992).

First of all, in terms of the analysis in Sections 2 and 3, Jacobson et al. (1997) focus on hysteresis which results from slow adjustment. As we have seen, slow adjustment, when approximated by zero adjustment (reduced rank in the counterfactual process), is what causes transitory impulses to have permanent effects. As the authors state in their introduction this is the kind of hysteresis they are concerned with. As opposed to the present analysis, their analysis is however based on a conditional model, i.e. a VARX (i.e. not the full system of observable variables), and they derive parameter restrictions that imply reduced rank in this model.[34] In particular, they do not model equilibrium unemployment in terms of *observable* exogenous variables (e.g. oil prices, or the replacement ratio), but rather unobserved variables. As a result, they do not focus directly on testing parameter restrictions implying equilibrium hysteresis, neither against, nor in conjunction with slow adjustment, which is a main focus here. This also implies that, the I(2) case or, in the words of Mosconi and Giannini (1992), the high instability case, is not considered. Another important difference between Jacobson et al. (1997) and the present analysis lies with the econometric approach: The authors adopt a so-called Common-Trends approach, and thus focus on the Moving Average Representation of the CVAR, in order to identify the structural parameters. In contrast, for that purpose, the analysis in Section 3 is based on testable restrictions on α and β. Moreover, the fact that the exogenous variables in their theoretical model are unobservable has the implication that some of the parameters that enter the characteristic roots, i.e. which are potentially crucial for the presence or absence of hysteresis, are in fact parameters from equations of unobserved variables. Moreover, the exogenous wage shock variable (ω in their equation 7), which resembles oil prices in the UK example above and which turns out to play a central role in their conclusion, is one of the unobservable variables. Finally, their analysis is based on a specific SVAR, as opposed to the SVAR in Section 2 which is defined in terms of block matrices. In this sense their analysis may thus be

[34] In their Footnote 9, they argue that the two observable exogenous variables that they condition on are not to be modelled endogenously.

more comparable to the five-variable SVAR in Section 4.

As mentioned, the econometric analysis in Sections 2 and 3 builds on Mosconi and Giannini (1992). However, there are some differences/extensions in the present work. Apart from the fact that these authors do not study unemployment, the most notable differences are that I state the block matrix formulation of the cointegration parameters, α and β, purely in terms of structural parameters, in order to make clear how the latter may be identified. Moreover, in terms of the taxonomy in Table 1, they are only concerned with Case III, and in particular, Case III_b. Finally, since they focus on cointegration properties only, the MA representation in block matrix notation, which I have derived in order to analyze the hysteresis equation (see Appendix A.3), is not analyzed.

6 Perspectives and concluding remarks

There are a vast number of empirical studies that fail to reject hysteresis in the form of unit root non-stationarity of unemployment. Typically, the result is that while a unit root in unemployment is rejected for US data, this cannot be rejected for most EU countries.[35] However, apart from a few pioneering multivariate econometric studies (Jacobson, Vredin, and Warne 1997), most empirical analyses have been based on *univariate* econometric methods, i.e. autoregressive models including unemployment only, and have therefore not been able to say much about *why* hysteresis occurs. Instead, there has been an excessive focus on detecting the empirical *existence* (or otherwise) of hysteresis, i.e. exact unit roots, for example by adopting various methods that increase the power of unit root tests relative to conventional Dickey-Fuller-type test (see footnote 4). Much less attention has been devoted towards uncovering the *sources* for hysteresis.

In light of this, the present paper has suggested a multivariate or system-based econometric approach to hysteresis based on a general Structural VAR model with exogenous variables and cointegration. This allowed the derivation of general testable parameter restrictions (in block matrix notation), in the form of classes of restricted CVAR models representing fundamentally different types of hysteresis. There were three overall types to be considered: One for which the only source of hysteresis is the slow mutual adjustment between the endogenous variables holding the exogenous variables fixed (slow adjustment), one for which hysteresis originates from a slowly moving exogenous determinant of equilibrium unemployment (equilibrium hysteresis), and, finally the case that combines these two pure sources. I focussed on deriving the block matrix CVARs (for the full system) for the cases for which the sub-systems, of respectively, the endogenous variables for fixed exogenous variables and of the exogenous variables, were at most I(1).

[35] For a survey see Røed (1997). As Ball (2009) notes there is not so much evidence to review for the period beyond this.

I then considered an example based on a SVAR with four endogenous variables (prices, wages unemployment and output) and one exogenous variable (oil prices). Although this small example is primarily meant as an illustration, it seems to provide clear evidence, that for the period 1988-2006, the non-stationarity of UK unemployment was *not* the result of slow adjustment. In particular, it was not the result of mechanisms introducing sluggishness in wage formation as is often emphasized by the hysteresis theories. Instead, these data support the case for which equilibrium unemployment has evolved hysteretically, as a result of a persistently changing exogenous determinant. That is, the evidence suggests that, for this sample, the level of oil prices shifted the Price Setting relation and thus equilibrium and actual unemployment in a persistent manner. Note that, these results are of course also consistent with "good hysteresis", i.e. where reductions in unemployment are equally persistent, and that they are based on the period after the two major oil price shocks in the 70s.

This paper provides scope for future research beyond applying the model in Sections 3 and 4 to other periods and/or countries. For example, one may focus on testing, on the one hand, hypotheses of various rigidities (e.g. menu costs) vis-a-vis the wage-formation-based hysteresis theories. Or one could test variants of the latter theories against each other, for example the hypotheses based on long-term unemployment in Layard and Nickell (1987) against the insider-outsider theories (Blanchard and Summers 1986, Lindbeck and Snower 1986). A potentially useful place to start, in order to pursue this further, would be to build a SVAR based on the framework in Layard et al. (2005), and Chapter 8, in particular. Their broad approach to unemployment has been widely adopted, and it allows one to incorporate all sorts of rigidities as well as both long-term unemployment - and insider-outsider-based explanations of sluggishly reacting labor markets. From a more econometric methodological point of view, one could extend the analysis in Section 3 to include I(2) in the two sub-systems of, respectively the counterfactual and the exogenous process, so that the classes of restricted CVARs corresponding to the remaining five Cases in Table 1 could be derived in block matrix representation.[36] There are many paths for future research to follow once the systemic aspect of unemployment hysteresis is taken seriously in the econometric analysis.

A Appendices

A.1 The partitioned Structural ECM and its ECM form

The point of departure is the SECM, (5),

$$A\Delta x_t = m_t + Fx_{t-1} - C\Delta x_{t-1} + \varepsilon_t. \tag{32}$$

[36]Some work on this is already in progress.

This has the following block representation,

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \Delta x_{1t} \\ \Delta x_{2t} \end{pmatrix} = \begin{pmatrix} m_{1t} \\ m_{2t} \end{pmatrix} + \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix} \begin{pmatrix} x_{1t-1} \\ x_{2t-1} \end{pmatrix} - \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix} \begin{pmatrix} \Delta x_{1t-1} \\ \Delta x_{2t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}, \tag{33}$$

corresponding to (1). The first line,

$$A_{11}\Delta x_{1t} = -A_{12}\Delta x_{2t} + m_{1t} + F_{11}x_{1t-1} + F_{12}x_{2t-1} - C_{11}\Delta x_{1t-1} - C_{12}\Delta x_{2t-1} + \varepsilon_{1t}, \tag{34}$$

gives the equations for the endogenous variables.

