Discussion Papers
Department of Economics
University of Copenhagen

No. 08-24

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http://www.econ.ku.dk

ISSN: 1601-2461 (online)

Decentralized Pricing in Minimum Cost Spanning Trees

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October 2008

Abstract

In the minimum cost spanning tree model we consider decentralized pricing rules, i.e. rules that cover at least the efficient cost while the price charged to each user only depends upon his own connection costs. We define a *canonical* pricing rule and provide two axiomatic characterizations. First, the canonical pricing rule is the smallest among those that improve upon the Stand Alone bound, and are either superadditive or piece-wise linear in connection costs. Our second, direct characterization relies on two simple properties highlighting the special role of the source cost.

Keywords: Pricing rules, Minimum cost spanning trees, Canonical pricing rule, Stand-alone cost, Decentralization.

1 Introduction

The notion of individual guarantees is as old as the discussion of Fair Division. My actual share of the pie will depend upon other agents' characteristics, but I am guaranteed a certain fraction of the pie no matter what these characteristics turn out to be. The higher this "worst case" share, the less risky my participation in the division rule. For instance Steinhaus ([23]), and many authors after him (see e.g., Brams and Taylor [7]), regards a division of the pie as "fair" if each participant receives a share worth (to him) at least $1/n$-th of the entire pie. In the pie division problem, the $1/n$-th guarantee can be implemented, and no higher share can.

This idea has been applied to virtually all formal models of Fair Division, including public decisions with side payments ([11],[24], [15]), the assignment of indivisible goods ([10], [4]) and cooperative production ([16]); see [17] for a systematic discussion.

Here we look for the best individual guarantees in a classic network connection problem, the *minimal cost spanning tree* problem ([3],[9],[22]). This is a cost sharing problem, therefore individual guarantees take the form of an *upper bound* on cost shares, and feasibility requires that the sum of these upper bounds cover at least the actual cost. In problems where costs are subadditive w.r.t. demands, the natural and much discussed upper bound is the *Stand Alone* upper bound (e.g., [21],[17]), namely the cost a given agent would incur to meet his own demand, irrespective of whether other users' demands are met or not. Its key feature is *decentralization*: the Stand Alone upper bound only depends upon the cost function and the individual demand of the agent in question. Thus it can be interpreted as a (non linear) *pricing rule*, that an agent can use to choose a level of demand. The interesting question is to find a feasible decentralized pricing rule that is as close as possible to cover the actual cost. For instance with a one-input one-output concave cost function C such that $C(0) = 0$, the Stand Alone cost is a feasible pricing rule that cannot be improved[1].

In the minimal cost spanning tree (thereafter mcst) problem, costs are subadditive w.r.t. demands, so the Stand Alone upper bound is a feasible pricing rule. However we define a *canonical* pricing rule that considerably improves it, and only depends upon the connection costs of the agent in question: it can be computed prior to any evaluation of *other* users connection costs. The canonical rule is given by a simple linear expression (Definition 2), closely related to an exact cost-sharing method in the mcst problem, dubbed the *Folk* solution in [5], and introduced independently by several authors ([13],[19],[6],[2]). Our canonical decentralized charge is bounded below by that of the Folk solution, with equality whenever the cost matrix is *irreducible*: see the discussion at the end of section 3.

[1] Writing x_i for agent i's demand, let $f(x_i)$ be a pricing rule improving upon the Stand alone upper bound: $f(x_i) \leq C(x_i)$ for all x_i. Combined with feasibility, $C(\sum_i x_i) \leq \sum_i f(x_i)$, this implies easily $f \equiv C$.

In section 4 we show a couple of natural examples where the ratio of the total charge collected by the canonical pricing rule to the efficient cost grows as $\log n$ in the number n of users. This compares favourably to the Stand Alone price, which in the same examples collects about n times the efficient cost. The canonical pricing rule has three desirable properties[2], pertaining to changes in connection costs and in the set of network users. The price I pay is a continuous function of all connecting costs, and is weakly increasing in any one of my own connection costs. If new users enter the network, this price decreases weakly. See Lemma 2.

