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Production in Incomplete Markets: Expectations Matter for Political Stability

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Production in incomplete markets: expectations matter for political stability

Hervé Crès* and Mich Tvede†

Abstract

In the present paper we study voting-based corporate control in a general equilibrium model with incomplete financial markets. Since voting takes place in a multi-dimensional setting, super-majority rules are needed to ensure existence of equilibrium. In a linear-quadratic setup we show that the endogenization of voting weights (given by portfolio holdings) can give rise to - through self-fulfilling expectations - dramatical political instability, i.e. Condorcet cycles of length two even for very high majority rules.

Keywords: Incomplete markets, super majority voting, political (in)stability, selfulfilling expectations.

JEL-classification: D21, D52, D71, D72.

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1 Introduction

In general equilibrium models with production and incomplete financial markets, agents (consumers/shareholders) trade assets, but at the market equilibrium, their gradients are typically not collinear: they disagree on the way to evaluate income streams outside the market span. Hence profit maximization is not a well defined objective for firms.[1]

A way to resolve these disputes between shareholders is based on majority voting in assemblies of shareholders.[2] Among others Drèze (1985) and DeMarzo (1993) propose the same concept of *majority-stable equilibria*: within each firm, the production plans of other firms remaining fixed, no alternative production plan should be able to rally a majority of the shares against the status quo. As Gevers (1974) already noted, the first problem this approach runs into is existence: Plott (1967) shows that in multi-dimensional voting models, a simple majority political equilibrium typically does not exist.[3] Super majority rules are a way to ensure existence: to defeat the status quo, a challenger should rally a proportion larger than 50% of the voting population.[4] The question of what a 'suitable' level a super majority is, arises: it should be high enough to ensure existence, and low enough not to be

[1]For details on standard general equilibrium models of production with incomplete markets and the roles of the firms, see, e.g., Magill & Quinzii (1996) and the references therein.

[2]The choice of a state contingent production plan in a publicly traded corporation is a genuine problem of social choice. This problem has been profoundly important in the history of economic thought as Arrow's impossibility theorem arose out of his effort to find mechanisms for solving disagreements in such cases.

[3]Benninga & Muller (1979) have shown that if production possibility frontiers are unidimensional, then 50% majority voting works. Another condition ensuring existence of 50% majority equilibria is that the degree of market incompleteness is equal to one, see Cres (2000) and Tvede & Crès (2005).

[4]To get existence, Drèze (1985) gives veto power to some shareholders. This result is generalized in Kelsey & Milne (1996) to encompass other voting rules, such as the generalized median voter rules, a special case of which has been applied to decision theory in firms by Sadanand & Williamson (1991).

too conservative. The standard way to proceed is to associate to each proposal its (Simpson-Kramer) *score*. The score of a proposal (the incumbent, or status quo) is the fraction of the voting population supporting, against this proposal, its most dangerous challenger, i.e., the alternative proposal that rallies the maximal fraction of voters against the incumbent. The most stable proposals are the ones with lowest score, the so-called *min-max*.

A central questions is: For which rate of super majority is the min-max stable? We illustrate the difficulty to answer this question through the investigation of an economy where consumers/investors have linear-quadratic utility functions. A nice observation is that the majority voting mechanism is likely to implement equilibria which have the nicest possible welfare properties one can hope for in an incomplete financial market environment. Indeed, looking at the first-order conditions of constrained Pareto optimality, Drèze (1974) argues that profit should be maximized with respect to shadow prices that average the idiosyncratic shadow prices of all shareholders; hence with respect to the shadow prices of the 'mean shareholder'. From Caplin & Nalebuff (1991) we know conditions under which the latter is a proxy to the min-max and is likely to be stable with respect to a rate of super majority inferior to 64%.

But even there things might not turn that simple. Since the electorate is endogenous, its composition is influenced by the agents' expectations. The classical concept of majority voting equilibrium supposes that shareholders have 'conservative' expectations: at equilibrium they expect that no challenger can defeat the status quo; therefore they believe that the status quo production plans are going to prevail in the future; so they stick to their current portfolios. In equilibrium, given these current portfolios, conservative expectations are self-fulfilling: no challenger can rally a high enough majority against the status quo. Hence voting equilibria may be viewed as plain Nash equilibria (see Drèze (1989), pp. 48-49).

But what happens if shareholders deviate from these conservative expectations? If they expect a challenger to defeat the status quo, they rebalance their portfolios; and it might be the case that, given the new distribution of voting weights/shares, the challenger rallies a high enough majority against the status quo and the expectations are fulfilled.

We expect that an equilibrium which is stable under conservative expectations might not be stable if agents' expectations deviate. Indeed, suppose a firm changes its production plan from y to y', then it changes the dividend matrix for investors/consumers. Therefore consumers whose investment needs are less covered by y' than by y might exit - at least partially - from the capital of the firm: they will sell shares to consumers whose needs are better covered by y' than by y. Hence deviating from conservative expectations might enlarge the voting weight of the consumers who are better off with the challenger y' and diminishes the voting weight of the consumers who are better off with the status quo y : This *exit effect* gives more voting weight in the corporate control mechanism to the shareholders who favor the challenger over the status quo. Clearly if these deviating expectations are confirmed at equilibrium, then the status quo is not stable. At equilibrium, deviation from conservative expectations should not be confirmed. We provide a new concept of equilibrium where such deviating expectations are never confirmed at equilibrium: we dub it majority *exit*-stable equilibrium.

It is shown that generically a (weakly) higher rate is necessary for the corporate charter to secure that a ρ-majority equilibrium is exit-stable. A robust example is provided where a strictly higher rate is needed. The extent to which the corporate charter needs to be increased to secure that a ρ-majority equilibrium is exit-stable depends on the case under consideration. We provide an example where no 50%-majority equilibrium is exit-stable for any rate of super majority. This example gives rise to Condorcet cycles between two alternatives, even for rates of super majority very close to unanimity. Since expectations, whether they are conservative or not, are significant for stability of equilibria, it is natural to think of stability as being influenced by a 'political sunspot'. In general, on the one hand conservative expectations should result in stability because majority equilibria exist for rather low rates of super majority and on the other hand non-conservative expectations should result in instability perhaps in the form of proxy fights and hostile takeovers.

