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Discounting Models for Outcomes over Continuous Time*

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Abstract

Events that occur over a period of time can be described either as sequences of outcomes at discrete times or as functions of outcomes in an interval of time. This paper presents discounting models for events of the latter type. Conditions on preferences are shown to be satisfied if and only if the preferences are represented by a function that is an integral of a discounting function times a scale defined on outcomes at instants of time.

Key words: continuous time, integral discounting, integral value or utility function.

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1. Introduction

An analyst who is developing an evaluation of alternatives in a decision model must judge whether to model the consequences of the alternatives as outcomes that occur at discrete times, to be called outcome sequences, or as outcomes that occur over intervals of time, to be called outcome streams. Outcome sequences are modeled as functions defined on a set of discrete times and outcome streams are modeled as functions defined on a set that is an interval of times. The judgment as to which type of model to use depends on the nature of the data, the nature of the consequences, and the proclivities of the analyst. Each type seems more appropriate under some circumstances.

This paper is concerned with outcome streams. The outcomes are numbers or vectors x, and the outcome streams are real- or vector-valued functions $\mathbf{x}$ defined on instants of time t in a planning period P. At an instant t, the value $x = \mathbf{x}(t)$ of an outcome stream $\mathbf{x}$ is a rate or an amount, or it is a vector of rates and amounts. For example, an outcome $\mathbf{x}(t)$ might be: the rate of usage of a natural resource, one or more rates of monetary costs and benefits, an amount that measures a type of environmental quality, or the amounts of multiple attributes that describe the health of an individual.

Comparisons between outcome streams will be modeled as a relation $\succsim$, that is, as a set of statements $\mathbf{x} \succsim \mathbf{y}$. We will regard a statement $\mathbf{x} \succsim \mathbf{y}$ as meaning that $\mathbf{x}$ is at least as good as $\mathbf{y}$ in some sense. Relations of strict preference, $\mathbf{x} \succ \mathbf{y}$, and of indifference, $\mathbf{x} \sim \mathbf{y}$, are to be defined in terms of $\succsim$. We will say that a function $V(\mathbf{x})$ represents a relation $\succsim$ on a set of outcome streams provided that: $V(\mathbf{x}) \geq V(\mathbf{y})$ if and only if $\mathbf{x} \succsim \mathbf{y}$ for any $\mathbf{x}, \mathbf{y}$ in the set. Following the usage of decision analysis rather than that of economics, we will call such a function a value function rather than a utility function.

A relation $\succsim$ defined on outcome streams can be interpreted either as comparisons of wellbeing due to the outcome streams, i.e., as hedonic comparisons, or as preferences between the outcome streams. Suppose, for example, that the outcomes, $x = (x_1, \dots, x_N)$, are average global concentrations of N greenhouse gases. Then, $\succsim$ can represent either comparisons of the effects (in some sense) of the outcome streams $\mathbf{x}$ or as the preferences (of some group) between the outcome streams. For brevity, we will call $\succsim$ a preference relation. In choosing this terminology, we must emphasize that in a particular application a preference relation may or may not have a preference interpretation.

For a set of alternatives of any type (outcome streams or some other type), we define a preference model as a mathematical result which shows that if a preference relation on the alternatives satisfies specified conditions then there exists a specified type of function that represents the relation. We prefer the term, preference model, to the term, representation theorem, for two reasons; it is limited to the study of preference relations (unlike the use of representation theorem in mathematics) and it gives weight to the other parts of the model, e.g., to the set of alternatives and to the conditions on the relation. By the above definition, a model that includes a function to represent a preference relation but does not state conditions on the relation that imply the representation is not a preference model.

There are well-known preference models for outcome sequences. On the one hand, there are linear models, i.e., models in which the value function is a linear function, $V(\mathbf{x}) = \sum_t a(t)\, \mathbf{x}(t)$, of the outcomes $\mathbf{x}(t)$; e.g., Williams and Nassar (1965). On the other hand, there are additive models, i.e., models in which the value function has the additive form, $V(\mathbf{x}) = \sum_t a(t)\, v(\mathbf{x}(t))$; e.g., Koopmans (1960, 1972), Diamond (1965), and Harvey (1986, 1995). And there are more special models (e.g., some of the models referenced above) in which the discount weights $a(t)$ form a negative-exponential sequence.

In linear models, the outcomes are single-attribute and $v(x)$ is the identity function. These models do not include such issues as: multiattribute tradeoffs, decreasing marginal utility, and equity between outcomes at different times and perhaps for different persons.

The results in this paper are preference models for outcome streams. They provide value functions that have the integral form:

$$V(\mathbf{x}) \ = \ \int_P a(t)\, v(\mathbf{x}(t))\, dt, \tag{1}$$

where $a(t)$ and $v(x)$ are functions that have specified properties. A preference model having this type of value function will be called an integral discounting model.

By analogy with the additive value function, $V(\mathbf{x}) = \sum a(t)\ v(\mathbf{x}(t))$, in a preference model for outcome sequences, we will call a value function (1) an integral value function. The function $v(x)$ will be called an outcome scale, and the function $a(t)$ will be called a discounting function. An outcome scale $v(x)$ represents preferences between outcomes x

at an instant of time, and a discounting function $a(t)$ represents tradeoffs between amounts of the outcome scale at different times; in particular, it compares an outcome scale amount at a future time to the same scale amount at the present time.

We present two integral discounting models. In the first model, the planning period P on which the outcome streams are defined is a bounded interval $[0, T]$, $T > 0$, and the value function is a Riemann integral, $V(\mathbf{x}) = \int_0^T a(t)\, v(\mathbf{x}(t))\, dt$. And in the second model, P is the unbounded interval $(0, \infty)$, and the value function is an improper Riemann integral, $V(\mathbf{x}) = \lim_{T \to \infty} \int_0^T a(t)\, v(\mathbf{x}(t))\, dt$. These models are intended for prescriptive applications. We expect that there are important behavioral violations of many of the conditions on a preference relation that are an essential part of the models.

In each model, an outcome stream $\mathbf{x}$ is a real-valued or vector-valued function that is piecewise continuous and bounded in the following sense: $\mathbf{x}$ is continuous except for at most a finite number of times, and there are outcomes a, b such that $a \le \mathbf{x}(t) \le b$ for any time t. If the outcomes are multiattribute, i.e., $x = (x_1, \dots, x_N)$, $N > 1$, the boundedness condition is to be satisfied for each attribute. This class of outcome streams seems to be large enough for essentially any application.

Each model implies that the outcome scale $v(x)$ is continuous and that the discounting function $a(t)$ is positive and non-increasing. Also, the model for an unbounded planning period implies that the integral, $\lim_{T \to \infty} \int_0^T a(t)\, dt$, is finite. Neither $a(t)$ nor $v(x)$ is otherwise restricted; for example, $a(t)$ is not required to be constant or to be a negative-exponential function, and $v(x)$ is not required to be a linear function (when the outcomes are single-attribute) or to be an additive function (when the outcomes are multiattribute).

On the one hand, this generality allows an analyst to include a variety of preference issues in applying one of these models. For example, in a public policy study with long range implications he can use a discounting function that decreases more slowly than an exponential function and thereby assigns appreciable importance to the distant future.

On the other hand, the generality of these models creates a need for special models in which there are additional conditions on a preference relation that it will satisfy if and only if one of the functions $a(t)$, $v(x)$ has a special form (e.g., $a(t)$ has an exponential form or $v(x)$ has an additive form). Harvey (1998a,b) discusses several such additional conditions.

Harvey (1998a,b) also discusses procedures for assessing a discounting function or an outcome scale and procedures for using a value function $V(\mathbf{x})$ to evaluate an outcome stream. Some of these conditions and procedures are analogous to well-known conditions and procedures for a discounting model for outcome sequences. But some are not; they utilize the fact that the planning period is a continuum rather than a discrete set of times.

A well-known evaluation procedure, both for outcome sequences and for outcome streams, is that of net present value. For a given sequence or stream, one uses the value function to calculate an indifferent sequence or stream in which the only non-zero outcome occurs at the present time. For an outcome stream $\mathbf{x}$, one can choose a unit interval, e.g., a year, and calculate an outcome x_0 such that $\mathbf{x}$ is indifferent to the outcome stream $\mathbf{x}_0$ defined by: $\mathbf{x}_0(t) = x_0$ for $0 \leq t \leq 1$ and $\mathbf{x}_0(t) = 0$ for $t > 1$. Another procedure, perhaps opposite to this one, is to calculate a constant outcome stream that is indifferent to $\mathbf{x}$.

Proofs are in Appendix A. They use only classical real analysis and do not involve abstract measure and integration theory.

2. Previous research

It is surprising that integral discounting models were not developed long ago—and many readers may assume that they have been. This section discusses previous research on related models and describes how they differ from integral discounting models. The discussion involves abstract mathematics and can be omitted without a loss of continuity.

There are three models, or more accurately groups of models, that are closely related to the models in this paper. Each of these models is not an integral discounting model—and for a different reason for each group of models. We are not aware of any previous integral discounting models, i.e., any preference models having a value function (1).

Using our notation and terminology as a lingua franca, the models are as follows.

(1) Weibull (1985) presents a model for outcome streams in which the value function has the linear form, $V(\mathbf{x}) = \int_P a(t)\, \mathbf{x}(t)\, d\mu$. The model is a preference model, but it is not sufficiently general to be an integral discounting model as we have defined the term. It corresponds to linear models for outcome sequences. Like those models, it excludes multivariable outcomes and types of preferences represented by an outcome scale.

Here, the triple $(P, \mathcal{F}, \mu)$ is an abstract measure space; the set X of outcomes is an interval $(-\infty, 0]$, $[0, \infty)$, or $(-\infty, \infty)$; and the set C of outcome streams $\mathbf{x}$ is a convex cone in the space $L^1(P, \mu)$ of Lebesgue integrable functions defined on P.

(2) Grodal and Mertens (1968) and Grodal (2003) present a model having a function, $V(\mathbf{x}) = \int_P a(t)\, v(\mathbf{x}(t))\, d\mu$, of the form (1). It is shown that if a preference relation $\succsim$ on a set C of outcome streams satisfies certain conditions then there exists such a function $V(\mathbf{x})$ such that: $V(\mathbf{x}) \geq V(\mathbf{y})$ implies $\mathbf{x} \succsim \mathbf{y}$ for $\mathbf{x}, \mathbf{y}$ in C. However, the model does not show the converse implication, that: $\mathbf{x} \succsim \mathbf{y}$ implies $V(\mathbf{x}) \geq V(\mathbf{y})$ for $\mathbf{x}, \mathbf{y}$ in C. Hence, $V(\mathbf{x})$ may fail to be a value function, and thus the model is not a preference model.

Here, $(P, \mathcal{F}, \mu)$ is an abstract measure space; the set of outcomes is a metric space; and the set C is a space of Lebesgue integrable functions that is closed under mixtures.

(3) Savage (1954, 1972), Fishburn (1970, 1982), Wakker (1985, 1989, 1993), Kopylov (2010) and others present preference models for what is often called subjective expected utility (SEU). Here, a decision maker chooses an alternative, called an act; a state of nature occurs; and the decision maker receives a consequence which is a function of his choice and the state of nature. A preference relation is defined on the acts, and conditions on the preference relation are introduced. In our notation, the models show that if the preference relation satisfies the conditions then there exists a probability set-function π defined on subsets of the set P of states of nature and a utility function $u(x)$ defined on consequences such that acts $\mathbf{x}$ having greater expected utility, $U(\mathbf{x}) = \int_P u(\mathbf{x}(t))\, d\pi$, are preferred.

The SEU models can be reinterpreted as models concerning outcome streams. Suppose that the states of nature are interpreted as times t in an interval P with Lebesgue measure λ and that the acts are interpreted as outcome streams $\mathbf{x}$. Then, the SEU models infer the existence of a value function of the form, $V(\mathbf{x}) = \int_P v(\mathbf{x}(t))\, d\pi$, where π is an infered set-function (without a probability interpretation) defined on a family of subsets of P and $v(x)$ is an infered function (without a utility interpretation) defined on a set X of outcomes.

The set-function π in an SEU model may or may not have a derivative $d\pi = a(t)\, dt$ for some function $a(t)$. The situation is that $V(\mathbf{x}) = \int_P v(\mathbf{x}(t))\, d\pi$ does not have the form, $V(\mathbf{x}) = \int_P a(t)\, v(\mathbf{x}(t))\, dt$, unless π is absolutely continuous with respect to λ and thus has a Radon-Nikodym derivative $a(t)$ with respect to λ. Therefore, the SEU models do not provide a discounting function $a(t)$, and thus they are not integral discounting models.

