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Abstract

Models of preferences between outcomes over continuous time are important for individual, corporate, and social decision making, e.g., medical treatment, infrastructure development, and environmental regulation. This paper presents a foundation for such models. It shows that conditions on preferences between real- or vector-valued outcomes over continuous time are satisfied if and only if the preferences are represented by a value function having an integral form.

Key words: continuous time, discounting, ordinal utility scale, value function, integral.

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1. Introduction

An analyst who is developing an evaluation of alternatives in a decision model must judge whether to model the consequences of the alternatives as outcomes that occur at discrete times or as outcomes that occur over continuous time. Outcomes occurring at discrete times are modeled as sequences defined on a finite or countable set of times and outcomes occurring over continuous time are modeled as functions defined on an interval of time, to be called a planning period. The judgment as to which type of model to use depends on the nature of the data, the nature of the consequences, and the proclivities of the analyst. Each type seems more appropriate under some circumstances.

This paper is concerned with outcomes that occur over continuous time, to be called outcome-streams. Thus, an outcome-stream is a function defined on an interval of time whose values are outcomes. At each instant of time, an outcome-stream is an amount or a rate—or more generally a vector of amounts and rates. For example, an outcome-stream at an instant of time might be: the rate of usage of a natural resource, one or more rates of monetary costs and benefits, a vector of indices that describe levels of environmental quality, or a double-subscripted vector of health characteristics that describe the health of the individuals in a population.

This paper develops models of preferences between outcome-streams. First, we develop models for a planning period with a finite horizon, and then we develop models for a planning period with an infinite horizon. Each model contains: outcome-streams that are real- or vector-valued functions defined on the planning period, and a preference relation defined on pairs of outcome-streams. We define conditions on the preference relation, and we show that it satisfies the conditions if and only if it is represented by an integral of a discounting function times a scale defined on outcomes at instants of time. This 'outcome scale' is ordinal but is cardinally unique. The integral will be called an integral value function, and the model will be called an integral-value model.

An outcome scale represents preferences between outcomes at a common instant of time, and a discounting function represents tradeoffs between amounts of the outcome scale at different instants. For a more detailed interpretation of these functions, see the working paper, Harvey and Østerdal (2005), on which this paper is based.

The integral-value models are developed by successively extending the family of outcome-streams on which the preference relation is defined. First, we develop a model for outcome-streams that are step functions defined on a bounded time interval. Here, a value of the integral value function reduces to a sum of discount weights times outcome scale amounts. The number of terms in a sum will depend on the outcome-stream.

Second, we develop a model by extending the family of step outcome-streams to a family of outcome-streams defined on the bounded interval that are component-wise Riemann integrable. The value function in this model is an integral as described above.

Third, we develop a model for a family of outcome-streams defined on the interval from zero to infinity. Here, the outcome-streams equal a specified 'null outcome' after some time and are component-wise Riemann integrable on the bounded interval from zero to that time. The time will vary from one outcome-stream to another.

Fourth, we develop a model for a family of outcome-streams that are defined on the unbounded interval that are Riemann integrable on each bounded subinterval. The family is defined in terms of the preference relation. Roughly speaking, an outcome-stream is in the family provided that it is arbitrarily unimportant in the sufficiently distant future.

A detailed discussion is needed to compare these models with previous models on preferences between outcomes at discrete times (e.g., Koopmans, 1960, 1972, Diamond, 1965, and Harvey, 1986, 1995) and on preferences between outcomes over continuous time (e.g., Grodal and Mertens, 1968, and Weibull, 1985). Hence, we defer a discussion to the end of the paper. Here, we discuss three features that distinguish the models in this paper from those in previous research.

(1) In the fourth model mentioned above, the family of comparable outcome-streams is defined in terms of the preference relation. Such a dependence allows instances of the model to have discounting functions with various behaviours at infinity. Previous models specify the family of outcome-streams that are comparable, and thus place restrictions on

the discounting function. This is so since the integral value function in such a model must have a finite value for each outcome-stream in the specified family.

Each of the models here allows nonconstant discounting, i.e., the discounting function can be neither constant nor exponential. It has been argued that only constant discounting should be used in a prescriptive or normative analysis; such arguments are based on the principles of 'temporal consistency' and 'economic efficiency.' But Harvey (1994), Ahlbrecht and Weber (1995), and Bleichrodt and Gafni (1996) argue to the contrary that nonconstant discounting can be reasonable for such a purpose. All three papers discuss the principle of temporal consistency; Harvey also discusses that of economic efficiency.

In particular, the models here allow a discounting function in which the discount rate tends to zero as time tends to infinity. Such a 'slow-discounting function' (Harvey, 1995) is greater than a negative-exponential function in the sufficiently distant future and thus assigns more importance to outcomes that occur then. A slow discounting function can provide insight in a policy study, e.g., a study on natural resources or on environmental quality, in which it is essential to consider the importance of outcomes in the distant future. An analyst can use a slow-discounting function and compare an evaluation based on it with an evaluation based on a negative-exponential discounting function.

(2) The models here allow vector-valued outcomes. Indeed, they allow certain cases in which some of the variables that define the outcomes are categorical variables rather than continuous variables. In the next section, we discuss this use of categorical variables.

If a utility scale has been previously specified, then one can introduce conditions on preferences between the induced utility-streams that imply an integral value function. However, the conditions cannot be interpreted unless the utility scale can be interpreted. In contrast to this approach, we do not assume that a utility scale has been previously specified, and we define conditions on preferences between the original outcome-streams.

(3) The outcome-streams here are (component-wise) Riemann integrable functions on bounded intervals of time. Riemann integrable functions are more elementary than the Lebesgue integrable functions in previous models. Thus, we can deduce integral value

functions by using elementary real analysis—while previous models deduce integral value functions by using existence results from measure theory and functional analysis.

Families of Riemann integrable functions are sufficiently large to include both step functions and continuous functions. Hence, one could verify conditions on preferences between hypothetical step outcome-streams that imply parametric families of discounting functions and outcome scales (see, Harvey 1998a,b). Then, one could use the resulting integral value function to compare the actual, continuous outcome-streams in a study.

The results in this paper are 'if and only if' results; they establish that a preference relation satisfies the conditions in a model if and only if it is represented by a function having the properties in the model. In this sense, we do not assume extra 'technical conditions' such as solvability or differentiability. Proofs are provided in the Appendix.

2. Components of the models

This section defines the components of the integral-value models presented in this paper. It therefore delineates the type of models that are included.

Outcomes and outcome-streams. Suppose that $N \geq 1$ real variables x_j, $j = 1,\ldots,N$, have been defined on sets X_j. Each variable x_j will be called a component variable, and each set X_j will be called a component set. A vector $x = (x_1,\ldots,x_N)$ in the product set $X = X_1 \times \ldots \times X_N$ will be called an outcome, and the set X will be called an outcome set.

We assume that each component set X_j is either an interval or a finite set of numbers. In the first case, x_j will be called a continuous variable, and in the second case x_j will be called a categorical variable.

A planning period will be a bounded interval $P = [0,T]$, $0 < T < \infty$, or the unbounded interval $P = [0,\infty)$. The upper endpoint, T or ∞, will be called the planning horizon.

An outcome-stream will be a real- or vector-valued function $\mathbf{x} = (\mathbf{x}_1,\ldots,\mathbf{x}_N)$ whose domain is a planning period P and whose values are in an outcome set X. Each real-valued function $\mathbf{x}_1,\ldots,\mathbf{x}_N$ in an outcome-stream $\mathbf{x}$ will be called a component-stream.

For outcome-streams and component-streams but not for other types of functions, we use bold type to distinguish between a function and its values. Thus, $x = \mathbf{x}(t)$ will denote the real- or vector value of an outcome-stream $\mathbf{x}$ at a time *t*, and $x_j = \mathbf{x}_j(t)$ will denote the value of a component-stream $\mathbf{x}_j$ at a time *t*.

Step outcome-streams. As a slightly imprecise notation, $\langle a,b \rangle$ will denote any interval that has the finite or infinite endpoints *a*, *b*. Thus, either *a* or *b* may or may not be in the interval. A <u>partition</u> p of a planning period $P = [0,T]$ will be a set of intervals, $\langle a_0, a_1 \rangle$, … , $\langle a_{m-1}, a_m \rangle$, where $0 = a_0 \le a_1 \le \ldots \le a_m = T$ and the intervals are pairwise disjoint with the union $[0,T]$.

A <u>step</u> <u>outcome</u>-<u>stream</u> based on a partition p will be an outcome-stream of the form, $\mathbf{x}(t) = x(i)$ for t in $\langle a_{i-1}, a_i \rangle$, $i = 1,\ldots,m$. Thus, an outcome-stream $\mathbf{x} = (\mathbf{x}_1,\ldots,\mathbf{x}_N)$ is a step outcome-stream if and only if each component-stream $\mathbf{x}_j$ in $\mathbf{x}$ is a step function with values in the component set X_j. The set of step outcome-streams based on a partition p will be denoted by S_p, and the union of the sets S_p will be denoted by S_T.

Outcome-streams of any type will be denoted by letters near the end of the alphabet, e.g., $\mathbf{x}$, $\mathbf{y}$, etc., and outcome-streams that are constant will be denoted by letters near the beginning of the alphabet, e.g., $\mathbf{a}$, $\mathbf{b}$, etc. An outcome-stream $\mathbf{a}$ is to have the value a for any time t, and so forth. For outcome-streams $\mathbf{x}, \mathbf{y}$ and a time interval $\langle\alpha,\beta\rangle$, $(\mathbf{x}_{\langle\alpha,\beta\rangle}, \mathbf{y})$ will denote the outcome-stream such that $(\mathbf{x}_{\langle\alpha,\beta\rangle}, \mathbf{y})(t) = \mathbf{x}(t)$ for t in $\langle\alpha,\beta\rangle$ and $(\mathbf{x}_{\langle\alpha,\beta\rangle}, \mathbf{y})(t) = \mathbf{y}(t)$ otherwise. And for outcome-streams $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and two disjoint time intervals $\langle\alpha,\beta\rangle$ and $\langle\alpha',\beta'\rangle$, $(\mathbf{x}_{\langle\alpha,\beta\rangle}, \mathbf{y}_{\langle\alpha',\beta'\rangle}, \mathbf{z})$ will have a similar meaning.