The corresponding reduced form ECM (see 6) becomes,

$$\begin{pmatrix} \Delta x_{1t} \\ \Delta x_{2t} \end{pmatrix} = \begin{pmatrix} \mu_{1t} \\ \mu_{2t} \end{pmatrix} + \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ 0 & \Pi_{22} \end{pmatrix} \begin{pmatrix} x_{1t-1} \\ x_{2t-1} \end{pmatrix} + \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix} \begin{pmatrix} \Delta x_{1t-1} \\ \Delta x_{2t-1} \end{pmatrix} + \begin{pmatrix} \upsilon_{1t} \\ \upsilon_{2t} \end{pmatrix}, \tag{35}$$

with,

$$\begin{aligned}
\mu_t &\equiv \begin{pmatrix} \mu_{1t} \\ \mu_{2t} \end{pmatrix} = \begin{pmatrix} A_{11}^{-1}\left(m_1 - A_{12}A_{22}^{-1}m_{2t}\right) \\ A_{22}^{-1}m_{2t} \end{pmatrix}, \\
\Pi &\equiv \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ 0 & \Pi_{22} \end{pmatrix} = \begin{pmatrix} A_{11}^{-1}F_{11} & A_{11}^{-1}\left(F_{12} - A_{12}A_{22}^{-1}F_{22}\right) \\ 0 & A_{22}^{-1}F_{22} \end{pmatrix}, \\
\Gamma_1 &\equiv \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix} = \begin{pmatrix} -A_{11}^{-1}C_{11} & -A_{11}^{-1}\left(C_{12} - A_{12}A_{22}^{-1}C_{22}\right) \\ 0 & -A_{22}^{-1}C_{22} \end{pmatrix},
\end{aligned}$$

and, Γ which is defined as $I - \Gamma_1$ is thus,

$$\Gamma \equiv \begin{pmatrix} I - \Gamma_{11} & -\Gamma_{12} \\ 0 & I - \Gamma_{22} \end{pmatrix} = \begin{pmatrix} I + A_{11}^{-1}C_{11} & A_{11}^{-1}\left(C_{12} - A_{12}A_{22}^{-1}C_{22}\right) \\ 0 & I + A_{22}^{-1}C_{22} \end{pmatrix}.$$

A.2 Analyzing the rank of Π when $r_1 < p_1$ and $r_2 < p_2$

The point of departure is (12), i.e.

$$\Pi \equiv \begin{pmatrix} \alpha_{11}\beta_{11}' & \Pi_{12} \\ 0 & \alpha_{22}\beta_{22}' \end{pmatrix},$$

for $r_1 < p_1$ and $r_2 < p_2$ (i.e. strictly reduced rank). Now, from basic linear algebra recall first that the rank of a matrix does not change when pre- and postmultiplying by full rank matrices (including elementary matrices). Therefore define the two $p \times p$ non-singular matrices,

$$K_1 = \begin{pmatrix} (\overline{\alpha}_{11}, \overline{\alpha}_{11\perp})' & 0 \\ 0 & (\overline{\alpha}_{22}, \overline{\alpha}_{22\perp})' \end{pmatrix} \text{ and } K_2 = \begin{pmatrix} (\overline{\beta}_{11}, \overline{\beta}_{11\perp}) & 0 \\ 0 & (\overline{\beta}_{22}, \overline{\beta}_{22\perp}) \end{pmatrix}, \tag{36}$$

where for a full rank matrix m_{ii}, $\overline{m}_{ii} = m_{ii}(m'_{ii}m_{ii})^{-1}$ with the same rank.

It then follows that,

$$K_1 \Pi K_2 = \begin{pmatrix} I_{r_1} & 0 & \overline{\alpha}'_{11} A_{11}^{-1}(F_{12}\overline{\beta}_{22} - A_{12}\alpha_{22}) & \overline{\alpha}'_{11} A_{11}^{-1} F_{12}\overline{\beta}_{22\perp} \\ 0 & 0 & \overline{\alpha}'_{11\perp} A_{11}^{-1}(F_{12}\overline{\beta}_{22} - A_{12}\alpha_{22}) & \overline{\alpha}'_{11\perp} A_{11}^{-1} F_{12}\overline{\beta}_{22\perp} \\ 0 & 0 & I_{r_2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{37}$$

Now make two elementary operations on this matrix, i.e. premultiply with E_1 to interchange the second and the third block line and then postmultiply with E_2 to interchange the second and last block column. From this it follows that,

$$r \equiv r(\Pi) = r(E_1 K_1 \Pi K_2 E_2) = r_1 + r_2 + s, \tag{38}$$

where, $s \equiv r(\overline{\alpha}'_{11\perp} A_{11}^{-1} F_{12}\overline{\beta}_{22\perp}) = r(\alpha'_{11\perp} A_{11}^{-1} F_{12}\beta_{22\perp}) = r(\delta'_{11\perp} F_{12}\beta_{22\perp})$.

Note that when $r_1 < p_1$ and $r_2 < p_2$ it always holds that, $r < p$. This is clear when $r(\alpha'_{11\perp} A_{11}^{-1} F_{12}\beta_{22\perp}) = 0$. When this is not the case it holds that $0 < r(\alpha'_{11\perp} A_{11}^{-1} F_{12}\beta_{22\perp}) \leq \min(p_1 - r_1, p_2 - r_2)$ which implies that $r \leq r_2 + p_1 < p_2 + p_1 = p$ for $p_1 - r_1 < p_2 - r_2$, while $r \leq r_1 + p_2 < p_1 + p_2 = p$, for $p_1 - r_1 > p_2 - r_2$.

A.3 The classes of CVARs for Cases I-III

A.3.1 The class of CVARs corresponding to pure slow adjustment (Case I)

Recall that $r = r(\Pi) = r_1 + p_2$, for $r_1 < p_1$. Since $\Pi_{11} = A_{11}^{-1} F_{11}$ and A_{11}^{-1} is non-singular the reduced rank restriction on Π_{11} is a reduced rank restriction on F_{11}. The latter can therefore be decomposed as $F_{11} = \delta_{11}\lambda'_{11}$, where δ_{11} and λ_{11} are $p_1 \times r_1$ of rank, r_1.

For $0 < r_1 < p_1$, Π can be decomposed as $\alpha\beta'$ with matrices given in (14). As for Π the rank of these matrices can be computed as the sum of the rank of the matrices on the diagonal since there is a non-singular matrix on the diagonal. Hence, $r(\alpha) = r(A_{11}^{-1}\delta_{11}) + r(\Pi_{22}) = r(\delta_{11}) + r(\Pi_{22}) = r_1 + p_2 = r$, and $r(\beta) = r(\lambda_{11}) + r(I_{p_2}) = r_1 + p_2 = r$.

The orthogonal complements can be computed as,

$$\alpha_\perp = \begin{pmatrix} I_{p_1} \\ -(\Pi_{12}\Pi_{22}^{-1})' \end{pmatrix} A'_{11}\delta_{11\perp} \text{ and } \beta_\perp = \begin{pmatrix} \lambda_{11\perp} \\ 0 \end{pmatrix}.$$

These are $p \times (p_1 - r_1)$ i.e. $p \times (p - r)$ and have full column rank, i.e. $r(\alpha_\perp) = r(\alpha'_\perp) = r(\delta'_{11\perp} A_{11}(I_{p_1}, -\Pi_{12}\Pi_{22}^{-1})) = r(\delta'_{11\perp}) = r(\delta_{11\perp}) = p_1 - r_1 = p - r$, which follows since $\delta_{11\perp}$ is the orthogonal complement of δ_{11} and $A_{11}(I_{p_1}, -\Pi_{12}\Pi_{22}^{-1})$ has full row rank. Defining the orthogonal complement of λ_{11} and (β_{11}), as $\lambda_{11\perp}$, it follows that, $r(\beta_\perp) = r(\lambda_{11\perp}) = p_1 - r_1 = p - r$.

Using that, $\alpha'_{11\perp} \equiv \delta'_{11\perp} A_{11}$, this implies that, $r(\alpha'_\perp(I - \Gamma_1)\beta_\perp) = r(\alpha'_{11\perp}(I - \Gamma_{11})\beta_{11\perp}) = p_1 - r_1 = p - r$, since, in this case the I(1) assumption for x_1^c holds (i.e. $r(\alpha'_{11\perp}(I - \Gamma_{11})\beta_{11\perp}) = p_1 - r_1$). In this case the I(1) assumption for x_1^c is thus necessary and sufficient for the the full process to be I(1).

