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A Necessary Moment Condition for the Fractional Functional Central Limit Theorem

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A necessary moment condition for the fractional functional central limit theorem*

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Abstract

We discuss the moment condition for the fractional functional central limit theorem (FCLT) for partial sums of $x_t = \Delta^{-d} u_t$, where $d \in (-1/2, 1/2)$ is the fractional integration parameter and u_t is weakly dependent. The classical condition is existence of $q > \max(2, (d+1/2)^{-1})$ moments of the innovation sequence. When d is close to $-1/2$ this moment condition is very strong. Our main result is to show that under some relatively weak conditions on u_t, the existence of $q \geq \max(2, (d+1/2)^{-1})$ is in fact necessary for the FCLT for fractionally integrated processes and that $q > \max(2, (d+1/2)^{-1})$ moments are necessary for more general fractional processes. Davidson and de Jong (2000) presented a fractional FCLT where only $q > 2$ finite moments are assumed, which is remarkable because it is the only FCLT where the moment condition has been weakened relative to the earlier condition. As a corollary to our main theorem we show that their moment condition is not sufficient.

Keywords: Fractional integration, functional central limit theorem, long memory, moment condition, necessary condition.

JEL Classification: C22.

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1 Introduction

The fractional functional central limit theorem (FCLT) is given in Davydov (1970) for partial sums of the fractionally integrated process $\Delta^{-d}\varepsilon_t$, where ε_t is i.i.d. with mean zero under a moment condition of the form $E|\varepsilon_t|^q < \infty$ for $q > q_0 = \max(2, (d + 1/2)^{-1})$.

This result has been extended and generalized in numerous directions. For example, Marinucci and Robinson (2000) replace ε_t by a class of linear processes, assuming the same moment condition. The latter authors proved FCLTs for so-called type II fractional processes, whereas Davydov (1970) discussed a type I fractional process, but the distinction between type I and type II processes is not relevant for our discussion of the moment condition.

Davidson and de Jong (2000, henceforth DDJ) state in their Theorem 3.1 that for some near-epoch dependent (NED) processes with uniformly bounded q'th moment the fractional FCLT holds, but with a much weaker moment condition than previous results, namely $q > 2$. To the best of our knowledge, Theorem 3.1 of DDJ is the only fractional FCLT for which the moment condition has been weakened relative to the earlier condition.

In the next section we give some definitions and construct an i.i.d. sequence and a fractional linear process which are central to our results. In Section 3 we present our main results which state that if the fractional FCLT holds for any class of processes $\mathcal{U}(q)$ containing these processes, then it follows that $q \geq q_0$ if the fractional FCLT is based on fractional integration coefficients and $q > q_0$ if the coefficients are more general. The proofs of both results are based on counter examples which are constructed in a similar way as a counter example in Wu and Shao (2006, Remark 4.1). In Section 4 we discuss the results and give two applications. In particular, it follows from our main result that if the FCLT holds for NED processes with uniformly bounded q moments, then $q \geq q_0$. Hence DDJ's Theorem 3.1 and all their subsequent results do not hold under the assumptions stated in their theorem.

Throughout, c denotes a generic finite constant, which may take different values in different places.

2 Definitions

Definition 1 *We assume that u_t is a zero mean covariance stationary stochastic process which satisfies the moment condition*

$$\sup_{-\infty<t<\infty} E|u_t|^q < \infty \text{ for some } q \geq 2 \tag{1}$$

and has long-run variance

$$\sigma_u^2 = \lim_{T\to\infty} T^{-1}E(\sum_{t=1}^{T} u_t)^2, \; 0 < \sigma_u^2 < \infty. \tag{2}$$

For such processes we define the two classes:

- *$\mathcal{U}_{lin}(q)$ is the class of linear processes $u_t = \sum_{n=0}^{\infty} \tau_n \varepsilon_{t-n}$, where $\sum_{n=0}^{\infty}\sum_{j=n}^{\infty} \tau_j^2 < \infty$ and ε_t is i.i.d. with mean zero and variance $\sigma_\varepsilon^2 > 0$.*

- *$\mathcal{U}_{NED}(q)$ is the class of processes u_t which are $\mathcal{L}_2$-NED of size $-\frac{1}{2}$ on v_t with $d_t = 1$, where v_t is either an α−mixing sequence of size $-q/(1-q)$ or a ϕ−mixing sequence of size $-q/(2(1-q))$; see Assumption 1 of DDJ.*

We base our results on the construction of the following two specific processes.