Our "canonical" terminology is vindicated by three axiomatic characterizations of this pricing rule. In Theorem 1 we borrow two functional properties of the mapping from the matrix of connection costs (for all users) to the efficient cost (that of an optimal spanning tree): this mapping is *superadditive* and *piece-wise linear.* Superadditivity w.r.t. connection costs conveys the designer's preference for flexibility: it is (weakly) cheaper to build an optimal network for today's cost matrix, and possibly another network for tomorrow's cost matrix, rather than a single network optimal for the sum of today and tomorrow's connecting costs. Piece-wise linearity is the fact that when the same network is optimal for two different cost matrices, then the optimal cost is linear in the cost matrix.

Theorem 1 states that the canonical pricing rule is the smallest one that improves upon the Stand Alone bound and is superadditive (or piece-wise linear) in the profile of connecting costs.

Theorem 2 offers an alternative characterization relying on two simple properties highlighting the special role of the source cost vis-a-vis the inter-agent connecting nodes.

2 The MCST Model

We recall the well known minimal cost spanning tree model (see e.g. [22]). Let $N \subset \mathbf{N} = \{1, 2, \dots\}$ be a finite set of agents where $|N| = n$. We consider networks with a *source* denoted by agent 0. The source can be considered as a firm supplying the agents in N. A network g over $N^0 = N \cup \{0\}$ is a set of unordered pairs ij where $i, j \in N^0$. We denote by $N^0(2)$ the set of such unordered pairs; its cardinality is $\frac{n(n+1)}{2}$. Sometimes we speak of $N^0(2)$ as the *complete network* on N^0. We write $G^0 = \{g | g \subset N^0(2)\}$ for the set of all networks of N^0.

Two agents i and j are *connected* in g if there is a path $i_1i_2, i_2i_3, \dots, i_{h-1}i_h$

[2] They are also satisfied by the Folk solution and the Stand Alone pricing rule.

such that $i_k i_{k+1} \in g$ for $1 \leq k \leq h-1$ where $i = i_1$ and $j = i_h$. A network g is said to be *connected* if i and j are connected in g for all $i, j \in N^0$. A path is called a *cycle* if it starts and ends with the same agent. A network is called a *tree* if it contains no cycles. A *spanning tree* is a tree connecting all agents in N^0. There are $(n+1)^{n-1}$ such spanning trees.

For each pair $ij \in N^0(2)$, there is a non-negative cost k_{ij} attached to the link between agents i and j. We think of such costs as the costs of establishing the link, maintenance costs or indirect costs such as congestion etc. The set of such costs is an element $K \in \mathbb{R}^{N^0(2)}$. We abuse notation by speaking of the cost *matrix* K.

A *minimum cost spanning tree* (mcst) is a spanning tree T where the total link cost $\sum_{ij \in T} k_{ij}$ is minimized over all spanning trees of N^0. We write this minimal cost as $v(N, K)$. Note that this is also the smallest cost over all networks, not necessarily trees, connecting the source to all agents. There is a unique mcst if all costs k_{ij} are different, but in general there may be more than one (up to $(n+1)^{n-1}$ if all costs k_{ij} are equal).

Two well known algorithms for finding a minimum cost spanning tree given K, are due to Kruskal ([14]) and Prim ([20]). We recall the latter, that will be useful below. There are n steps: in step 1 we pick a cheapest link between the source and one agent; in step t we add one of the cheapest links between the set M_{t-1} of agents already connected to the source, and $N \setminus M_{t-1}$.

3 The Canonical Pricing Rule: definition

As explained in the Introduction, we wish to cover at least the efficient cost while charging each user of the network in a way that only depends upon his or her "local" costs.