The hysteresis equation is based on the structural long-run impact matrix which can be computed as,

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1 & -\mathcal{C}_1 F_{12} F_{22}^{-1} \\ 0 & 0 \end{pmatrix} \tag{39}$$

where $\mathcal{C}_1 \equiv \lambda_{11\perp} M^{-1} \delta'_{11\perp}$, for $M \equiv \delta'_{11\perp} A_{11}(I_{p_1} - \Gamma_{11})\lambda_{11\perp}$. Note that, $\mathcal{C}_1 = \beta_{11\perp} M^{-1} \alpha'_{11\perp} A_{11}^{-1}$, resembling the notation for the full system analysis (see 8).

For $r_1 = 0$ (and still $r_2 = p_2$), the orthogonal complements are $\beta_\perp = (I_{p_1}, 0)'$ and $\alpha_\perp = (I_{p_1}, -\Pi_{12}\Pi_{22}^{-1})'$ which both have full column rank, p_1. The long-run impact matrix is given by (39) with $\mathcal{C}_1 = (I_{p_1} - \Gamma_{11})^{-1} A_{11}^{-1}$.

A.3.2 The class of CVARs corresponding to equilibrium hysteresis (Case II)

In this case $r_1 = p_1$ and $0 \leq r_2 < p_2$, so that $r = p_1 + r_2$. Resembling the decomposition of Π_{11} in Appendix A.3.1, it is now Π_{22} that is decomposed as $\Pi_{22} = A_{22}^{-1}\delta_{22}\lambda'_{22}$.

For $0 < r_2 < p_2$, it follows that $r(\alpha) = r(A_{11}^{-1}) + r(A_{22}^{-1}\delta_{22}) = p_1 + r_2 = r$ and $r(\beta) = r(\beta') = r(F_{11}) + r(\lambda'_{22}) = p_1 + r_2 = r$. The orthogonal complements (of dimensions $p \times p_2 - r_2$) becomes,

$$\alpha_\perp = \begin{pmatrix} 0 \\ A'_{22}\delta_{22\perp} \end{pmatrix}, \beta_\perp = \begin{pmatrix} -F_{11}^{-1} F_{12} \lambda_{22\perp} \\ \lambda_{22\perp} \end{pmatrix}, \tag{40}$$

where $\delta_{22\perp}$ and $\lambda_{22\perp}$ are the orthogonal complements of δ_{22} and λ_{22}, respectively. These have full column rank, $p_2 - r_2$.

This implies that, $\alpha'_\perp(I - \Gamma_1)\beta_\perp = \alpha'_{22\perp}(I - \Gamma_{22})\beta_{22\perp}$, which has full rank, $p_2 - r_2$, by the I(1) assumption for x_2. In this case it is thus the latter assumption that is necessary and sufficient for the full process to be I(1).

Based on (40) the structural long-run impact matrix becomes,

$$\mathcal{C} = \begin{pmatrix} 0 & -F_{11}^{-1} F_{12} \mathcal{C}_2 \\ 0 & \mathcal{C}_2 \end{pmatrix}, \tag{41}$$

where $\mathcal{C}_2 = \lambda_{22\perp}(\delta'_{22\perp}A_{22}(I-\Gamma_{22})\lambda_{22\perp})^{-1}\delta'_{22\perp}$, which is the long-run impact matrix for the exogenous process. Using the above partitioning for the $p \times 1$ vectors, ST_t and e_u, the hysteresis equation (17) is implied.

For $r_2 = 0$, $\Pi_{22} = F_{22} = 0$ and $r = p_1$. Clearly the rank of $\alpha = ((A_{11}^{-1})', 0)'$ and $\beta = (F_{11}, F_{12})'$ is p_1.[37] In this case the orthogonal complements ($p \times p_2$) are given by (40) with $\delta_{22\perp} = \lambda_{22\perp} = I_{p_2}$, and the long-run impact matrix becomes (41) with $\mathcal{C}_2 = (I - \Gamma_{22})^{-1}A_{22}^{-1}$.

A.3.3 The combinations of slow adjustment and equilibrium hysteresis (Case III)

A.3.3.1 Case III$_a$ - The I(1) case To ease notation I do not use the decomposition of Π_{ii} where F_{ii} is decomposed as $\delta_{ii}\lambda'_{ii}$ for $i = 1, 2$ as Appendix A.3.1. To revert to this notation, simply insert, $\alpha_{ii} \equiv A_{ii}^{-1}\delta_{ii}$, $\beta_{ii} \equiv \lambda_{ii}$, $\alpha_{ii\perp} \equiv A'_{ii}\delta_{ii\perp}$ and $\beta_{ii\perp} \equiv \lambda_{ii\perp}$ for $i = 1, 2$, in the formulas below.

To compute Π (which now has rank $r = r_1 + r_2$) follow the approach in Appendix A.2. In (37) define $\widetilde{\Pi} \equiv K_1 \Pi K_2$, which has reduced rank, $r < p$, and can be decomposed as $\widetilde{\alpha}\widetilde{\beta}'$. Thus $\Pi \equiv K_1^{-1}\widetilde{\Pi}K_2^{-1} = K_1^{-1}\widetilde{\alpha}\widetilde{\beta}'K_2^{-1}$, from which, α and β in (19) can be found as $\alpha = K_1^{-1}\widetilde{\alpha}$ and $\beta' \equiv \widetilde{\beta}'K_2^{-1}$.

To see this, recalling (38) note first that, $s = 0 \Leftrightarrow \alpha'_{11\perp}A_{11}^{-1}F_{12}\beta_{22\perp} = 0 \Leftrightarrow \overline{\alpha}'_{11\perp}A_{11}^{-1}F_{12}\overline{\beta}_{22\perp} = 0$, and insert this into (37). This implies that $\widetilde{\alpha}$ and $\widetilde{\beta}'$ (which are both $p_1 + p_2 \times r_1 + r_2$) are,

$$\widetilde{\alpha} = \begin{pmatrix} I_{r_1} & 0 \\ 0 & *_3 \\ 0 & I_{r_2} \\ 0 & 0 \end{pmatrix}, \widetilde{\beta}' = \begin{pmatrix} I_{r_1} & 0 & *_1 & *_2 \\ 0 & 0 & I_{r_2} & 0 \end{pmatrix}, \tag{42}$$

where $*_1 \equiv \overline{\alpha}'_{11}A_{11}^{-1}(F_{12}\overline{\beta}_{22} - A_{12}\alpha_{22})$, $*_2 \equiv \overline{\alpha}'_{11}A_{11}^{-1}F_{12}\overline{\beta}_{22\perp}$ and $*_3 \equiv \overline{\alpha}'_{11\perp}A_{11}^{-1}(F_{12}\overline{\beta}_{22} - A_{12}\alpha_{22})$.

Now, from (42) it follows that $r(\widetilde{\alpha}) = r(I_{r_1}) + r(I_{r_2}) = r_1 + r_2 = r$ which follows from the elementary operation of interchanging the third and fourth block rows (which does not change the rank). For $\widetilde{\beta}$, it holds that $r(\widetilde{\beta}) = r(\widetilde{\beta}') = r(I_{r_1}) + r((0, I_{r_2}, 0)) = r_1 + r_2 = r$. Using the above definitions of $\widetilde{\alpha}$ and $\widetilde{\beta}$, this establishes that $r(\alpha) = r(\beta) = r$.