One can argue to the contrary that integral discounting models are a variation of the SEU models since the class of value functions of the form, $V(\mathbf{x}) = \int_P v(\mathbf{x}(t))\, d\pi$, includes the class of value functions of the form, $V(\mathbf{x}) = \int_P a(t)\, v(\mathbf{x}(t))\, dt$, namely those in which the measure π is absolutely continuous with respect to Lebesgue measure. But to derive a discounting model from an SEU model, one must define additional conditions on the preference relation in the SEU model and show that the previous and new conditions together imply the existence of a value function that has the discounting form (1).

In conclusion, the development of abstract integral discounting models appears to be an open research question.

The models (1)–(3) also differ from those in this paper with respect to an important feature not mentioned above. If the set P in a model (1)–(3) is chosen as an interval of time and the set C of outcome streams is chosen to contain the continuous, bounded outcome streams, then C will also contain some outcome streams that are discontinuous at an uncountable number of times. Appendix B provides a discussion of this feature.

The feature is important for applications because it may be impossible to visualize an outcome stream that is discontinuous at an uncountable number of times (e.g., the index function for a Cantor set). Thus, it would be impossible to judge whether the conditions on preferences are satisfied by such functions.

For the special case in which preferences satisfy the condition of stationarity, Kopylov (2010) constructs an exponential discounting model for a set C of so-called step outcome streams. (Harvey, 1995 presents a similar model and a model with a general function $a(t)$ for sets of outcome sequences.) Kopylov also suggests that the stationarity condition can be added to his SEU model for larger sets C. Like his SEU model, such a model would have the feature described above.

3. Outcome streams and preference relations

This section defines the objects that are assumed in an integral value model, namely outcomes, outcome streams, and preference relations on outcomes and outcome streams.

3.1. Outcome streams

An outcome is a vector $x = (x_1, \ldots, x_N)$ of amounts of one or more attributes. Each variable x_j is defined on an interval X_j, $j = 1, \ldots, N$, and the set of outcomes is the product set, $X = X_1 \times \ldots \times X_N$. The set X will be called an outcome set.

A planning period P is an interval of times t. We will consider bounded intervals of the form, $P = [0, T]$, $0 < T < \infty$, and the unbounded interval, $P = [0, \infty)$.

Definition 1a. An outcome stream is a real- or vector-valued function $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$, $N \geq 1$, whose domain is a planning period P, whose values are in an outcome set X, and which has the following properties:

(i) $\mathbf{x}$ is piecewise continuous, i.e., for each $j = 1, \ldots, N$ the component function $\mathbf{x}_j$ is continuous except for at most a finite number of times t.

(ii) $\mathbf{x}$ is bounded, i.e., for each $j = 1, \ldots, N$ there are amounts a_j, b_j in the interval X_j such that $a_j \leq \mathbf{x}_j(t) \leq b_j$ for any time t.

Constant outcome streams will be denoted by letters at the beginning of the alphabet. Thus, an outcome stream $\mathbf{a}$ is to have the outcome a for any time, and so forth.

A subinterval of P with endpoints s, s' will be denoted by $\langle s, s' \rangle$. Here, an endpoint $0 \leq s \leq s' \leq \infty$ may or may not be in $\langle s, s' \rangle$. For two outcome streams $\mathbf{x}, \mathbf{y}$ and an interval $\langle s, s' \rangle$, $(\mathbf{x}_{\langle s, s' \rangle}, \mathbf{y})$ will denote the outcome stream such that $(\mathbf{x}_{\langle s, s' \rangle}, \mathbf{y})(t) = \mathbf{x}(t)$ for t in $\langle s, s' \rangle$ and $(\mathbf{x}_{\langle s, s' \rangle}, \mathbf{y})(t) = \mathbf{y}(t)$ otherwise. And for outcome streams $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and disjoint intervals $\langle s, s' \rangle$, $\langle t, t' \rangle$, $(\mathbf{x}_{\langle s, s' \rangle}, \mathbf{y}_{\langle t, t' \rangle}, \mathbf{z})$ will have a similar meaning. An outcome stream $(\mathbf{x}_{\langle s, s' \rangle}, \mathbf{y})$ will be called a splicing of $\mathbf{x}$ and $\mathbf{y}$, and a set C of outcome streams such that $(\mathbf{x}_{\langle s, s' \rangle}, \mathbf{y})$ is in C for any $\mathbf{x}, \mathbf{y}$ in C will be called closed under splicing.

Definition 1b. An outcome stream set is a set of outcome streams that contains the constant outcome streams and is closed under splicing.

The distance between two outcomes x, y will be defined as, $|x - y| = \Sigma_j |x_j - y_j|$, and the distance between two outcome streams $\mathbf{x}, \mathbf{y}$ for a bounded planning period $P =$ $[0, T]$ will be defined as, $\int |\mathbf{x} - \mathbf{y}| = \Sigma_j \int_0^T |x_j(t) - y_j(t)|\, dt$.

We assume that there is a distinguished outcome which can be interpreted as a zero amount or rate or as a vector of zero amounts or rates. This null outcome will be denoted by o, and the constant outcome stream with this outcome will be denoted by $\mathbf{o}$. In an application, o can be defined in a manner that is specific for that application.

3.2. Preference relations

A pair of preference relations, one on outcomes and the other on outcome streams, can be defined in two ways: (i) Define a relation $\succsim$ on outcome streams and then define a relation $\succsim^X$ on outcomes in terms of $\succsim$, or (ii) Define two relations, $\succsim^X$ and $\succsim$, and then require $\succsim$ to agree with $\succsim^X$ in some sense. The two methods can be shown to be equivalent. We will use the second method because outcome streams are defined in terms of outcomes and thus it seems more natural to begin with outcomes.

Definition 2. Suppose that X is an outcome set and that C is an outcome stream set with outcomes in X. A preference relation $\succsim^X$ on X is a set of statements $x \succsim^X y$ for x, y in X, and a preference relation $\succsim$ on C is a set of statements $\mathbf{x} \succsim \mathbf{y}$ for $\mathbf{x}, \mathbf{y}$ in C. The pair $(X, \succsim^X)$ will be called an <u>outcome space</u> provided that it has non-indifferent outcomes. Then, the pair $(C, \succsim)$ will be called an <u>outcome stream space</u>.

4. Conditions on preferences

This section presents conditions on preferences in an outcome stream space $(C, \succsim)$. In constructing the integral discounting models, C will be a variety of sets. Definition 2 above enables us to state the conditions once rather than several times.

(A) $\succsim$ <u>agrees</u> with $\succsim^X$ on C: For any $\mathbf{x}, \mathbf{y}$ in C,

(a) If $\mathbf{x}(t) \succsim^X \mathbf{y}(t)$ for all t in P except for at most a finite number, then $\mathbf{x} \succsim \mathbf{y}$.

(b) If $\mathbf{x}(t) \succsim^X \mathbf{y}(t)$ for all t in P except for at most a finite number and $\mathbf{x}(t) \succ^X \mathbf{y}(t)$ on a non-point interval in P, then $\mathbf{x} \succ \mathbf{y}$.

By a non-point interval, we mean an interval $\langle s, s' \rangle$ such that $s < s'$. In the case that $s = s'$, $\langle s, s' \rangle$ is either a point interval $[s, s]$ or the empty interval.

(B) $\succsim^X$ is <u>complete</u> and <u>transitive</u> on X, and $\succsim$ is <u>complete</u> and <u>transitive</u> on C:

(a) For any x, y, z in X: $x \succsim^X y$ or $y \succsim^X x$, and if $x \succsim^X y$ and $y \succsim^X z$ then $x \succsim^X z$.

(b) For any $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in C: $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$, and if $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{z}$ then $\mathbf{x} \succsim \mathbf{z}$.

(C) $\succsim$ is <u>continuous</u> on C: For any $\mathbf{x}$ in C and any constant outcome stream $\mathbf{a}$,

(a) If $\mathbf{a} \prec \mathbf{x}$, then there is a $\delta > 0$ such that $|c - a| < \delta$ implies $\mathbf{c} \prec \mathbf{x}$ for all c in X.

(b) If $\mathbf{a} \succ \mathbf{x}$, then there is a $\delta > 0$ such that $|c - a| < \delta$ implies $\mathbf{c} \succ \mathbf{x}$ for all c in X.

(D) $\succsim$ is <u>preferentially independent</u> on C: For any $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ in C and any bounded interval $\langle s, s' \rangle$, $(\mathbf{w}_{\langle s, s' \rangle}, \mathbf{x}) \succsim (\mathbf{w}_{\langle s, s' \rangle}, \mathbf{y})$ implies $(\mathbf{z}_{\langle s, s' \rangle}, \mathbf{x}) \succsim (\mathbf{z}_{\langle s, s' \rangle}, \mathbf{y})$.

Condition (D) states that if two outcome streams are equal during an interval $\langle s, s' \rangle$, then the common outcome stream in $\langle s, s' \rangle$ can be changed to another common outcome stream in $\langle s, s' \rangle$ without changing the comparison.

Conditions analogous to (D) play an essential role in additive value models such as those of Debreu (1960) and Gorman (1968). In brief, imagine that a planning period $P = [0, T]$ is partitioned into subintervals, $\langle t_{i-1}, t_i \rangle$, $i = 1, \ldots, m$, such that $\langle t_{i-1}, t_i \rangle = \langle s, s' \rangle$ for some i. By (D), preferences between two step outcome streams (see Appendix) with the same outcome in $\langle s, s' \rangle$ do not depend on the common outcome in $\langle s, s' \rangle$. Thus, the subintervals in the partition have the same role as attributes in an additive value model.

The conditions below involve outcome streams with only two or three outcomes. As above, we denote constant outcome streams by letters at the beginning of the alphabet.

For two disjoint, non-point intervals, $\langle s, s' \rangle$ and $\langle t, t' \rangle$, two outcomes $a^- \prec^X a^+$ in $\langle s, s' \rangle$ and two outcomes $b^- \prec^X b^+$ in $\langle t, t' \rangle$ will be called <u>tradeoffs pairs</u> provided that $(\mathbf{a}^+_{\langle s, s' \rangle}, \mathbf{b}^-_{\langle t, t' \rangle}, \mathbf{o}) \sim (\mathbf{a}^-_{\langle s, s' \rangle}, \mathbf{b}^+_{\langle t, t' \rangle}, \mathbf{o})$. Intuitively, a person is just willing to receive a^- instead of a^+ in order to receive b^+ instead of b^-.

For three outcomes $a_0 \prec^X a_{1/2} \prec^X a_1$ in $\langle s, s' \rangle$, $a_{1/2}$ will be called a <u>mid-outcome</u> of a_0, a_1 provided that there are outcomes $b^- \prec^X b^+$ in $\langle t, t' \rangle$ such that $a_0, a_{1/2}$ and b^-, b^+ are tradeoffs pairs and $a_{1/2}, a_1$ and b^-, b^+ are tradeoffs pairs with respect to $\langle s, s' \rangle$ and $\langle t, t' \rangle$. Intuitively, a person is willing to worsen the outcome in $\langle t, t' \rangle$ by the same amount in order to improve the outcome in $\langle s, s' \rangle$ either from a_0 to $a_{1/2}$ or from $a_{1/2}$ to a_1.

(E) $\succsim$ is <u>mid-outcome independent</u> on C: For any disjoint, non-point intervals $\langle s, s' \rangle$ and $\langle t, t' \rangle$ and outcomes $a_0 \prec^X a_1$, if the pair a_0, a_1 in $\langle s, s' \rangle$ has a mid-outcome with respect to outcomes in $\langle t, t' \rangle$, and the pair a_0, a_1 in $\langle t, t' \rangle$ has a mid-outcome with respect to outcomes in $\langle s, s' \rangle$, then a_0, a_1 has the same mid-outcomes in each case.

Condition (E) implies that the outcome scale $v(x)$ does not depend on time. A variety of similar conditions in additive models for multiattribute outcomes or outcome sequences are described in Fishburn (1970), Krantz et al. (1972, p. 305), and Harvey (1986, 1995).

(F) For any outcomes $a \succsim^X b$ and any times $s \leq t$, $(\mathbf{a}_{\langle s,\, s+\Delta\rangle}, \mathbf{b}_{\langle t,\, t+\Delta\rangle}, \mathbf{o}) \succsim (\mathbf{b}_{\langle s,\, s+\Delta\rangle}, \mathbf{a}_{\langle t,\, t+\Delta\rangle}, \mathbf{o})$ for disjoint intervals $\langle s, s+\Delta\rangle, \langle t, t+\Delta\rangle$.

Intuitively, condition (F) states that a person prefers a better event a to occur sooner rather than later. Indifference is not excluded. Similar conditions for preferences between outcome sequences are defined in Koopmans (1972) and Harvey (1986, 1995).