Given a mcst problem (N, K) and an agent i, we write $k[i]$ for the $(n-1)$ dimensional vector $(k_{ij}, j \in N \setminus \{i\})$ of this agent's own connection costs to other agents.

Definition 1 *Fix the set N^0 of users and the source. A decentralized pricing rule is a mapping $f : \mathbb{R}_+ \times \mathbb{R}_+^{n-1} \to \mathbb{R}_+$ such that for all $x \geq 0$, the mapping $y \to f(x; y)$ is symmetric in the $n-1$ coordinates of y; and for all cost matrices K we have*

$$\sum_{i \in N} f(k_{0i}; k[i]) \geq v(N, K). \tag{1}$$

The simplest example of a decentralized pricing rule is the *stand alone cost* $sa(k_{0i}; k[i]) = k_{0i}$ for which inequality (1) is obvious. The *canonical* pricing rule defined below is unanimously preferred to sa, because it charges less than sa to every user, often significantly so. Yet this rule too overcharges in most problems, as do all decentralized pricing rules.

Lemma 1 *No decentralized pricing rule can cover costs exactly for all problems.*

Proof. Consider a problem (N, K) where $k_{jl} = 0$ for all $j, l \in N^0$ and let $f(k_{0i}, k[i]) = \alpha$ be the common charge of every agent. Sequentially changing the source cost from 0 to 1 for all agents except for agent 1 and using budget-balance repeatedly, we see that the price charged to each agent cannot change. Next consider $\bar{K}$ where $\bar{k}_{0h} = 1$ for all $h \in N$ and $\bar{k}_{jl} = 0$ otherwise. We have $f(\bar{k}_{0j}, \bar{k}[j]) = \alpha$ for all $j \neq 1$ and by budget-balance $f(\bar{k}_{0i}, \bar{k}[i]) = \alpha + 1$. As the choice of 1 was arbitrarily in the symmetric problem $(N, \bar{K})$, we have a contradiction. ■

We now define the decentralized pricing rule that is the object of this paper in two equivalent ways. In equation (2) we denote by Π_N the set of orderings of N, and given $\pi \in \Pi_N$, by $\mathcal{P}(i, \pi)$ denote the union of the source and the set of agents prior to agent i in the order π, i.e. $\mathcal{P}(i, \pi) = \{0\} \cup \{j \in N | \pi(j) < \pi(i)\}$.

In equation (3) we arrange the $n - 1$ numbers $k[i]$ increasingly as $k_i^t, 1 \leq t \leq n - 1$, so that $k_i^1 \leq \cdots \leq k_i^{n-1}$.

Definition 2 *Given N^0, the Canonical Pricing Rule is defined as*

$$can(k_{0i}; k[i]) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \min_{j \in \mathcal{P}(i,\pi)} \{k_{ij}\} \tag{2}$$

It is equivalently computed as

$$can(k_{0i}; k[i]) = \frac{1}{n} k_{0i} + \sum_{t=1}^{n-1} \frac{1}{t(t+1)} \min\{k_i^t, k_{0i}\} \tag{3}$$

We have

$$can(k_{0i}; k[i]) \leq sa(k_{0i}; k[i]) \tag{4}$$

Equation (3) is a closed form expression of the charge $can(k_{0i}; k[i])$. Therefore it is preferred to (2) when we compute the canonical price for a specific problem.

On the other hand equation (2) gives an intuitive interpretation of the canonical price, as the expected marginal cost of adding a given agent to a (random, and typically inefficient) spanning tree. Pick an unbiased random ordering π of the agents, and construct a spanning tree T_π as follows. Start by connecting the first agent to the source and charging him the corresponding cost; connect next the second agent to either the source or the first agent, whichever is cheaper, and charge him that new cost; ..; charge to the t-th agent the cost of the cheapest link to one of its predecessors or the source; and so on. This is similar to the Prim algorithm, with the crucial difference that in the latter, the t-th agent is not selected at random: instead it is the cheapest to connect with the $t-1$ first agents and the source. In particular for any π the cost of T_π is no less than $v(N,K)$, so inequality (1) holds.