The paper is constructed as follows. Section 2 sets up the model. Section 3 and 4 define the concept of ρ-majority equilibrium, provides computations and links efficiency to stability. Section 5 introduces the concept of *exit-stability*, and, through a simple geometric example, it explores the possible occurrence of political sunspots, and of Condorcet cycles of length two for any rate of super majority.

2 Setup

Consider an economy with 2 dates, $t \in \{0, 1\}$, 1 state at the first date $s = 0$, and S states at the second date $s \in \{1, \dots, S\}$. The probability distribution over the set of states at date 1 is $\pi = (\pi^1, \dots, \pi^S)$ where $\pi^s > 0$. There are: 1 commodity at every state, a continuum of consumers $\phi \in \Phi$ where Φ is the set of characteristics of consumers, and J firms $j \in \{1, \dots, J\}$.

Consumers are characterized by their initial endowments $\omega = (\omega^0, \omega^1, \dots, \omega^S)$ where $\omega^s \in \mathbb{R}$ and preference parameters $\gamma > 0$. Utility functions are of the linear-quadratic type, so the utility of the consumption bundle $x = (x^0, x^1, \dots, x^S)$ where $x^s \in \mathbb{R}$ of a consumer with preference parameter γ is

$$u_\gamma(x) \;=\; x^0 + \sum_{s=1}^{S} \pi^s \left(\gamma x^s - \frac{1}{2}(x^s)^2 \right)$$

Therefore preferences are of mean-variance type with identical linear risk tolerances. The distribution of consumers is described by a probability measure on the product of the set of initial endowments and the set of preference paremeters $\phi = (\omega, \gamma) \in \mathbb{R}^{S+1} \times \mathbb{R}_{++}$. The set of characteristics Φ is supposed to be endowed with the Borel σ-algebra. The probability measure on the set of characteristics is supposed to have compact and convex support $\Phi \subset \mathbb{R}^{S+1} \times \mathbb{R}_{++}$ and to be described by a continuous density $f : \Phi \to \mathbb{R}_+$.

Let $\Omega = (\Omega^0, \Omega^1, \ldots, \Omega^S)$ where

$$\Omega \ = \ \int_\Phi \omega \ f(\phi) d\phi$$

be the mean of initial endowment vectors and let Γ where

$$\Gamma \ = \ \int_\Phi \gamma \ f(\phi) d\phi$$

be the mean of the preference parameters.

Firms are characterized by their production sets $Y_j \subset \mathbb{R}^{S+1}$. Production sets are supposed to be convex and the set of efficient production plans Z_j, where Z_j is defined by

$$Z_j \ = \ \{y_j \in Y_j | (\{y_j\} + \mathbb{R}^{S+1}_+) \cap Y_j = \{y_j\}\},$$

is supposed to be compact. It is assumed that if $y_j \in \text{co } Z_j$ for all j, then the dividend matrix

$$y_\mathbf{1} \ = \ \begin{pmatrix} y_1^1 & \cdots & y_J^1 \\ \vdots & & \vdots \\ y_1^S & \cdots & y_J^S \end{pmatrix}$$

has rank J.

Consumers have shares in firms and the distribution of shares is described by an integrable function $\delta = (\delta_1, \ldots, \delta_J) : \Phi \to \mathbb{R}^J_+$ such that

$$\int_\Phi \delta_j(\phi) \ f(\phi) d\phi \ = \ 1.$$

It is of no importance whether shares are assumed to be non-negative or not. Here shares are assumed to be non-negative.

Let $q = (q_1, \ldots, q_J)$ be the price vector, then the problem of consumer ϕ is

$$\max_{(x,\theta)} \ \ x^0 + \sum_{s=1}^S \pi^s \left(\gamma x^s - \frac{1}{2}(x^s)^2 \right)$$

$$\text{s.t.} \ \begin{cases} x^0 - \omega^0 \ = \ \sum_j q_j \delta_j(\phi) - \sum_j (q_j - y_j^0) \theta_j \\ \\ x^s - \omega^s \ = \ \sum_j y_j^s \theta_j \text{ for all } s \geq 1. \end{cases}$$

3 Stock Market Equilibrium

Definition 1 *A* ***stock market equilibrium*** *(with fixed production plans) is an integrable consumption map, an integrable portfolio map, a price vector and a list of production plans* (q^*, x^*, θ^*, y) *where* $x^* : \Phi \to \mathbb{R}^{S+1}$*,* $\theta^* : \Phi \to \mathbb{R}^J$*,* $q^* \in \mathbb{R}^J$ *and* $y = (y_1, \ldots, y_J)$ *such that:*

- *consumers maximize utilities, so* $(x^*(\phi), \theta^*(\phi))$ *is a solution to the problem of consumer* ϕ *given* y *and* q^**, and;*
- *markets clear, so for all* $j \in \{1, \ldots, J\}$

$$\begin{aligned} \textstyle\int_\Phi x^*(\phi)\ f(\phi)d\phi &= \textstyle\int_\Phi \omega\ f(\phi)d\phi \\ \textstyle\int_\Phi \theta^*(\phi)\ f(\phi)d\phi &= \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \end{aligned}$$

Suppose that (θ^*, x^*, q^*, y) is a stock market equilibrium, then the first-order condition for the problem of consumer ϕ is

$$D_x u_\gamma(x^*(\phi)) \begin{pmatrix} y_1^0 - q_1 & \cdots & y_J^0 - q_J \\ y_1^1 & \cdots & y_J^1 \\ \vdots & & \vdots \\ y_1^S & \cdots & y_J^S \end{pmatrix} = 0.$$

Therefore let subscript $\mathbf{1}$ denote the last S coordinates of a vector, let $\mathbf{1}_S$ ($\mathbf{1}_J$) denote the row-vector with S (J) coordinates where all coordinates are equal to 1 and let Π denote the $S \times S$ matrix with π in the diagonal, then as a function of the characteristics and the portfolio of a consumer the equilibrium price vector is

$$q^* = y^0 + (\gamma \mathbf{1}_S - \omega_{\mathbf{1}} - y_{\mathbf{1}} \theta^*(\phi))^T \Pi y_{\mathbf{1}}.$$

Hence the equilibrium price vector is

$$q^* = y^0 + (\Gamma \mathbf{1}_S - \Omega_{\mathbf{1}} - y_{\mathbf{1}} \mathbf{1}_J)^T \Pi y_{\mathbf{1}}.$$