In order to find α and β note that the inverses of the transformation matrices, K_1

[37] For example, the latter follows since $r(F_{11}) = r(\Pi_{11}) = p_1$ so that F_{11} spans all of R^{p_1}. This means that F_{12} can be written as $F_{12} = F_{11}S$ for some $p_1 \times p_2$ matrix, S. Thus $r(\beta) = r(\beta') = r(F_{11}(I_{p_1}, S)) = p_1$ since F_{11} has full rank and (I_{p_1}, S) is an echelon matrix.

and K_2 in (36) can be written as,

$$K_1^{-1} = \begin{pmatrix} (\alpha_{11}, \alpha_{11\perp}) & 0 \\ 0 & (\alpha_{22}, \alpha_{22\perp}) \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{11\perp} & 0 & 0 \\ 0 & 0 & \alpha_{22} & \alpha_{22\perp} \end{pmatrix},$$

and,

$$K_2^{-1} = \begin{pmatrix} \begin{pmatrix} \beta'_{11} \\ \beta'_{11\perp} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \beta'_{22} \\ \beta'_{22\perp} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \beta'_{11} & 0 \\ \beta'_{11\perp} & 0 \\ 0 & \beta'_{22} \\ 0 & \beta'_{22\perp} \end{pmatrix}.$$

Using, $\alpha = K_1^{-1}\widetilde{\alpha}$ and $\beta' \equiv \widetilde{\beta}' K_2^{-1}$, it thus follows that,

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix} \text{ and } \beta' = \begin{pmatrix} \beta'_{11} & \beta'_{21} \\ 0 & \beta'_{22} \end{pmatrix}, \tag{43}$$

where $\alpha_{12} \equiv \alpha_{11\perp}\overline{\alpha}'_{11\perp}A_{11}^{-1}(F_{12}\overline{\beta}_{22} - A_{12}\alpha_{22})$ and $\beta'_{21} \equiv \overline{\alpha}'_{11}A_{11}^{-1}(F_{12} - A_{12}\alpha_{22}\beta'_{22})$. Multiplying α and β' gives Π in (12) as $\alpha_{11}\beta'_{11} = A_{11}^{-1}\delta_{11}\lambda'_{11} = \Pi_{11}$, $\Pi_{22} = \alpha_{22}\beta'_{22} = A_{22}^{-1}\delta_{22}\lambda'_{22}$ and $\Pi_{12} = \alpha_{11}\beta'_{21} + \alpha_{12}\beta'_{22} = A_{11}^{-1}(F_{12} - A_{12}\alpha_{22}\beta'_{22})$.

Inserting $\alpha_{ii} \equiv A_{ii}^{-1}\delta_{ii}$, $\beta_{ii} \equiv \lambda_{ii}$, $\alpha_{ii\perp} \equiv A'_{ii}\delta_{ii\perp}$ and $\beta_{ii\perp} \equiv \lambda_{ii\perp}$ and the implied $\overline{\alpha}_{ii}$, $\overline{\alpha}_{ii\perp}$ and $\overline{\beta}_{ii\perp}$ for $i = 1, 2$, into (43) gives (19) in the text.

Defining $Z_0 \equiv F_{12}\overline{\beta}_{22}\overline{\alpha}'_{22} - A_{12}$ and $Z_1 \equiv \overline{\beta}_{11}\overline{\alpha}'_{11}A_{11}^{-1}F_{12}$, the orthogonal complements of α and β in (43) can be written as,

$$\alpha'_\perp = \begin{pmatrix} \overline{\alpha}'_{11\perp} & -\overline{\alpha}'_{11\perp}A_{11}^{-1}Z_0 \\ 0 & \overline{\alpha}'_{22\perp} \end{pmatrix} \text{ and } \beta_\perp = \begin{pmatrix} \overline{\beta}_{11\perp} & -Z_1\overline{\beta}_{22\perp} \\ 0 & \overline{\beta}_{22\perp} \end{pmatrix}, \tag{44}$$

which can be found by finding $\widetilde{\alpha}_\perp$ and $\widetilde{\beta}_\perp$ first based on the definitions of $\widetilde{\alpha}$ and $\widetilde{\beta}$. It is straightforward to show that $\alpha'_\perp\alpha = 0$, $\beta'_\perp\beta = 0$ and that $r(\alpha_\perp) = r(\beta_\perp) = p - r$, so that these are in fact the orthogonal complements.

To verify that x_t is I(1) when $s = 0$, compute first, $\alpha'_\perp(I - \Gamma_1)\beta_\perp$, which must thus have full rank. This matrix product product can be written as,

$$\alpha'_\perp(I - \Gamma_1)\beta_\perp = \begin{pmatrix} \overline{\alpha}'_{11\perp}(I - \Gamma_{11})\overline{\beta}_{11\perp} & -\overline{\alpha}'_{11\perp}W\overline{\beta}_{22\perp} \\ 0 & \overline{\alpha}'_{22\perp}(I - \Gamma_{22})\overline{\beta}_{22\perp} \end{pmatrix}, \tag{45}$$

where $W \equiv (I - \Gamma_{11})Z_1 + \Gamma_{12} + A_{11}^{-1}Z_0(I - \Gamma_{22})$. Taking determinants gives $\det(\alpha'_\perp(I - \Gamma_1)\beta_\perp) = \det(\overline{\alpha}'_{11\perp}(I - \Gamma_{11})\overline{\beta}_{11\perp})\det(\overline{\alpha}'_{22\perp}(I - \Gamma_{22})\overline{\beta}_{22\perp}) \neq 0$ where the latter follows as for this case the I(1) assumptions for x^c_{1t} and x_{2t} in (18) hold. I.e. this shows that $s = 0$ is sufficient for $x_t \sim$ I(1). Necessity (in addition to sufficiency) is shown in Appendix A.3.3.2.

From this the structural long-run impact matrix in (8) can be computed as,

$$\mathcal{C} \equiv \begin{pmatrix} \mathcal{C}_1 & \breve{\mathcal{C}}_{12} \\ 0 & \mathcal{C}_2 \end{pmatrix}, \tag{46}$$

where $\mathcal{C}_1 \equiv C_1 A_{11}^{-1}$, is the structural long-run impact matrix (see equation 8) for the x_1^c process, with $C_1 \equiv \beta_{11\perp} \left(\alpha'_{11\perp}(I - \Gamma_{11})\beta_{11\perp}\right)^{-1} \alpha'_{11\perp}$, and where $\breve{\mathcal{C}}_{12} \equiv \mathcal{C}_1 A_{11} W \mathcal{C}_2 - \mathcal{C}_1 F_{12} \overline{\beta}_{22} \overline{\alpha}'_{22} A_{22}^{-1} - \overline{\beta}_{11} \overline{\alpha}'_{11} A_{11}^{-1} F_{12} \mathcal{C}_2$, with $\mathcal{C}_2$ is defined accordingly for x_{2t}.[38]

The Hysteresis equation therefore becomes,

$$u_t = \mathcal{U}_t + \gamma'_1 ST_{1t} + \gamma'_2 ST_{2t}, \tag{47}$$

where $\gamma'_1 \equiv e'_{1u} \mathcal{C}_1$ and $\gamma'_2 \equiv e'_{1u} \breve{\mathcal{C}}_{12}$.

Inserting the definitions of α_{ii} and β_{ii} etc. (see the above) the hysteresis equation (20) can be derived.

The hysteresis equations for the pure Cases I and II are special cases of this. To see this note that for Case I $\mathcal{C}_2 = 0$, $\beta_{22} = I_{p_2}$, $\alpha_{22} = \Pi_{22}$ (full rank) so that $\overline{\alpha}'_{22} A_{22}^{-1} = F_{22}^{-1}$. Inserting this into (46) gives (39), and the result follows. For Case II, $\alpha_{11} = \Pi_{11}$, (full rank), $\beta_{11} = I_{p_1}$ so that $\overline{\alpha}'_{11} = F_{11}^{-1} A_{11}$ and $\mathcal{C}_1 = 0$. Inserting this into (46) gives (41), and the result follows.

To find the parameters, α and β, and the hysteresis equation when $r_2 = 0$, while $0 < r_1 < p_1$, set $\Pi_{22} = 0$ $(p_2 \times p_2)$ and use the transformation matrices K_i in (36) where $(\overline{\alpha}_{22}, \overline{\alpha}_{22\perp})'$ and $(\overline{\beta}_{22}, \overline{\beta}_{22\perp})$ are each replaced by I_{p_2}.