Definition 2 *Let ε_t be i.i.d. with mean zero, variance $\sigma_\varepsilon^2 > 0$, and finite q'th moment for some $q \geq 2$ to be chosen later. For such ε_t we define the two processes:*

- $u_{1t} = \varepsilon_t$.
- $u_{2t} = \varepsilon_t + \Delta^{1+d}\varepsilon_t$.

For these two processes we note the following connection with the classes $\mathcal{U}_{lin}(q)$ and $\mathcal{U}_{NED}(q)$. Because $q \geq q_0$ is only stronger than $q \geq 2$ when $d < 0$ we consider only this case.

Lemma 1 *For $d \in (-1/2, 0)$ and for $i = 1, 2$, $u_{it} \in \mathcal{U}_{lin}(q) \cap \mathcal{U}_{NED}(q)$ and the long-run variance of u_{it} is σ_ε^2.*

Proof. Clearly, $u_{1t} = \varepsilon_t$ is contained in both $\mathcal{U}_{lin}(q)$ and $\mathcal{U}_{NED}(q)$ and has long-run variance σ_ε^2.

Let $b_j(d) = (-1)^j\binom{-d}{j}$ denote the coefficients in the binomial expansion of $(1-z)^{-d}$, which satisfy $|b_j(d)| \leq cj^{d-1}$. The process $u_{2t} = \varepsilon_t + \Delta^{1+d}\varepsilon_t = \varepsilon_t + \sum_{j=0}^{\infty} b_j(-d-1)\varepsilon_{t-j}$ is a linear process and

$$\sum_{n=0}^{\infty}\sum_{j=n}^{\infty} b_j(-d-1)^2 \leq c\sum_{n=0}^{\infty}\sum_{j=n}^{\infty} j^{-2d-4} \leq c\sum_{n=0}^{\infty} n^{-2d-3} \leq c \text{ for } d \in (-1/2, 0),$$

so that u_{2t} is in $\mathcal{U}_{lin}(q)$. To see that u_{2t} is in $\mathcal{U}_{NED}(q)$, we calculate

$$||u_{2t} - E(u_{2t}|\varepsilon_{t-m}, \ldots, \varepsilon_{t+m})||_2 = ||\sum_{n=m+1}^{\infty} b_n(-d-1)\varepsilon_{t-n}||_2 \leq c(\sum_{n=m+1}^{\infty} n^{-2d-4})^{1/2} \leq cm^{-d-3/2}.$$

Because $3/2 + d > 1/2$ for $d \in (-1/2, 0)$, this shows that u_{2t} is $\mathcal{L}_2$-NED of size $-1/2$ on ε_t, and hence u_{2t} is also in $\mathcal{U}_{NED}(q)$. The generating function for u_{2t} is $f(z) = 1 + (1-z)^{1+d}$ and for $z = 1$ we find because $1 + d > 0$ that $f(1) = 1$. Therefore the long-run variance of u_{2t} is $\lim_{T\to\infty} T^{-1}E(\sum_{t=1}^{T}(\varepsilon_t + \Delta^{1+d}\varepsilon_t))^2 = f(1)^2 Var(\varepsilon_t) = \sigma_\varepsilon^2$. ■

We next give a general formulation of the FCLT for fractional processes. For any process u_t which satisfies (1) and (2), we construct a fractional process by defining

$$x_t = \Delta^{-d}u_t = \sum_{j=0}^{\infty} b_j(d)u_{t-j} \text{ for } -1/2 < d < 0. \tag{3}$$

This process is well defined because, with $||x||_2$ denoting the $\mathcal{L}_2$-norm, we have from (1) that

$$||x_t||_2 \leq c\sum_{j=0}^{\infty} j^{d-1}||u_{t-j}||_2 \leq c \text{ for } d < 0.$$

We also define the scaled partial sum process

$$X_T(\xi) = \sigma_T^{-1}\sum_{t=1}^{[T\xi]} x_t, \; 0 \leq \xi \leq 1, \tag{4}$$

where $\sigma_T^2 = E(\sum_{t=1}^{T} x_t)^2$ and $[z]$ is the integer part of the real number z.