Thus at a stock market equilibrium, for consumer ϕ the portfolio, the consumption at date 1 and the gradient of consumption at date 1 are

$$\begin{aligned}
\theta^{*T} &= (y^0 - q^* + (\gamma \mathbf{1}_S - \omega_{\mathbf{1}})\Pi y_{\mathbf{1}})(y_{\mathbf{1}}^T \Pi y_{\mathbf{1}})^{-1} \\
&= \mathbf{1}_J + ((\gamma - \Gamma)\mathbf{1}_S + (\Omega_{\mathbf{1}} - \omega_{\mathbf{1}}))\Pi y_{\mathbf{1}}(y_{\mathbf{1}}^T \Pi y_{\mathbf{1}})^{-1} \\
x_{\mathbf{1}}^* &= \omega_{\mathbf{1}} + \theta^{*T} y_{\mathbf{1}}^T \\
&= \omega_{\mathbf{1}} + \mathbf{1}_J y_{\mathbf{1}}^T + ((\gamma - \Gamma)\mathbf{1}_S + (\Omega_{\mathbf{1}} - \omega_{\mathbf{1}}))\Pi y_{\mathbf{1}}(y_{\mathbf{1}}^T \Pi y_{\mathbf{1}})^{-1} y_{\mathbf{1}}^T \\
D_{x_{\mathbf{1}}} u_\gamma(x^*) &= (\gamma \mathbf{1}_S - \omega_{\mathbf{1}} - \mathbf{1}_J y_{\mathbf{1}}^T)\Pi \\
&\quad -((\gamma - \Gamma)\mathbf{1}_S + (\Omega_{\mathbf{1}} - \omega_{\mathbf{1}}))\Pi y_{\mathbf{1}}(y_{\mathbf{1}}^T \Pi y_{\mathbf{1}})^{-1} y_{\mathbf{1}}^T \Pi
\end{aligned}$$

It is assumed that for all consumers the consumption at date 1 is below the bliss-point, so $x_{\mathbf{1}}^{*s}(\phi) < \gamma$ for all ϕ and $s \geq 1$.

4 Majority Stable Equilibria

The problem of firm j is more complicated to state than the problem of consumer ϕ because in firms consumers vote over production plans. Let (q^*, x^*, θ^*, y) be a stock market equilibrium and let y'_j be a challenger to y_j, then consumer ϕ votes for the challenger if and only if consumer ϕ is better off

$$u_\gamma(x^*(\phi) + (y'_j - y_j)\theta_j^*(\phi)) \;>\; u_\gamma(x^*(\phi)).$$

Let $\Phi(x^*, \theta^*, y, y'_j) \subset \Phi$ be the set of consumers that vote for the challenger. Since the dimension of the set of alternatives may be higher than one, super-majority rules may be needed to ensure political stability in firms, so let $\rho \in [0, 1]$ be the majority rule. Then at a stock market equilibrium (q^*, x^*, θ^*, y) the preferred set of firm j is defined by

$$P_j^\rho(x^*, \theta_j^*, y_j) \;=\; \left\{ y'_j \in Y_j \;\middle|\; \frac{\int_{\Phi(x^*, \theta^*, y, y'_j)} \max\{\theta_j^*(\phi), 0\}\; f(\phi) d\phi}{\int_\Phi \max\{\theta_j^*(\phi), 0\}\; f(\phi) d\phi} > \rho \right\}.$$

Therefore a challenger production plan is preferred to a status quo production plan if and only if $\rho \times 100$ percent of the consumers are better off with the change. The production plan y_j is a solution to the problem of firm j if and only if $P_j^\rho(x^*, \theta_j^*, y) = \emptyset$.

Definition 2 *A* ρ***-majority stable equilibrium*** *is an integrable consumption map, a integrable portfolio map, a price vector and a list of production plans* $(q^*, x^*, \theta^*, y^*)$ *such that:*

- $(q^*, x^*, \theta^*, y^*)$ *is a stock market equilibrium, and;*
- y_j^* *is a solution to the problem of firm* j *so* $P_j^\rho(x^*, \theta_j^*, y_j^*) = 0$.

Since utility functions are quasi-concave, if

$$u_\gamma(x^*(\phi) + (y_j' - y_j)\theta^*(\phi)) \ > \ u_\gamma(x^*(\phi)),$$

then for all $\tau \in [0, 1[$

$$u_\gamma(x^*(\phi) + ((1-\tau)y_j' + \tau y_j - y_j)\theta^*(\phi)) \ > \ u_\gamma(x^*(\phi)).$$

Therefore if the distance between the challenger and the status quo decreases, then the fraction of consumers who are better off with the challenger compared to the status quo increases.

Observation 1 *Suppose that* (q^*, x^*, θ^*, y) *is a stock market equilibrium and that* U_j *is an open neighborhood relative to* Y_j *of* y_j*, then* $P_j^\rho(x^*, \theta_j^*, y_j) \cap U_j = \emptyset$ *if and only if* $P_j^\rho(x_j, \theta_j^*, y_j) = \emptyset$.

According to the minimal differentiation principle only infinitesimal changes of production plans need to be considered and for infinitesimal changes first-order approximations of utility function can be used to evaluate changes in utility; At a stock market equilibrium (q^*, x^*, θ^*, y) consumer ϕ is better off with a change in direction v of the production plan in firm j if and only if there exists $\tau > 0$ such that

$$u_\gamma(x^*(\phi) + \tau v \theta_j^*(\phi)) \ > \ u_\gamma(x^*(\phi)).$$

so consumer ϕ is better off with a change in direction v if and only if

$$Du_\gamma(x^*(\phi))v\theta_j^*(\phi) \;>\; 0.$$

In a general setup, Tvede & Crès (2005) shows that $(S-J)/(S-J+1)$ is the lowest rate for which majority stable equilibria exist where $(S-J)$ is the number of missing markets. The drawback of this result is that in case markets are very incomplete (i.e., $(S-J)$ is large), one needs a super-majority close to unanimity to ensure existence of equilibria. In the present parametric setup, a dimension-free rate can be given for existence of stable equilibria. It comes as a straightforward application of Theorem 1 in Caplin & Nalebuff (1991), an application whose possibility is mentioned in Caplin & Nalebuff (1991) Example 4.2.