The distance between two outcomes x, y is defined as their Euclidean distance $|x - y| = [\sum_{j=1}^{N}(x_j - y_j)^2]^{1/2}$. The distance between two outcome-streams $\mathbf{x}$, $\mathbf{y}$ in a set S_p is defined as the integral, $\Delta(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{m}(a_i - a_{i-1})\,|\,x(i) - y(i)\,|$, of $|\,\mathbf{x}(t) - \mathbf{y}(t)\,|$. And the distance between two component-streams $\mathbf{x}_j$, $\mathbf{y}_j$ in $\mathbf{x}$, $\mathbf{y}$ is defined as the integral, $\int |\,\mathbf{x}_j - \mathbf{y}_j\,| = \sum_{i=1}^{m}(a_i - a_{i-1})\,|\,x_j(i) - y_j(i)\,|$, of $|\,\mathbf{x}_j(t) - \mathbf{y}_j(t)|$. The integral distances are the same for any set S_p that contains the step outcome-streams $\mathbf{x}$, $\mathbf{y}$, and we have the inequalities: $\int |\,\mathbf{x}_k - \mathbf{y}_k\,| \le \Delta(\mathbf{x}, \mathbf{y}) \le \sum_{j=1}^{N} \int |\,\mathbf{x}_j - \mathbf{y}_j\,|$ for $k = 1,\ldots,N$.

Preferences and their measurement. By the term preferences, we mean either hedonic comparisons, i.e., comparisons of what a person or group experiences, or the preferences, either descriptive or prescriptive, of a person or group.

For two outcome-streams **x**, **y** in a set *C*, the statement that **x** is at least as preferred as **y** will be denoted by $\mathbf{x} \succsim \mathbf{y}$. Other types of relations will be defined in terms of $\mathbf{x} \succsim \mathbf{y}$ in the usual manner; e.g., $\mathbf{x} \sim \mathbf{y}$ will mean that $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{x}$, and $\mathbf{x} \succ \mathbf{y}$ will mean that $\mathbf{x} \succsim \mathbf{y}$ and not $\mathbf{y} \succsim \mathbf{x}$. A set of statements, $\mathbf{x} \succsim \mathbf{y}$ with **x**, **y** in *C*, will be denoted by $\succsim$. With this interpretation, $\succsim$ will be called a <u>preference</u> <u>relation</u> on the set *C*.

For two outcomes *x*, *y* in an outcome set *X*, the statement that *x* is at least as preferred as *y* will be denoted by $x \succsim_X y$, and $\succsim_X$ will denote a set of such statements.

A function $V(\mathbf{x})$ defined on a set *C* of outcome-streams will be called a <u>value</u> <u>function</u> for a preference relation $\succsim$ on *C* provided that $V(\mathbf{x}) \geq V(\mathbf{y})$ if and only if $\mathbf{x} \succsim \mathbf{y}$ for any **x**, **y** in *C*. A similar definition applies for a function $v(x)$ defined on an outcome set *X*. For purposes of distinction, such a function $v(x)$ will be called an <u>outcome</u> <u>scale</u>.

The present paper defines conditions on a preference relation $\succsim$ for a variety of sets *C* and shows that $\succsim$ satisfies the conditions if and only if there exist functions $a(t)$ and $v(x)$ such that the following integral is a value function for $\succsim$:

$$V(\mathbf{x}) = \int_P a(t)\, v(\mathbf{x}(t))\, dt\,.$$

A function of this form will be called an <u>integral</u> <u>value</u> <u>function</u>, and a model of this type will be called an <u>integral</u>-<u>value</u> <u>model</u>. In each model, the function $a(t)$ is a discounting function, and the function $v(x)$ is an outcome scale.

Coherence between $\succsim$ and $\succsim_X$. How should a preference relation on outcome-streams be related to a preference relation on outcomes? One method is to define the preference relations and then introduce assumptions that connect them. A second method is to derive preferences between outcomes from preferences between outcome-streams. We will use the second method. In our opinion, preferences between outcome-streams have a direct meaning and preferences between outcomes are based on such preferences. In brief, our reason is that outcomes must occur over time in order to be experienced.

To formalize the situation, suppose that a time $\tau > 0$ in the planning period *P* and an outcome o in the outcome set *X* are specified and that $C_{\tau,\mathrm{o}}$ denotes the set of outcome-streams of the form, $(\mathbf{a}_{[0,\tau]}, \mathbf{o})$, *a* in *X*. Suppose, moreover, that a preference relation $\succsim$ has been defined on a set *C* of outcome-streams. We make the following assumption.

Assumption 1. The set *C* includes the set $C_{\tau,\mathrm{o}}$ for a specified time $\tau > 0$ and a specified outcome o, and a preference relation $\succsim_X$ is defined on the outcome set X by:

$$a \succsim_X b \text{ if and only if } (\mathbf{a}_{[0,\tau]}, \mathbf{o}) \succsim (\mathbf{b}_{[0,\tau]}, \mathbf{o}).$$

The integral-value models include conditions on preferences which imply that the preference relation $\succsim_X$ does not depend on the choice of τ and o. In the models with $P = [0, \infty)$, we regard o as a 'null outcome' and we define the set *C* in terms of o.

Modeling assumptions on $\succsim_X$. The models in this paper include three assumptions on the preference relation $\succsim_X$. First, we assume that $\succsim_X$ is non-trivial, that is, there exist outcomes x_j, x'_j in X_j that are not indifferent according to $\succsim_X$. The purpose of this assumption is to avoid discussing an uninteresting special case. The assumption implies that at least one of the sets X_j is non-point (i.e., it contains more than one number).

Second, we assume that $\succsim_X$ is weakly increasing in each component variable. For an index $j = 1, \ldots, N$, suppose that $\overline{x}_j$ denotes a combination of amounts of the variables x_k, $k \neq j$, and $x = (x_j, \overline{x}_j)$ denotes an outcome where x_j and the amounts x_k, $k \neq j$, are suitably arranged. In this notation, the condition states that for each $j = 1, \ldots, N$ and any $x = (x_j, \overline{x}_j)$ and $x' = (x'_j, \overline{x}_j)$ in X: $x_j \geq x'_j$ implies $x \succsim_X x'$. Typically, the condition can be satisfied by a suitable choice of the component variables.

The third assumption seems the most important. For our method of proof to succeed, we need a guarantee that the range of any continuous outcome scale $v(x)$ is an interval. The additive-value model of Debreu (1960) is similar in this regard. It needs a guarantee that each function $v_i(x_i)$ in an additive value function $V(x_1, \ldots, x_n) = \sum_{i=1}^{n} v_i(x_i)$ has an interval range. Debreu assumes that the domain D_i on which a function $v_i(x_i)$ is defined is topologically connected. This condition is stronger than is needed; a set is topologically connected if and only if every continuous function defined on it has an interval range.

Harvey (2006) introduces a weaker condition and shows that it suffices for Debreu's additive-value model. As a general definition, he calls a set S with a preference relation $\succsim$ preferentially connected provided that S cannot be divided into two non-empty sets A and B such that each is open as a subset of S and $a \succ b$ for any elements a in A and b in B. He shows that a pair $(S, \succsim)$ is preferentially connected if and only if any continuous function defined on S that is a value function for $\succsim$ has an interval range.

As a basis for the models in this paper, we use an additive-value model in which each set D_i is a common outcome set X. Both here and in the rest the paper, the method of proof is the same whether X is assumed to be topologically connected or is assumed to be preferentially connected. Hence, we assume only preferential connectedness.

Whereas topological connectedness of an outcome set requires that every component variable is a continuous variable, preferential connectedness permits some of the variables to be categorical. Because of its special structure, an outcome set X is preferentially connected if and only if the subsets of X defined by fixing the values of the categorical variables can be ordered such that any two adjacent subsets contain indifferent outcomes (Harvey, 2006).

The above three assumptions can be combined into the following statement.

Assumption 2. The preference relation $\succsim_X$ defined on an outcome set X is non-trivial and is weakly increasing in each component variable, and X is preferentially connected.

Definition 1. A pair $(C, \succsim)$ will be called an outcome-stream space and the related pair $(X, \succsim_X)$ will be called an outcome space provided that Assumptions 1, 2 are satisfied.

3. Conditions on preferences

This section presents conditions on preferences in an outcome-stream space $(C, \succsim)$. It also presents several implications of the conditions for the outcome space $(X, \succsim_X)$.

In contrast with Assumptions 1, 2, the conditions (A)-(E) below are 'if and only if' requirements in each integral-value model, that is, the conditions (A)-(E) both imply and are implied by the existence of an integral value function that has the stated properties.

Assumptions 1, 2 correspond to the assumption in Harvey (1998a,b) that the outcome set X is a non-point interval in which greater amounts are preferred, and conditions (A)-(E) correspond to conditions with the same labels in those papers.

(A) $\succsim$ concurs with $\succsim_X$ on C: For any $\mathbf{x}$, $\mathbf{y}$ in C,

(a) If $\mathbf{x}(t) \succsim_X \mathbf{y}(t)$ almost everywhere (a.e.) for t in P, then $\mathbf{x} \succsim \mathbf{y}$.

(b) If $\mathbf{x}(t) \succsim_X \mathbf{y}(t)$ a.e. in P and $\mathbf{x}(t) \succ_X \mathbf{y}(t)$ on a non-point interval, then $\mathbf{x} \succ \mathbf{y}$.

(B) $\succsim$ is transitive on C: For any $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in C, if $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{z}$, then $\mathbf{x} \succsim \mathbf{z}$.

$\succsim$ is complete on C: For any $\mathbf{x}$, $\mathbf{y}$ in C, either $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$.

(C) $\succsim$ is continuous on C with respect to S_T: For any $\mathbf{x}$ in C and any $\mathbf{w}$ in $S_T \cap C$,

(a) If $\mathbf{w} \prec \mathbf{x}$, then there exists a $\delta > 0$ such that $\Delta(\mathbf{z}, \mathbf{w}) < \delta$ implies that $\mathbf{z} \prec \mathbf{x}$ for any $\mathbf{z}$ in $S_T \cap C$.

(b) If $\mathbf{w} \succ \mathbf{x}$, then there exists a $\delta > 0$ such that $\Delta(\mathbf{z}, \mathbf{w}) < \delta$ implies that $\mathbf{z} \succ \mathbf{x}$ for any $\mathbf{z}$ in $S_T \cap C$.

(D) $\succsim$ is tradeoffs independent on C: Suppose that $\langle a, b \rangle$ is a bounded interval in P and that the following outcome-streams are in C. Then, $(\mathbf{x}_{\langle a,b \rangle}, \mathbf{x}) \succsim (\mathbf{x}_{\langle a,b \rangle}, \mathbf{y})$ implies that $(\mathbf{z}_{\langle a,b \rangle}, \mathbf{x}) \succsim (\mathbf{z}_{\langle a,b \rangle}, \mathbf{y})$.