To see why equations (2) and (3) are equivalent, fix $i=1$ for simplicity and observe that if $(k_{01};k[1])$ remains in the cone where the ordering of the n numbers $k_{01}, k_{1i}, 2 \leq i \leq n$ does not change, the right hand side of either equation is a linear function of $(k_{01};k[1])$[3]. A basis of such a cone is made of vectors $(k_{01};k[1])$ with all coordinates equal to 0 or 1, thus it is enough to check the equivalence for such vectors. If $k_{01}=0$, both equations give $can(k_{01};k[1])=0$; if $k_{01}=1$ but $k[1]=0$, both equations give $can(k_{01};k[1])=\frac{1}{n}$. Assume next $k_{01}=1, k_{1i}=0$ for $2 \leq i \leq t, k_{1j}=1$ for $t+1 \leq j \leq n$. Then $\min_{j \in \mathcal{P}(1,\pi)}\{k_{1j}\}=1$ if and only if all predecessors of 1 in π are not smaller than $t+1$, which happens with probability $\frac{1}{t}$, so (2) yields $can(k_{01};k[1])=\frac{1}{t}$. It is clear that (3) gives the same conclusion.

We turn to the relationship between the canonical pricing rule and the (budget-balanced) Folk solution of the mcst problem mentioned in the introduction. The latter uses a *reduction* of the cost matrix introduced by Bird ([3], [1]). Given (N,K), the *reduced cost matrix* K^* is the smallest matrix K' such that $v(N,K)=v(N,K')$. Equivalently k^*_{ij} is the largest number z such that any path from i to j contains at least one edge with cost at least z. The matrix K is called *irreducible* if $K=K^*$. A matrix K is irreducible *if and only if* for all i,j,l we have $\max\{k_{ij},k_{jl}\} \geq k_{il}$, which is easy to recognize numerically. Moreover for any K, irreducible or not, the reduced matrix K^* is irreducible.

Proposition 1 in [5] shows that for any irreducible cost matrix K, the canonical charges $(can(k_{0i};k[i]), i \in N)$ cover the costs exactly. Hence the following equation

$$\varphi_i(N,K)=can(k^*_{0i};k^*[i]) \text{ for all } N,K \text{ and } i$$

defines a budget-balanced solution φ, and this is the *Folk solution*. Because $K^* \leq K$, and the mapping $K \rightarrow can(k_{0i};k[i])$ is monotonic, the Folk solution is always bounded above by the canonical upper bound:

$$\varphi_i(N,K) \leq can(k_{0i};k[i]) \text{ for all } i, \text{ all } K$$

[3]This is the piece-wise linearity property formally defined in section 4.

with equality if the matrix K is irreducible. Note however that the equality $\varphi_i(N, K) = can(k_{0i}; k[i])$ for all i does not imply that K is irreducible[4].

4 The canonical pricing rule: examples and further properties

In the next two examples, we compute how far the profile of canonical upper bounds $(can(k_{0i}; k[i]), i \in N)$ is above budget-balance, and how far below the profile of stand alone upper bounds $(sa(k_{0i}; k[i]), i \in N)$ (inequality (4)).

Example 1: Consider the linear tree $0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow n$, with connecting costs corresponding to distances, namely $k_{ij} = |i - j|$. Assume n is even to fix ideas, and write $\mathcal{H}_n$ for the harmonic number $\mathcal{H}_n = \sum_{j=1}^{n} \frac{1}{j}$. Then compute from (3):

$$can_i = \sum_{j=1}^{i} \frac{1}{2j-1} \text{ for } 1 \leq i \leq \frac{n}{2}$$

$$can_i = \sum_{j=1}^{n-i} \frac{1}{2j-1} + \sum_{j=2(n-i)+1}^{n} \frac{1}{j} = \mathcal{H}_n - \sum_{j=1}^{n-i} \frac{1}{2j} \text{ for } \frac{n}{2} \leq i \leq n$$

from which one deduces easily

$$\sum_{i=1}^{n} can_i \simeq \frac{n}{2}\mathcal{H}_n \Leftrightarrow \frac{\sum_{i=1}^{n} can_i}{v(N,K)} \simeq \frac{1}{2}\log n$$