Theorem 1 in Caplin & Nalebuff (1991) states that, in an n-dimensional subspace, there is no way to cut linearly a compact and convex support endowed with a ν-concave distribution through its centroid so that one of the two resulting pieces is larger than $100r(n+1/\nu)$ percent of the weight.

All ingredients are present to apply Theorem 1 in Caplin & Nalebuff (1991); indeed, at stock market equilibria: 1. portfolios $\theta^*(\phi)$ depend linearly on characteristics ϕ; 2. the set of characteristics where shares in firm j are non-negative is convex; 3. voting weights $\max\{\theta_j^*(\phi), 0\}$ depend linearly on characteristics in the set of characteristics where shares in firm j are non-negative, and; 4. gradients $Du_\gamma(x^*(\phi))$ depend linearly on characteristics ϕ.

The dimensionality here is given by the number of parameters in ϕ, hence $(n =)$ $S+2$. If we restrict our study to the set of ϕ's where shares in firm j are positive (which is, according to the latter point 3., compact and convex) then consumer ϕ is better off with a change in direction v if and only if $Du_\gamma(x^*(\phi))v > \; 0$; one therefore sees that the subset of consumers/parameters ϕ which are better off with a change in direction v is defined (thanks to the latter point 4) as a linear cutting of a compact convex support through the centroid of the considered distribution; moreover the density of the distribution of voting weights on $v_j(y^*)$ is $\theta_j(y^*, \phi)f(\phi)$ where

θ_j is 1-concave and f is σ-concave; so according to Lemma 1 in the appendix the density is $\sigma/(\sigma+1)$-concave. Hence according to Theorem 1 in Caplin & Nalebuff (1991) $(q^*, (x^*, \theta^*), y^*)$ where $q^* = q(y^*)$, $x^*(\phi) = x(y^*, \phi)$ and $\theta^*(\phi) = \theta(y^*, \phi)$ is a ρ-majority equilibrium for $\rho \geq r(S + 3 + 1/\sigma)$.

Observation 2 *Let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by*

$$r(a) \;=\; 1 - \left(\frac{a}{a+1}\right)^a .$$

Suppose that $f : \Phi \to \mathbb{R}_+$ is σ-concave, so for all $\phi, \phi' \in \Phi$ and $\tau \in [0,1]$

$$f((1-\tau)\phi + \tau\phi')^\sigma \;\geq\; (1-\tau)f(\phi)^\sigma + \tau f(\phi')^\sigma .$$

Then for all $\rho \geq r(S + 3 + 1/\sigma)$ there exist ρ-majority equilibria.

The proof of the observation is postponed to the appendix. The proof reveals that a stock market equilibria where the profit of firm j is maximized with respect to the average gradient of the consumers with positive shares in firm j exists and is majority stable. Therefore the production plan for each firm satisfies the Drèze criterion and the Drèze criterion ensures that the first-order conditions of constrained Pareto optimality (for consumers with positive shares) are satisfied. This is as close as we can get to Pareto optimality given the current state of the art. We find it remarkable that the Drèze criterion can be supported by majority voting.

Two important properties of the linear-quadratic utility functions are central for Observation 2: 1. the distribution of initial shares does not matter because utility functions are linear in consumption at date 0, and; 2. gradients and shares are linear functions of the parameters.

A sufficient condition for existence of majority stable equilibria for $\rho = 0.5$ is that the set of characteristics is one-dimensional because then the median voter theorem can be applied. (In a multi-dimensional setup it is known that symmetry conditions on the set of characteristics where shares are non-negative and on distribution of gradients and portfolios are needed - see

Grandmont (1978).) In DeMarzo (1993) two other sufficient conditions for existence for $\rho = 0.5$ are provided: 1. production sets are one-dimensional, or; 2. the number of missing markets $S - J$ is equal to 1 - see also Tvede & Crès (2005). A simple example that is used in the sequel of the paper is provided.

A geometric example

There are two states of nature and only one firm, so $S = 2$ and $J = 1$. The probability distribution on the set of states is symmetric so $\pi^1 = \pi^2 = 0.5$. Consumers have identical utility functions and initial endowments are 0 at date 0, so consumers only differ with respect to their initial endowments at date 1. Endowments at date 1 are distributed on the line between $a = (-1, 1)$ and $b = (1, -1)$ such that there is no aggregate risk so $\Omega_{\mathbf{1}} = (0, 0)$. The set of efficient production plans $Z \subset \mathbb{R}^S$ is supposed to be defined by

$$Z \;=\; \{y \in \mathbb{R}^3_+ | y^0 = 1 \text{ and } \|(y^1, y^2)\| = 1\}.$$

At a stock market equilibrium (q^*, x^*, θ^*, y), for consumer ϕ the portfolio is $\theta^*(\phi) = 1 - \omega_{\mathbf{1}} \cdot y_{\mathbf{1}}$ so $\theta^*(\phi) \geq 0$ for all $\omega_{\mathbf{1}}$ on the line between a and b and $y \in Z$, and the consumption at date 1 is $x^*_{\mathbf{1}}(\phi) = \omega_{\mathbf{1}} + \theta^*(\phi) y_{\mathbf{1}}$. Therefore if $y^2 > y^1$ ($y^2 < y^1$), then $\theta^*(\phi)$ is increasing (decreasing) from a to b and $x^*_{\mathbf{1}}(\phi)$ is the orthogonal projection of $\omega_{\mathbf{1}}$ on the line through $y_{\mathbf{1}}$ othogonal to Z at $y_{\mathbf{1}}$.

For a challenger $y' \in Z$ consumer ϕ is better off with the challenger if and only if $u(x^*(\phi) + \theta^*(\phi)(y' - y)) > u(x^*(\phi))$ or equivalently

$$\gamma \mathbf{1}_S \cdot (y'_{\mathbf{1}} - y_{\mathbf{1}}) \; > \omega_{\mathbf{1}} \cdot (y'_{\mathbf{1}} - y_{\mathbf{1}}).$$

because $x^*_{\mathbf{1}}(\phi) = \omega_{\mathbf{1}} + \theta^*(\phi) y_{\mathbf{1}}$. Therefore consumers with $\omega_{\mathbf{1}}$ below the line $\ell(y, y')$ defined by $(\gamma \mathbf{1}_S - \omega_{\mathbf{1}}) \cdot (y' - y) = 0$ are better off with the challenger y' and consumers with $\omega_{\mathbf{1}}$ above the line are better off with the status quo y. The problem is sketched in a Hotelling-like model in Figure 1 below.