Condition (D) states that if two outcome-streams are equal during an interval $\langle \alpha, \beta \rangle$ (so that a comparison depends on outcomes at other times), then the common outcome-stream in $\langle \alpha, \beta \rangle$ can be changed to another common outcome-stream in $\langle \alpha, \beta \rangle$ without changing the comparison. Condition (D) can also be interpreted as stating that tradeoffs (see below) at times not in $\langle \alpha, \beta \rangle$ do not depend on the outcome-stream in $\langle \alpha, \beta \rangle$.

Conditions analogous to (D) play an essential role in additive-value models: e.g., Debreu (1960) and Gorman (1968). Such conditions usually are called 'preferential independence.' However, we prefer the term tradeoffs independence to emphasize the interpretation of (D) in terms of tradeoffs between outcomes at different times.

Two outcome pairs a, b and c, d will be called tradeoffs pairs with respect to a pair of intervals $\langle\alpha,\beta\rangle$ and $\langle\gamma,\delta\rangle$ in P provided that $\langle\alpha,\beta\rangle$, $\langle\gamma,\delta\rangle$ are bounded and disjoint and $(\mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{d}_{\langle\gamma,\delta\rangle}, \mathbf{o})$ is indifferent to $(\mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{c}_{\langle\gamma,\delta\rangle}, \mathbf{o})$. An outcome $\hat{a}$ will be called a tradeoffs midvalue of an outcome pair $\underline{a}, \overline{a}$ on an interval $\langle\alpha,\beta\rangle$ provided that there exist an interval $\langle\gamma,\delta\rangle$ and an outcome pair c, d such that $\underline{a}, \hat{a}$ and c, d are tradeoffs pairs and $\hat{a}, \overline{a}$ and c, d are tradeoffs pairs with respect to the intervals $\langle\alpha,\beta\rangle$ and $\langle\gamma,\delta\rangle$.

Condition (E) below is a requirement on preferences between outcome-streams of the form $(\mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{b}_{\langle\gamma,\delta\rangle}, \mathbf{o})$. Thus, we use it only for sets C that include any such outcome-stream. A variety of analogous conditions for vectors and discrete-time consequences are described in Fishburn (1970), Krantz et al. (1972, page 305), and Harvey (1986, 1995).

(E) $\succsim$ is midvalue independent on C: For any bounded intervals $\langle\alpha,\beta\rangle$, $\langle\alpha',\beta'\rangle$ in P, if an outcome pair $\underline{a}, \overline{a}$ has tradeoffs midvalues both on $\langle\alpha,\beta\rangle$ and on $\langle\alpha',\beta'\rangle$, then the outcome pair $\underline{a}, \overline{a}$ has the same tradeoffs midvalues on $\langle\alpha,\beta\rangle$ and $\langle\alpha',\beta'\rangle$.

We present below several implications of conditions (A)-(C) for an outcome space $(X, \succsim_X)$. To do so, we need the following definitions. For two outcomes x, y, $x \geq y$ will state that $x_j \geq y_j$ for $j = 1,\ldots,N$. A preference relation $\succsim_X$ will be called weakly increasing provided that $x \geq y$ implies $x \succsim_X y$, and a real-valued function $v(x)$ defined on X will be called weakly increasing provided that $x \geq y$ implies $v(x) \geq v(y)$. Hence, any weakly increasing preference relation $\succsim_X$ or function $v(x)$ is weakly increasing in each component variable. Finally, a preference relation $\succsim_X$ will be called continuous provided that for any outcomes $x \succ w$ there exists a $\delta > 0$ such that for any outcome z: $|z - w| < \delta$ implies $z \prec x$, and $|z - x| < \delta$ implies $z \succ w$.

Lemma 1. Suppose that an outcome-stream space $(C, \succsim)$ satisfies condition (B). Then:

(i) The preference relation $\succsim_X$ is transitive, complete, and weakly increasing.

(ii) The outcome set X contains outcomes x, y such that $x \succ_X y$.

(iii) Any outcome scale for the outcome space $(X, \succsim_X)$ is weakly increasing and has a non-point range, and any continuous outcome scale for $(X, \succsim_X)$ has an interval range.

(iv) If the space $(C, \succsim)$ satisfies condition (C), then the preference relation $\succsim_X$ is continuous and there exists a continuous outcome scale for $(X, \succsim_X)$.

(v) If the space $(C, \succsim)$ satisfies condition (A), then the preference relation $\succsim_X$ does not depend on the choice of τ and o, that is, for any time $\tau' > 0$ and for any outcomes o' and a, b, if the outcome-streams below are in C, then:

$$(\mathbf{a}_{[0,\tau]}, \mathbf{o}) \succsim (\mathbf{b}_{[0,\tau]}, \mathbf{o}) \text{ if and only if } (\mathbf{a}_{[0,\tau']}, \mathbf{o}') \succsim (\mathbf{b}_{[0,\tau']}, \mathbf{o}').$$

4. Models for a bounded planning period

This section presents two integral-value models for outcome-streams defined on a bounded planning period $P = [0, T]$. First, we present a model for step outcome-streams on $[0, T]$, and then we extend this result to present a model for a set of outcome-streams on $[0, T]$ whose component-streams are Riemann integrable functions on $[0, T]$.

Step outcome-streams. As defined in Section 2, a step outcome-stream on $[0, T]$ (i.e., an outcome-stream $\mathbf{x}$ in S_T) has the form, $\mathbf{x}(t) = x(i)$ for t in $\langle a_{i-1}, a_i \rangle$, $i = 1, \ldots, m$, where $p : \langle a_0, a_1 \rangle, \ldots, \langle a_{m-1}, a_m \rangle$ is a partition of the interval $[0, T]$. Such an outcome-stream is piecewise constant with a finite number of values.

Theorem 1. An outcome-stream space $(S_T, \succsim)$, $T > 0$, satisfies conditions (A)-(E) if and only if it has a value function of the form

$$V(\mathbf{x}) = \int_0^T a(t) v(\mathbf{x}(t))\, dt, \quad \mathbf{x} \text{ in } S_T \tag{1}$$

such that the Lebesgue integral (1) exists for any $\mathbf{x}$ in S_T and:

(a) The function $v(x)$ defined on the outcome set X is continuous, weakly increasing, has a non-point interval range, and is an outcome scale for the outcome space $(X, \succsim_X)$.

(b) The function $a(t)$ defined on the planning period $P = [0, T]$ is non-negative and Lebesgue integrable.

(c) The function $A(t) = \int_0^t a(s)\,ds$ defined on $[0,T]$ is strictly increasing, absolutely continuous, and has the value $A(0) = 0$.

(d) The function $V(\mathbf{x})$ is continuous on S_T, that is, for any $\mathbf{x}$ in S_T and any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\Delta(\mathbf{z},\mathbf{x}) < \delta$ implies $|V(\mathbf{z}) - V(\mathbf{x})| < \varepsilon$ for any $\mathbf{z}$ in S_T.

Moreover, the function $v(x)$ is unique up to a positive affine transformation, and the function $A(t)$ is unique up to a positive multiple.

The function $a(t)$ can be interpreted as a discounting function, and the indefinite integral $A(t)$ is then a cumulative discounting function. The fact that $a(t)$ is not required to be Riemann integrable is of practical importance since a Riemann integrable function must be bounded. In particular, the model allows a discounting function to be unbounded near the present, $t = 0$. The most common such discounting functions are the so-called power discounting functions. They correspond to the functions $A(t) = t^k$ where the parameter k is in the range $0 < k < 1$. Then, $a(t) = k\,t^{k-1}$ for $t > 0$, and thus $a(t)$ is unbounded near $t = 0$. These discounting functions are used in descriptive models of choice behavior (e.g., Ainslie, 1992) and in prescriptive models of quality-adjusted life years (QALYs) (e.g., Pliskin et al., 1980).

The properties (a)-(c) of the functions $v(x)$, $a(t)$, and $A(t)$ do not imply the joint-continuity property (d) of the function $V(\mathbf{x})$. For a counterexample, see Harvey (1998b).

Since an outcome-stream $\mathbf{x}$ in S_T is in a set S_p, i.e., $\mathbf{x}$ is a step outcome-stream with respect to some partition p, the integral $V(\mathbf{x})$ in (1) reduces to a finite sum

$$V(\mathbf{x}) = \sum_{i=1}^{m} \left(A(a_i) - A(a_{i-1}) \right) v(x_i), \quad \mathbf{x} \text{ in } S_p . \tag{1$'$}$$

It follows that $V(\mathbf{x})$ has the same value for any partition p with $\mathbf{x}$ in S_p.

While a sum $V(\mathbf{x})$ has a finite number of terms, the number of terms varies from one step outcome-stream to another. Indeed, there is no upper bound on the number of terms in a sum. Hence, the model here is not a finite additive-value model.

Our method of proof proceeds in a direction opposite to the above derivation of (1′) from (1). First, we construct an additive-value model with a value function (1′) for a set

S_P; next we extend this result to construct a model for the union S_T of the sets S_P; and then we show that the value function (1′) can be written as a Lebesgue integral (1).

Riemann outcome-streams. A real-valued function $f(t)$ defined on an interval $[0,T]$ is said to be Riemann integrable provided that, roughly speaking, any sequence of sums $\sum_{i=1}^{m} f(t_i)(a_i - a_{i-1})$ based on partitions of $[0,T]$ converges to the same amount as the maximum lengths of the intervals $\langle a_{i-1}, a_i \rangle$ tend to zero. A function $f(t)$ on $[0,T]$ is Riemann integrable if and only if it is bounded and is continuous almost everywhere.

Here, we define a family of outcome-streams $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ on $[0,T]$ by requiring that each component-stream $\mathbf{x}_j$ in $\mathbf{x}$ is Riemann integrable and has bounds that are in the component set X_j. Since each $\mathbf{x}_j$ is continuous at a time t if and only if $\mathbf{x}$ is continuous at t, and each $\underline{x}_j \le \mathbf{x}_j(t) \le \overline{x}_j$ if and only if $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_N) \le \mathbf{x}(t) \le \overline{x} = (\overline{x}_1, \ldots, \overline{x}_N)$, we can define the family by requiring properties of the outcome-stream $\mathbf{x}$ itself.