Thus the relative excess charge $\frac{\sum_{i=1}^{n} can_i}{v(N,K)}$ becomes arbitrarily large, but much slower than under the Stand Alone upper bound:

$$\sum_{i=1}^{n} sa_i = \frac{n(n+1)}{2} \Leftrightarrow \frac{\sum_{i=1}^{n} sa_i}{v(N,K)} \simeq \frac{1}{2}n$$

Note that $\frac{n}{2}$ is also the expected cost of a spanning tree chosen uniformly among all $(n+1)^{n-1}$ spanning trees, independently of any cost consideration. The canonical charges are much closer to the efficient cost.

We conjecture that for any cost matrix satisfying the triangular property $k_{ij} \leq k_{il} + k_{lj}$, the relative excess charge never grows faster than $\frac{1}{2}\log n$.

[4] An example is $N = \{1, 2\}, k_{12} = 2, k_{0i} = 1, i = 1, 2$. Here $v(n, K) = 2 = can_1 + can_2$, yet $k_{12}^* = 1$.

Example 2: Consider the random cost matrix K where all entries k_{ij} are IID with uniform distribution on $[0,1]$. Using the algorithm leading to equation (2) it is easy to compute

$$E\{can_i\} = \frac{1}{n}(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}) \simeq \frac{\log n}{n}; \text{ and } E\{sa_i\} = \frac{1}{2}$$

Up to a factor 2, the limit in n of $\frac{E\{can_i\}}{E\{sa_i\}}$ is as in Example 1.

Moreover Theorem 6.21 in [8] shows that

$$E\{v(N,K)\} \simeq \sum_{j=1}^{\infty} \frac{1}{j^3} \simeq 1.202$$

so the relative excess charge of the canonical rule is, again, of the order $\log n$. Notice that the choice of the uniform distribution is not important in this example. Any cumulative distribution with positive derivative at zero would give the same asymptotic comparison, because the efficient cost would again have a finite limit. We omit the details.

We now define three properties that play a leading role in the axiomatic discussion of the mcst and other fair division problems (see e.g., [12],[2],[5]). These properties are defined directly for a decentralized pricing rule, although it is clear that they apply to general solutions, budget-balanced or not.

The first two are compelling regularity properties:

- **Continuity:** *$f_i(N, k_{0i}, k[i])$ is continuous in $(k_{0i}, k[i])$.*
- **Cost Monotonicity:** *$f_i(N, k_{0i}, k[i])$ is weakly increasing in each cost k_{ij}, k_{0i}.*

Violation of Continuity means that a tiny measurement error may have dramatic consequences on individual charges. Violation of Cost Monotonicity opens the door to artificial inflation of one's costs.

Our third axiom compares a decentralized pricing rule across problems involving different sets of users. For any subset S of N we use the notation $k[i,S]$ for the $|S|$-dimensional vector $(k_{ij}, j \in S)$.

- **Population Monotonicity:** *for any profile of costs $(k_{0i}, k[i])$, any $S \subset N$ and $i \in S$ we have*

$$f(N, k_{0i}, k[i]) \leq f(S, k_{0i}, k[i,S]).$$

This says that the addition of a new user is never detrimental to any of the existing users. It is a strengthening of the Stand Alone upper bound, provided we assume that in a one agent problem, the pricing rule is simply $f(\{i\}, k_{0i}) = k_{0i}$. Population Monotonicity generates clean incentives in the game where agents must decide whether or not to request connection to the source, based on their willingness to pay for such service (see [18]).