(G) For any time $s > 0$, there exist outcomes $a \prec^X b \prec^X c$ such that $(\mathbf{c}_{[s-\Delta,\, s]}, \mathbf{a}) \succsim (\mathbf{b}_{[0,\,\Delta]}, \mathbf{a})$ for $0 < \Delta < \frac{1}{2} s$.

Intuitively, condition (G) states that the greater improvement from a to c at the future time s is preferred to the lesser improvement from a to b at the present time. Similar conditions in models for outcome sequences are not used because the discount weights $a(t)$ in a sum, $V(\mathbf{x}) = \sum_t a(t)\, v(\mathbf{x}(t))$, are finite and positive.

Conditions (F) and (G) exclude opposite extremes in discounting. (F) excludes valuing the present less than a future time, and (G) excludes valuing the present infinitely more than a future time. Each of these exclusions seems appropriate for prescriptive purposes.

Condition (F) implies that the discounting function $a(t)$ is non-increasing, and (G) implies that it is bounded at the origin. These conditions can be omitted—as is done in Harvey and Østerdal (2007). If (G) is omitted, then the Riemann integral, $\int_0^T a(t)\, v(\mathbf{x}(t))\, dt$, generalizes to an improper Riemann integral, $\lim_{s \to 0} \int_s^T a(t)\, v(\mathbf{x}(t))\, dt$. And if both (F) and (G) are omitted, then $\int_0^T a(t)\, v(\mathbf{x}(t))\, dt$ generalizes to a Lebesgue integral.

5. Model for a bounded planning period

This section presents the following integral discounting model for outcome streams defined on a bounded planning period $P = [0, T]$, $0 < T < \infty$.

Theorem 1. An outcome stream space $(X_T, \succsim)$ satisfies conditions (A)-(G) if and only if it has a value function of the form,

$$V(\mathbf{x}) = \int_0^T a(t)\, v(\mathbf{x}(t))\, dt, \quad \mathbf{x} \text{ in } X_T. \tag{2}$$

such that the Riemann integral (2) exists for any $\mathbf{x}$ in X_T and:

(a) The function $v(x)$, x in X, is a continuous value function for $(X, \succsim^X)$ which has a non-point interval range and the value $v(\text{o}) = 0$.

(b) The function $a(t)$, $0 \leq t \leq T$, is positive and non-increasing.

(c) The function $A(t) = \int_0^t a(s)\, ds$, $0 \leq t \leq T$, is strictly increasing.

Moreover, each of the functions $v(x)$, $A(t)$ is unique up to a positive multiple.

As remarked in the introduction, $v(x)$ will be called an outcome scale and $a(t)$ will be called a discounting function. $A(t)$ will be called a cumulative discounting function.

Harvey (1998a,b) and Harvey and Østerdal (2007) present related models in which condition (C) is stronger and (F), (G) are not present. In those models, the discounting function $a(t)$ is Lebesgue integrable and thus the integral $V(\mathbf{x})$ is a Lebesgue integral.

As presented in the Appendix, we construct a value function (2) by two extensions of the set of outcome streams on which the preference relation is defined. This construction is parallel to the construction described below for Riemann integration itself.

Suppose that a planning period $P = [0, T\,]$ is partitioned into disjoint subintervals, $\langle t_0, t_1 \rangle, \ldots, \langle t_{m-1}, t_m \rangle$, where $0 = t_0 \leq t_1 \leq \ldots \leq t_{m-1} \leq t_m = T$. An outcome stream that is constant on each subinterval $\langle t_{i-1}, t_i \rangle$, $i = 1, \ldots m$, will be called a <u>step outcome stream</u>.

To construct our model, we first interpret the additive model of Debreu (1960) as a model with an additive value function $V_p(\mathbf{x})$ for the set of step outcome streams defined by a single partition p. The first extension is to extend this additive value model to a model for the larger set S_T of all step outcome streams defined by any partition.

The second extension is to extend the model for the set S_T to a model for the set X_T. This extension relies on approximating an outcome stream by step outcome streams. More specifically, it relies on the squeeze property of Riemann integration, i.e., the equivalence of Riemann's definition in 1854 of the integral of a function $f(t)$ and Darboux' definition in 1875 of the integral of $f(t)$ as the common limit of sequences of sums associated with step functions $\ell_n(t)$, $u_n(t)$, $n = 1, 2, \ldots$, which are defined such that $\ell_n(t) \leq f(t) \leq u_n(t)$ for all t and the distances $\int |u_n - \ell_n|$ tend to zero as n tends to infinity.

6. Model for an unbounded planning period

This section presents an integral discounting model for outcome streams defined on the unbounded planning period, $P = [0, \infty)$. As part of the model, we introduce the following condition on a preference space $(X_\infty, \succsim)$.

(H) For any $\mathbf{x}$ in X_∞, any non-point interval $[s, s']$, and any $a \prec^X b$, there is a $T \geq s'$ such that $(\mathbf{a}_{[s,s']}, \mathbf{x}) \precsim (\mathbf{b}_{[s,s']}, \mathbf{x}, \mathbf{o}_{(t,\infty)})$ and $(\mathbf{a}_{[s,s']}, \mathbf{x}, \mathbf{o}_{(t,\infty)}) \precsim (\mathbf{b}_{[s,s']}, \mathbf{x})$ for all $t > T$.

Intuitively, condition (H) states that an outcome stream is arbitrarily unimportant in the sufficiently distant future. Thus, (H) is a counterpart to condition (G) on the importance of an outcome stream in the very near future.

Theorem 2. An outcome stream space $(X_\infty, \succsim)$ satisfies conditions (A)-(H) if and only if it has a value function of the form,

$$V(\mathbf{x}) \;=\; \lim_{T\to\infty} \int_0^T a(t)\, v(\mathbf{x}(t))\, dt \tag{3}$$

such that the improper Riemann integral (2) exists for any $\mathbf{x}$ in X_∞ and the functions $v(x)$, $a(t)$, and $A(t)$ have the properties (a)-(c) in Theorem 1 for all $0 \leq t < \infty$.

Moreover, each of the functions $v(x)$ and $A(t)$ is unique up to a positive multiple.

In this model, a discounting function $a(t)$ has a finite integral, $\lim_{T\to\infty} \int_0^T a(t)\, dt$, since X_∞ includes the constant outcome streams. Therefore, the model includes most types of discounting but excludes discounting in which $a(t)$ decreases very slowly as a function of time. For example, it includes constant discounting, i.e., discounting with a negative-exponential discounting function, $a(t) = \exp(-r\, t)$, $r > 0$, but it excludes non-discounting, i.e., discounting with the constant function, $a(t) = 1$.

The model includes some but not all types of discounting in which the discount rate, $r(t) = -\,a'(t) \,/\, a(t)$, is decreasing. Consider, for example, the discounting functions, $a(t) = (1 + b\, t)^{-k}$, with the parameters $b > 0$, $k > 0$, discussed in Harvey (1998a,b). The model includes such discounting functions with $k > 1$ but not with $k \leq 1$.

Harvey (1998a,b) and Harvey and Østerdal (2007) also present models for $P = [0, \infty)$. There, the set C of outcome streams to be compared is not the same for any preference relation (e.g., the set X_∞ in Theorem 2) but is defined in terms of the preference relation.

Appendix: Proofs of Results

Lemma A1. If an outcome stream space $(C, \succsim)$ satisfies conditions (A)-(C), then:

(a) Either there exists an outcome $a \succ^X o$ or there exists an outcome $a \prec^X o$.

(b) For any outcomes a, b, $a \succsim^X b$ if and only if $(\mathbf{a}_{\langle s, s' \rangle}, \mathbf{x}) \succsim (\mathbf{b}_{\langle s, s' \rangle}, \mathbf{x})$ for any $\mathbf{x}$ in C and any non-point interval $\langle s, s' \rangle$.

(c) $\succsim^X$ is continuous in the sense that for any outcomes $a \prec^X b$ there exists a $\delta > 0$ such that: $|c - a| < \delta$ implies $c \prec^X b$ and $|c - b| < \delta$ implies $c \succ^X a$ for all outcomes c.

(d) There exists a continuous value function for $\succsim^X$.

(e) Any continuous value function for $\succsim^X$ has a non-point interval range.

(f) For any $\mathbf{x}$ in C, there exists a constant outcome stream $\mathbf{a}$ with $\mathbf{a} \sim \mathbf{x}$.

Proof. Definition 2 implies that there is an outcome a that is not indifferent to o. Then, $a \succ^X o$ or $a \prec^X o$ since $\succsim^X$ is complete by condition (B).

To show (b), first assume $a \succsim^X b$. Then, $(\mathbf{a}_{\langle s, t \rangle}, \mathbf{x}) \succsim (\mathbf{b}_{\langle s, t \rangle}, \mathbf{x})$ by condition (A.a). Next, assume $(\mathbf{a}_{\langle s, t \rangle}, \mathbf{x}) \succsim (\mathbf{b}_{\langle s, t \rangle}, \mathbf{x})$ for any $\mathbf{x}$ in C. By Definition 1, $(\mathbf{a}_{\langle s, t \rangle}, \mathbf{x})$ and $(\mathbf{a}_{\langle s, t \rangle}, \mathbf{x})$ are in C for any $\mathbf{x}$ in C. If $b \succ^X a$, then $(\mathbf{b}_{\langle s, t \rangle}, \mathbf{x}) \succ (\mathbf{a}_{\langle s, t \rangle}, \mathbf{x})$ by condition (A.b). Thus, $b \succ^X a$ is false. Hence, $a \succsim^X b$ since $\succsim^X$ is complete by condition (B).

To show (c), consider $a \prec^X b$. By condition (C.a), there is a $\delta > 0$ such that $|c - a| < \delta$ implies $c \prec^X b$, and by (C.b), there is a $\delta > 0$ such that $|c - b| < \delta$ implies $c \succ^X a$.

To show (d) and (e), note that: (i) X is a product of intervals, and (ii) $\succsim^X$ is complete and transitive by condition (B), and continuous by (c). By Debreu (1954), it follows that $\succsim^X$ has a continuous value function. The range of any such function is a non-point interval since the set X is connected and contains outcomes that are not indifferent.

To show (f), note that $\mathbf{x}$ has values in a compact subset $\ell_j \le \mathbf{x}_j(t) \le u_j$, $j = 1, \dots, N$, of X. By (d), there is a continuous value function $v(x)$ for $\succsim^X$. Such a function has a maximum value v^* and a minimum value v_* for the compact set, and thus $v_* \le v(\mathbf{x}(t)) \le v^*$ for all t. Choose outcomes a^*, a_* such that $v(a^*) = v^*$ and $v(a_*) = v_*$. Then, $v(a_*) \le v(\mathbf{x}(t)) \le v(a^*)$ for all t which implies $a_* \precsim^X \mathbf{x}(t) \precsim^X a^*$ for all t which implies $\mathbf{a}_* \precsim \mathbf{x} \precsim \mathbf{a}^*$. If $\mathbf{x} \sim \mathbf{a}_*$ or $\mathbf{x} \sim \mathbf{a}^*$, we are through. Otherwise, the sets $\{b$ in X: $\mathbf{b} \succ \mathbf{x}\}$ and $\{b$ in X: $\mathbf{b} \prec \mathbf{x}\}$ are non-empty. They are open by condition (C), and thus their union is not all of the connected set X. Hence, $\mathbf{a} \sim \mathbf{x}$ for some $\mathbf{a}$ since $\succsim$ is complete.

A partition p of an interval $[0, T]$ is a set of subintervals, $\langle t_0, t_1\rangle, \dots, \langle t_{m-1}, t_m\rangle$, where $0 = t_0 \le t_1 \le \dots \le t_{m-1} \le t_m = T$ and the subintervals are disjoint with the union $[0, T]$. A step outcome stream based on a partition p is an outcome stream that is constant on each subinterval $\langle t_{i-1}, t_i\rangle$, $i = 1, \dots, m$. By a slight abuse of notation, we will denote the values of a step outcome stream $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$ by $\mathbf{x}(t) = (\mathbf{x}_1(t), \dots, \mathbf{x}_N(t)) = x(i) = (x_1(i), \dots, x_N(i))$ for t in $\langle t_{i-1}, t_i\rangle$, $i = 1, \dots, m$.

The set of step outcome streams based on a partition p will be denoted by S_p, and the set of step outcome streams for all partitions of $[0, T]$ will be denoted by S_T. A step outcome stream $\mathbf{x}$ in S_p can be regarded as a vector $(x(1), \dots, x(m))$ of outcomes, and the set S_p can be regarded as the product set $X\times\dots\times X$ of the vectors $(x(1), \dots, x(m))$.