Definition 2. An outcome-stream $\mathbf{x}$ defined on $P = [0,T]$ will be called a <u>Riemann outcome-stream</u> on $[0,T]$ provided that:

(i) $\mathbf{x}$ is continuous almost everywhere on $[0,T]$.

(ii) There exist outcomes $\underline{x}$, $\overline{x}$ such that $\underline{x} \le \mathbf{x}(t) \le \overline{x}$ for any t in $[0,T]$.

The set of Riemann outcome-streams on $[0,T]$ will be denoted by R_T.

Any outcome-stream on $[0,T]$ that is piecewise continuous (e.g., a step outcome-stream or a continuous outcome-stream) satisfies (i), (ii) above and thus is a Riemann outcome-stream on $[0,T]$. Hence, the set R_T of Riemann outcome-streams on $[0,T]$ seems to be sufficiently inclusive for typical applications.

Theorem 2. An outcome-stream space $(R_T, \succsim)$, $T > 0$, satisfies conditions (A), (B) on the set R_T, satisfies condition (C) on the pair of sets R_T, S_T, and satisfies conditions (D), (E) on the set S_T if and only if it has a value function of the form

$$V(\mathbf{x}) = \int_0^T a(t)\, v(\mathbf{x}(t))\, dt, \quad \mathbf{x} \text{ in } R_T \tag{2}$$

such that the Lebesgue integral (2) exists for any $\mathbf{x}$ in R_T and the functions $v(x)$, $a(t)$, $A(t) = \int_0^t a(s)\,ds$, and $V(\mathbf{x})$ have the properties (a)-(d) in Theorem 1.

Moreover, the function $v(x)$ is unique up to a positive affine transformation, and the function $A(t)$ is unique up to a positive multiple.

A real-valued function $f(t)$ defined on $[0, T]$ is Riemann integrable if and only if it is Darboux integrable, that is, there exists monotone sequences $\{\overline{s}^{(n)}\}_{n=1}^{\infty}$, $\{\underline{s}^{(n)}\}_{n=1}^{\infty}$ of step functions such that $\underline{s}^{(n)} \leq f(t) \leq \overline{s}^{(n)}$, t in $[0, T]$, and the distances $\int | \overline{s}^{(n)} - \underline{s}^{(n)} |$ tend to zero as n tends to infinity. The proof of Theorem 2 uses this equivalence. We show that for any Riemann outcome-stream **x** there exist monotone sequences $\{\overline{\mathbf{s}}^{(n)}\}_{n=1}^{\infty}$, $\{\underline{\mathbf{s}}^{(n)}\}_{n=1}^{\infty}$ of step outcome-streams such that $\underline{\mathbf{s}}^{(n)}(t) \leq \mathbf{x}(t) \leq \overline{\mathbf{s}}^{(n)}(t)$, t in $[0, T]$, and the distances $\Delta(\overline{\mathbf{s}}^{(n)}, \underline{\mathbf{s}}^{(n)})$ tend to zero as n tends to infinity, and we use this 'squeeze property' of Riemann outcome-streams to extend the integral-value model in Theorem 1 for step outcome-streams to an integral-value model for Riemann outcome-streams.

5. Models for an unbounded planning period

This section presents two models for outcome-streams defined on the planning period $P = [0, \infty)$. Like Theorems 1, 2, the first model is a steppingstone to the second model.

In the second model, the set of outcome-streams that are comparable (i.e., the set on which the preference relation is complete) is specified in terms of the preference relation. In this sense among others, the model differs from all previous continuous-time models and from most previous discrete-time models. See Section 6 for details.

The discrete-time models in Harvey (1986, 1995)—and models in Wakker (1993) for discrete probability distributions—do assume completeness of a preference relation on a set that depends on the relation. Harvey argues that this comparability dependence permits an arbitrary sequence of discount weights, and Wakker argues that it is the crucial change in the axioms of Savage (1954) that permits an unbounded utility function.

Finite outcome-streams. Here, we present a model in which each outcome-stream equals the null outcome o (see Assumption 1) after a time that depends on the outcome-stream.

Definition 3. An outcome-stream $\mathbf{x}$ on the planning period $P=[0,\infty)$ will be called a finite outcome-stream provided that there exists a horizon $T>0$ such that the restriction of $\mathbf{x}$ to $[0,T]$ is a Riemann outcome-stream on $[0,T]$ and $\mathbf{x}(t)=\mathrm{o}$ for any $t>T$.

The set of finite outcome-streams will be denoted by R_f.

An outcome-stream $\mathbf{x}$ in a set R_T, $T>0$, will be identified with the outcome-stream $(\mathbf{x}_{[0,T]}, \mathbf{o}_{(T,\infty)})$ in the set R_f. Thus, R_f is the union of the sets R_T. And for $T'>T$, an outcome-stream $\mathbf{x}$ in R_T will be identified with the outcome-stream $(\mathbf{x}_{[0,T]},\ \mathbf{o}_{(T,T']})$ in the set $R_{T'}$, and thus, R_T is a subset of $R_{T'}$.

Theorem 3. An outcome-stream space $(R_f, \succsim)$ satisfies conditions (A), (B) on each set R_T, $T>0$, satisfies condition (C) on each pair R_T, S_T, $T>0$, and satisfies conditions (D), (E) on each set S_T, $T>0$, if and only if $(R_f, \succsim)$ has a value function of the form

$$V(\mathbf{x}) = \lim_{T\to\infty} \int_0^T a(t)\,v(\mathbf{x}(t))\,dt, \quad \mathbf{x} \text{ in } R_f \tag{3}$$

such that the improper Lebesgue integral $V(\mathbf{x})$ exists for any $\mathbf{x}$ in R_f and:

(a) The function $v(x)$ defined on the set X is continuous, weakly increasing, has a non-point interval range, is an outcome scale for the space $(X, \succsim_X)$, and $v(\mathrm{o})=0$.

(b) The function $a(t)$ defined on the interval $[0,\infty)$ is non-negative and is Lebesgue integrable on each interval $[0,T]$, $T>0$.

(c) The function $A(t)=\int_0^t a(s)\,ds$ defined on the interval $[0,\infty)$ is strictly increasing and is absolutely continuous on each interval $[0,T]$, $T>0$, and $A(0)=0$.

(d) For each $T>0$, the function $V(\mathbf{x})$ is continuous at each $\mathbf{w}$ in S_T in that for any $\varepsilon>0$ there exists a $\delta>0$ such that $\Delta(\mathbf{z},\mathbf{w})<\delta$ implies $|V(\mathbf{z})-V(\mathbf{w})|<\varepsilon$ for $\mathbf{z}$ in S_T.

Moreover, each of the functions $v(x)$ and $A(t)$ is unique up to a positive multiple.

Comparable outcome-streams. The model in Theorem 3 can be extended to a model for a set of outcome-streams for which the improper integral (3) converges. We define a set of outcome-streams on $[0, \infty)$, and in terms of a preference relation on this set we define a smaller set and construct a model for the preference relation restricted to this smaller set.

Definition 4. An outcome-stream $\mathbf{x}$ on $[0,\infty)$ will be called a Riemann outcome-stream on $[0,\infty)$ provided that for any horizon $T > 0$ the restriction of $\mathbf{x}$ to $[0,T]$ is in R_T.

The set of Riemann outcome-streams on $[0,\infty)$ will be denoted by R_∞.

Suppose that a preference relation $\succsim$ is defined on a set R_∞. We do not assume that $\succsim$ is complete on R_∞; instead, we will define a subset of R_∞ in terms of $\succsim$ and assume that $\succsim$ is complete on the subset. Roughly speaking, the subset is to contain the outcome-streams in R_∞ that become arbitrarily unimportant in the sufficiently distant future.

To make this idea precise, consider tradeoffs between the immediate future period $[0, 1]$ and an unbounded future period (t, ∞), $t \geq 1$. Then, for an outcome-stream $\mathbf{x}$ we can compare changes between two outcomes a and b in the period $[0, 1]$ with changes between $\mathbf{x}$ and the null outcome-stream $\mathbf{o}$ in the period (t, ∞).

Definition 5. A Riemann outcome-stream $\mathbf{x}$ on $[0,\infty)$ will be called comparable provided that for any outcomes $a \prec_X b \prec_X c$ there exists a horizon $T \geq 1$ such that for any $t \geq T$:

$$(\mathbf{a}_{[0,1]}, \mathbf{o}) \prec (\mathbf{b}_{[0,1]}, \mathbf{o}, \mathbf{x}_{(t,\infty)}) \prec (\mathbf{c}_{[0,1]}, \mathbf{o}).$$

The set of comparable outcome-streams will be denoted by R_C.

First, we show two circumstances in which an outcome-stream on $[0, \infty)$ satisfies the above condition of comparability.

Lemma 2. Suppose that the pair $(R_\infty, \succsim)$ is an outcome-stream space. Then:

(a) Any finite outcome-stream is comparable.

(b) For any two outcome-streams $\mathbf{x}, \mathbf{y}$ in R_∞, if $\mathbf{x}$ is comparable and there exists a horizon $U > 0$ such that $\mathbf{x}(t) = \mathbf{y}(t)$ for all $t > U$, then $\mathbf{y}$ is comparable.

Below, we present an integral-value model for an outcome-stream space $(R_C, \succsim)$. In this model, the improper integral (4) converges for any outcome-stream in the subset R_C of R_∞. One may ask whether, conversely, any outcome-stream in R_∞ such that the integral (4) converges is in the subset R_C. This statement is true if and only if $\succsim$ satisfies an additional condition; see Harvey (1998b).

Theorem 4. An outcome-stream space $(R_\infty, \succsim)$ satisfies conditions (A), (B), and (D) on the set R_C, satisfies condition (C) on each pair of sets R_T, S_T, $T > 0$, and satisfies condition (E) on each set S_T, $T > 0$, if and only if the outcome-stream space $(R_C, \succsim)$ has a value function of the form

$$V(\mathbf{x}) = \lim_{T\to\infty} \int_0^T a(t)v(\mathbf{x}(t))\,dt, \ \ \mathbf{x} \text{ in } R_C \tag{4}$$

such that the improper Lebesgue integral $V(\mathbf{x})$ exists for any $\mathbf{x}$ in R_C and the functions $v(x)$, $a(t)$, $A(t)$, and $V(\mathbf{x})$ have the properties (a)-(d) in Theorem 3.

Moreover, each of the functions $v(x)$ and $A(t)$ is unique up to a positive multiple.