Lemma 2 *The canonical pricing rule (and the Stand Alone rule) are Continuous, Cost Monotonic and Population Monotonic.*
Proof. It is enough to check that for all $t, 1 \leq t \leq n-1$, the mapping $(k_{0i}, k[i]) \to k_i^t$ meets all three properties. ■

5 Main characterization

We introduce two properties pertaining to addition and positive linear combinations of cost profiles.

- **Superadditivity:** *for any two* $(k_{0i}^1, k^1[i])$ *and* $(k_{0i}^2, k^2[i])$*, we have*

$$f(N, k_{0i}^1 + k_{0i}^2, k^1[i] + k^2[i]) \geq f(N, k_{0i}^1, k^1[i]) + f(N, k_{0i}^2, k^2[i]).$$

Recall that the efficient cost $v(N, K)$ is superadditive in the matrix K of connecting costs: if K changes over several periods and we must pay connection costs in each period, it is advantageous to adjust optimally the spanning tree in each period. The axiom imposes the same property for the pricing rule.

For our next axiom, given a permutation σ of $\{1, \ldots, p\}$ we write $C_\sigma = \{x \in \mathbf{R}_+^p | x_{\sigma(1)} \leq \cdots \leq x_{\sigma(p)}\}$ for the cone in $\mathbf{R}_+^p$ such that the relative ordering of the coordinates is constant and given by σ. We say that the real valued function g with domain $\mathbf{R}_+^p$ is *piece-wise linear* if for any σ its restriction to C_σ is positively linear (respects positive linear combinations). Key observation: the efficient cost $v(N, K)$ is piece-wise linear in $\mathbf{R}_+^{\frac{n(n+1)}{2}}$.

- **Piece-wise Linearity:** *the pricing rule* $(k_{0i}, k[i]) \to f(k_{0i}, k[i])$ *is piece-wise linear in* $\mathbf{R}_+^n$.

Like Superadditivity above (or additivity in the axiomatic cost sharing literature), this axiom wants the solution to share some key structural property of the efficient cost. Its main justification is informational parsimony: a piece-wise linear pricing rule is entirely determined by its value over n coordinate vectors in one of the cones C_σ.

Theorem 1: *The canonical pricing rule (and the Stand Alone rule) , are superadditive and piece-wise linear.*

Conversely, if a decentralized pricing rule f is superadditive or piece-wise linear, we have

$$\{f(k_{0i}, k[i]) \leq sa(k_{0i}, k[i]) \text{ for all } K\} \Rightarrow \{can(k_{0i}, k[i]) \leq f(k_{0i}, k[i]) \text{ for all } K\}.$$

Proof. *Step 1. can is superadditive and piece-wise linear*

In the sum (2) defining $can(k_{0i}; k[i])$, each term $\min_{j\in\mathcal{P}(i,\pi)}\{k_{ij}\}$ is superadditive in $(k_{0i}, k[i])$. If the relative ordering of the n numbers $(k_{0i}, k_{ij}, j \in N \setminus \{i\})$ remains fixed, each term $\min_{j\in\mathcal{P}(i,\pi)}\{k_{ij}\}$ is positively linear in $(k_{0i}, k[i])$.

Step 2. Assume f is superadditive and $f \leq sa$, prove $can \leq f$

Notation: we write γ^t for the n-dimensional vector $\gamma^t = (1; \overbrace{0, \cdots, 0}^{t-1}, \overbrace{1, \cdots, 1}^{n-t})$. Fix $x \geq 0$ and consider the cost matrix K with $k_{ij} = k_{0i} = x$ for all i, j. Feasibility implies $nx \leq n\, f(x\gamma^1)$; moreover by assumption $f(x\gamma^1) \leq sa(x\gamma^1) = x$. Thus $f(x\gamma^1) = x$. Next fix any $t, 2 \leq t \leq n-1$, and consider the cost matrix

$$k_{0i} = x \text{ for all } i;\ k_{ij} = 0 \text{ if } 1 \leq i, j \leq t;\ k_{ij} = 1 \text{ otherwise.}$$