For two step outcome streams $\mathbf{x}, \mathbf{y}$, the distances $\int|\mathbf{x}_j - \mathbf{y}_j|$ and $\int|\mathbf{x} - \mathbf{y}|$ defined in Section 3.1 are, $\int|\mathbf{x}_j - \mathbf{y}_j| = \sum_i |x_j(i) - y_j(i)|\,(t_i - t_{i-1})$ and $\int|\mathbf{x} - \mathbf{y}| = \sum_j \int|\mathbf{x}_j - \mathbf{y}_j|$. We will define the distance between two outcomes x, y as, $|x - y| = \sum_j |x_j - y_j|$. Then, the distance between two outcome streams is also, $\int|\mathbf{x} - \mathbf{y}| = \sum_{i,j}|x_j(i) - y_j(i)|\,(t_i - t_{i-1})$.

A partition p with at least three non-point intervals will be called proper. Then, the set S_p and the space $(S_p, \succsim)$ also will be called proper.

The additive value model due to Debreu (1960) and Gorman (1968) can be interpreted as an additive value model for a proper space $(S_p, \succsim)$. The following result does so.

Lemma A2. If a proper space $(S_p, \succsim)$ satisfies conditions (A)-(D), then:

(a) If a subinterval $\langle s, s'\rangle = \langle t_{i-1}, t_i\rangle$ is non-point, then it is essential, i.e., there exists $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in S_p such that $(\mathbf{x}_{\langle s, s'\rangle}, \mathbf{z})$ and $(\mathbf{y}_{\langle s, s'\rangle}, \mathbf{z})$ are not indifferent.

(b) If a subinterval $\langle s, s'\rangle = \langle t_{i-1}, t_i\rangle$ is a point or is empty, then it is inessential, i.e., $(\mathbf{x}_{\langle s, s'\rangle}, \mathbf{z})$ and $(\mathbf{y}_{\langle s, s'\rangle}, \mathbf{z})$ are indifferent for any $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in S_p.

(c) $\succsim$ is component independent on S_p, i.e., for any $\langle s, s'\rangle = \langle t_{i-1}, t_i\rangle$ and any $\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}$ in S_p: $(\mathbf{x}_{\langle s, s'\rangle}, \mathbf{w}) \succsim (\mathbf{y}_{\langle s, s'\rangle}, \mathbf{w})$ if and only if $(\mathbf{x}_{\langle s, s'\rangle}, \mathbf{z}) \succsim (\mathbf{y}_{\langle s, s'\rangle}, \mathbf{z})$.

Part (c) implies that for each subinterval $\langle s, s'\rangle = \langle t_{i-1}, t_i\rangle$ the relation $\succsim$ defines a relation $\succsim_i$ on the set X by: $a \succsim_i b$ if and only if $(\mathbf{a}_{\langle s, s'\rangle}, \mathbf{w}) \succsim (\mathbf{b}_{\langle s, s'\rangle}, \mathbf{w})$.

(d) If a subinterval $\langle t_{i-1}, t_i\rangle$ is essential, then the relation $\succsim_i$ on X is the relation $\succsim^X$. And if $\langle t_{i-1}, t_i\rangle$ is inessential, then $a \sim_i b$ for all a, b in X.

(e) The space $(S_p, \succsim)$ has an additive value function, $V(\mathbf{x}) = \sum_{i \in E} v_i(x_i)$, where:

(i) E denotes the set of indices for the essential subintervals $\langle t_{i-1}, t_i \rangle$.

(ii) $v_i(x)$, $i \in E$, are continuous value functions for the outcome space $(X, \succsim^X)$.

(f) A function $V(\mathbf{x})$ is cardinally unique, i.e., a function $V^*(\mathbf{x})$ is also an additive value function for $(S_p, \succsim)$ if and only if $V^*(\mathbf{x}) = a\, V(\mathbf{x}) + b$ for some constants $a > 0$ and b.

Proof. By Lemma A1.a there exist outcomes $a \succ^X b$. If an interval $\langle s, s' \rangle$ is non-point, then, $(\mathbf{a}_{\langle s, s' \rangle}, \mathbf{o}) \succ (\mathbf{b}_{\langle s, s' \rangle}, \mathbf{o})$ by condition (A.b). And if $\langle s, s' \rangle$ is a point set or empty, then $(\mathbf{x}_{\langle s, s' \rangle}, \mathbf{z}) \sim (\mathbf{y}_{\langle s, s' \rangle}, \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z}$ by condition (A.a).

To show (c), suppose that $[0, s\rangle$ denotes the interval of times that are less than those in $\langle s, s' \rangle$ and $\langle s', T]$ denotes the interval of times that are greater than those in $\langle s, s' \rangle$. Then, for example, $(\mathbf{x}_{\langle s, s' \rangle}, \mathbf{w}) = (\mathbf{w}_{[0, s\rangle}, \mathbf{x}_{\langle s, s' \rangle}, \mathbf{w}_{\langle s', T]})$. Note that by condition (D), $(\mathbf{w}_{[0, s\rangle}, \mathbf{x}_{\langle s, s' \rangle}, \mathbf{w}_{\langle s', T]}) \succsim (\mathbf{w}_{[0, s\rangle}, \mathbf{y}_{\langle s, s' \rangle}, \mathbf{w}_{\langle s', T]})$ implies that $(\mathbf{z}_{[0, s\rangle}, \mathbf{x}_{\langle s, s' \rangle}, \mathbf{w}_{\langle s', T]}) \succsim (\mathbf{z}_{[0, s\rangle}, \mathbf{y}_{\langle s, s' \rangle}, \mathbf{w}_{\langle s', T]})$ which implies that $(\mathbf{z}_{[0, s\rangle}, \mathbf{x}_{\langle s, s' \rangle}, \mathbf{z}_{\langle s', T]}) \succsim (\mathbf{z}_{[0, s\rangle}, \mathbf{y}_{\langle s, s' \rangle}, \mathbf{z}_{\langle s', T]})$.

To show (d), note that if $\langle t_{i-1}, t_i \rangle$ is essential then $\succsim_i$ is $\succsim^X$ by Lemma A1.b. And if $\langle t_{i-1}, t_i \rangle$ is inessential then $a \sim_i b$ for all a, b by condition (A.a).

To show (e) and (f), regard the set S_p as the set $X \times \ldots \times X$ vectors $(x(1), \ldots, x(m))$ of outcomes. The set X is a product set of intervals, and thus $X \times \ldots \times X$ is a product of intervals. The relation $\succsim$ on $X \times \ldots \times X$ is complete, transitive, and continuous by conditions (B), (C), and it is preferentially independent by condition (D). Since S_p is proper, at least three components of S_p are essential. Thus, the additive value model of Debreu (1960) can be interpreted as a model for $(S_p, \succsim)$ as described in (e) and (f). In particular, a component function $v_i(x_i)$ is a continuous value function for $\succsim_i$ and thus $v_i(x_i)$ is a continuous value function for $\succsim^X$ if i is in E.

Since the functions $v_i(x_i)$, $i \in E$, in part (e) above are ordinal value functions for the relation $\succsim^X$, they are <u>ordinally equal</u>, i.e., $v_j(x) = f_{ij}(v_i(x))$ for some strictly increasing function $f_{ij}(v)$. Below, we show that condition (E) implies that the functions $v_i(x_i)$ are <u>cardinally equal</u>, i.e., $v_j(x) = a_{ij}\, v_i(x) + b_{ij}$ for some constants $a_{ij} > 0$ and b_{ij}. Harvey (1986) presents a similar result for the case of outcome sequences.

Lemma A3. If a space $(S_p, \succsim)$ as described in Lemma A2 also satisfies condition (E), then it has a value function of the form, $V(\mathbf{x}) = \sum_i d(i)\, v(x_i)$, such that:

(a) The function $v(x)$, x in X, is a continuous value function for $(X, \succsim^X)$ which has a non-point interval range and the value $v(o) = 0$.

(b) A weight $d(i)$ is positive if the interval $\langle t_{i-1}, t_i \rangle$ is non-point and is zero otherwise.

Moreover, the function $v(x)$ and the weights $d(i)$ are unique up to positive multiples.

Proof. Suppose that $v(x)$, $w(x)$ denote two functions $v_i(x)$, $v_j(x)$, $i \neq j$, in an additive value function, $V(\mathbf{x}) = \sum_{i \in E} v_i(x_i)$, as described in Lemma A2.e and that I_v, I_w denote their ranges. As the major part of the proof, we show that $v(x)$, $w(x)$ are cardinally equal.

Each function $v(x)$, $w(x)$ is a continuous value function for $\succsim^X$ by Lemma A2.e, and thus each range I_v, I_w is a non-point interval by Lemma A1.e. Moreover, $v(x)$, $w(x)$ are ordinally equal, i.e., $w(x) = f(v(x)$ for some strictly increasing function $f(v)$. The range of $f(v)$ is an interval (namely I_w), and so $f(v)$ is continuous.

Suppose that ℓ_v and ℓ_w denote the lengths of I_v and I_w. Then, $\ell_v, \ell_w > 0$ (and $\ell_v, \ell_w \leq \infty$). An outcome pair $x^- \prec^X x^+$ in $\langle t_{i-1}, t_i \rangle$ and an outcome pair $y^- \prec^X y^+$ in $\langle t_{j-1}, t_j \rangle$ are tradeoffs pairs if and only if $v(x^+) - v(x^-) = w(y^+) - w(y^-)$. It follows that if $v(x^+) - v(x^-) < \ell_w$, then $x^- \prec^X x^+$ has a tradeoffs pair in $\langle t_{j-1}, t_j \rangle$, and if $w(y^+) - w(y^-) < \ell_v$, then $y^- \prec^X y^+$ has a tradeoffs pair in $\langle t_{j-1}, t_j \rangle$.

Next, we show that for any $a_0 \prec^X a_{1/2} \prec^X a_1$: If $v(a_{1/2}) - v(a_0) = v(a_1) - v(a_{1/2})$, then $w(a_{1/2}) - w(a_0) = w(a_1) - w(a_{1/2})$. First, assume that: $v(a_{1/2}) - v(a_0)$, $v(a_1) - v(a_{1/2}) < \ell_w$ and $w(a_{1/2}) - w(a_0)$, $w(a_1) - w(a_{1/2}) < \ell_v$. Then, each of the pairs $a_0, a_{1/2}$ and $a_{1/2}, a_1$ in $\langle t_{i-1}, t_i \rangle$ has a tradeoffs pair in $\langle t_{j-1}, t_j \rangle$, and each of the pairs $a_0, a_{1/2}$ and $a_{1/2}, a_1$ in $\langle t_{j-1}, t_j \rangle$ has a tradeoffs pair in $\langle t_{i-1}, t_i \rangle$. If $v(a_{1/2}) - v(a_0) = v(a_1) - v(a_{1/2})$, then the pairs $a_0, a_{1/2}$ and $a_{1/2}, a_1$ in $\langle t_{i-1}, t_i \rangle$ have the same tradeoffs pairs in $\langle t_{j-1}, t_j \rangle$, and thus $a_{1/2}$ is a mid-outcome of a_0, a_1 in $\langle t_{i-1}, t_i \rangle$. Hence, by condition (E), $a_{1/2}$ is also a mid-outcome of a_0, a_1 in $\langle t_{j-1}, t_j \rangle$, and thus $w(a_{1/2}) - w(a_0) = w(a_1) - w(a_{1/2})$.

The general case can be reduced to this special case as follows. For the three outcomes $a_0 \prec^X a_{1/2} \prec^X a_1$ in $\langle t_{i-1}, t_i \rangle$, define $\Delta = v(a_{1/2}) - v(a_0) = v(a_1) - v(a_{1/2}) > 0$. For any even integer $n \geq 2$, there exists a sequence $a_0 \prec^X a_{1/n} \prec^X \ldots \prec^X a_{(n-1)/n} \prec^X a_1$ in $\langle t_{i-1}, t_i \rangle$

that includes $a_0, a_{1/2}, a_1$ such that $v(a_{k/n}) - v(a_{(k-1)/n}) = \Delta/n$, $k = 1, \ldots, n$. If $n \geq \Delta/\ell_w$, then $\Delta/n \leq \ell_w$ and each pair $a_{(k-1)/n}, a_{k/n}$ in $\langle t_{i-1}, t_i \rangle$ has a tradeoffs pair in $\langle t_{j-1}, t_j \rangle$.

The continuous function $f(v)$ is uniformly continuous on the interval $[v(a_0), v(a_1)]$ in I_v since it is continuous on I_v, and thus there exist sufficiently large even integers n such that $f(v(a_{k/n})) - f(v(a_{(k-1)/n})) = w(a_{k/n}) - w(a_{(k-1)/n}) \leq \ell_v$, $k = 1, \ldots, n$. Then, each pair $a_{(k-1)/n}, a_{k/n}$, $k = 1, \ldots, n$, in $\langle t_{j-1}, t_j \rangle$ has a tradeoffs pair in $\langle t_{i-1}, t_i \rangle$. Thus, by the above argument, $w(a_{(k+1)/n}) - w(a_{k/n}) = w(a_{k/n}) - w(a_{(k-1)/n})$, $k = 1, \ldots, n$. These equations imply that $w(a_{1/2}) - w(a_0) = w(a_1) - w(a_{1/2})$.