6. Relationships with previous research

It is surprising that the models developed here were not developed long ago—at least for the case of a single outcome variable—and many readers may assume that they have been. In reflecting on our work, we cannot avoid the thought that one reason for this lack of prior research may be the difficulty of the proofs. We were unable to derive the models as corollaries of known mathematical results, and we leave it as an open question whether such an approach is possible.

However, a variety of continuous-time models have been developed, and thus we need to explain how they differ from those in this paper. For completeness, we also mention a few discrete-time models. Loewenstein (1992) provides a broader history of discounting.

Samuelson (1937) defined a continuous-time model in which the outcomes are rates x of a person's consumption, the outcome-streams are consumption streams $x = \mathbf{x}(t)$ defined on an interval P, and preferences between the outcome-streams are represented by an integral, $V(\mathbf{x}) = \int_P e^{-rt}\, v(\mathbf{x}(t))\,dt$, where $r > 0$ is an instantaneous discount rate and $v(x)$ is the cardinal utility of a rate x of consumption.

Samuelson's model is not a measurement theory model, that is, he did not deduce his integral value function from a list of conditions on preferences. Samuelson's purpose for the model was to infer a person's cardinal utility function for consumption rates from the person's choices of optimal outcome-streams.

A variety of measurement-theory models with discrete time have been developed. Williams and Nassar (1966) developed a model in which the outcomes are net gains and the outcome-streams are cash flows $\mathbf{x} = (x_0, \dots, x_m)$ for a fixed m. They establish that preferences satisfy certain conditions if and only if they are represented by a function of the form, $V(\mathbf{x}) = \sum_{t=0}^{m} a_t\, x_t$. This model does not allow a nonlinear utility function $v(x)$.

Koopmans (1960, 1972), Koopmans et al. (1964), and Diamond (1965) developed models in which the outcomes are in a connected subset of a space R^n and the outcome-streams are sequences $\mathbf{x} = (x_0, x_1, \dots)$ of outcomes at equally-spaced points of time, e.g., outcomes during annual periods. In each model, preferences satisfy certain conditions if and only if they are represented by a sum, $V(\mathbf{x}) = \sum_{t=0}^{\infty} (1+r)^{-t} v(x_t)$ where $r > 0$ is an annual discount rate and $v(x)$ is the cardinal utility of an outcome x.

In each of these discrete-time models, categorical variables are not allowed, and the set of comparable outcome-streams does not depend on the preference relation. The finite-period model by Williams and Nassar allows non-constant discounting while the infinite-period models allow a nonlinear utility function $v(x)$.

Harvey (1986, 1995) developed discrete-time models in which the outcomes are in an interval, the outcome-streams are sequences of outcomes, and preferences are represented by a function of the form, $V(\mathbf{x}) = \sum_{t=0}^{\infty} a_t\, v(x_t)$. Here, the set of comparable outcome-streams depends on the preference relation and non-constant discounting is allowed.

Two types of measurement-theory models with continuous-time have been developed. Grodal (2003, Section 12.3 and Note 12.5.1) presents models in which the outcomes are in a connected separable metric space X, the outcome-streams are Lebesgue measurable functions defined on an interval P with values in X, and preferences are represented by a function of the form, $V(\mathbf{x}) = \int_P a(t)\, v(\mathbf{x}(t))\, d\mu(t)$ where μ is a measure on P. The models are based on a working paper by Grodal and Mertens (1968).

These models do not allow categorical outcome variables or a dependence of the set of comparable outcome-streams on the preference relation. In particular, constant outcome-streams are assumed to be comparable. Thus, the models exclude non-discounting and certain types of so-called slow discounting (see, e.g., Harvey, 1986, 1995).

Moreover, the models are incomplete in two ways. They establish that 'if **x** is preferred to **y** then $V(\mathbf{x}) > V(\mathbf{y})$,' but they do not establish that 'if $V(\mathbf{x}) > V(\mathbf{y})$ then **x** is preferred to **y**.' Therefore, it could happen that $V(\mathbf{x}) > V(\mathbf{y})$ while **x** and **y** are indifferent. In this sense, $V(\mathbf{x})$ only partially represents the preference relation. Second, the models establish that conditions on preferences imply that the preferences are partially represented by a function $V(\mathbf{x})$ as described, but they do not establish the converse implication.

Weibull (1985) developed a second type of continuous-time model. In this model, the outcomes are real numbers and the set of outcome-streams is a convex cone C in a space $L^1(\mu)$ of measurable functions. By means of the Riesz Representation Theorem for affine functionals on $L^1(\mu)$, he shows that preferences satisfy certain conditions if and only if they are represented by a function of the form, $V(\mathbf{x}) = \int_P a(t)\,\mathbf{x}(t)\,d\mu(t)$.

Weibull's model differs from those in this paper in five respects. First, it allows only a single continuous outcome variable. Second, the set C of outcome-streams does not depend on the preference relation. The set C may be too small for many applications since any consequence in C has a finite non-discounted value, $\int_P \mathbf{x}(t)\,d\mu(t)$. Hence, outcome-streams that are constant on an unbounded planning period are excluded. By contrast, the approach in this paper allows outcome-streams that lack finite non-discounted values.

Third, the set C of outcomes will be unbounded above whenever the outcome variable has positive values and unbounded below whenever it has negative values. By contrast, the approach in this paper allows component sets to be bounded or semi-bounded intervals or even finite sets. Such component sets may be needed in a variety of applications.

The fourth difference is that as in the Williams and Nassar model, a nonlinear utility function $v(x)$ is not allowed. Thus, the model excludes issues of preferences such as decreasing marginal utility and intertemporal equity.

Fifth, the set C consists of Lebesgue integrable functions rather than Riemann outcome-streams (whose component functions are therefore continuous almost everywhere). It seems likely that in any application the outcome-streams will be continuous almost everywhere. And in such an application, assumptions on preferences would be far more difficult to envision for Lebesgue outcome-streams than for Riemann outcome-streams.

Appendix: Proofs of Results

Proof of Lemma 1. By Assumption 1, the one-to-one correspondence between outcomes a and outcome-streams $(\mathbf{a}_{[0,\tau]},\mathbf{o})$ defines the preference relation $\succsim_X$ in terms of the preference relation $\succsim$ in the subspace $(C_{\tau,\mathrm{o}}, \succsim)$ of the outcome-stream space $(C, \succsim)$.

To show parts (i), (ii), first observe that condition (B) implies that $\succsim_X$ is transitive and complete. To show that $\succsim_X$ is weakly increasing, consider two outcomes $x \geq y$. For each $k = 0,\ldots,N$, define $x^{(k)}$ as the vector with the components y_j for $j = 0,\ldots,k$ and x_j for $j = k+1,\ldots,N$. Then, $x^{(0)} = x$, $x^{(N)} = y$, and each vector $x^{(k)}$ is in the product set X. For each $k = 1,\ldots,N$, the outcomes $x^{(k-1)}$, $x^{(k)}$ can differ only in their k-th components and $x^{(k-1)} \geq x^{(k)}$. Hence, Assumption 2 implies that $x^{(k-1)} \succsim_X x^{(k)}$ for each k, and thus $x \succsim_X y$ by transitivity. Assumption 2 also states that not all outcomes are indifferent. Since $\succsim_X$ is complete, it follows that there exist outcomes $x \succ_X y$.

To show part (iii), suppose that $v(x)$ is an outcome scale for $(X, \succsim_X)$. Then, $x \geq y$ implies that $x \succsim_X y$ by part (i) which implies that $v(x) \geq v(y)$. Thus, $v(x)$ is weakly increasing. By Assumption 2, there exist outcomes x, y with $x \succ_X y$. Thus, $v(x) > v(y)$ which implies that the outcome scale $v(x)$ has a non-point range. And since $(X, \succsim_X)$ is preferentially connected by Assumption 2, a result in Harvey (2006) implies that any continuous outcome scale has an interval range.

For part (iv), suppose that the space $(C, \succsim)$ satisfies condition (C). Then, the preference relation $\succsim_X$ is continuous since $\Delta((\mathbf{a}_{[0,\tau]}, \mathbf{o}), (\mathbf{b}_{[0,\tau]}, \mathbf{o})) = \tau|a-b|$ for a, b in X, and thus a result in Debreu (1954, 1964) implies that $\succsim_X$ has a continuous outcome scale (since the set X with the metric of Euclidean distance is a separable metric space).

Part (v) is implied by the following more detailed result.

Lemma A1. If an outcome-stream space $(C, \succsim)$ satisfies conditions (A), (B), then:

(i) The comparison of outcome-streams in C that are constant on an interval does not depend on the common outcome-stream at other times, that is, for any interval $\langle\alpha,\beta\rangle$, any outcomes a, b, and any outcome-streams $\mathbf{y}$, $\mathbf{y}'$:

$$(\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y}) \succsim (\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y}) \text{ if and only if } (\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y}') \succsim (\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y}').$$

(ii) The comparison of outcome-streams in C that are constant on a non-point interval does not depend on the common interval, that is, for any non-point intervals $\langle\alpha,\beta\rangle$ and $\langle\alpha',\beta'\rangle$, any outcomes a, b, and any outcome-stream $\mathbf{y}$:

$$(\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y}) \succsim (\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y}) \text{ if and only if } (\mathbf{a}_{\langle\alpha',\beta'\rangle},\mathbf{y}) \succsim (\mathbf{b}_{\langle\alpha',\beta'\rangle},\mathbf{y}).$$

<u>Proof</u>. For both parts, we show that $a \succsim_X b$ if and only if $(\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y}) \succsim (\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y})$ for any outcomes a, b, any non-point interval $\langle\alpha,\beta\rangle$, and any outcome-stream $\mathbf{y}$ in the set C. First, assume that $a \succsim_X b$. Then, $(\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y})(t) \succsim_X (\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y})(t)$ for any t in the planning period P. Hence, condition (A) implies that $(\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y}) \succsim (\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y})$. Next, assume that $a \succsim_X b$ is false. Then, $b \succ_X a$ since $\succsim_X$ is complete by Lemma 1(i). Thus, $(\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y})(t) \succsim_X (\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y})(t)$ for any t and $(\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y})(t) \succ_X (\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y})(t)$ for t in the non-point interval $\langle\alpha,\beta\rangle$. Hence, $(\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y}) \succ (\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y})$ by condition (A), and thus $(\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y}) \succsim (\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y})$ is false.

Since $\succsim$ is transitive, the above result implies (i) and (ii) for any non-point interval $\langle\alpha,\beta\rangle$. If $\langle\alpha,\beta\rangle$ is a point interval, then $(\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y})(t) \sim (\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y})(t)$ a.e. which implies $(\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{y}) \sim (\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{y})$ by condition (A).