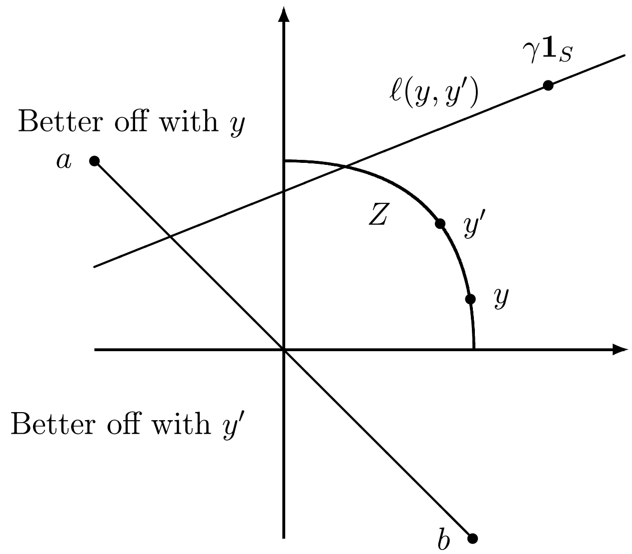

Figure 1: The status quo y vs. the challenger y'

As the challenger y' moves towards the status quo y, the line $\ell(y, y')$ turns clockwise around $\gamma \mathbf{1}_S$. Hence if the distance between the challenger and the status quo decreases, then the fraction of consumers who are better off with the challenger compared to the status quo increases as stated in Observation 1 on the minimal differentiation principle. Clearly if the distribution of consumers on the line from a to b is symmetric, then y^* where $y^{2*} = y^{1*}$ is a ρ-majority stable equilibrium for $\rho = 0.5$, as the distribution of shares is uniformly distributed on the line from a to b so $\theta^*(\phi) = 1$ for all ϕ. Moreover at the ρ-majority stable equilibrium where $y^{2*} = y^{1*}$ the median voter is the Drèze mean shareholder. Obviously for all $y \in Z$ such that $y^1, y^2 > 0$, there exists a symmetric distribution of consumers such that y is a ρ-majority stable equilibrium for $\rho = 0.5$.

5 'Exit' expectations

At majority stable equilibria as in the standard equilibrium concepts of the literature consumers are supposed to have *conservative* expectations in the sense that they expect status quo production plans to be stable. Therefore consumers expect that no challenger is able to defeat status quo. In equilibrium these conservative expectations are confirmed: no challenger defeats status quo. But what happens if consumers deviate from conservative expectations?

For a list of status quo production plans $y = (y_1, \ldots, y_J)$ suppose that consumers expect the challenger y'_j to defeat the status quo y_j in a proxy fight in firm j. Then all consumers should trade as if the production plan of firm j is y'_j rather than y_j. Hence if (q', x', θ', y') where $y' = (y_1, \ldots, y_{j-1}, y'_j, y_{j+1}, \ldots, y_J)$, is a stock market equilibrium, then the outcome should be $(q', \tilde{x}, \theta', y)$ where $\tilde{x} : \Phi \to \mathbb{R}^{S+1}$ is defined by

$$\begin{cases} \tilde{x}^0(\phi) & = & \omega^0 + \sum_j \delta_j(\phi) q'_j - \sum_j \theta_j(\phi)(q'_j - y_j^0) \\ \\ \tilde{x}^s(\phi) & = & \omega^s + \sum_j \theta_j(\phi) y_j^s \text{ for all } s \geq 1 \end{cases}$$

so prices and portfolios adjust to expectations: the challenger y'_j defeats the status quo y_j. Hence the change of production plan from y_j to y'_j should turn the outcome into a stock market equilibrium.

Clearly the change from the status quo y_j to the challenger y'_j changes the dividend matrix, so y'_j compared to y_j offers new insurance opportunities. Thus consumers whose insurance needs are less (more) covered by y'_j than by y_j will exit (enter) - at least partially - from the capital of the firm. Hence consumers whose needs are less covered by y'_j than by y_j sell shares to consumers whose needs are more covered by y'_j than by y_j. Thus exit expectations enlarge the voting weight of the consumers who are better off with the challenger y'_j and diminish the voting weight of the consumers who are better off with y_j. Clearly if exit expectations are confirmed at equilibrium, then the status quo is not stable. At equilibrium, exit expectations should not be confirmed. Therefore another equilibrium concept is proposed.

Definition 3 *A* ***ρ-majority exit-stable equilibrium*** *is an integrable consumption map, an integrable portfolio map, a price vector and a list of production plans $(q^*, x^*, \theta^*, y^*)$ such that:*

- *$(q^*, x^*, \theta^*, y^*)$ is a stock market equilibrium, and;*
- *if $(q, x, \theta, (y^*_{-j}, y_j))$ is a stock market equilibrium and $\tilde{x} : \Phi \to \mathbb{R}^{S+1}$ is defined by*

$$\begin{cases} \tilde{x}^0(\phi) &= \omega^0 + \sum_k \delta_k(\phi) q_k - \sum_k \theta_k(\phi)(q_k - y_k^{*0}) \\ \tilde{x}^s(\phi) &= \omega^s + \sum_k \theta_k(\phi) y_k^{*s} \text{ for all } s \geq 1 \end{cases}$$

  then $P_k^\rho(\tilde{x}, \theta, y_k^*) = \emptyset$.

It should be expected that a ρ-majority exit-stable equilibrium is a ρ-majority stable equilibrium as shown - at least partially - in the following proposition.

Proposition 1 *Let $(q^*, x^*, \theta^*, y^*)$ be a ρ-majority exit-stable equilibrium and suppose that for all $(y_n)_{n \in \mathbb{N}}$ there exists $(q_n, x_n, \theta_n)_{n \in \mathbb{N}}$ such that $(q_n, x_n, \theta_n, y_n)$ is a stockmarket equilibrium and if $y_n \to y^*$, then $(x^n, \theta^n, q^n) \to (x^*, \theta^*, q^*)$ in the sup-norm. Then $(q^*, x^*, \theta^*, y^*)$ is a ρ-majority stable equilibrium.*

The proof of the proposition is postponed to the appendix.