A.3.3.2 Case III$_b$ - The I(2) case The remaining sub-cases of Case III are defined for, $0 < s \leq \min(p_1 - r_1, p_2 - r_2)$. As before I analyze the model first based on, α_{ii}, β_{ii} etc. and then at the end all expressions are stated in terms of the parameters, δ_{ii} and λ_{ii} etc. (see the above).

To find α and β for $0 < s < \min(p_1 - r_1, p_2 - r_2)$ and $0 < r_i < p_i$, $i = 1, 2$, recall first (37) in Appendix A.2 for which the $(p_1 - r_1) \times (p_2 - r_2)$ matrix, $\overline{\alpha}'_{11\perp} A_{11}^{-1} F_{12} \overline{\beta}_{22\perp}$, is no longer zero but has now rank $s > 0$. This can therefore be decomposed into,

$$\overline{\alpha}'_{11\perp} A_{11}^{-1} F_{12} \overline{\beta}_{22\perp} = HG' \tag{48}$$

where H is $(p_1 - r_1) \times s$ and G is $(p_2 - r_2) \times s$, both with rank, s, obeying $0 < s \leq \min(p_1 - r_1, p_2 - r_2)$. Following the approach in Appendix A.3.3.1 the $\widetilde{\Pi}$ matrix in (37) that results

[38] In this derivation it has been used that $\det(\overline{\alpha}'_{ii\perp}(I - \Gamma_{ii})\overline{\beta}_{ii\perp}) \neq 0$, that $\left(\overline{\alpha}'_{ii\perp}(I - \Gamma_{ii})\overline{\beta}_{ii\perp}\right)^{-1} = \beta'_{ii\perp} C_i \alpha_{ii\perp}$ for $i = 1, 2$, that $C_1 \alpha_{11\perp} \overline{\alpha}'_{11\perp} = C_1$ and $\overline{\beta}_{22\perp} \beta'_{22\perp} C_2 = C_2$.

when (48) is inserted can be decomposed into $\widetilde{\alpha}\widetilde{\beta}'$ with, $*_3 \equiv \overline{\alpha}'_{11\perp} A_{11}^{-1}(F_{12}\overline{\beta}_{22} - A_{12}\alpha_{22})$.

$$\widetilde{\alpha} = \begin{pmatrix} I_{r_1} & 0 & 0 \\ 0 & *_3 & H \\ 0 & I_{r_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \widetilde{\beta}' = \begin{pmatrix} I_{r_1} & 0 & *_1 & *_2 \\ 0 & 0 & I_{r_2} & 0 \\ 0 & 0 & 0 & G' \end{pmatrix}, \tag{49}$$

which are both $p \times r = p \times r_1 + r_2 + s$, and full rank, $r_1 + r_2 + s$. As in Appendix A.3.3.1 α and β' can then be computed as, $\alpha = K_1^{-1}\widetilde{\alpha}$ and $\beta' \equiv \widetilde{\beta}' K_2^{-1}$, i.e.,

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{11\perp}\overline{\alpha}'_{11\perp}A_{11}^{-1}Z_0\alpha_{22} & \alpha_{11\perp}H \\ 0 & \alpha_{22} & 0 \end{pmatrix} \text{ and} \tag{50}$$

$$\beta' = \begin{pmatrix} \beta'_{11} & \overline{\alpha}'_{11}A_{11}^{-1}(F_{12} - A_{12}A_{22}^{-1}F_{22}) \\ 0 & \beta'_{22} \\ 0 & G'\beta'_{22\perp} \end{pmatrix},$$

which both have full rank (Appendix A.3.3.1). Inserting $\alpha_{ii} \equiv A_{ii}^{-1}\delta_{ii}$, $\beta_{ii} \equiv \lambda_{ii}$, $\alpha_{ii\perp} \equiv A'_{ii}\delta_{ii\perp}$ and $\beta_{ii\perp} \equiv \lambda_{ii\perp}$ and the implied $\overline{\alpha}_{ii}$, $\overline{\alpha}_{ii\perp}$ and $\overline{\beta}_{ii\perp}$ for $i = 1, 2$, into (43) gives (21) in the text.[39]

Defining the orthogonal complements of H and G, as respectively, $H_\perp$ ($p_1 - r_1 \times p_1 - r_1 - s$) and $G_\perp$ ($p_2 - r_2 \times p_2 - r_2 - s$) both full rank matrices, the orthogonal complements of α and β become,

$$\alpha'_\perp = \begin{pmatrix} H'_\perp\overline{\alpha}'_{11\perp} & -H'_\perp\overline{\alpha}'_{11\perp}A_{11}^{-1}Z_0 \\ 0 & \alpha'_{22\perp} \end{pmatrix} \text{ and } \beta_\perp = \begin{pmatrix} \beta_{11\perp} & -Z_1\overline{\beta}_{22\perp}G_\perp \\ 0 & \overline{\beta}_{22\perp}G_\perp \end{pmatrix}, \tag{51}$$

where Z_0 and Z_1 are defined as in Appendix A.3.3.1. It is straightforward to show orthogonality. To see that $r(\beta_\perp) = p - r$ note that $r(\beta_\perp) \leq (p_1 - r_1) + (p_2 - r_2 - s)$, i.e. the number of columns, and that due to the zero block in the lower left corner of $\beta_\perp$ it also holds that, $r(\beta_\perp) \geq r(\beta_{1\perp}) + r(\overline{\beta}_{22\perp}G_\perp) = (p_1 - r_1) + (p_2 - r_2 - s)$. Thus, $r(\beta_\perp) = (p_1 - r_1) + (p_2 - r_2 - s) = (p_1 + p_2) - (r_1 + r_2 + s) = p - r$. This argument also holds for $\alpha_\perp$ and so $r(\alpha_\perp) = p - r$.

To show that x_t is no longer I(1) and in fact I(2) note first that it follows from (51) that,

$$\alpha'_\perp(I - \Gamma_1)\beta_\perp = \begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}, \tag{52}$$

where $N_1 \equiv H'_\perp\overline{\alpha}'_{11\perp}(I - \Gamma_{11})\beta_{11\perp}$, $N_2 \equiv -H'_\perp\overline{\alpha}'_{11\perp}W\overline{\beta}_{22\perp}G_\perp$, $N_3 \equiv \alpha'_{22\perp}(I - \Gamma_{22})\overline{\beta}_{22\perp}G_\perp$ and W which is defined in Appendix A.3.3.1. To compute the rank of $\alpha'_\perp(I - \Gamma_1)\beta_\perp$,

[39] Note that, $\overline{\alpha}'_{11}A_{11}^{-1}(F_{12} - A_{12}A_{22}^{-1}F_{22}) = \overline{\alpha}'_{11}\Pi_{12}$.

note that by definition of this case, the I(1) assumption for x_{1t}^c and x_{2t} (i.e. condition (18) for $i = 1$ and 2, respectively) holds. This implies, respectively, that, $r(N_1) = r(H'_\perp) = p_1 - r_1 - s$ and $r(N_3) = r(G_\perp) = p_2 - r_2 - s$. Next, define the elementary matrix $\bar{\mathsf{E}} = \begin{pmatrix} E & 0 \\ 0 & I_{p_2-r_2-s} \end{pmatrix}$, where $E = (E_1, E_{1\perp})$ with E_1 and $E_{1\perp}$ being respectively $p_1 - r_1 \times p_1 - r_1 - s$ and $p_1 - r_1 \times s$ both with full rank, and E_1 is constructed such that $N_1 E_1$ consists of the $p_1 - r_1 - s$ linearly independent columns of N_1.