Lemma A2. If an outcome-stream space $(S_T, \succsim)$ satisfies conditions (A)-(B), then:

(i) For any point interval $\langle a_{i-1},a_i\rangle$ in $[0,T]$ and any $\mathbf{x}$, $\mathbf{y}$ in S_T: if $\mathbf{x}(t)=\mathbf{y}(t)$ for t not in $\langle a_{i-1},a_i\rangle$, then $\mathbf{x} \sim \mathbf{y}$. (In this sense, any point interval is 'inessential.')

(ii) For any non-point interval $\langle a_{i-1},a_i\rangle$ in $[0,T]$, there exist $\mathbf{x}$, $\mathbf{y}$ in S_T such that $\mathbf{x}(t)=\mathbf{y}(t)$ for t not in $\langle a_{i-1},a_i\rangle$ but $\mathbf{x} \sim \mathbf{y}$ is false. (In this sense, any non-point interval is 'essential.')

<u>Proof</u>. For part (i), consider two outcome-streams $\mathbf{x}$, $\mathbf{y}$ as described. Then, $\mathbf{x}(t)=\mathbf{y}(t)$ a.e., and thus $\mathbf{x}(t) \sim_X \mathbf{y}(t)$ a.e. which implies that $\mathbf{x} \sim \mathbf{y}$ by condition (A).

For part (ii), note that since $\succsim_X$ is non-trivial and complete, there exist outcomes a, b such that $a \succ_X b$. Then, condition (A) implies that $(\mathbf{a}_{\langle\alpha,\beta\rangle},\mathbf{o}) \succ (\mathbf{b}_{\langle\alpha,\beta\rangle},\mathbf{o})$.

Lemma A3. If an outcome-stream space $(S_T, \succsim)$ satisfies conditions (A)-(C), then for any non-point, disjoint intervals $\langle\alpha, \beta\rangle$, $\langle\alpha', \beta'\rangle$ in $[0,T]$ and any outcomes a, b such that $a \prec_X b$, there exist outcomes a^+, b^- such that: $a \prec_X a^+$, $b^- \prec_X b$, and

$$(\mathbf{a}^+_{\langle\alpha,\beta\rangle}, \mathbf{a}_{\langle\alpha',\beta'\rangle}, \mathbf{o}) \prec (\mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{b}_{\langle\alpha',\beta'\rangle}, \mathbf{o}) \prec (\mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{b}^-_{\langle\alpha',\beta'\rangle}, \mathbf{o}).$$

<u>Proof</u>. We show the existence of an outcome a^+ as described. The arguments for the existence of b^- are similar and thus can be omitted.

Define $A^- = \{x \text{ in } X : x \prec_X a\}$, $A^0 = \{x \text{ in } X : x \sim_X a\}$, and $A^+ = \{x \text{ in } X : x \succ_X a\}$. These sets are pairwise disjoint, and since $\succsim_X$ is complete their union is X. Moreover, the sets A^0 and A^+ are nonempty. We will use the assumption that X is preferentially connected to show that there exist an a^0 in A^0 that is in the closure of A^+.

The set A^+ is open since $\succsim_X$ is continuous. Since X is preferentially connected, it follows that A^+ is not closed. However, the set $A^0 \cup A^+$ is closed (again since $\succsim_X$ is continuous), and thus the closure of A^+ is a subset of $A^0 \cup A^+$. Hence, there exists an outcome a^0 in A^0 that is in the closure of A^+. It follows that there exists a sequence $\{a_n^+\}_{n=1}^{\infty}$ of outcomes in A^+ such that $| a_n^+ - a^0 |$ tends to zero as n tends to infinity.

Condition (A) implies $(\mathbf{a}^0_{\langle\alpha,\beta\rangle}, \mathbf{a}_{\langle\alpha',\beta'\rangle}, \mathbf{o}) \sim (\mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{a}_{\langle\alpha',\beta'\rangle}, \mathbf{o}) \prec (\mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{b}_{\langle\alpha',\beta'\rangle}, \mathbf{o})$ since $a^0 \sim_X a \prec_X b$. Thus, condition (C) implies that there exists a $\delta > 0$ such that for any a^+: $\Delta\big((\mathbf{a}^+_{\langle\alpha,\beta\rangle}, \mathbf{a}_{\langle\alpha',\beta'\rangle}, \mathbf{o}), (\mathbf{a}^0_{\langle\alpha,\beta\rangle}, \mathbf{a}_{\langle\alpha',\beta'\rangle}, \mathbf{o})\big) < \delta$ implies $(\mathbf{a}^+_{\langle\alpha,\beta\rangle}, \mathbf{a}_{\langle\alpha',\beta'\rangle}, \mathbf{o}) \prec (\mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{b}_{\langle\alpha',\beta'\rangle}, \mathbf{o})$. By the above result, $| a^+ - a^0 | < \delta(\beta - \alpha)^{-1}$ for some a^+ in A^+, and it follows that $\Delta\big((\mathbf{a}^+_{\langle\alpha,\beta\rangle}, \mathbf{a}_{\langle\alpha',\beta'\rangle}, \mathbf{o}), (\mathbf{a}^0_{\langle\alpha,\beta\rangle}, \mathbf{a}_{\langle\alpha',\beta'\rangle}, \mathbf{o})\big) = (\beta - \alpha) | a^+ - a^0 | < \delta$.

Suppose that $p : \langle a_{i-1}, a_i\rangle$, $i = 1, \dots, m$, and $q : \langle b_{j-1}, b_j\rangle$, $j = 1, \dots, n$, denote two partitions of a planning period $[0,T]$. Since the intersections $\langle a_{i-1}, a_i\rangle \cap \langle b_{j-1}, b_j\rangle$ are pairwise disjoint, they form another partition of $[0,T]$. We will refer to this partition as the <u>conjunction</u> of *p* and *q*, and we will denote it by *pq*. The sets S_p and S_q are subsets of S_{pq} since, for example, an outcome-stream that is in S_p is constant on each interval $\langle a_{i-1}, a_i\rangle$ and thus is constant on each interval $\langle a_{i-1}, a_i\rangle \cap \langle b_{j-1}, b_j\rangle$.

Lemma A4. Suppose that an outcome-stream space $(S_T, \succsim)$ satisfies conditions (A)-(E). Suppose, moreover, that a partition $p: \langle a_{i-1}, a_i \rangle$, $i = 1, \ldots, m$, of $[0, T]$ contains at least three non-point intervals. Then, the subspace $(S_p, \succsim)$ has a value function of the form

$$V_p(\mathbf{x}) = \Sigma_{i=1}^{m} a_{i,p} v_p(x(i)), \quad \mathbf{x} \text{ in } S_p \tag{A1}$$

such that:

(a) The function $v_p(x)$ defined on X is continuous, weakly increasing, and has a non-point interval range. Moreover, it is an outcome scale for the outcome space $(X, \succsim_X)$.

(b) A coefficient $a_{i,p}$ is positive if $\langle a_{i-1}, a_i \rangle$ is a non-point interval and zero otherwise.

Moreover, the function $v_p(x)$ is unique up to a positive affine transformation and the coefficients $a_{i,p}$ are unique up to a common positive multiple.

Proof. Assume that $(S_T, \succsim)$ satisfies conditions (A)-(E). The set S_p of outcome-streams $\mathbf{x}$ of the form $\mathbf{x}(t) = x(i)$ for t in $\langle a_{i-1}, a_i \rangle$, corresponds to the set $X^m = X \times \ldots \times X$ of outcome-vectors $\mathbf{x}^m = (x(1), \ldots, x(m))$. Thus, $\succsim$ induces a preference relation $\succsim^m$ on X^m, and the space $(X^m, \succsim^m)$ can be identified with the space $(S_p, \succsim)$. We will define the distance between two outcome-vectors $\mathbf{x}^m, \mathbf{y}^m$ in X^m as the distance $\Delta(\mathbf{x}, \mathbf{y}) = \Sigma_{i=1}^{m}(a_i - a_i) \,|\, x(i) - y(i) \,|$ between the corresponding outcome-streams $\mathbf{x}, \mathbf{y}$ in S_p.

Lemma A2 implies that an i-th component set in X^m is essential if it corresponds to a non-point interval $\langle a_{i-1}, a_i \rangle$ in the partition p and is inessential if it corresponds to a point interval $\langle a_{i-1}, a_i \rangle$ in p. Thus, X^m has at least three essential component sets.

Lemmas A1, A2 imply that the preference relation $\succsim^m$ induces a preference relation on each essential component set that coincides with $\succsim_X$ and induces a preference relation on each inessential component set that regards any outcomes as indifferent. Condition (B) implies that $\succsim^m$ is transitive and complete, condition (C) implies that $\succsim^m$ is continuous, and condition (D) implies that $\succsim^m$ is tradeoffs independent.

The outcome space $(X, \succsim_X)$ is preferentially connected. Hence, the extension of Debreu's additive-value model in Harvey (2006) implies that $(X^m, \succsim^m)$ has an additive value function, $V^m(\mathbf{x}^m) = \Sigma_{i \in M} \, a_i \, v_i(x(i))$, where M denotes the set of indices of the essential component sets. Here, each component function $v_i(x)$ is a continuous value function for $(X, \succsim_X)$, and each coefficient a_i is positive. Lemma 1 implies that each function $v_i(x)$ is weakly increasing and has a non-point interval range. Moreover, the functions $v_i(x)$, i in M, are unique up to a common positive linear transformation and the coefficients a_i, i in M, are unique up to a positive multiple.

Condition (E) implies that $\succsim^m$ satisfies the condition of 'equal tradeoffs midvalues' defined in Harvey (1986). By use of an argument there, the functions $v_i(x)$ can be chosen as a common function, which we will denote by $v(x)$. Thus, the space $(X^m, \succsim^m)$ has a value function of the form, $V^m(\mathbf{x}^m) = \sum_{i \in M} a_i \, v(x(i))$. By defining $a_i = 0$ for each inessential component set, it follows that the space $(S_p, \succsim)$ has a value function of the form, $V_p(\mathbf{x}) = \sum_{i=1}^m a_{i,p} v_p(x(i))$, where the function $v_p(x)$ and the coefficients $a_{i,p}$ have the properties (a), (b). In particular, each coefficient for an inessential component set must be zero since the function $v_p(x)$ is not constant, and thus the coefficients $a_{i,p}$ as described in (b) are unique up to a positive multiple.