Fractional FCLT for $\mathcal{U}(q)$: We say that the functional central limit theorem (FCLT) for fractional processes holds for a set $\mathcal{U}(q)$ of processes if, for $u_t \in \mathcal{U}(q)$, it holds that

$$X_T(\xi) \xrightarrow{D} X(\xi) \text{ in } D[0,1], \tag{5}$$

where $X(\xi)$ is fractional Brownian motion.

Here, $\xrightarrow{D}$ denotes convergence in distribution (weak convergence) in $D[0,1]$ endowed with a suitable metric, see Billingsley (1968).

3 The necessity result

Our main result is the following theorem.

Theorem 1 *Let $X_T(\xi)$ be defined by (3) and (4) with $-1/2 < d < 0$ and let $\mathcal{U}(q)$ be any class containing u_{1t} and u_{2t}. If the fractional FCLT holds for $\mathcal{U}(q)$ for some $q \geq 2$, then $q \geq q_0$.*

Proof. We prove the theorem by assuming that there is a $q_1 \in [2, q_0)$ for which the FCLT holds for $\mathcal{U}(q_1)$, and show that this leads to a contradiction by a careful construction of ε_t and therefore u_{1t} and u_{2t}.

For $u_{it}, i = 1, 2$, we define x_{it} and X_{iT} by (3) and (4), and because u_{it} is in $\mathcal{U}(q_1)$ the fractional FCLT holds by the maintained assumption for u_{it} and hence $X_{iT}(\xi)$ converges in distribution to fractional Brownian motion.

(i) The normalizing variance for X_{1T}. The variance of $\sum_{t=1}^T x_{1t} = \sum_{t=1}^T \Delta^{-d} u_{1t} = \sum_{t=1}^T \Delta^{-d} \varepsilon_t$ can be found in Davydov (1970), see also Lemma 3.2 of DDJ,

$$\sigma_{1T}^2 = E(\sum_{t=1}^T x_{1t})^2 \sim \sigma_\varepsilon^2 V_d T^{2d+1}, \tag{6}$$

where $V_d = \frac{1}{\Gamma(d+1)^2}(\frac{1}{2d+1} + \int_0^\infty ((1+\tau)^d - \tau^d)^2 d\tau)$ is a constant and "$\sim$" means that the ratio of the left- and right-hand sides converges to one.

(ii) The normalizing variance for X_{2T}. We write x_{2t} and X_{2T} in terms of x_{1t} and X_{1T}, using (3) and (4),

$$\begin{aligned} x_{2t} &= \Delta^{-d} u_{2t} = x_{1t} + \varepsilon_t - \varepsilon_{t-1} && (7) \\ X_{2T}(\xi) &= \sigma_{2T}^{-1} \sum_{t=1}^{[T\xi]} x_{2t} = \sigma_{1T}\sigma_{2T}^{-1} X_{1T}(\xi) + \sigma_{2T}^{-1}(\varepsilon_{[T\xi]} - \varepsilon_0), && (8) \end{aligned}$$

We next find that the variance of $\sum_{t=1}^T x_{2t} = \sum_{t=1}^T \Delta^{-d} u_{2t} = \sum_{t=1}^T (x_{1t} + \varepsilon_t - \varepsilon_{t-1})$ is

$$\begin{aligned} \sigma_{2T}^2 &= E(\sum_{t=1}^T x_{2t})^2 = E(\sum_{t=1}^T (x_{1t} + \varepsilon_t - \varepsilon_{t-1}))^2 = E(\varepsilon_T - \varepsilon_0 + \sum_{t=1}^T x_{1t})^2 \\ &= E(\varepsilon_T - \varepsilon_0)^2 + E(\sum_{t=1}^T x_{1t})^2 + 2E(\sum_{t=1}^T \Delta^{-d}\varepsilon_t(\varepsilon_T - \varepsilon_0)). \end{aligned}$$