Now, consider any $v_0 < v_1$ in I_v. There are outcomes $a_0, a_{1/2}, a_1$ such that $v(a_0) = v_0$, $v(a_1) = v_1$, and $v(a_{1/2}) = ½\, v_0 + ½\, v_1$. Thus, $v(a_{1/2}) - v(a_0) = v(a_1) - v(a_{1/2})$ which implies $w(a_{1/2}) - w(a_0) = w(a_1) - w(a_{1/2})$ which implies $w(a_{1/2}) = ½\, w(a_0) + ½\, w(a_1)$. But $v(a_{1/2}) = ½\, v_0 + ½\, v_1$ implies $f(v(a_{1/2})) = f(½\, v_0 + ½\, v_1)$, and $w(a_{1/2}) = ½\, w(a_0) + ½\, w(a_1)$ implies that $f(v(a_{1/2})) = ½\, f(v_0) + ½\, f(v_1)$. Hence, the function $f(v)$ satisfies Jensen's equation, $f(½\, v_0 + ½\, v_1) = ½\, f(v_0) + ½\, f(v_1)$. It follows that $f(v) = a\, v + b$ for some constants a, b (e.g., Aczél, 1966). Here, $a > 0$ since $f(v)$ is strictly increasing.

In conclusion, the function $V(\mathbf{x})$ can be written as, $V(\mathbf{x}) = \sum_{i \in E} (\, d(i)\, v(x_i) + b_i\,) = \sum_{i \in E} d(i)\, v(x_i) + \sum_{i \in E} b(i)$, where $d(i) > 0$ and $v(x_i)$ is one of the functions $v_i(x_i)$. By the cardinal uniqueness of $V(\mathbf{x})$, we can omit the constant $\sum_{i \in E} b(i)$. Finally, by defining $d(i) = 0$ for the inessential intervals we can write $V(\mathbf{x})$ as, $V(\mathbf{x}) = \sum_i d(i)\, v(x_i)$.

Property (a) is implied by Lemmas A2.e and A1.e; property (b) is implied by Lemma A2.d; and the uniqueness properties of $v(x)$ and $d(i)$ are implied by Lemma A2.f.

Next, we extend the model for a set S_p to a model for the union S_T of the sets S_p. The key idea is as follows. Suppose that p: $\langle s_{i-1}, s_i \rangle$, $i = 1, \ldots, m$, and q: $\langle t_{j-1}, t_j \rangle$, $j = 1, \ldots, n$, are two partitions of $[0, T]$. Since the intersections $\langle s_{i-1}, s_i \rangle \cap \langle t_{j-1}, t_j \rangle$ are pairwise disjoint, they form a partition of $[0, T]$. We will denote this partition by pq and call it the conjunction of p and q. An outcome stream in S_p or S_q is constant on each interval $\langle s_{i-1}, s_i \rangle \cap \langle t_{j-1}, t_j \rangle$. Thus, the sets S_p and S_q are subsets of S_{pq}, and for the same reason the function (A1) below is well-defined, i.e., if an outcome stream $\mathbf{x}$ is in two sets S_p, S_q, then $V(\mathbf{x}) = \sum_i \big(A(s_i) - A(s_{i-1})\big)\, v(x(i)) = \sum_j \big(A(t_j) - A(t_{j-1})\big)\, v(x(j))$.

Theorem A1. If an outcome stream space $(S_T, \succsim)$, $T > 0$, satisfies conditions (A)-(E), then it has a value function of the form,

$$V(\mathbf{x}) = \Sigma_i \left(A(s_i) - A(s_{i-1})\right) v(x(i)), \quad \mathbf{x} \text{ in } S_T. \tag{A1}$$

such that:

(a) The function $v(x)$, x in X, is a continuous value function for $(X, \succsim^X)$ which has a non-point interval range and the value $v(o) = 0$.

(b) The function $A(t)$, $0 \le t \le T$, is strictly increasing and has the value $A(0) = 0$.

Moreover, each of the functions $v(x)$, $A(t)$ is unique up to a positive multiple.

Proof. If a set S_p or S_q is proper, then the conjunction S_{pq} is proper. Hence, S_T is a union of proper sets, and it suffices to consider only proper sets.

The functions and weights $v(x)$, $d(i)$, $V(\mathbf{x})$ that are associated with a (proper) space $(S_p, \succsim)$ will be written as $v_p(x)$, $d_p(i)$, and $V_p(\mathbf{x})$. Thus, $V_p(\mathbf{x}) = \Sigma_i\, d_p(i)\, v_p(x(i))$.

Assume that the weights $d_p(i)$ are normalized such that $\Sigma_i\, d_p(i) = 1$. If there is an outcome $a \succ^X o$, assume that such an outcome has been chosen and the function $v_p(x)$ is normalized such that $v_p(o) = 0$ and $v_p(a) = 1$. Otherwise, there is an outcome $a \prec^X o$. In this case, assume that such an outcome is chosen and $v_p(x)$ is normalized such that $v_p(o) = 0$ and $v_p(a) = -1$. Then, the quantities $v_p(x)$, $d_p(i)$, and $V_p(\mathbf{x})$ are unique.

Since a space $(S_p, \succsim)$ is a subspace of a space $(S_{pq}, \succsim)$, a value function $V_{pq}(\mathbf{x}) = \Sigma_{i,j}\, d_{pq}(i,j)\, v_{pq}(x(i,j))$ for $(S_{pq}, \succsim)$ is also a value function for $(S_p, \succsim)$. An outcome stream $\mathbf{x}$ in S_p has the outcome $x(i)$ in the intervals $\langle s_{i-1}, s_i\rangle \cap \langle t_{j-1}, t_j\rangle$, $j = 1, \ldots, n$, and thus $V_{pq}(\mathbf{x}) = \Sigma_i \left(\Sigma_j\, d_{pq}(i,j)\right) v_{pq}(x(i))$ for $\mathbf{x}$ in S_p. But $v_{pq}(x)$ and $d_{pq}(i,j)$ are normalized, and thus $v_{pq}(x) = v_p(x)$ for x in X and $\Sigma_j\, d_{pq}(i,j) = d_p(i)$, $i = 1, \ldots, m$. By the same argument, $v_{pq}(x) = v_q(x)$ for x in X and $\Sigma_i\, d_{pq}(i,j) = d_q(j)$, $j = 1, \ldots, n$.

It follows that $v_p(x) = v_q(x)$ for x in X, and thus the normalized functions $v_p(x)$ are equal. Suppose that $v(x)$ denotes the common function.

Next, we show that an weight for an interval depends only on its endpoints, i.e., if $d_p(h)$, $d_q(k)$ are the weights for intervals $\langle s_{h-1}, s_h\rangle$, $\langle t_{k-1}, t_k\rangle$ in two partitions p, q and $s_{h-1} = t_{k-1}$, $s_h = t_k$, then $d_p(h) = d_q(k)$. For the interval $\langle s_{h-1}, s_h\rangle \cap \langle t_{k-1}, t_k\rangle$ in pq also has these endpoints, and thus the intersection intervals $\langle s_{i-1}, s_i\rangle \cap \langle t_{k-1}, t_k\rangle$, $i \neq h$, and $\langle s_{h-1}, s_h \rangle \cap \langle t_{j-1}, t_j \rangle$, $j \neq k$, are point sets or empty. Hence, $d_{pq}(i, k) = 0$, $i \neq h$ and $d_{pq}(h,j) = 0$, $j \neq k$. It follows that $d_p(h) = \sum_j d_{pq}(h,j) = d_{pq}(h,j) = \sum_i d_{pq}(i,k) = d_q(k)$.

As shown, a weight $d_p(h)$ is a function, $d_p(h) = f(s_{h-1}, s_h)$, of the endpoints s_{h-1}, s_h of the interval $\langle s_{h-1}, s_h \rangle$. But what type of function? Suppose that p has adjacent intervals $\langle s_{h-1}, s_h \rangle$, $\langle s_h, s_{h+1} \rangle$ and q has an interval $\langle t_{k-1}, t_k \rangle = \langle s_{h-1}, s_h \rangle \cup \langle s_h, s_{h+1} \rangle$. Then, $d_p(h) = d_{pq}(h, k)$, $d_p(h+1) = d_{pq}(h+1, k)$, and thus $f(s_{h-1}, s_{h+1}) = f(t_{k-1}, t_k) = d_q(k) = d_{pq}(h, k) + d_{pq}(h+1, k) = d_p(h) + d_p(h+1) = f(s_{h-1}, s_h) + (s_h, s_{h+1})$.

The endpoints are not restricted, and thus $f(s, u) = f(s, t) + f(t, u)$ for any $s \leq t \leq u$ in $[0, T]$. Define $A(t) = f(0, t)$, $0 \leq t \leq T$. Then, $f(t, u) = A(u) - A(t)$ and $A(0) = 0$.

A function $V_p(\mathbf{x})$ can now be written as, $V_p(\mathbf{x}) = \sum_i \big(A(s_i) - A(s_{i-1})\big)\, v(x(i))$ where the functions $v(x)$ and $A(t)$ are independent of the partition p. Therefore, $V(\mathbf{x})$ can be defined as equal to $V_p(\mathbf{x})$ for outcome streams $\mathbf{x}$ in the set S_p.

Since $\sum_i A(s_i) - A(s_{i-1}) = \sum_i d_p(i) = 1$, an amount $V(\mathbf{x})$ is a weighted average of amounts $v(x)$. Thus, the range of $V(\mathbf{x})$ is the non-point interval range of $v(x)$.

$V(\mathbf{x})$ is a value function for S_T. For consider $\mathbf{x}, \mathbf{y}$ in S_T. Then, $\mathbf{x}$ is in S_p and $\mathbf{y}$ is in S_q for some p, q. Hence, $\mathbf{x}$ and $\mathbf{y}$ are in S_{pq}, and thus $V(\mathbf{x}) = V_{pq}(\mathbf{x})$ and $V(\mathbf{y}) = V_{pq}(\mathbf{y})$. Therefore, $\mathbf{x} \succsim \mathbf{y}$ if and only if $V_{pq}(\mathbf{x}) \geq V_{pq}(\mathbf{y})$ if and only if $V(\mathbf{x}) \geq V(\mathbf{y})$.

Next, we show that the functions $v(x)$, $A(t)$ have the properties (a), (b). Lemma A3 implies that $v(x)$ has the properties in (a). It also implies that $A(t)$ is strictly increasing since a weight $d_p(i) = A(s_i) - A(s_{i-1})$ for a non-point interval is positive.

It remains to show the uniqueness properties of $v(x)$ and $A(t)$. Suppose that $V_1(\mathbf{x}) = \sum_i \left(A_1(s_i) - A_1(s_{i-1}) \right) v_1(x(i))$ and $V_2(\mathbf{x}) = \sum_i \left(A_2(s_i) - A_2(s_{i-1}) \right) v_2(x(i))$ are value functions for S_T with the properties (a), (b). Then, for a partition p, $V_1(\mathbf{x})$ and $V_2(\mathbf{x})$ are value functions for the subset S_p of S_T. Lemma A3 states that: (i) $v_2(x) = a_p\, v_1(x)$ where $a_p > 0$, and (ii) $A_2(s_i) - A_2(s_{i-1}) = c_p \left(A_1(s_i) - A_1(s_{i-1}) \right)$, $i = 1, \ldots, m$, where $c_p > 0$. Equation (ii) implies by addition that $A_2(s_i) = c_p\, A_1(s_i)$ for $i = 1, \ldots, m$. In particular, $A_2(T) = c_p\, A_1(T)$ since $s_m = T$. But $v_1(x), v_2(x) \neq 0$ for some outcome x and $A_1(T), A_1(T) \neq 0$, and thus the constants a_p and c_p are independent of p.

Conversely, if $V_1(\mathbf{x})$ is a value function and $v_2(x) = a\, v_1(x)$, $A_2(t) = c\, A_1(t)$ where $a, c > 0$, then $V_2(\mathbf{x}) = \sum_i \left(A_2(s_i) - A_2(s_{i-1}) \right) v_2(x) = c\, a\, V_1(\mathbf{x})$ is also a value function.

Lemma A4. Suppose that an outcome stream space $(S_T, \succsim)$ as described in Theorem A1 satisfies the conditions (F), (G). Then, $A(t)$, $0 \le t \le T$, has the additional properties:

(a) $A(t)$, $0 \le t \le T$, is concave (i.e., $A\big(p\,t + (1\!-\!p)\,t'\big) \ge p\,A(t) + (1\!-\!p)\,A(t')$ for t, t' in $[0, T]$ and $0 \le p \le 1$) and is absolutely continuous.