Proof of Theorem 1. A partition with at least three non-point intervals will be called proper. The conjunction pq of a proper partition p and any partition q is proper. Thus, the set S_T is the union of the sets S_p such that p is a proper partition.

To show the forward implications, we normalize the value functions in Lemma A4 and paste together the normalized functions to construct a value function of the form (1).

By Lemma 1, there exist outcomes $a^1 \succ_X a^{-1}$. Assume that for a proper partition p, the outcome scale $v_p(x)$ and the coefficients $a_{i,p}$ in a value function $V_p(\mathbf{x})$ in Lemma A4 are normalized such that $v_p(a^{-1}) = -1$, $v_p(a^1) = 1$, and $\sum_{i=1}^m a_{i,p} = 1$. The resulting scale $v_p(x)$, coefficients $a_{i,p}$, and function $V_p(\mathbf{x}) = \sum_{i=1}^m a_{i,p} v_p(x(i))$ are unique.

For two proper partitions p, q, suppose that $V_{pq}(\mathbf{x}) = \sum_{i=1}^m \sum_{j=1}^n a_{ij,pq} v_{pq}(x(i, j))$ is the normalized value function for the conjunction pq. Then, $V_{pq}(\mathbf{x})$ is a value function for the subset S_p of S_{pq}, and $V_{pq}(\mathbf{x}) = \sum_{i=1}^m (\sum_{j=1}^n a_{ij,pq}) v_{pq}(x(i))$ for $\mathbf{x}$ in S_p. Since $v_{pq}(a^{-1}) = -1$, $v_{pq}(a^1) = 1$, and $\sum_{i=1}^m (\sum_{j=1}^n a_{ij,pq}) = 1$, $V_{pq}(\mathbf{x})$ is normalized as a value function for S_p. Thus, $v_{pq}(x) = v_p(x)$ for x in X and $\sum_{j=1}^{n} a_{ij,pq} = a_{i,p}$ for $i = 1, \ldots, m$ by uniqueness. The same arguments apply for $V_{pq}(\mathbf{x})$ restricted to the subset S_q of S_{pq}.

We will show that the normalized function $v_p(x)$ and the normalized coefficients $a_{i,p}$ associated with a proper partition p do not depend on p. First, note that for any two proper partitions p and q: $v_p(x) = v_{pq}(x) = v_q(x)$, x in X. We will denote the common function by $v(x)$.

Next, we show that a normalized coefficient $a_{i,p}$ associated with a proper partition p is a function, $a_{i,p} = f(a_{i-1}, a_i)$, of the endpoints a_{i-1}, a_i of the interval $\langle a_{i-1}, a_i \rangle$. For suppose that p is a proper partition with an interval $\langle a_{h-1}, a_h \rangle$ and q is a proper partition with an interval $\langle b_{k-1}, b_k \rangle$ such that $a_{h-1} = b_{k-1}$, $a_h = b_k$. Then, the interval $\langle a_{h-1}, a_h \rangle \cap \langle b_{k-1}, b_k \rangle$ in the conjunction pq also has these endpoints. Therefore, the intervals $\langle a_{i-1}, a_i \rangle \cap \langle b_{k-1}, b_k \rangle$, $i \neq h$, and $\langle a_{h-1}, a_h \rangle \cap \langle b_{j-1}, b_j \rangle$, $j \neq k$, are point intervals, and hence $a_{ik,pq} = 0$ for $i \neq h$ and $a_{hj,pq} = 0$ for $j \neq k$. It follows that $a_{hk,pq} = \sum_{i=1}^{m} a_{ik,pq} = a_{k,q}$ and $a_{hk,pq} = \sum_{j=1}^{n} a_{hj,pq} = a_{h,p}$, and thus $a_{k,q} = a_{h,p}$.

Suppose that p is a proper partition with adjacent intervals $\langle a_{h-1}, a_h \rangle$, $\langle a_h, a_{h+1} \rangle$ and q is a proper partition with an interval $\langle b_{k-1}, b_k \rangle = \langle a_{h-1}, a_h \rangle \cup \langle a_h, a_{h+1} \rangle$. For $h' = h+1$: $a_{h,p} = \sum_{j=1}^{n} a_{hj,pq} = a_{hk,pq}$, $a_{h',p} = \sum_{j=1}^{n} a_{h'j,pq} = a_{h'k,pq}$, and $a_{k,q} = \sum_{i=1}^{m} a_{ik,pq} = a_{hk,pq} + a_{h'k,pq}$ Thus, $a_{k,q} = a_{h,p} + a_{h',p}$. It follows that $f(a_{h-1}, a_{h+1}) = f(a_{h-1}, a_h) + f(a_h, a_{h+1})$, and thus $f(a,c) = f(a,b) + f(b,c)$ for any $a \le b \le c$ in the interval $[0,T]$.

To solve this functional equation, define $A(t) = f(0,t)$. Then, $f(b,c) = A(c) - A(b)$ and $A(0) = f(0,0) = A(0) - A(0) = 0$. See, e.g., Aczél (1966, pp. 223-224) for references.

The value function $V_p(\mathbf{x}) = \sum_{i=1}^{m} a_{i,p} v_p(x(i))$ for a set S_p can now be written as:

$$V_p(\mathbf{x}) = \sum_{i=1}^{m} \left(A(a_i) - A(a_{i-1}) \right) v(x(i)), \ \mathbf{x} \text{ in } S_p,$$

where the functions $A(t)$ and $v(x)$ are independent of the proper partition p.

If an outcome-stream $\mathbf{x}$ is in the sets S_p, S_q for different proper partitions p, q, then $\mathbf{x}$ is in S_{pq} and $V_p(\mathbf{x}) = V_{pq}(\mathbf{x}) = V_q(\mathbf{x})$. Thus, for any $\mathbf{x}$ in S_T, the amount $V(\mathbf{x})$

in (2) is well-defined as the common amount $V_p(\mathbf{x})$ for any proper partition p such that $\mathbf{x}$ is in S_p.

The function $V(\mathbf{x})$ is a value function for the space $(S_T, \succsim)$. For consider any $\mathbf{x}$, $\mathbf{y}$ in S_T. Then, $\mathbf{x}$ is in S_p and $\mathbf{y}$ is in S_q for some proper partitions p and q. Hence, $\mathbf{x}$ and $\mathbf{y}$ are both in S_{pq}, and thus $V(\mathbf{x}) = V_{pq}(\mathbf{x})$ and $V(\mathbf{y}) = V_{pq}(\mathbf{y})$. Therefore, $\mathbf{x} \succsim \mathbf{y}$ if and only if $V_{pq}(\mathbf{x}) \geq V_{pq}(\mathbf{y})$ if and only if $V(\mathbf{x}) \geq V(\mathbf{y})$.

The normalizations, $\sum_{i=1}^{m} a_{i,p} = 1$, imply that $A(T) = 1$. Hence, $V(\mathbf{a}) = v(a)$ for any outcome a. Moreover, an amount $V(\mathbf{x})$ is a weighted average of amounts $v(x(i))$. Thus, the range of the function $V(\mathbf{x})$ equals the non-point interval range of the function $v(x)$.

The normalization, $v(a^{\pm 1}) = \pm 1$, of the function $v(x)$ implies that the common range of the functions $v(x)$ and $V(\mathbf{x})$ includes the interval $[-1, 1]$. Thus, for any $-1 \leq r \leq 1$ there exists an outcome a^r such that $v(a^r) = r$ and $V(\mathbf{a}^r) = r$.

Next, we show that the functions $v(x)$, $A(t)$, $V(\mathbf{x})$, have properties (a)-(c). Lemma A4 implies that $v(x)$ has the properties in (a) since $v(x) = v_p(x)$ for any proper partition p.

The function $A(t)$ is strictly increasing on $[0, T]$ since by Lemma A4 any coefficient $a_{i,p}$ for a nonpoint interval is positive. Moreover, $A(0) = 0$ as shown above.

To show that $A(t)$ is absolutely continuous on $[0, T]$ it suffices to show that for any $0 < \varepsilon < 1$ there is a $\delta > 0$ such that $\sum_{i=1}^{n}(a_i - a_{i-1}) < \delta$ implies $\sum_{i=1}^{n}(A(a_i) - A(a_{i-1})) < \varepsilon$ for any pairwise disjoint intervals (a_{i-1}, a_i), $i = 1, \ldots, n$, in the interval $[0, T]$. Here, the union of the intervals (a_{i-1}, a_i) can be any subset of $[0, T]$.

For intervals (a_{i-1}, a_i) as described, define a step outcome-stream $\mathbf{z}$ by $\mathbf{z}(t) = a^1$ if t is in the union of the intervals (a_{i-1}, a_i) and $\mathbf{z}(t) = a^0$ otherwise. Then, $\Delta(\mathbf{z}, \mathbf{a}^0) = \sum_{i=1}^{n}(a_i - a_{i-1}) \,|\, a^1 - a^0 \,|$ and $V(\mathbf{z}) = \sum_{i=1}^{n}(A(a_i) - A(a_{i-1}))$.

Consider an outcome-stream $\mathbf{a}^{\varepsilon}$, $0 < \varepsilon < 1$. Then, $\mathbf{a}^0 \prec \mathbf{a}^{\varepsilon}$. Condition (C) implies that there exists a $\delta > 0$ such that $\Delta(\mathbf{z}, \mathbf{a}^0) < \delta$ implies $\mathbf{z} \prec \mathbf{a}^{\varepsilon}$ for any $\mathbf{z}$ in S_T. Define $\delta' = \delta \,|\, a^1 - a^0 \,|^{-1}$. Then, $\sum_{i=1}^{n}(a_i - a_{i-1}) < \delta'$ implies $\Delta(\mathbf{z}, \mathbf{a}^0) < \delta$ implies $\mathbf{z} \prec \mathbf{a}^{\varepsilon}$ implies $V(\mathbf{z}) < V(\mathbf{a}^{\varepsilon})$ implies $\sum_{i=1}^{n}(A(a_i) - A(a_{i-1})) < \varepsilon$.