A geometric example - continued

Consider the ρ-majority stable equilibrium $(q^*, x^*, \theta^*, y^*)$ where $\rho = 0.5$ and $y^{2*} = y^{1*}$ exhibited at the end of Section 4. Suppose exit expectations occur. There are two effects working in opposite direction when measuring the support of the challenger against the status quo. The first effect is the Hotelling effect depicted on Figure 1: the closer the challenger is to the status quo, the more consumers are better off with the challenger. Hence a classical centripetal force influences the position of the challenger (at the source of the minimum differentiation principal in the conservative expectations regime).

But there is another effect which rests on the fact that, under the exit expectations regime, the distribution of voting shares changes with the position of the challenger; one then has the exit effect: the further away the challenger is from the status quo, the more shares do consumers, who are better off with the challenger, have. Hence a centrifugal force influences the position of the challenger. Whether the Hotelling effect or the exit effect dominates depends on the parameters of the model.

We are now presenting examples of distributions where the exit effect dominates the Hotelling effect. In these examples the median voter disapears. It can even be the case that in this one-dimensional problem, no equilibria exists for rates of super-majority strictly smaller than unanimity.

Let endowment at date 1 be parametrized by $\tau \in [-1, 1]$ such that at date 1 the endowment of consumer ϕ is τ in state 1 and $-\tau$ in state 2. Let efficient production plans be parametrized by $v \in [0, 1]$ such that at date 1 the production of the firm is v in state 1 and $\sqrt{1-v^2}$ in state 2. Suppose that $\gamma = 1$.

Suppose that consumers are uniformly distributed on the line between $a = (-1, 1)$ and $b = (1, -1)$. For y^* where $y^{2*} = y^{1*} = 1/\sqrt{2}$ under conservative expectations, portfolios are uniformly distributed and consumers with less (more) endowment in state 1 than in state 2 are better off with a change of production plan to a production plan with more (less) output in state 1 and less in state 2. Therefore y is stable for the majority rule $\rho = 0.5$. For y where $y^1 = y^2 = 1/\sqrt{2}$ under exit expectations where consumers expect z with $z^1 = v$ and $z^2 = \sqrt{1-v^2}$, the portfolio of consumer ϕ is $1 - \tau(v - \sqrt{1-v^2})$ and consumer ϕ is better off with a change of production plan from y to z if and only if

$$\tau \ < \ \frac{v + \sqrt{1-v^2} - \sqrt{2}}{v - \sqrt{1-v^2}}$$

for $v \in]1/\sqrt{2}, 1]$. Hence the voting weight of the consumers who are better off with z, where $z^1 = v$, $z^2 = \sqrt{1-v^2}$ and $v \in]1/\sqrt{2}, 1]$, than with y is

$$\int_{-1}^{\frac{v+\sqrt{1-v^2}-\sqrt{2}}{v-\sqrt{1-v^2}}} \frac{1}{2}(1 - \tau(v - \sqrt{1-v^2}))\ d\tau$$

$$= \ \frac{1}{2}\frac{(2v-\sqrt{2})(1-\sqrt{1-v^2}) + \sqrt{2}v - 1}{v - \sqrt{1-v^2}}$$

The relation between v and the voting weight of the consumers who are better off with z than with y is shown in Figure 1 where it is seen that y is exit-stable for the majority rule $\rho \approx 0.53$. Thus in order to ensure exit stability the majority rule has to be increased from 0.5 to approximately 0.53. But with 0.53 as majority rule only conservative expectations are self-fulfilling and quite paradoxically no challenger is supported by more than 50% of the shares against the status quo so the rate of super majority is not reached.

Figure 2

Clearly for any distribution of endowments there exists a γ such that the Hotelling effect dominates the exit effect. Indeed the exit effect depends on θ and θ does not depend on γ because $\theta = 1 - \omega_{\mathbf{1}} \cdot y_{\mathbf{1}}$ and the Hotelling effect increases with γ. Hence as γ tends to infinity the rate of super majority needed to ensure exit-stability converges to 0.5. However if the exit effect dominates as in Figure 1, then the minimal differentiation principle does not apply to consumers with exit expectations.

Suppose that the distribution of consumers on the line between a and b is described by a density $f_\alpha : [-1, 1] \to \mathbb{R}_+$, where $\alpha \geq 0$, defined by

$$f_\alpha(\tau) = \begin{cases} \alpha(-\tau)^{2\alpha} + \dfrac{1}{2(\alpha+1)} & \text{for } \tau \in [-1, 0[\\ \\ \alpha(\tau)^{2\alpha} + \dfrac{1}{2(\alpha+1)} & \text{for } \tau \in [0, 1] \end{cases}$$

Then the mass of consumers is one for all α. Moreover it is the uniform distribution for $\alpha = 0$ and it converges to the distribution where all the mass is equally split between a and b as α tends to ∞. For a production plan y where $y^1 = v$ and $y^2 = \sqrt{1 - v^2}$ and $v \in [1/\sqrt{2}, 1]$, under conservative expectations the voting weight of the consumers who are better off with a production plan with a marginally larger production than v state 1 and a marginally smaller production than $\sqrt{1-v^2}$ in state 2 is

$$\begin{aligned} W_\alpha(v) &= \int_{-1}^{\frac{\sqrt{1-v^2}-v}{\sqrt{1-v^2}+v}} (1-\tau(v-\sqrt{1-v^2}))\left(\alpha(-\tau)^{2\alpha}+\frac{1}{2(\alpha+1)}\right)\,d\tau \\ &= \frac{1}{2(\alpha+1)}\left(1+\frac{\sqrt{1-v^2}-v}{\sqrt{1-v^2}+v}\right) \\ &\quad +\frac{v-\sqrt{1-v^2}}{4(\alpha+1)}\left(1-\left(\frac{\sqrt{1-v^2}-v}{\sqrt{1-v^2}+v}\right)^2\right) \\ &\quad +\frac{\alpha}{2\alpha+1}\left(1-\left(-\frac{\sqrt{1-v^2}-v}{\sqrt{1-v^2}+v}\right)^{2\alpha+1}\right) \\ &\quad +\frac{\alpha(v-\sqrt{1-v^2})}{2(\alpha+1)}\left(1-\left(-\frac{\sqrt{1-v^2}-v}{\sqrt{1-v^2}+v}\right)^{2(\alpha+1)}\right). \end{aligned}$$

Clearly $W_\alpha : [1/\sqrt{2}, 1] \to [0, 1]$ is continuous, $W_\alpha(1/\sqrt{2}) = 0.5$ and $W_\alpha(1) = 0$ and $\lim_{\alpha\to\infty} W_\alpha(v) = 1/2 + (v - \sqrt{1-v^2})/2 > 0.5$ for all $v \in]1/\sqrt{2}, 1[$. Therefore there exists a sequence $(\alpha_n, v_n)_{n\in\mathbb{N}}$, where $\lim_{n\to\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} v_n = 1$ with $v_n \in]1/\sqrt{2}, 1[$ for all n, such that $W_{\alpha_n}(v_n) = 0.5$ for all n. Hence under conservative expectations v_n and by symmetry $\sqrt{1-v_n^2}$ are stable for $\rho = 0.5$ for the distribution f_{α_n}.