(b) The left derivative, $A'_{-}(t) = \lim_{\Delta \downarrow 0} ((A(t) - A(t-\Delta)) / \Delta$, exists for $0 < t \le T$ and is bounded, non-increasing, and positive. (It follows that $\lim_{t \downarrow 0} A'_{-}(t)$ exists and that $A'_{-}(t) \ge \lim_{t \downarrow 0} A'_{-}(t)$ for $0 < t \le T$.)

(c) $A(t)$, $0 \le t \le T$, is an indefinite Riemann integral, $A(t) = \int_0^{\,t} a(s)\, ds$, where the function $a(t)$ is defined as $a(0) = \lim_{t \downarrow 0} A'_{-}(t)$ and $a(t) = A'_{-}(t)$ for $0 < t \le T$.

(d) The value function $V(\mathbf{x})$ in (A1) can be written as the Riemann integral,

$$V(\mathbf{x}) \;=\; \int_0^{\,T} a(t)\, v(\mathbf{x}(t))\, dt, \quad \mathbf{x} \text{ in } S_T. \tag{A2}$$

Proof. Define $u = ½\,(s + t)$ for any $s < t$. Condition (F) implies that $V(\mathbf{a}_{\langle s,\, u\rangle}, \mathbf{b}_{\langle u,\, t\rangle}, \mathbf{o}) \ge V(\mathbf{b}_{\langle s,\, u\rangle}, \mathbf{a}_{\langle u,\, t\rangle}, \mathbf{o})$ for any outcomes $a \succsim^X b$. This implies (by algebraic manipulation) that $(A(u) - ½\,A(s) - ½\,A(t))\,(v(a) - v(b)) \ge 0$. But $v(a) > v(b)$ whenever $a \succ^X b$, and thus $A(u) \ge ½\,A(s) + ½\,A(t)$, that is, $A(t)$, $0 \le t \le T$, is midpoint concave.

Since $A(t)$, $0 \le t \le T$, is strictly increasing, it follows that it is concave. (The proof of this result but not the statement of it is in Hardy et al., 1934, pp. 72, 73.)

Therefore, $A(t)$ is continuous for $0 < t < T$, (see, e.g., Stromberg, 1981, p. 199). It is straightforward to infer from $A(t)$ is continuous at $t = T$ and that $A(t)$, $0 \le t \le T$, is bounded (since $A(0) = 0$). Next, we will show that condition (G) implies that $A(t)$ is continuous at $t = 0$. Then, $A(t)$ is continuous for all $0 \le t \le T$. Hence, it is absolutely continuous for $0 \le t \le T$ (see, e.g., Stromberg, 1981, p. 202).

Condition (G) implies that for $s > 0$ there exist $a \prec^X b \prec^X c$ such that $V(\mathbf{c}_{[s-\Delta,\, s]}, \mathbf{a}) \ge V(\mathbf{b}_{[0,\, \Delta]}, \mathbf{a})$ for all $0 < \Delta < ½\,s$. Then, $(A(s) - A(s-\Delta))\,(v(c) - v(a)) \ge (A(\Delta) - A(0))\,(v(b) - v(a))$ which implies that $A(\Delta) - A(0) \le K_s\,(A(s) - A(s-\Delta))$ where $K_s = (v(c) - v(a)) / (v(b) - v(a)) > 1$. Thus, $A(t)$ is continuous at $t = 0$ since it is continuous at $t = s$.

To show parts (b)-(d), we will use the following properties of a function $A(t)$ that is concave on an interval $0 < t < T$. See, e.g., Stromberg (1981, pp. 129, 199) for proofs.

(i) The left derivative $A'_-(t) = \lim_{\Delta \downarrow 0} ((A(t) - A(t - \Delta)) / \Delta$ and the right derivative $A'_+(t) = \lim_{\Delta \downarrow 0} ((A(t + \Delta) - A(t)) / \Delta$ exist for $0 < t < T$.

(ii) $((A(t) - A(t - \Delta)) / \Delta \geq A'_-(t) \geq A'_+(t) \geq ((A(t + \Delta) - A(t)) / \Delta$ for t, Δ such that $0 < t - \Delta < t < t + \Delta < T$.

(iii) $A'_-(t)$, $0 < t < T$, is non-increasing. (It follows that $A'_-(t)$ is continuous except for at most countably many points.)

(iv) $A'_-(t) = A'_+(t)$ $0 < t < T$ (i.e., the derivative $A'(t) = A'_-(t) = A'_+(t)$ exists) except for at most countably many points.

Since $A(t)$ is strictly increasing, we also have the property that $A'_+(t)$ and $A'_-(t)$ are positive for $0 < t < T$. For by choosing $\Delta > 0$ such that $t < t + \Delta < T$ we have, $A'_-(t) \geq A'_+(t) \geq ((A(t + \Delta) - A(t)) / \Delta > 0$ by (ii).

We next investigate the cases, $t = 0$ and $t = T$. Since $A(t)$ is concave on $0 \leq t \leq T$, it has the following properties (see, e.g., Stromberg (1981, p. 199): (1) $A(\Delta) - A(0) \geq A(½T) - A(½T - \Delta) \geq A(½T + \Delta) - A(½T) \geq A(T) - A(T - \Delta)$, and (2) $(A(T) - A(T - \Delta')) / \Delta' \leq A(T) - A(T - \Delta) / \Delta$ for $\Delta' < \Delta$. And as shown above, condition (G) with $s = T$ implies that: (3) $A(\Delta) - A(0) \leq K_T (A(T) - A(T - \Delta))$ for $0 < \Delta < ½T$.

First, consider $t = 0$. The inequalities (1), (3) imply that $A'_-(\Delta) \leq (A(\Delta) - A(0)) / \Delta \leq K_T (A(T) - A(T - \Delta)) / \Delta \leq K_T (A(½T + \Delta) - A(½T)) / \Delta \leq K_T A'_+(½T)$. Thus, $A'_-(t)$ is bounded above. It is also non-increasing, and thus $\lim_{t \downarrow 0} A'_-(t)$ exists. Second, consider, $t = T$. The inequalities (1), (3) imply that $(A(T) - A(T - \Delta)) / \Delta \geq K_T^{-1}(A(\Delta) - A(0)) / \Delta \geq K_T^{-1}(A(½T) - A(½T - \Delta)) / \Delta \geq K_T^{-1} A'_-(½T)$. Thus, $(A(T) - A(T - \Delta)) / \Delta$ has a positive lower bound. By (2), it is non-increasing as Δ decreases, and thus $A'_-(T) > 0$ exists.

Now, define $a(t)$, $0 \leq t \leq T$, as in part (c). Then, $a(t)$ is bounded and is continuous except for at most countably many points, and thus it is Riemann integrable on $[0, T]$.

Since $A(t)$ is absolutely continuous and $A(0) = 0$, the fundamental theorem of calculus implies that $A'(t)$ is Lebesgue integrable on $[0, T]$ and that $A(t) = \int_0^t A'(s)\, ds$ for $0 \leq t \leq T$. But $A'(t) = A'_-(t)$ wherever $A'(t)$ exists, and thus $A(t) = \int_0^t a(s)\, ds$.

For any $\mathbf{x}$ in S_T, $V(\mathbf{x}) = \sum_i (A(s_i) - A(s_{i-1}))\, v(x(i)) = \sum_i \big(\int_{\langle s_{i-1}, s_i \rangle} a(t)\, dt\big)\, v(x(i)) = \sum_i \big(\int_{\langle s_{i-1}, s_i \rangle} a(t)\, v(x(i))\, dt\big) = \int_0^T a(t)\, v(\mathbf{x}(t))\, dt$, and thus $V(\mathbf{x})$ has the form (A2).

Proof of Theorem 1. For the forward part of the proof, assume that an outcome stream space $(X_T, \succsim)$ satisfies the stated conditions. Then, $\succsim$ restricted to the set S_T satisfies the conditions in Theorem A1 and Lemma A4. Hence, there exist functions $v(x)$, $a(t)$, $A(t)$ with the properties stated there such that $V(\mathbf{x})$ is a value function for $(S_T, \succsim)$.

By Lemma A1, the range of $v(x)$, x in X, is an interval I_v. By Definition 1, for any $\mathbf{x}$ in X_T, each function $\mathbf{x}_j$ has bounds, $a_j \le \mathbf{x}_j(t) \le b_j$, in the interval X_j, $j = 1, \ldots, N$. Thus, the range of $\mathbf{x}$ is a subset of the product set $S = [a_1, b_1] \times \ldots \times [a_N, b_N]$. Since the set S is compact and the function $v(x)$ is continuous, the image of S is a compact subinterval $[\ell, u]$ of I_v, that is, $v(S) = [\ell, u]$. Thus, $\ell \le v(\mathbf{x}(t)) \le u$, for all $0 \le t \le T$.

For $\mathbf{x}$ in X_T, the function $v(\mathbf{x}(t))$ is Riemann integrable since $v(x)$ is continuous and $\mathbf{x}(t)$ satisfies Definition 1. But $a(t)$ is Riemann integrable by Lemma A4. Hence, $a(t)\, v(\mathbf{x}(t))$ is Riemann integrable, that is, $V(\mathbf{x}) = \int_0^T a(t)\, v(\mathbf{x}(t))\, dt$ is well-defined.

Since $\succsim$ is complete by condition (B), the properties (2), (3) below suffice to show that $V(\mathbf{x})$ is a value function for the space $(X_T, \succsim)$.

(1) If $V(\mathbf{x}) < V(\mathbf{y})$, then there exist step outcome streams $\mathbf{w}, \mathbf{z}$ such that: $\mathbf{x} \precsim \mathbf{w}$, $\mathbf{z} \precsim \mathbf{y}$ and $|\,V(\mathbf{w}) - V(\mathbf{x})\,| < \frac{1}{2}\varepsilon$, $|\,V(\mathbf{y}) - V(\mathbf{z})\,| < \frac{1}{2}\varepsilon$ where $\varepsilon = V(\mathbf{y}) - V(\mathbf{x}) > 0$.

Proof. Suppose that p_n, $n = 1, 2, \ldots$, is a sequence of partitions of the interval $[0, T]$ into subintervals $\langle t_{i-1}^{(n)}, t_i^{(n)} \rangle$ such that, $\lim_{n\to\infty} \max_i |\, t_i^{(n)} - t_{i-1}^{(n)} \,| = 0$. For each partition p_n, define $u_{(n)}(i) = \sup\{ v(\mathbf{x}(t))\colon\ t \text{ in } \langle t_{i-1}^{(n)}, t_i^{(n)} \rangle \}$ and $\ell^{(n)}(i) = \inf\{ v(\mathbf{x}(t))\colon\ t \text{ in } \langle t_{i-1}^{(n)}, t_i^{(n)} \rangle \}$, and then define upper and lower step functions $u^{(n)}(t)$, $\ell^{(n)}(t)$ for the function $v(\mathbf{x}(t))$ by $u^{(n)}(t) = u_{(n)}(i)$ and $\ell^{(n)}(t) = \ell_{(n)}(i)$ for t in $\langle t_{i-1}^{(n)}, t_i^{(n)} \rangle$.

Then, for each $n = 1, 2, \ldots$: $\ell^{(n)}(t) \le v(\mathbf{x}(t)) \le u^{(n)}(t)$ for all $0 \le t \le T$. Moreover, $\lim_{n\to\infty} \int_0^T u^{(n)}(t)\, dt = \lim_{n\to\infty} \int_0^T \ell^{(n)}(t)\, dt = \int_0^T v(\mathbf{x}(t))\, dt$, since $v(\mathbf{x}(t))$ is Riemann integrable. But, $\int_0^T a(t)\, u^{(n)}(t)\, dt - \int_0^T a(t)\, \ell^{(n)}(t)\, dt \le a(0)\,(\int_0^T u^{(n)}(t)\, dt - \int_0^T \ell^{(n)}(t)\, dt)$, and thus: $\lim_{n\to\infty} \int_0^T a(t)\, u^{(n)}(t)\, dt = \lim_{n\to\infty} \int_0^T a(t)\, \ell^{(n)}(t)\, dt = \int_0^T a(t)\, v(\mathbf{x}(t))\, dt$.