Now, since $r(\bar{\mathsf{E}}) = p_1 - r_1 + p_2 - r_2 - s = p - r$, it follows that $r(\alpha'_\perp(I - \Gamma_1)\beta_\perp) = r(\alpha'_\perp(I - \Gamma_1)\beta_\perp \bar{\mathsf{E}})$, i.e. that,

$$r(\alpha'_\perp(I - \Gamma_1)\beta_\perp) = r(\begin{pmatrix} N_1 E_1 & (N_1 E_{1\perp}, N_2) \\ 0 & (0, N_3) \end{pmatrix}), \tag{53}$$

which has the non-singular block, $N_1 E_1$, on the diagonal, implying that,

$$r(\alpha'_\perp(I - \Gamma_1)\beta_\perp) = r(N_1 E_1) + (0, N_3) = p - r - s < p - r, \tag{54}$$

for $s > 0$ recalling that $r = r_1 + r_2 + s$. This shows that, as is already shown in Appendix A.3.3.1, not only is $s = 0$ (i.e. $\delta'_{11\perp} F_{12} \lambda_{22\perp} = 0$) sufficient for $x_t \sim$ I(1) it is also necessary, since according to (54) if $s > 0$ then $r(\alpha'_\perp(I - \Gamma_1)\beta_\perp) < p - r$ and so I(1) is lost. Note that it holds that $p - r - s \leq r(\alpha'_\perp(I - \Gamma_1)\beta_\perp) \leq p - r$.

The following Theorem summarizes the main findings so far.

Theorem 1 *For the general Case III with,* $0 < r_i < p_i$, *and for which (18) holds, for* $i = 1, 2$, *and no roots are inside the complex unit disc, the condition defining Case* III_a, $s = 0$ *i.e.* $\delta'_{11\perp} F_{12} \lambda_{22\perp} = 0$, *is equivalent to* x_t *being I(1) with cointegration parameters given in (19).*

It can be shown that the contents of this theorem resemble the analysis in Mosconi and Rahbek (1997). In particular, their equation (8) can be written as, $A_{11}^{-1} F_{12} = \alpha_{11} \varphi' \beta'_{22\perp} + a_{12} \beta'_{22}$, where $\varphi' \equiv \overline{\alpha}'_{11} A_{11}^{-1} F_{12} \overline{\beta}_{22\perp}$ and $a_{12} = A_{11}^{-1} F_{12} \overline{\beta}_{22}$, using the present notation. It is immediately seen that this restriction is sufficient for $\overline{\alpha}'_{11\perp} A_{11}^{-1} F_{12} \overline{\beta}_{22\perp} = 0$ (and $\delta'_{11\perp} F_{12} \lambda_{22\perp} = 0$), i.e. $s = 0$ (see Appendix A.2). To prove that the reverse is also true one can use the two orthogonal projections, $\alpha_{11\perp} \overline{\alpha}'_{11\perp} + \alpha_{11} \overline{\alpha}'_{11} = I_{p_1}$ and $\beta_{22\perp} \overline{\beta}'_{22\perp} + \beta_{22} \overline{\beta}'_{22} = I_{p_2}$. From this it follows that $A_{11}^{-1} F_{12} = (\alpha_{11\perp} \overline{\alpha}'_{11\perp} + \alpha_{11} \overline{\alpha}'_{11}) A_{11}^{-1} F_{12} (\beta_{22\perp} \overline{\beta}'_{22\perp} + \beta_{22} \overline{\beta}'_{22})$, which reduces to $\alpha_{11} \varphi' \beta'_{22\perp} + a_{12} \beta'_{22}$, when $s = 0$, i.e. $\overline{\alpha}'_{11\perp} A_{11}^{-1} F_{12} \overline{\beta}_{22\perp} = 0$.

Thus, for $s > 0$, $x_t \sim I(d)$ with $d > 1$, and in fact $d = 2$ for this particular model. Thus, since the above I(1) parametrization is no longer valid, the purpose is now to derive the I(2) parameters to be tested, including the MA representation (from which the hysteresis equation can be derived). I follow the notation in Johansen (1996) with the *warning* that there, s denotes the rank of $\alpha'_\perp(I - \Gamma_1)\beta_\perp$ while here $s \equiv r(\overline{\alpha}'_{11\perp} A_{11}^{-1} F_{12} \overline{\beta}_{22\perp})$.

When deriving the I(2) parametrization note first that α and β are still given by (50) and their respective orthogonal complements by (51). Since $\alpha'_\perp(I-\Gamma_1)\beta_\perp$ has reduced rank $= p-r-s < p-r$ (cf. 54) this is decomposed as, $\alpha'_\perp(I-\Gamma_1)\beta_\perp = \xi\eta'$ where ξ and η are both $p-r \times p-r-s$ with full column rank. Corresponding to this the orthogonal complements are $\xi_\perp$ and $\eta_\perp$ both $p-r \times s$ and full column rank. Given these parameters the remaining I(2) parameters can be derived: In particular, $a_1 \equiv \overline{\alpha}_\perp\xi$, $a_2 \equiv \alpha_\perp\xi_\perp$, $b_1 \equiv \overline{\beta}_\perp\eta$ and $b_2 \equiv \beta_\perp\eta_\perp$.[40]

Using the N_i notation above, one finds that,

$$\xi = \begin{pmatrix} I_{p_1-r_1-s} & -H'_\perp\overline{\alpha}'_{11\perp}(I-\Gamma_{11})Z_1\overline{\beta}_{22\perp}G_\perp \\ 0 & N_3 \end{pmatrix}, \text{ and}$$

$$\eta' = \begin{pmatrix} N_1 & \eta'_{21} \\ 0 & I_{p_2-r_2-s} \end{pmatrix},$$

where $\eta'_{21} \equiv -H'_\perp\overline{\alpha}'_{11\perp}(W-(I-\Gamma_{11})Z_1)\overline{\beta}_{22\perp}G_\perp$. As just shown $r(N_3) = r(G_\perp) = p_2 - r_2 - s$ and $r(N_1) = r(H'_\perp) = p_1 - r_1 - s$ which imply, respectively, that $r(\xi) = r(I_{p_1-r_1-s}) + r(N_3) = p-r-s$ and $r(\eta) = r(N_1) + r(I_{p_2-r_2-s}) = p-r-s$. From this it follows that, $\xi'_\perp = (0, G'Z_2)$ and $\eta'_\perp = (H'Z_3, 0)$, where $Z_2 \equiv (\beta'_{22\perp}\beta_{22\perp})(\alpha'_{22\perp}(I_{p_2}-\Gamma_{22})\beta_{22\perp})^{-1}$ and $Z'_3 \equiv (\alpha'_{11\perp}(I_{p_1}-\Gamma_{11})\beta_{11\perp})^{-1}(\alpha'_{11\perp}\alpha_{11\perp})$. Thus, $r(\xi_\perp) = r(G) = s$ and $r(\eta_\perp) = r(H) = s$.

From these matrices it follows that $a'_2 = (0, G'\beta'_{22\perp}\mathcal{C}_2 A_{22})$ and $b'_2 = (H'\alpha'_{11\perp}A'_{11}\mathcal{C}'_1, 0)$.

To compute a_1 and b_1, $\overline{\alpha}_\perp$ and $\overline{\beta}_\perp$ are needed, respectively. The latter parameters can be written as,

$$\overline{\beta}_\perp = \begin{pmatrix} \overline{\beta}_{11\perp} & -Z_1W_1Z_5^{-1} \\ 0 & W_1Z_5^{-1} \end{pmatrix}, \tag{55}$$

where $W_1 \equiv \overline{\beta}_{22\perp}G_\perp$ and $Z_5 \equiv G'_\perp\overline{\beta}'_{22\perp}(I_{p_2} + Z'_1Z_1)\overline{\beta}_{22\perp}G_\perp$ whereas,

$$\overline{\alpha}_\perp = \begin{pmatrix} V_1Z_6^{-1} & V_1Z_6^{-1}V'_1V_2\overline{\alpha}_{22\perp} \\ -P_{\alpha_{22}}V'_2V_1Z_6^{-1} & \overline{\alpha}_{22\perp} - P_{\alpha_{22}}V'_2V_1Z_6^{-1}V'_1V_2\overline{\alpha}_{22\perp} \end{pmatrix},$$

where $V_1 \equiv \overline{\alpha}_{11\perp}H_\perp$, $V_2 \equiv A_{11}^{-1}Z_0$ and $Z_6 \equiv V'_1(I_{p_1} + V_2P_{\alpha_{22}}V'_2)V_1$, with the projection matrix, $P_{\alpha_{22}} \equiv \overline{\alpha}_{22}\alpha'_{22}$.