Under exit expectations, suppose that $\sqrt{1-v^2}$ where $v \in]1/\sqrt{2}, 1]$, is the status quo and that v is the challenger so consumers expect v to defeat $\sqrt{1-v^2}$, then the voting weight of the consumers who are better off with v than with $\sqrt{1-v^2}$ is

$$\begin{aligned} W_\alpha^e(v) &= \int_{-1}^{0} (1-\tau(v-\sqrt{1-v^2}))\left(\alpha(-\tau)^{2\alpha}+\frac{1}{2(\alpha+1)}\right)\,d\tau \\ &= \frac{1}{2(\alpha+1)}+\frac{v-\sqrt{1-v^2}}{4(\alpha+1)}+\frac{\alpha}{2\alpha+1}+\frac{\alpha(v-\sqrt{1-v^2})}{2(\alpha+1)} \end{aligned}$$

Therefore $\lim_{n\to\infty} W_{\alpha_n}^e(v_n) = 1$ because $\lim_{n\to\infty} v_n = 1$. Hence for all $\rho < 1$ there exists n that v_n and $\sqrt{1-v_n^2}$ is a 2-cycle in the sense that if $\sqrt{1-v_n^2}$

resp. v_n is the status quo, but consumers expect the challenger v_n resp. $\sqrt{1-v_n^2}$ to defeat the status quo, then the majority for challenger is larger than ρ.

Under exit expectations, for every $\rho \in [0,1[$, there exist $\bar{\alpha}, \varepsilon > 0$, such that for $\alpha > \bar{\alpha}$: 1. if $v \in [0,\varepsilon] \cup [1-\varepsilon, 1]$, then the voting weight of the consumers who are better off with $\sqrt{1-v^2}$ is larger than ρ, and; 2. if $v \in [\varepsilon, 1-\varepsilon]$, then the voting weight of the consumers who are better off with either 0 or 1 is larger than ρ. Therefore for every $\rho \in [0,1[$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, then no production plan is exit stable.

Under conservative expectations, there exist stable production plans for the simple majority rule. Under exit expectations for all super-majority rules there exist distributions of consumers such that: no production plan is exit-stable, and; there exist 2-cycles of stable production plans (v and $\sqrt{1-v^2}$ are stable under conservative expectations and if consumers expect v to defeat $\sqrt{1-v^2}$, then v defeats $\sqrt{1-v^2}$ and if consumers expect $\sqrt{1-v^2}$ to defeat v, then $\sqrt{1-v^2}$ defeats v).

6 Final remarks

For conservative expectations we have shown in our setup that if shares are traded *before* production plans are decided as in Drèze (1974), then the initial distribution of shares is without importance for stability. However if shares are traded *after* production plans are decided as in Grossman & Hart (1979), then the initial distribution of shares matter. Indeed in order to apply Caplin & Nalebuff (1991) the initial distribution of shares has to be assumed to be σ-concave. Therefore, in our setup, markets have an important role in smoothing shares.

For exit expectations we have shown in our setup that if shares are traded before production plans are decided, then markets may be destabilizing in the sense that stability cannot be ensured because the challenger is expected to defeat the status quo. If share are traded after production plans are decided, then expectations are without importance. Therefore there seems to be a tradeoff between on the one hand the smoothing effect of markets and on the other hand the destabilizing effect of markets.

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Appendix

The product of a α-concave and a β-concave distribution

Lemma 1 *If $G : K \to \mathbb{R}_+$ is α-concave and $H : K \to \mathbb{R}_+$ is β-concave. Then $F : K \to \mathbb{R}_+$ defined by $F(a) = G(a)H(a)$ for all $a \in K$ is ν-concave for all*

$$\nu \ \leq \ \frac{\alpha\beta}{\alpha+\beta}.$$

Proof: It follows from the definition of ν -concavity that if

$$
\begin{aligned}
&(((1-t)G(a)^\alpha + tG(b)^\alpha)^{1/\alpha}((1-t)H(a)^\beta + tH(b)^\beta)^{1/\beta})^\nu \\
\geq\ & (1-t)(G(a)H(a))^\nu + t(G(b)H(b))^\nu
\end{aligned}
$$

for all $a, b \in K$ and $t \in [0,1]$ then $F : K \to \mathbb{R}_+$ is ν-concave.

Let $g, h : [0,1] \to \mathbb{R}_+$ be defined by

$$
\begin{aligned}
g(t) &= ((1-t)G(a)^\alpha + tG(b)^\alpha)^{1/\alpha} \\
h(t) &= ((1-t)H(a)^\beta + tH(b)^\beta)^{1/\beta}
\end{aligned}
$$

then g is α-concave and h is β-concave. Let $f : [0,1] \to \mathbb{R}_+$ be defined by $f(t) = (g(t)h(t))^\nu$ then the second-order derivative is

$$
\begin{aligned}
D^2 f &= \nu(gh)^{\nu-2}((\nu-1)((gDf)^2 + (fDg)^2) + 2\nu(gDf)(fDg) \\
&\quad + fg(gD^2f + fD^2g)) \\
&\leq \nu(gh)^{\nu-2}((\nu-\alpha)(gDf)^2 + 2\nu(gDf)(fDg) + (\nu-\beta)(fDg)^2).
\end{aligned}
$$

The "$\leq$" follows from the fact that g being α-concave is equivalent to g^α being concave so $D^2 g^\alpha = \alpha g^{\alpha-2}((\alpha-1)(Dg)^2 + gD^2g) \leq 0$ implying $gD^2g \leq (1-\alpha)(Dg)^2$ – similarly for h and β.