The first term is constant, the next is σ_{1T}^2, and letting $1_{\{A\}}$ denote the indicator function of the event A, the last term consists of

$$E(\sum_{t=1}^{T}\Delta^{-d}\varepsilon_t\varepsilon_T) = \sum_{t=1}^{T}\sum_{k=0}^{\infty}b_k(d)E(\varepsilon_{t-k}\varepsilon_T) = \sigma_\varepsilon^2\sum_{t=1}^{T}\sum_{k=0}^{\infty}b_k(d)1_{\{k=t-T\}} = \sigma_\varepsilon^2 b_0(d) = \sigma_\varepsilon^2$$

and

$$E(\sum_{t=1}^{T}\Delta^{-d}u_{1t}\varepsilon_0) = \sum_{t=1}^{T}\sum_{k=0}^{\infty}b_k(d)E(\varepsilon_{t-k}\varepsilon_0) = \sigma_\varepsilon^2\sum_{t=1}^{T}\sum_{k=0}^{\infty}b_k(d)1_{\{k=t\}} \leq c\sum_{t=1}^{T}t^{d-1} \leq c \text{ for } d<0.$$

Therefore,

$$\sigma_{2T}^2 \sim \sigma_{1T}^2 + c. \tag{9}$$

(iii) The contradiction. We now construct the i.i.d. process ε_t so that it has no moment higher than q_1, that is $E|\varepsilon_t|^q = \infty$ for $q > q_1$, by choosing the tail to satisfy

$$P(|\varepsilon_t|^{q_1} \geq c) \sim \frac{1}{c(\log c)^2} \text{ as } c \to \infty. \tag{10}$$

In this case we still have $E|\varepsilon_t|^{q_1} < \infty$. We then find

$$\begin{aligned} P(\sigma_{1T}^{-1}\max_{1\leq t\leq T}|\varepsilon_t| < c) &= P(\sigma_{1T}^{-1}|\varepsilon_1| < c)^T = P(|\varepsilon_1|^{q_1} < c^{q_1}\sigma_{1T}^{q_1})^T \\ &= (1 - P(|\varepsilon_1|^{q_1} \geq c^{q_1}T^{q_1/q_0}))^T \\ &\sim \left(1 - \frac{1}{c^{q_1}T^{q_1/q_0}(q_1(\log c + q_0^{-1}\log T))^2}\right)^T \\ &\sim \exp(-\frac{T^{1-q_1/q_0}}{c^{q_1}(q_1(\log c + q_0^{-1}\log T))^2}) \to 0 \end{aligned}$$

as $T \to \infty$ because $q_1 < q_0$. Thus, $\sigma_{1T}^{-1}\max_{1\leq t\leq T}|\varepsilon_t| \xrightarrow{P} \infty$ because the normalizing constant $\sigma_{1T} = \sigma_\varepsilon V_d^{1/2}T^{1/q_0} = \sigma_\varepsilon V_d^{1/2}T^{1/2+d} < \sigma_\varepsilon V_d^{1/2}T^{1/q_1}$ is too small to normalize $\max_{1\leq t\leq T}|\varepsilon_t|$ correctly.