The range of $v(\mathbf{x}(t))$ is a subset of the interval $[\ell, u]$ defined above. Thus, there exist outcomes $a^{(n)}(i), b^{(n)}(i)$ such that $v(a^{(n)}(i)) = \ell^{(n)}(i)$ and $v(b^{(n)}(i)) = u^{(n)}(i)$. Define step outcome streams $\mathbf{a}^{(n)}(t)$ and $\mathbf{b}^{(n)}(t)$ by: $\mathbf{a}^{(n)}(t) = a^{(n)}(i)$ and $\mathbf{b}^{(n)}(t) = b^{(n)}(i)$ for t in $\langle t_{i-1}^{(n)}, t_i^{(n)} \rangle$. Then, $v(\mathbf{a}^{(n)}(t)) = \ell^{(n)}(t)$ and $v(\mathbf{b}^{(n)}(t)) = u^{(n)}(t)$.

Thus, $v(\mathbf{a}^{(n)}(t)) \le v(\mathbf{x}(t)) \le v(\mathbf{b}^{(n)}(t))$ for $0 \le t \le T$, and $\lim_{n\to\infty} \int_0^T a(t)\, v(\mathbf{a}^{(n)}(t))\, dt =$

$\lim_{n\to\infty} \int_0^T a(t)\, v(\mathbf{b}^{(n)}(t))\, dt = \int_0^T a(t)\, v(\mathbf{x}(t))\, dt$. In particular, the inequalities imply that, $\mathbf{x}(t) \precsim^X \mathbf{b}^{(n)}(t)$ for $0 \le t \le T$ and each $n = 1, 2, \ldots$ which implies that $\mathbf{x} \precsim \mathbf{b}^{(n)}$ for each $n = 1, 2, \ldots$ by condition (A). And the limits imply that for any $\varepsilon > 0$ there exists an n such that $| \int_0^T a(t)\, v(\mathbf{b}^{(n)}(t))\, dt = \int_0^T a(t)\, v(\mathbf{x}(t))\, dt | < ½\, \varepsilon$.

Choose $\mathbf{w}$ above as such a step outcome steam $\mathbf{b}^{(n)}$. By a similar argument, we can show the existence of a step outcome stream $\mathbf{z}$ that satisfies the other statements in (1).

(2) $V(\mathbf{x}) < V(\mathbf{y})$ implies $\mathbf{x} \prec \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$ in X_T.

Proof. The inequalities in (1) imply that $V(\mathbf{w}) < V(\mathbf{z})$. Thus, $\mathbf{w} \prec \mathbf{z}$ since $V(\mathbf{x})$ is a value function for S_T. Hence, $\mathbf{x} \precsim \mathbf{w} \prec \mathbf{z} \precsim \mathbf{y}$ which implies that $\mathbf{x} \prec \mathbf{y}$ by condition (B).

(3) $\mathbf{x} \prec \mathbf{y}$ implies $V(\mathbf{x}) < V(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$ in X_T.

Proof. By Lemma A1, there exist constant outcome streams $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} \sim \mathbf{x}$ and $\mathbf{b} \sim \mathbf{y}$. Thus, $V(\mathbf{a}) = V(\mathbf{x})$, $V(\mathbf{b}) = V(\mathbf{y})$ by (2). But, $\mathbf{a} \sim \mathbf{x} \prec \mathbf{y} \sim \mathbf{b}$ implies $\mathbf{a} \prec \mathbf{b}$ by condition (B). Hence, $V(\mathbf{a}) < V(\mathbf{b})$ since $\mathbf{a}, \mathbf{b}$ are in S_T, and thus $V(\mathbf{x}) < V(\mathbf{y})$.

To show that $v(x), a(t), A(t)$ have the properties (a)-(c), note that $v(x), A(t)$ have the properties (a), (c) by Theorem A1 and $a(t)$ has the property (b) by Lemma A4.

The uniqueness properties of $v(x)$ and $A(t)$ are implied by their uniqueness properties in Theorem A1 since S_T is a subset of X_T.

The converse implications, namely that the existence of a value function as described implies conditions (A)-(G), are straightforward to verify except for the case of condition (A.b). In this case, the implication is a conseqence of the following stronger result.

(4) For any $\mathbf{x}, \mathbf{y}$ in X_T, if $\mathbf{x}(t) \prec^X \mathbf{y}(t)$ for a non-finite number of times t, then there exists a non-point interval $[s, s']$ such that $\int_{[s,s']} a(t)\, v(\mathbf{x}(t))\, dt < \int_{[s,s']} a(t)\, v(\mathbf{y}(t))\, dt$.

Proof. Choose a time t_c such that $\mathbf{x}(t), \mathbf{y}(t)$ are continuous and $\mathbf{x}(t) \prec^X \mathbf{y}(t)$ at t_c. Since $v(\mathbf{y}(t_c)) - v(\mathbf{x}(t_c)) > 0$, t_c is in a non-point interval $[s, s']$ such that, $\sup\{v(\mathbf{x}(t))\text{: } t \text{ in } [s, s']\} < \inf\{v(\mathbf{y}(t))\text{: } t \text{ in } [s, s']\}$, and thus, $\int_{[s,s']} a(t)\, v(\mathbf{x}(t))\, dt < \int_{[s,s']} a(t)\, v(\mathbf{y}(t))\, dt$.

A set X_T of outcome streams $\mathbf{x}$ will be identified with the subset of X_∞ containing the outcome streams of the form $(\mathbf{x}_{[0,\, T]}, \mathbf{o})$. The union of these subsets of X_∞ will be denoted by X_f. For $T < T'$, the set X_T will also be identified with the subset of $X_{T'}$ containing the outcome streams of the form $(\mathbf{x}_{[0,\, T]}, \mathbf{o}_{(T,\, T']})$.

Theorem A2. If an outcome stream space $(X_f, \succsim)$ satisfies conditions (A)-(G), then it has a value function of the form, $V(\mathbf{x}) = \lim_{T\to\infty} \int_0^T a(t)\, v(\mathbf{x}(t))\, dt$, such that the improper Riemann integral $V(\mathbf{x})$ exists for any $\mathbf{x}$ in X_f and:

(a) The function $v(x)$, x in X, is a continuous value function for $(X, \succsim^X)$ which has a non-point interval range and the value $v(\text{o}) = 0$.

(b) The function $a(t)$, $0 \le t < \infty$, is positive and non-increasing, and it is Riemann integrable on each interval $[0, T]$, $T > 0$.

(c) The function $A(t) = \int_0^t a(s)\, ds$, $0 \le t < \infty$, is bounded and strictly increasing and has the value $A(0) = 0$.

Moreover, each of the functions $v(x)$ and $A(t)$ is unique up to a positive multiple.

<u>Proof</u>. Since X_T is a subset of $X_{T'}$ for $T' > T$, X_f is the union of the sets X_T, $T \ge 1$. By Lemma A1.a there exists an outcome $a \succ^X \text{o}$ or an outcome $a \prec^X \text{o}$. The arguments are the same in both cases, so it suffices to assume that there is an outcome $a \succ^X \text{o}$.

The assumptions here imply those in Theorem 1 for any $T > 0$. Thus, for any $T \ge 1$ there is a value function, $V_T(\mathbf{x}) = \int_0^T a_T(t)\, v_T(\mathbf{x}(t))\, dt$, as described in Theorem 1 for the subspace $(X_T, \succsim)$ of $(X_f, \succsim)$. We will assume that the functions $v_T(x)$, $A_T(t)$ are normalized such that $v_T(\text{o}) = 0$, $v_T(a) = 1$, $A_T(0) = 0$, $A_T(1) = 1$. Hence, $v_T(x)$ and $A_T(t)$ are unique. Then, for any $T' > T \ge 1$, $V_T(\mathbf{x})$ and $V_{T'}(\mathbf{x})$ are normalized value functions for X_T, and thus $v_T(x) = v_{T'}(x)$, x in X, and $A_T(t) = A_{T'}(t)$, $0 \le t \le T$. It follows that the left derivatives of $A_T(t)$ and $A_{T'}(t)$ are equal for $0 \le t \le T$ and that they have the same limit as t tends to zero.

Hence, the following functions are well-defined: the function $v(x)$ defined by $v(x) = v_T(x)$ for x in X, the function $A(t)$, $0 \le t < \infty$, defined by $A(t) = A_T(t)$ for $0 \le t \le T$, the function $a(t)$, $0 \le t < \infty$, defined as the left derivative of $A_T(t)$ for $0 \le t \le T$, and the function $V(\mathbf{x})$, $\mathbf{x}$ in X_f, defined as $V(\mathbf{x}) = V_T(\mathbf{x}) = \int_0^T a_T(t)\, v_T(\mathbf{x}(t))\, dt$ for $\mathbf{x}$ in X_T. Since $v(\text{o}) = 0$, the last definition implies that $V(\mathbf{x}) = \lim_{T\to\infty} \int_0^T a(t)\, v(\mathbf{x}(t))\, dt$, $\mathbf{x}$ in X_f.

Theorem 1 implies that $V(\mathbf{x})$ is a value function for any set X_T, $T \ge 1$, and that the functions $v(x)$, $A(t)$, $a(t)$ have properties (a)-(c). Moreover, $V(\mathbf{x})$ is a value function for $(X_f, \succsim)$. For if $\mathbf{x}, \mathbf{y}$ are in X_f, then $\mathbf{x}$ is in X_T and $\mathbf{y}$ is in $X_{T'}$ for some $T, T' \ge 1$.

Define $T^* = \max\{T, T'\}$. Then, $\mathbf{x}$ and $\mathbf{y}$ are in X_{T^*} and thus can be compared by the normalized function $V_{T^*}(\mathbf{x})$. Hence, $\mathbf{x} \succsim \mathbf{y}$ if and only if $V_{T^*}(\mathbf{x}) \geq V_{T^*}(\mathbf{y})$ if and only if $V(\mathbf{x}) \geq V(\mathbf{y})$ since $V(\mathbf{x}) = V_{T^*}(\mathbf{x})$ and $V(\mathbf{y}) = V_{T^*}(\mathbf{y})$.

To show that each function $A(t)$, $v(x)$ is unique up to a positive multiple, consider two value functions, $V_1(\mathbf{x}) = \lim_{T\to\infty} \int_0^T a_1(t)\, v_1(\mathbf{x}(t))\, dt$ and $V_2(\mathbf{x}) = \lim_{T\to\infty} \int_0^T a_2(t)\, v_2(\mathbf{x}(t))\, dt$ as described. Then, $v_1(o) = v_2(o) = 0$ and $A_1(0) = A_2(0) = 0$. Since $V_1(\mathbf{x})$ and $V_2(\mathbf{x})$ are value functions for any set X_T, Theorem 1 implies that there are constants $\alpha_T > 0$, $\gamma_T > 0$ such that $v_2(x) = \alpha_T v_1(x)$, x in X, and $A_2(t) = \gamma_T A_1(t)$, $0 \leq t \leq T$. For $T \leq T'$: $\alpha_T\, v_1(x) = v_2(x) = \alpha_{T'} v_1(x)$, x in X and $\gamma_T A_1(t) = A_2(t) = \gamma_{T'} A_1(t)$, $0 \leq t \leq T$. But $v(a) \neq 0$ and $A(T) \neq 0$, and thus $\alpha_T = \alpha_{T'}$ and $\gamma_T = \gamma_{T'}$. Hence, α_T and γ_T are independent of $T > 0$.

Proof of Theorem 2. For the forward part of the proof, assume that an outcome stream space $(X_\infty, \succsim)$ satisfies the stated conditions. Then, $\succsim$ restricted to the set X_f satisfies the conditions in Theorem A2. Hence, there exist functions $v(x)$, $a(t)$, $A(t)$ with the stated properties such that $V(\mathbf{x})$ in (A2) is a value function for $(X_f, \succsim)$.

To show that $V(\mathbf{x})$ converges for any $\mathbf{x}$ in X_∞, it suffices to show that for any $\varepsilon > 0$ there exists a time $T > 0$ such that $|\, V((\mathbf{x}_{[0,s]}, \mathbf{o})) - V((\mathbf{x}_{[0,s']}, \mathbf{o}))\,| < \varepsilon$ for all $s, s' > T$.