It can be shown that $\beta'_\perp\overline{\beta}_\perp = I_{p-r}$ and that $\overline{\beta}_\perp\beta'_\perp$ is symmetric, so that $\overline{\beta}_\perp$ is the Moore-Penrose inverse (henceforth the MP inverse) of $\beta'_\perp$ which is unique and has the form $\beta_\perp(\beta'_\perp\beta_\perp)^{-1}$. A similar reasoning goes for $\overline{\alpha}_\perp$.

It follows that,

$$b_1 = \begin{pmatrix} \overline{\beta}_{1\perp}N'_1 - Z_1W_1Z_5^{-1}\eta_{21} & -Z_1W_1Z_5^{-1} \\ W_1Z_5^{-1}\eta_{21} & W_1Z_5^{-1} \end{pmatrix}, \tag{56}$$

[40] Note that, a_i and b_i are denoted α_i and β_i, repectively, in Johansen (1996), p. 57.

and that,

$$a_1 = \begin{pmatrix} V_1 Z_6^{-1} & V_1 Z_6^{-1} V_1' \Lambda W_1 \\ -P_{\alpha_{22}} V_2' V_1 Z_6^{-1} & [P_{\alpha_{22\perp}}(I_{p_2} - \Gamma_{22}) - P_{\alpha_{22}} V_2' V_1 Z_6^{-1} V_1' \Lambda] W_1 \end{pmatrix}, \tag{57}$$

where $\Lambda \equiv V_2 P_{\alpha_{22\perp}}(I_{p_2} - \Gamma_{22}) - (I - \Gamma_{11}) Z_1$.[41]

In order to derive the I(2) MA representation and eventually the hysteresis equation, one must check the I(2) condition (see p. 58 in Johansen 1996) which is here that,

$$a_2' \theta b_2, \tag{58}$$

has full rank (equal to s) and where $\theta \equiv (I - \Gamma_1)\overline{\beta}\overline{\alpha}'(I - \Gamma_1) + \Gamma_1$. Thus one needs to compute $\overline{\beta}$ and $\overline{\alpha}$ (i.e. the unique MP inverses).

I first derive $\overline{\beta}$ in sufficient detail and then $\overline{\alpha}$, since apart from a few elementary operations, the structure of α identical to that of β (see below). Since $\overline{\beta} \equiv \beta(\beta'\beta)^{-1}$ where β is given in (50) compute first $\beta'\beta$. This becomes a 3×3 block matrix which can be partitioned into a 2×2 block matrix instead given by,

$$\beta'\beta = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}, \tag{59}$$

where $\mathfrak{A} \equiv \beta_{11}'\beta_{11} + \beta_{21}'\beta_{21}$, $\mathfrak{B} \equiv (\beta_{21}'\beta_{22}, \beta_{21}'\beta_{22\perp}G)$, $\mathfrak{C} \equiv \mathfrak{B}'$ and $\mathfrak{D}$ is a block diagonal matrix with, respectively $\beta_{22}'\beta_{22}$ (upper left block) and $G'\beta_{22\perp}'\beta_{22\perp}G$ as its diagonal blocks. The matrix β_{21} is defined as in Case III$_a$ (see equation 19). Define, $\mathfrak{F} \equiv \mathfrak{A} - \mathfrak{B}\mathfrak{D}^{-1}\mathfrak{C} = \beta_{11}'\beta_{11} + \beta_{21}' V \beta_{21}$, with $V \equiv \overline{\beta}_{22\perp} G_\perp [G_\perp'(\beta_{22\perp}'\beta_{22\perp})^{-1} G_\perp]^{-1} G_\perp' \overline{\beta}_{22\perp}'$ noting that $\mathfrak{D}$ is non-singular. Then $\mathfrak{F}$ $(r_1 \times r_1)$ is also non-singular. To see this note that V is idempotent and symmetric implying that $\mathfrak{F}$ can be written as $\mathfrak{F} = \mathfrak{J}'\mathfrak{J}$ with $\mathfrak{J}' \equiv (\beta_{11}', \beta_{21}'V')$. It thus follows that $r(\mathfrak{F}) = r(\mathfrak{J}'\mathfrak{J}) = r(\mathfrak{J}) = r(\mathfrak{J}') = r_1$. The inverse of (59) can thus be computed using the formulae in e.g. Abadir and Magnus (2005) exercise 5.16, and premultiplying this with β' gives,

$$\overline{\beta} = \begin{pmatrix} \beta_{11}\mathfrak{F}^{-1} & -\beta_{11}\mathfrak{F}^{-1}\beta_{21}'\overline{\beta}_{22} & -\beta_{11}\mathfrak{F}^{-1}\beta_{21}'\overline{X} \\ V\beta_{21}\mathfrak{F}^{-1} & (I_{p_2} - V\beta_{21}\mathfrak{F}^{-1}\beta_{21}')\overline{\beta}_{22} & (I_{p_2} - V\beta_{21}\mathfrak{F}^{-1}\beta_{21}')\overline{X} \end{pmatrix}, \tag{60}$$

where $\overline{X} \equiv X(X'X)^{-1}$ with $X \equiv \beta_{22\perp}G$. [42]

[41] It is straightforward to show that b_1, b_2 and β are mutually orthogonal and span all of $\mathbb{R}^p$. The same holds for a_1, a_2 and α.

[42] As the computations may become rather cumbersome when computing $\overline{\beta}$ from the formulae $\beta(\beta'\beta)^{-1}$, it is sometimes easier simply to guess some "$\overline{\beta}$ matrix", $\overline{\beta}_{Guess}$, based on e.g. $\beta'\overline{\beta}_{Guess} = I_r$ and then make sure that this is the MP inverse of β'. If this is so then since we know that the latter is unique and has the form $\beta(\beta'\beta)^{-1}$ in this case we are sure that $\overline{\beta}_{Guess} = \overline{\beta} \equiv \beta(\beta'\beta)^{-1}$. Note that, if $\beta'\overline{\beta}_{Guess} = I_r$ it only remains to show that $\overline{\beta}_{Guess}\beta'$ is symmetric to ensure that $\overline{\beta}_{Guess}$ is the Moore-Penrose inverse of β'.

To compute $\overline{\alpha}$, the idea is to exploit, that, when making a few elementary operations on α, we obtain a matrix which "looks like β". We may then modify the formulae in (60) to compute $\overline{\alpha}$. To make sure that one has found $\alpha(\alpha'\alpha)^{-1}$ one can then check whether $\overline{\alpha}$ is the MP inverse of α' (See footnote 42). It follows that,

$$\overline{\alpha} = \begin{pmatrix} \overline{\alpha}_1 & \underline{V}\alpha_{12}\underline{\mathfrak{F}}^{-1} & (I_{p_1} - \underline{V}\alpha_{12}\underline{\mathfrak{F}}^{-1}\alpha_{12}')\overline{Y} \\ 0 & \alpha_{22}\underline{\mathfrak{F}}^{-1} & -\alpha_{22}\underline{\mathfrak{F}}^{-1}\alpha_{12}'\overline{Y} \end{pmatrix}, \tag{61}$$

where $\overline{Y} \equiv Y(Y'Y)^{-1}$ with $Y \equiv \alpha_{11\perp}H$ and $\underline{\mathfrak{F}} = \alpha_{22}'\alpha_{22} + \alpha_{12}'\underline{V}\alpha_{12}$ with $\alpha_{12} \equiv \alpha_{11\perp}\overline{\alpha}_{11\perp}'A_{11}^{-1}Z_0\alpha_{22}$ and $\underline{V} \equiv \overline{\alpha}_{11\perp}H_\perp(H_\perp'(\alpha_{11\perp}'\alpha_{11\perp})^{-1}H_\perp)^{-1}H_\perp'\overline{\alpha}_{11\perp}'$. From this one can show that $\alpha'\overline{\alpha} = I_r$ and that $\overline{\alpha}\alpha'$ is symmetric so that $\overline{\alpha}$ is the MP inverse of α' and thus equal to $\alpha(\alpha'\alpha)^{-1}$.