Finally $(\nu-\alpha)(gDf)^2 + 2\nu(gDf)(fDg) + (\nu-\beta)(fDg)^2) \leq 0$ for all values of gDf and fDg if and only if $\nu \leq \alpha\beta/(\alpha+\beta)$. Hence, $F : K \to \mathbb{R}_+$ is ν-concave for all $\nu \leq \alpha\beta/(\alpha+\beta)$.

Q.E.D

Proof of Observation 2

Let the maps $q : \text{co } Z \to \mathbb{R}^J$, $\theta : \text{co } Z \times \Phi \to \mathbb{R}^J$ and $x : \text{co } Z \times \Phi \to \mathbb{R}^{S+1}$ be defined by

$$\begin{aligned} q(y) &= y^0 + (\Gamma \mathbf{1}_S - \Omega_{\mathbf{1}} - y_{\mathbf{1}} \mathbf{1}_J)^T \Pi y_{\mathbf{1}} \\ \theta(y,\phi) &= (y^0 - q^* + (\gamma \mathbf{1}_S - \omega_{\mathbf{1}}) \Pi y_{\mathbf{1}})(y_{\mathbf{1}}^T \Pi y_{\mathbf{1}})^{-1} \\ &= \mathbf{1}_J + ((\gamma - \Gamma)\mathbf{1}_S + (\Omega_{\mathbf{1}} - \omega_{\mathbf{1}}))\Pi y_{\mathbf{1}} (y_{\mathbf{1}}^T \Pi y_{\mathbf{1}})^{-1} \\ x(y,\phi) &= (\omega^0 + q(y)\delta(\phi) - (q(y) - y^0)\theta(y,\phi), \omega_{\mathbf{1}} + \theta(y,\phi) y_{\mathbf{1}}^T) \end{aligned}$$

Then all maps are continuous and θ and x are linear in characteristics.

Let the correspondences $v_1, \ldots, v_J : \text{co } Z \to \Phi$ be defined by

$$v_j(y) = \{\phi | \theta_j(y,\phi) \geq 0\}.$$

Then all correspondences are continuous and convex valued. Let the maps $\pi_1, \ldots, \pi_J : \text{co } Z \to \mathbb{R}^{S+1}$ be defined by

$$\pi_j(y) = \int_{v_j(y)} Du_\gamma(x(y,\phi)) \, \theta_j(y,\phi) f(\phi) d\phi.$$

Then all maps are continuous.

Let the correspondences $p_j : \text{co } Z \to \text{co } Z_j$ be defined by

$$p_j(y) = \{y_j' | \forall z_j' \in \text{co } Z_j : \pi_j(y) z_j' \leq \pi_j(y) y_j\}.$$

Then all correspondences are upper hemi-continuous. Therefore according to Kakutani's fixed point theorem there exists y^* such that $y_j^* \in p_j(y^*)$ for all j and $y_j^* \in Z_j$ because by assumption $Du_\gamma(x(y,\phi)) \in \mathbb{R}_{++}^{S+1}$ for all ϕ.

Q.E.D

Proof of Proposition 1

Let $(q^*, x^*, \theta^*, y^*)$ be a ρ-majority exit-stable equilibrium and suppose that for all $(y_n)_{n \in \mathbb{N}}$ there exists $(q_n, x_n, \theta_n)_{n \in \mathbb{N}}$ such that $(q_n, x_n, \theta_n, y_n)$ is a stock-market equilibrium and if $y_n \to y^*$, then $(x^n, \theta^n, q^n) \to (x^*, \theta^*, q^*)$ in the sup-norm. Therefore if $(1/\|y_j^n - y_j^*\|)(y_j^n - y_j^*) \to v_j$ and $\theta^*(\phi) Du_\gamma(x^*(\phi)) \cdot v_j > 0$, then there exists N such that if $n \geq N$, then $u_\gamma(x_n(\phi) + \theta_j(\phi)(y_j^n - y_j^*)) > u_\gamma(x_n(\phi))$, because $(q_n, x_n, \theta_n) \to (q^*, x^*, \theta^*)$ in the sup-norm and because the utility function is continuous. Hence if a ρ-majority exit-stable equilibrium, then it is a ρ-majority stable equilibrium.

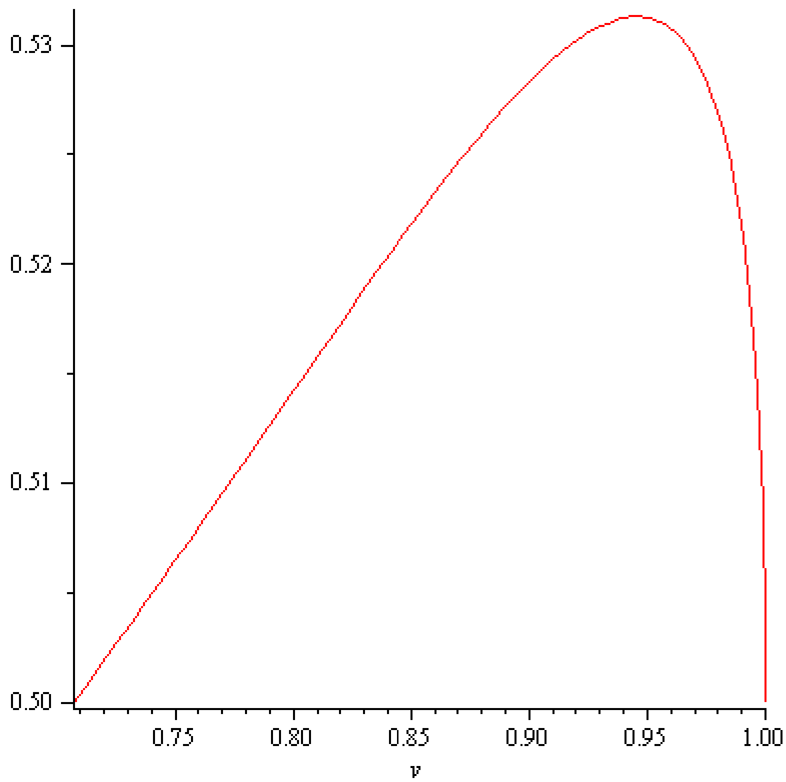

Figure 2: Rate of super-majority