The definition (8) implies the evaluation

$$\max_{0\leq\xi\leq 1}|\varepsilon_{[T\xi]}| \leq \max_{0\leq\xi\leq 1}|\varepsilon_{[T\xi]} - \varepsilon_0| + |\varepsilon_0| \leq \max_{0\leq\xi\leq 1}|\sigma_{2T}X_{2T}(\xi)| + \max_{0\leq\xi\leq 1}|\sigma_{1T}X_{1T}(\xi)| + |\varepsilon_0|$$

such that

$$\sigma_{1T}^{-1}\max_{0\leq\xi\leq 1}|\varepsilon_{[T\xi]}| \leq \max_{0\leq\xi\leq 1}|\sigma_{1T}^{-1}\sigma_{2T}X_{2T}(\xi)| + \max_{0\leq\xi\leq 1}|X_{1T}(\xi)| + \sigma_{1T}^{-1}|\varepsilon_0|. \tag{11}$$

We have seen in (6) and (9) that $\sigma_{2T}^2 \sim \sigma_{1T}^2 + c$ and $\sigma_{1T}^2 \sim \sigma_\varepsilon^2 V_d T^{1+2d} \to \infty$ for $d > -1/2$, so that $\sigma_{1T}\sigma_{2T}^{-1} \to 1$. Therefore, both $\sigma_{1T}^{-1}\sigma_{2T}X_{2T}(\xi)$ and $X_{1T}(\xi)$ converge in distribution by the previous results and it follows from (11) that $\sigma_{1T}^{-1}\max_{0\leq\xi\leq 1}|\varepsilon_{[T\xi]}|$ is $O_P(1)$. This contradicts that $\sigma_{1T}^{-1}\max_{1\leq t\leq T}|\varepsilon_t| \xrightarrow{P} \infty$, and hence completes the proof of Theorem 1. ■

The proof of Theorem 1 implies that the issue is that the rate of convergence, $T^{-(d+1/2)}$, of $\sum_{t=1}^{[T\xi]}\Delta^{-d}u_{1t}$ can be very slow for d close to $-1/2$. Thus, more control on the tail-behavior of the u_t sequence is needed when $d \in (-1/2, 0)$, and this is achieved through the moment condition (1).

We end this section by giving a complementary result that shows when the moment condition $q > q_0$ is necessary instead of $q \geq q_0$. The former is the moment condition applied by Davydov (1970) and Marinucci and Robinson (2000), and indeed all other fractional FCLT results of which we are aware (with the exception of DDJ).

Define coefficients $a_j(d)$ which satisfy $a_j(d) \sim c\ell(j)j^{d-1}$, where $\ell(j)$ is a (normalized) slowly varying function, see Bingham, Goldie, and Teugels (1989, p. 15). Note that the $b_j(d)$ coefficients from the fractional difference filter are a special case of $a_j(d)$. We now define the general fractional process,

$$x_t = \sum_{j=0}^{\infty} a_j(d) u_{t-j} \text{ for } -1/2 < d < 0, \tag{12}$$

and let the partial sum process $X_T(\xi)$ be defined in (4) as before. We then obtain the following result.

Theorem 2 *Let $X_T(\xi)$ be defined by (12) and (4) with $-1/2 < d < 0$ and let $\mathcal{U}(q)$ be any class containing u_{1t} and u_{2t}. If the fractional FCLT holds for $\mathcal{U}(q)$ for some $q \geq 2$, then $q > q_0$.*

Proof. We assume that there is a $q_1 \in [2, q_0]$ for which the FCLT holds for $\mathcal{U}(q_1)$ and show that this leads to a contradiction. For $u_{it}, i = 1, 2$, we define x_{it} and X_{iT} by (12) and (4) and use the proof of Theorem 1 with the following modifications.

(i) From Karamata's Theorem, see Bingham, Goldie, and Teugels (1989, p. 26), we find that the normalizing variance is $\sigma_{1T}^2 \sim c\ell(T)^2 T^{2d+1} = c\ell(T)^2 T^{1/q_0}$.