Observe that the canonical price is $f(x\gamma^1) = x$ for agents $t+1, \cdots, n$ and $f(x\gamma^t)$ for agents $1, \cdots, t$. Feasibility implies

$$tf(x\gamma^t) + (n-t)x \geq (n-t+1)x \Rightarrow f(x\gamma^t) \geq \frac{x}{t}$$

Next we pick a n-dimensional vector of costs $(a; b_2, b_3, .., b_n)$, where a is the cost to the source, and

$$b_2 \leq \cdots \leq b_p \leq a \leq b_{(p+1)} \leq \cdots \leq b_n \tag{5}$$

We decompose the vector $(a; b_2, b_3, .., b_n)$ as follows

$$(a; b_2, b_3, .., b_n) = b_2\gamma^1 + \{\sum_{t=2}^{p-1}(b_{t+1} - b_t)\gamma^t\} + (a - b_p)\gamma^p + (0; \overbrace{0, .., 0}^{p-1}, b_{(p+1)} - a, .., b_n - a) \tag{6}$$

By the argument above

$$f(b_2\gamma^1) = b_2;\ f((b_{t+1} - b_t)\gamma^t) \geq \frac{1}{t}(b_{t+1} - b_t) \text{ for } 2 \leq t \leq p-1;\ f((a - b_p)\gamma^p) \geq \frac{1}{p}(a - b_p)$$

Now superadditivity and $f \geq 0$ imply

$$f(a; b_2, b_3, .., b_n) \geq b_2 + \frac{(b_3 - b_2)}{2} + \frac{(b_4 - b_3)}{3} + .. + \frac{(a - b_p)}{p}$$

the desired conclusion, upon checking that the right-hand-side is precisely $can(a; b_2, b_3, .., b_n)$.

Step 3. Assume f is piece-wise linear and $f \leq sa$, prove $can \leq f$

Both sides of the desired inequality are piece-wise linear in $(k_{0i}, k[i])$ and symmetric in $k[i]$. Thus it is enough to prove $can(a; b_2, b_3, .., b_n) \leq f(a; b_2, b_3, .., b_n)$ when $(a; b_2, b_3, .., b_n)$ is in one of the cones C^p defined by (5) for $2 \leq p \leq n$, or in C^1 defined by $a \leq b_2 \leq \cdots \leq b_n$. Fix $p, 2 \leq p \leq n$. By assumption, in C^p the function f takes the form

$$f(a; b) = \lambda_1 a + \sum_{s=2}^{n} \lambda_s b_s$$

for some fixed numbers λ_s (depending on p).

We use the notation γ^t in step 1, as well as $\delta^t = (0; \overbrace{0, \cdots, 0}^{t-1}, \overbrace{1, \cdots, 1}^{n-t})$. Note that $\gamma^t \in C^p$ iff $t \leq p$, while $\delta^t \in C^p$ iff $t \geq p$.

From the stand alone upper bound at δ^t for $t \geq p$, we get $\sum_{s=t+1}^{n} \lambda_s \leq 0$. Our assumption that f is non negative then implies $\lambda_s = 0$ for $s \geq p + 1$[5].

Next we consider the cost matrix $k_{ij} = k_{0i} = 1$ for all i, j: feasibility and $f \leq sa$ imply $f(\gamma^1) = 1 \Leftrightarrow \sum_{s=1}^{p} \lambda_s = 1$.