For $\mathbf{x}$ in X_∞ and $\varepsilon > 0$, choose outcomes $a \prec^X b \prec^X c$ with $A(1)(v(c) - v(a)) < \varepsilon$. Condition (H) implies that there is a time $T \geq 1$ such that $(\mathbf{a}_{[0,1]}, \mathbf{x}, \mathbf{o}_{(s,\infty)}) \precsim (\mathbf{b}_{[0,1]}, \mathbf{x})$ and $(\mathbf{b}_{[0,1]}, \mathbf{x}) \precsim (\mathbf{c}_{[0,1]}, \mathbf{x}, \mathbf{o}_{(s',\infty)})$ for all $s, s' > T$. Then, $(\mathbf{a}_{[0,1]}, \mathbf{x}, \mathbf{o}_{(s,\infty)}) \precsim (\mathbf{c}_{[0,1]}, \mathbf{x}, \mathbf{o}_{(s',\infty)})$ by transitivity, and $V((\mathbf{a}_{[0,1]}, \mathbf{x}, \mathbf{o}_{(s,\infty)})) \leq V((\mathbf{c}_{[0,1]}, \mathbf{x}, \mathbf{o}_{(s',\infty)}))$ since $V(\mathbf{x})$ is a value function for X_f. Thus, $A(1)v(a) + V((\mathbf{x}_{[0,s]}, \mathbf{o})) \leq A(1)v(c) + V((\mathbf{x}_{[0,s']}, \mathbf{o}))$ which implies that $V((\mathbf{x}_{[0,s]}, \mathbf{o})) - V((\mathbf{x}_{[0,s']}, \mathbf{o})) \leq A(1)(v(c) - v(a)) < \varepsilon$. Since this argument is valid with s and s' interchanged, $|\, V((\mathbf{x}_{[0,s]}, \mathbf{o})) - V((\mathbf{x}_{[0,s']}, \mathbf{o}))\,| < \varepsilon$ for $s, s' > T$.

Since $\succsim$ is complete by condition (B), the properties (2), (3) below suffice to show that $V(\mathbf{x})$ is a value function for $(X_\infty, \succsim)$.

(1) If $V(\mathbf{x}) < V(\mathbf{y})$, then there exist a non-point interval $[s, s']$ and outcomes $a \prec^X b$ such that: $\mathbf{x} \precsim (\mathbf{a}_{[s,s']}, \mathbf{x})$, $(\mathbf{b}_{[s,s']}, \mathbf{y}) \precsim \mathbf{y}$ and $|\, V((\mathbf{a}_{[s,s']}, \mathbf{x})) - V((\mathbf{x})\,| < 1/4\, \varepsilon$, $|\, V((\mathbf{y}) - V((\mathbf{b}_{[s,s']}, \mathbf{y}))\,| < 1/4\, \varepsilon$ where $\varepsilon = V(\mathbf{y}) - V(\mathbf{x})$.

Proof. $V(\mathbf{x}) < V(\mathbf{y})$ implies that $V((\mathbf{x}_{[0,T]}, \mathbf{o})) < V((\mathbf{y}_{[0,T]}, \mathbf{o}))$ for some $T > 0$ which implies that $(\mathbf{x}_{[0,T]}, \mathbf{o})) \prec (\mathbf{y}_{[0,T]}, \mathbf{o})$ since $V(\mathbf{x})$ is a value function for X_f. By the proof of (4) in the proof of Theorem 1, there exist a non-point interval $[s, s']$ such that: $0 < \int_{[s,s']} a(t)\, v_* \,dt - \int_{[s,s']} a(t)\, v(\mathbf{x}(t))\, dt < 1/4\, \varepsilon$ and $0 < \int_{[s,s']} a(t)\, v(\mathbf{y}(t))\, dt - \int_{[s,s']} a(t)\, v^*\, dt < 1/4\, \varepsilon$ where $v_* = \sup\{v(\mathbf{x}(t))\text{: } t \text{ in } [s, s']\} < v^* = \inf\{v(\mathbf{y}(t))\text{: } t \text{ in } [s, s']\}$. The conclusions in (1) are true for any outcomes a, b such that $v(a) = v_*$ and $v(b) = v^*$.

(2) If $V(\mathbf{x}) < V(\mathbf{y})$, then $\mathbf{x} \prec \mathbf{y}$.

Proof. Assume the situation in (1) . Since $b \succ^X a$, there exists an outcome $a^+ \succ^X a$ such that $\int_{[s,s']} a(t)\, v(a^+)\, dt - \int_{[s,s']} a(t)\, v(a)\, dt < 1/8\, \varepsilon$. Then, $|\, V((\mathbf{a}^+_{[s,s']}, \mathbf{x})) - V((\mathbf{a}_{[s,s']}, \mathbf{x}))\,| < 1/8\, \varepsilon$. By condition (H), there is a time T_1 such that $(\mathbf{a}_{[s,s']}, \mathbf{x}) \precsim (\mathbf{a}^+_{[s,s']}, \mathbf{x}, \mathbf{o}_{(t,\infty)})$ for $t > T_1$. There is also a T_1' such that $|\, V((\mathbf{a}^+_{[s,s']}, \mathbf{x})) - V((\mathbf{a}^+_{[s,s']}, \mathbf{x}, \mathbf{o}_{(t,\infty)}))\,| < 1/8\, \varepsilon$ for $t > T_1'$. Define $M_1 = \max\{T_1, T_1'\}$. Then, by addition of inequalities, $|\, V((\mathbf{a}^+_{[s,s']}, \mathbf{x}, \mathbf{o}_{(t,\infty)})) - V((\mathbf{a}_{[s,s']}, \mathbf{x}))\,| < 1/4\, \varepsilon$ for $t > M_1$.

By a similar argument, there exist an outcome $b^- \prec^X b$ and a time M_2 such that $(\mathbf{b}^-_{[s,s']}, \mathbf{y}, \mathbf{o}_{(t,\infty)}) \precsim (\mathbf{b}_{[s,s']}, \mathbf{y})$ and $|\, V((\mathbf{b}^-_{[s,s']}, \mathbf{y}, \mathbf{o}_{(t,\infty)})) - V((\mathbf{b}_{[s,s']}, \mathbf{y}))\,| < 1/4\, \varepsilon$ for $t > M_2$. Since $\varepsilon = V(\mathbf{y}) - V(\mathbf{x})$, it follows from the two inequalities in (1) and the above two inequalities that $V((\mathbf{b}^-_{[s,s']}, \mathbf{y}, \mathbf{o}_{(t,\infty)})) > V((\mathbf{a}^+_{[s,s']}, \mathbf{x}, \mathbf{o}_{(t,\infty)}))$ for $t > \max\{M_1, M_2\}$.

Hence, $(\mathbf{b}^-_{[s,s']}, \mathbf{y}, \mathbf{o}_{(t,\infty)}) \succ (\mathbf{a}^+_{[s,s']}, \mathbf{x}, \mathbf{o}_{(t,\infty)})$ for $t > \max\{M_1, M_2\}$ since $V(\mathbf{x})$ is a value function for $(X_f, \succsim)$. In summary, the above arguments yield the preferences: $\mathbf{x} \precsim (\mathbf{a}_{[s,s']}, \mathbf{x}) \precsim (\mathbf{a}^+_{[s,s']}, \mathbf{x}, \mathbf{o}_{(t,\infty)}) \prec (\mathbf{b}^-_{[s,s']}, \mathbf{y}, \mathbf{o}_{(t,\infty)}) \precsim (\mathbf{b}_{[s,s']}, \mathbf{y}) \precsim \mathbf{y}$. Therefore, $\mathbf{x} \prec \mathbf{y}$.

(3) $\mathbf{x} \prec \mathbf{y}$ implies $V(\mathbf{x}) < V(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$ in X_T.

Proof. Define $A(\infty) = \lim_{T\to\infty} \int_0^T a(t)\, dt < \infty$. The value of a constant outcome stream $\mathbf{a}$ is, $V(\mathbf{a}) = A(\infty)\, v(a)$. By Lemma A1, there exist constant outcome streams $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} \sim \mathbf{x}$, $\mathbf{b} \sim \mathbf{y}$. Thus, $V(\mathbf{a}) = V(\mathbf{x})$, $V(\mathbf{b}) = V(\mathbf{y})$ by (2). But, $\mathbf{a} \sim \mathbf{x} \prec \mathbf{y} \sim \mathbf{b}$ implies $\mathbf{a} \prec \mathbf{b}$ by condition (B). Hence, $V(\mathbf{a}) = A(\infty)\, v(a) < V(\mathbf{b}) = A(\infty)\, v(b)$, and thus, $V(\mathbf{x}) < V(\mathbf{y})$.

To show that $v(x)$, $a(t)$, $A(t)$ have the properties (a)-(c), note that $v(x)$, $A(t)$ have the properties (a), (c) by Theorem A1 and $a(t)$ has the property (b) by Lemma A4.

The uniqueness properties of $v(x)$ and $A(t)$ follow from those in Theorem A1 since X_f is a subset of X_∞. And it is straightforward to verify the converse implications.

Appendix B: Comments on related models

Here, we substantiate a claim made at the end of Section 2. The claim is that whenever the set C of outcome streams in a model (1)–(3) discussed in Section 2 contains the continuous, bounded outcome streams, then C also contains some outcome streams that are discontinuous at an uncountable number of times. We will show a stronger result, namely that for any proportion, $p < 1$, the set C contains some outcome streams $\mathbf{x}$ such that the proportion of times at which $\mathbf{x}$ is discontinuous is greater than p.

By a model (1)–(3), we will mean the special case of such a model in which P is an interval of times and the set X is a product of intervals. In such a model, there is a field (= algebra) $\mathcal{F}$ of subsets of P such that for any partition of P into sets in $\mathcal{F}$ the set C contains any outcome stream that is constant on each set in the partition, i.e., any simple outcome stream. A set C that also contains the continuous, bounded outcome streams will be called a proper set. It suffices to show the following property of a model (1)–(3).

(I) If the set C in a model (1)–(3) is proper, then the field $\mathcal{F}$ in the model contains any closed subset P' of P.

The reason that (I) suffices is as follows. The Cantor subsets of $[0, 1]$ are closed, and for any $p < 1$ there is a Cantor set whose Lebesgue measure is greater than p. For $P = [0, T]$ or $P = [0, \infty)$, we can choose P' as a union of shifted Cantor sets. If $x_j < x'_j$ are amounts in a component interval X_j for the outcome space X, then an outcome stream $\mathbf{x}$ such that $\mathbf{x}_j(t)$ is constant for $k \neq j$, $\mathbf{x}_j(t) = x_j$ for t in P' and $\mathbf{x}_j(t) = x'_j$ otherwise will be in the set C, and $\mathbf{x}$ will be discontinuous at every time t in the Cantor set P'.

In a model (1), (2), $\mathcal{F}$ is a σ-field that contains the Borel sets. Thus, $\mathcal{F}$ contains the closed sets, and (I) is established.

In an SEU model (3), however, $\mathcal{F}$ may or may not be a σ-field, and different arguments are needed for different models. Here, we present arguments for two such models.

First, consider Theorem V.4.6 in Wakker (1989, p. 100). Here, the set X has a field Δ of subsets that contains the open subsets of X, and any outcome stream $\mathbf{x}$ in C is $(\mathcal{F}, \Delta)$–measureable, i.e., $\mathbf{x}^{-1}(Y)$ is in $\mathcal{F}$ for any Y in Δ. The field Δ is assumed to contain the open subsets of X (p. 94).

Choose an outcome stream $\mathbf{x}$ such that a component function $\mathbf{x}_j$ is continuous, bounded, and strictly increasing, and the other component functions are constant. For such an outcome stream, the image $\mathbf{x}(A)$ of any open subset A of P is an open subset of X, and $\mathbf{x}^{-1}(\mathbf{x}(A)) = A$. Thus, A is in the field $\mathcal{F}$. Since any closed subset of P is the complement of an open subset, it follows that any closed subset of P is in $\mathcal{F}$.

As a second example, consider Theorem 1 in Kopylov (2010). The assumptions (I) and (III) in his paper imply with a short argument that there exists a field $\mathcal{F}$ of subsets of P such that the simple outcome streams with respect to $\mathcal{F}$ are in the set C.

We will assume that for some $j = 1, \ldots, N$ greater amounts x_j are strictly preferred for some fixed amounts x_k, $k \neq j$. (This assumption can be weakened, but doing so seems unnecessary since in applications at least one attribute will be strictly monotonic.)

For any closed subset A of P and amounts $x_j < x'_j$ in X_j, there is a continuous function $\mathbf{x}_j$ such that $x_j \leq \mathbf{x}_j(t) \leq x'_j$ for t in P and $\{ t : \mathbf{x}_j(t) \leq x_j \} = A$. For example, define $\mathbf{x}_j(t) = (x'_j - x_j)\,\big(d(t, A) / (1 + d(t, A)) \big)$ where $d(t, A)$ is the distance function, $d(t, A) = \inf\{ |t - a| : a \text{ in } A \}$. Suppose that $\mathbf{x}$ denotes the outcome stream such that its j-th component function is $\mathbf{x}_j$ and $\mathbf{x}_k(t) = x_k$, $k \neq j$, for the amounts x_k mentioned above. Also, suppose that $\mathbf{x}_c$ denotes the constant outcome stream, $\mathbf{x}_c(t) = (x_j ; x_k, k \neq j)$

The set $\{ t : \mathbf{x}(t) \precsim \mathbf{x}_c(t) \} = \{ t : \mathbf{x}_j(t) \leq x_j \} = A$. Thus, assumption (IV) states that the closed set A is in the field $\mathcal{F}$ mentioned above.

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