(iii) We choose the tail of ε_t as in (10) in the proof of Theorem 1 and take $\ell(T) = (\log T)^{-1}$ and find

$$\begin{aligned} P(\sigma_{1T}^{-1} \max_{1 \leq t \leq T} |\varepsilon_t| < c) &= P(\sigma_{1T}^{-1} |\varepsilon_1| < c)^T = P(|\varepsilon_1|^{q_1} < c^{q_1} \sigma_{1T}^{q_1})^T \\ &= (1 - P(|\varepsilon_1|^{q_1} \geq c^{q_1} T^{q_1/q_0} \ell(T)^{q_1}))^T \\ &\sim \left(1 - \frac{1}{c^{q_1} T^{q_1/q_0} \ell(T)^{q_1} (q_1(\log c + q_0^{-1} \log T + \log \ell(T)))^2}\right)^T \\ &\sim \exp(-\frac{T^{1-q_1/q_0} \ell(T)^{-q_1}}{c^{q_1}(q_1(\log c + q_0^{-1} \log T + \log \ell(T)))^2}) \to 0 \end{aligned}$$

as $T \to \infty$ because $q_1 \leq q_0$. Note that even with $q_1 = q_0$ (and $q_0 > 2$ because $d < 0$) we have the factor $\exp(-c(\log T)^{q_1 - 2}) \to 0$ which ensures the convergence to zero. The contradiction follows exactly as in the proof of Theorem 1. ■

4 Discussion

In this section we present two corollaries which demonstrate how our results apply to the processes in Marinucci and Robinson (2000) and to those in DDJ, respectively, and we discuss some implications for the results of DDJ.

Corollary 1 *Let $X_T(\xi)$ be defined by (12) and (4) with $-1/2 < d < 0$. If the fractional FCLT holds for $\mathcal{U}_{lin}(q)$ then $q > q_0$. Thus, the moment condtion (1) with $q > q_0$ is necessary for Theorem 1 of Marinucci and Robinson (2000).*

Proof. The first statement follows from Theorem 2 because u_{1t} and u_{2t} are in $\mathcal{U}_{lin}(q)$ by Lemma 1. The second statement follows because the univariate version of Assumption A of Marinucci and Robinson (2000) (translated to type I processes) was in fact used to define the class $\mathcal{U}_{lin}(q)$. ■

It follows from Corollary 1 that the moment condition applied by Marinucci and Robinson (2000) is in fact necessary for their results. That is, using the coefficients $a_j(d)$ to define a general fractional process where u_t is a linear process, our results show that $q > q_0$ is necessary for the fractional FCLT. However, it does not follow from our results that $q \geq q_0$ is necessary for the FCLT when u_t is an i.i.d. or ARMA process because the process u_{2t} needed in the construction is neither i.i.d. nor ARMA.

We next discuss the implications of Theorem 1 for the results of DDJ who state in their Theorem 3.1 that the fractional FCLT (5) holds for $\mathcal{U}_{NED}(q)$ if $q > 2$. It is noteworthy that $\mathcal{U}_{NED}(q)$ allows u_t to have a very general dependence structure through the NED assumption, but in particular that DDJ assume only that $\sup_t E|u_t|^q < \infty$ for $q > 2$, which is much weaker than (1) if $d < 0$.

The following corollary to Theorem 1 shows how our result applies to DDJ.

Corollary 2 *Let $X_T(\xi)$ be defined by (3) and (4) with $-1/2 < d < 0$. If the fractional FCLT holds for $\mathcal{U}_{NED}(q)$ then $q \geq q_0$. Thus, the moment condtion (1) with $q \geq q_0$ is necessary for Theorem 3.1 of DDJ.*

Proof. >From Lemma 1 we know that u_{1t} and u_{2t} are in $\mathcal{U}_{NED}(q)$ which by Theorem 1 proves the first statement. The last statement follows because Assumption 1 of DDJ was used to define $\mathcal{U}_{NED}(q)$. ■

It follows from Corollary 2 that Theorem 3.1 of DDJ (and their subsequent results relying on Theorem 3.1) does not hold under their Assumption 1. It is well known, e.g. Billingsley (1968, chp. 15), that the fractional FCLT holds upon proving convergence of the finite-dimensional distributions and tightness (stochastic equicontinuity). Since finite-dimensional convergence holds from standard central limit theorems for $\mathcal{U}_{NED}(q)$, and in particular holds with only $q > 2$ moments, it is the proof of tightness that fails in DDJ and requires the stronger moment condition.

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