Finally we fix $t \leq p$ and consider the cost matrix K^t

$$k_{0i} = 1 \text{ for all } i;\ k_{ij} = 0 \text{ if } i, j \leq t,\ k_{ij} = 1 \text{ otherwise}$$

$$\Rightarrow (k_{0i}, k[i]) = \gamma^t \text{ if } i \leq t;\ \ (k_{0i}, k[i]) = \gamma^1 \text{ if } i \geq t + 1$$

Here feasibility gives

$$v(N, K^t) = n - t + 1 \leq t f(\gamma^t) + (n - t) f(\gamma^1) = t(\lambda_1 + \sum_{s=t+1}^{p} \lambda_s) + (n - t)$$

$$\Leftrightarrow f(\gamma^t) \geq \frac{1}{t} = can(\gamma^t) \text{ for } 1 \leq t \leq p$$

In C^p, both $f(a; b)$ and $can(a; b)$ are linear combinations of $(a; b_2, \cdots, b_p)$ only (recall $\lambda_s = 0$ for $s \geq p + 1$), and each vector in the cone $\{b_2 \leq \cdots \leq b_p \leq a\}$

[5] Note that even absent the assumption $f \geq 0$, these equalities follow the feasibility property (1).

is a positive linear combination of the vectors $\gamma^t, 1 \le t \le p$. Therefore the above inequalities conclude the proof. $\blacksquare$

Example 3: Here is a three agent example of a superadditive and piecewise linear decentralized pricing rule not bounded below by the canonical rule. Let

$$f(a; b_1, b_2) = \frac{1}{3}a + \min\{a, b_1, b_2\}$$

One check first that inequality (1) holds, and that f is not bounded above by the Stand Alone rule. Moreover f is continuous and cost monotonic. Finally, we have

$$f(4; 1, 5) = \frac{7}{3} < can(4; 1, 5) = \frac{1}{3}4 + \frac{1}{2} + \frac{1}{6}4 = \frac{5}{2}.$$

6 A direct characterization

We report an additional characterization of the canonical pricing rule, based on the following two axioms on pricing rules.

- **Relevance:** *for any two agents $i, j \in N$ and profile of costs $(k_{0i}, k[i])$*

$$\{k_{ij} \ge k_{0i}\} \Rightarrow f(N, k_{0i}, k[i]) = f(N \setminus \{j\}, k_{0i}, k[i, N \setminus \{j\}]).$$

If $k_{ij} \ge k_{0i}$, the ij link is not relevant for agent i when linking to any network. Relevance then stipulates that the *irrelevant* cost k_{ij} has no impact on the price charged to agent i.

Next we have a decentralized version of an axiom analyzed in [2]:

- **Equal Share of Extra Costs:** *for any profile of costs $(k_{0i}, k[i])$, agent i and number $\delta > 0$*

$$\{k_{ij} \le k_{0i} \text{ for all } j\} \Rightarrow f(k_{0i} + \delta, k[i]) = f(k_{0i}, k[i]) + \frac{\delta}{n} \text{ for all } i.$$

This axiom applies only to profiles where all nodes j are relevant to i's charge, because it is cheaper to connect to any one of them than to the source. Then it requires to charge to agent i a fair share of any additional cost of connecting to the source.

Theorem 2: *The Canonical Pricing Rule is uniquely characterized by Relevance, Equal Share of Extra Costs, and the stand alone upper bound: $f(k_{0i}, k[i]) \le k_{0i}$.*

Proof. (Sketch) It is easy to check that the axioms are satisfied by *can*. We show that they lead to a unique solution. By Relevance all agents with higher link costs than i's source cost can be disregarded. Next, among the remaining agents, rank their link costs in increasing order and consider the highest cost k^* below k_{0i}. By Equal Share of Extra Costs the difference $k^* - k_{0i}$ is shared equally between the remaining agents. Using Relevance the agent(s) with link cost k^* can then be removed and Equal Share of Extra Costs can be used to share the difference between k^* and the second highest cost and so forth until only agent i is left to be connected to the source. By feasibility and stand alone upper bound agent i must then pay his source cost. Thus, the price is uniquely determined by the axioms. ■

We note that in the special case of 2-agent pricing problems the canonical pricing rule is uniquely characterized by feasibility (1), Stand Alone upper bound and Equal Share of Extra Costs. We omit the straightforward argument.

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