The next step is to compute (58), which becomes, $a_2'\theta b_2 = G'\beta_{22\perp}'\mathcal{C}_2A_{22}\theta_3\mathcal{C}_1A_{11}\alpha_{11\perp}H$, where θ_3 is the $p_2 \times p_1$ sub-matrix of θ in the lower left corner, which is thus the only part of θ that needs to be computed. Defining $\mathfrak{v}_2$ as the second block row of $\overline{\beta}$ (see 60) and $\mathfrak{w}_1$ as the first block row of $\overline{\alpha}$ (see 61) it follows that $\theta_3 = (I_{p_2} - \Gamma_{22})\mathfrak{v}_2\mathfrak{w}_1'(I_{p_1} - \Gamma_{11})$ and hence, that $a_2'\theta b_2$ can be written as the product of, $G'\beta_{22\perp}'\mathcal{C}_2A_{22}(I_{p_2} - \Gamma_{22})\mathfrak{v}_2$ and $\mathfrak{w}_1'(I_{p_1} - \Gamma_{11})\mathcal{C}_1A_{11}\alpha_{11\perp}H$. Computations are facilitated by first simplifying each of these terms (block matrices) separately. It follows that,

$$a_2'\theta b_2 = I_s, \tag{62}$$

and therefore that for this case the process is always I(2), i.e. independently of the remaining parameters.

From (62) it follows that the structural long-run impact matrix (of which "C_2" in Johansen (1996), p. 58 is the corresponding reduced form matrix) corresponding to the I(2) trend $(\sum_{s=1}^{t}\sum_{i=1}^{s}\varepsilon_i)$ shock has the simple form,

$$\Psi = b_2a_2'A^{-1} = \begin{pmatrix} 0 & \mathcal{C}_1F_{12}\mathcal{C}_2 \\ 0 & 0 \end{pmatrix}. \tag{63}$$

A full analytic solution (i.e. as a function of the above block matrices) of the matrix corresponding to the I(1) stochastic trends, which I denote Φ (of which "C_1" in Johansen (1996), p. 58 is the corresponding reduced form matrix) can be found from using the second and third relations after (4.28) in Johansen (1996), p. 58. This is however rather cumbersome and not pursued further here since this matrix is not estimated directly but rather depends on the above parameters which are estimated. We can, however, characterize this matrix in sufficiently detail. In particular, due to the exogenous assumption of x_{2t}, Φ will have the form,

$$\Phi = \begin{pmatrix} \mathcal{C}_1 & \Phi_{12} \\ 0 & \mathcal{C}_2 \end{pmatrix}, \tag{64}$$

where Φ_{12} can be found from solving the equations in Johansen (1996) referred to above.[43]

Finally, the hysteresis equation can be derived. To distinguish between the integration order (here 1 and 2) of a stochastic trend at time t, introduce the notation, $ST_{j1t} \equiv \sum_{i=1}^{t} \varepsilon_{ji}$, for the first order stochastic trend (I(1)), and, $ST_{j2t} \equiv \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_{ji}$, for the second order stochastic trend (I(2)), for $j = 1, 2$ is the variable index. The γ coefficients will resemble this notation, i.e. γ_{j1} and γ_{j2} correspond to ST_{j1t} and ST_{j2t}, respectively. Using this and (63) and (64) the hysteresis equation will have the form as stated in (22) in the text.

A.4 The UK example

The matrices

In terms of the matrices in the structural ECM form, (5), the example

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\phi_2 & 1 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 1 & 0 & 0 \\ 0 & 0 & -\omega_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{65}$$

$$F = \begin{pmatrix} \kappa_1 + \kappa_2 - 1 & \rho_1 & \rho_{21} + \rho_{22} & \rho_5 + \rho_6 & \rho_3 + \rho_4 \\ \phi_2 + \phi_3 + \phi_4 & \kappa_3 + \kappa_4 - 1 & 0 & 0 & 0 \\ 0 & \lambda_1 + \lambda_2 + \lambda_3 & \kappa_5 + \kappa_6 - 1 & 0 & 0 \\ \omega_5 + \omega_6 & 0 & \omega_3 + \omega_4 & \omega_1 + \omega_2 - 1 & 0 \\ 0 & 0 & 0 & 0 & \pi_2 + \pi_3 - 1 \end{pmatrix},$$

$$C = \begin{pmatrix} \kappa_2 & 0 & \rho_{22} & \rho_6 & \rho_4 \\ \phi_4 & \kappa_4 & 0 & 0 & 0 \\ 0 & \lambda_3 & \kappa_6 & 0 & 0 \\ \omega_6 & 0 & 0 & \omega_2 & 0 \\ 0 & 0 & 0 & 0 & \pi_3 \end{pmatrix} \text{ and } m_t = \begin{pmatrix} \rho_0 - g_A t + z_t^p \\ \phi_1 + z_t^d \\ \lambda_0 + z_t^u \\ \omega_0 + z_t^w \\ \pi_1 + z_t^o \end{pmatrix}.$$

From this the reduced form parameters of (6) and the block matrices used throughout the text can be computed straightforwardly. Note that, $F_{22} = \pi_2 + \pi_3 - 1 = \Pi_{22}$ (since $A_{22} = 1$) and that $C_{22} = \pi_3$.

[43] That the first block element (Φ_{11}) is $\mathcal{C}_1$ (corresponding to the x_1^c process) must hold which can be realized from setting all ε_{2i} shocks to zero for all i.

Accumulated multiplier notation

To simplify the main expressions in the text the following accumulated-mulitplier notation may be used.

Table 5: Accummulated multipliers corresponding to the UK example.

Definition of multiplier	Expected sign:
$M_{p,y} \equiv \frac{\rho_1}{1-\kappa_1-\kappa_2}$	?
$M_{p,u} \equiv \frac{\rho_{21}+\rho_{22}}{1-\kappa_1-\kappa_2}$	?
$M_{p,p^o} \equiv \frac{\rho_3+\rho_4}{1-\kappa_1-\kappa_2}$	> 0 ($= 1$ under homogeneity)
$M_{p,w} \equiv \frac{\rho_5+\rho_6}{1-\kappa_1-\kappa_2}$	> 0 ($= 1$ under homogeneity)
$M_{y,p} \equiv \frac{\phi_2+\phi_3+\phi_4}{1-\kappa_3-\kappa_4}$	< 0
$M_{u,y} \equiv \frac{\lambda_1+\lambda_2+\lambda_3}{1-\kappa_5-\kappa_6}$	< 0
$M_{w,p} \equiv \frac{\omega_5+\omega_6}{1-\omega_1-\omega_2}$	> 0 ($= 1$ under homogeneity)
$M_{w,u} \equiv \frac{\omega_3+\omega_4}{1-\omega_1-\omega_2}$	< 0

Description of the data

Table 6: Descrption of the UK data.

Name and definition	Details:	Source:
p	log. of GDP deflator	Office for National Statistics, UK (Ecowin)
y	log. of real GDP	Office for National Statistics, UK (Ecowin)
u	log. of unemployment rate	IMF - International Financial Statistics
w	log. of hourly wage rate	IMF - International Financial Statistics
p^{poil}	log. of crude oil prices (Dubai spot)*	OECD

* The annual series were used to construct quarterly series (by simple interpolation)

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