To show property (c), consider an $\mathbf{x}$ in S_T and an $\varepsilon > 0$. As the primary case, assume that there exist $\mathbf{x}^-$, $\mathbf{x}^+$ in S_T with $V(\mathbf{x}^-) < V(\mathbf{x}) < V(\mathbf{x}^+)$. Since the function $V(\mathbf{x})$ has an interval range, $V(\mathbf{x})-\varepsilon < V(\mathbf{x}^{-\varepsilon}) < V(\mathbf{x}) < V(\mathbf{x}^{\varepsilon}) < V(\mathbf{x})+\varepsilon$ for some $\mathbf{x}^{-\varepsilon}$, $\mathbf{x}^{\varepsilon}$ in S_T. Thus, by condition (C) there exist $\delta_1, \delta_2 > 0$ such that $\Delta(\mathbf{z}, \mathbf{x}) < \delta_1$ implies $\mathbf{z} \succ \mathbf{x}^{-\varepsilon}$, and $\Delta(\mathbf{z}, \mathbf{x}) < \delta_2$ implies $\mathbf{z} \prec \mathbf{x}^{\varepsilon}$. Define $\delta = \min\{\delta_1, \delta_2\}$. Then, $\Delta(\mathbf{z}, \mathbf{x}) < \delta$ implies $\mathbf{x}^{-\varepsilon} \prec \mathbf{z} \prec \mathbf{x}^{\varepsilon}$ implies $V(\mathbf{x})-\varepsilon < V(\mathbf{z}) < V(\mathbf{x})+\varepsilon$.

As a second case, assume that $V(\mathbf{z}) \le V(\mathbf{x})$ for any $\mathbf{z}$ in S_T. Then, there exists an $\mathbf{x}^{-\varepsilon}$ in S_T such that $V(\mathbf{x})-\varepsilon < V(\mathbf{x}^{-\varepsilon}) < V(\mathbf{x})$. Thus, by condition (C) there exists a $\delta > 0$ such that $\Delta(\mathbf{z}, \mathbf{x}) < \delta$ implies $\mathbf{z} \succ \mathbf{x}^{-\varepsilon}$ for $\mathbf{z}$ in S_T. Thus, $\Delta(\mathbf{z}, \mathbf{x}) < \delta$ implies $V(\mathbf{z}) > V(\mathbf{x}^{-\varepsilon})$ implies $V(\mathbf{x})-\varepsilon < V(\mathbf{z}) \le V(\mathbf{x})$. The arguments are similar and thus can be omitted for the remaining case that $V(\mathbf{z}) \ge V(\mathbf{x})$ for any $\mathbf{z}$ in S_T.

For the converse part of the proof, assume that an outcome-stream space $(S_T, \succsim)$ has a value function $V(\mathbf{x})$ of the form (1) with properties (a)-(c). Then, it is straightforward to show that $(S_T, \succsim)$ satisfies conditions (A), (B), (D), and (E).

To show that $(S_T, \succsim)$ satisfies the continuity condition (C), consider any $\mathbf{x}, \mathbf{y}$ in S_T with $\mathbf{x} \prec \mathbf{y}$ and thus $V(\mathbf{x}) < V(\mathbf{y})$. Define $\varepsilon = V(\mathbf{y}) - V(\mathbf{x}) > 0$. By property (c), there is a $\delta > 0$ such that $\Delta(\mathbf{z}, \mathbf{x}) < \delta$ implies $|V(\mathbf{z}) - V(\mathbf{x})| < \varepsilon$ for $\mathbf{z}$ in S_T. Thus, $\Delta(\mathbf{z}, \mathbf{x}) < \delta$ implies $V(\mathbf{z}) < V(\mathbf{y})$ which implies $\mathbf{z} \prec \mathbf{y}$. By a similar argument, there is a $\delta > 0$ such that $\Delta(\mathbf{z}, \mathbf{y}) < \delta$ implies $\mathbf{z} \succ \mathbf{x}$. Hence, condition (C) is satisfied.

It remains to show the uniqueness properties of the functions $v(x)$ and $A(t)$. Suppose that $V(\mathbf{x}) = \sum_{i=1}^{m} (A(a_i) - A(a_{i-1}))\, v(x(i))$ and $\hat{V}(\mathbf{x}) = \sum_{i=1}^{m} (\hat{A}(a_i) - \hat{A}(a_{i-1}))\, \hat{v}(x(i))$ are value functions for $(S_T, \succsim)$ with the properties (a)-(d). Then, for any proper partition p, $V(\mathbf{x})$ and $\hat{V}(\mathbf{x})$ are value functions for the subset S_p of S_T. Lemma A4 implies that $\hat{v}(x) = \alpha_1^p\, v(x) + \beta_1^p$, x in X, where $\alpha_1^p > 0$. Since $v(x)$ has a non-point range, α_1^p, β_1^p are independent of p, and thus $\hat{v}(x)$ is a positive linear transformation of $v(x)$.

Lemma A4 also implies that $\hat{A}(a_i) - \hat{A}(a_{i-1}) = \alpha_2^p\,(A(a_i) - A(a_{i-1}))$, $i = 1, \ldots, m$, where $\alpha_2^p > 0$. By adding these equations, it follows that $\hat{A}(T) = \alpha_2^p A(T)$, and thus α_2^p is independent of p. Hence, $\hat{A}(a_1) = \hat{A}(a_1) - \hat{A}(a_0) = \alpha_2\,(A(a_1) - A(a_0)) = \alpha_2\, A(a_1)$ where α_2 is the common value of α_2^p. But a_1 can be any time in $[0, T]$, and thus $\hat{A}(t)$ is a positive multiple of $A(t)$.

Conversely, if $V(\mathbf{x}) = \sum_{i=1}^{m} (A(a_i) - A(a_{i-1}))\, v(x(i))$ is a value function for $(S_T, \succsim)$ and $\hat{V}(\mathbf{x}) = \sum_{i=1}^{m} (\hat{A}(a_i) - \hat{A}(a_{i-1}))\, \hat{v}(x(i))$ where $\hat{v}(x) = \alpha_1 v(x) + \beta_1$ and $\hat{A}(t) = \alpha_2 A(t)$, $\alpha_1, \alpha_2 > 0$, then $\hat{V}(\mathbf{x}) = \alpha_1 \alpha_2 \hat{V}(\mathbf{x}) + \alpha_1 \beta_1 A(T)$ and thus $\hat{V}(\mathbf{x})$ is also a value function.

Lemma A5. An outcome-stream $\mathbf{x}$ is a Riemann outcome-stream on $[0,T]$ if and only if there exist two sequences $\{\overline{\mathbf{x}}^{(n)}\}_{n=1}^{\infty}$ and $\{\underline{\mathbf{x}}^{(n)}\}_{n=1}^{\infty}$ of step outcome-streams such that:

$$\underline{\mathbf{x}}^{(1)}(t) \le \underline{\mathbf{x}}^{(2)}(t) \le \ldots \le \mathbf{x}(t) \le \ldots \le \overline{\mathbf{x}}^{(2)}(t) \le \overline{\mathbf{x}}^{(1)}(t)$$

for t in $[0,T]$, and $\lim_{n\to\infty} \Delta(\overline{\mathbf{x}}^{(n)}, \underline{\mathbf{x}}^{(n)}) = 0$.

<u>Proof</u>. For the forward implication, assume that **x** is a Riemann outcome-stream on $[0,T]$. Then, each component-stream $\mathbf{x}_j$ is continuous a.e. and $\underline{x}_j \le \mathbf{x}_j(t) \le \overline{x}_j$ for t in $[0,T]$.

Choose a nested sequence of partitions $\{p^{(n)}\}_{n=1}^{\infty}$ so that $\lim_{n\to\infty} \max_i (a_i^{(n)} - a_{i-1}^{(n)}) = 0$. For each $j = 1,\ldots,N$, define two sequences $\{\overline{\mathbf{x}}_j^{(n)}\}_{n=1}^{\infty}$, $\{\underline{\mathbf{x}}_j^{(n)}\}_{n=1}^{\infty}$ of step functions by: $\underline{\mathbf{x}}_j^{(n)}(t) = \inf\{\mathbf{x}_j(t) : t \in \langle a_{i-1}^{(n)}, a_i^{(n)} \rangle\}$ and $\overline{\mathbf{x}}_j^{(n)}(t) = \sup\{\mathbf{x}_j(t) : t \in \langle a_{i-1}^{(n)}, a_i^{(n)} \rangle\}$ for t in the subinterval $\langle a_{i-1}^{(n)}, a_i^{(n)} \rangle$ in the partition $p^{(n)}$. Since a component set X_j is finite or an interval, and $\underline{x}_j \le \mathbf{x}_j(t) \le \overline{x}_j$, the 'inf' and 'sup' values of the step functions $\underline{\mathbf{x}}_j^{(n)}(t)$, $\overline{\mathbf{x}}_j^{(n)}(t)$, are in X_j. Hence, the values of the corresponding vector-valued functions $\overline{\mathbf{x}}^{(n)}$, $\underline{\mathbf{x}}^{(n)}$ are in the outcome set X, and thus $\overline{\mathbf{x}}^{(n)}$ and $\underline{\mathbf{x}}^{(n)}$ are step outcome-streams.

The sequences $\{\overline{\mathbf{x}}^{(n)}\}_{n=1}^{\infty}$, $\{\underline{\mathbf{x}}^{(n)}\}_{n=1}^{\infty}$ satisfy the stated inequalities since the sequence $\{p^{(n)}\}_{n=1}^{\infty}$ of partitions is nested. Moreover, $\lim_{n\to\infty} \int (\overline{\mathbf{x}}_j^{(n)} - \underline{\mathbf{x}}_j^{(n)}) = 0$ for $j = 1,\ldots,N$ since a real-valued function is continuous a.e. and bounded if and only if it is so-called Darboux integrable. Hence, $\lim_{n\to\infty} \Delta(\overline{\mathbf{x}}^{(n)}, \underline{\mathbf{x}}^{(n)}) = 0$ since $\Delta(\mathbf{x},\mathbf{y}) \le \sum_{i=1}^{m} \int |\mathbf{x}_j - \mathbf{y}_j|$.

The proof of the converse implication is essentially the above arguments in reverse.

Lemma A6. Suppose that a function $a(t)$ on $[0,T]$ is non-negative, Lebesgue integrable, and its indefinite integral $A(t) = \int_0^t a(s)\,ds$ is strictly increasing, and that a function $v(x)$ on X is continuous and weakly increasing. Then, for any $\mathbf{x}$, $\mathbf{y}$ in R_T:

(a) The function $v(\mathbf{x}(t))$ is Riemann integrable on $[0,T]$ and the function $a(t)\,v(\mathbf{x}(t))$ is Lebesgue integrable on $[0,T]$.

(b) For any sequences $\{\overline{\mathbf{x}}^{(n)}\}_{n=1}^{\infty}$ and $\{\underline{\mathbf{x}}^{(n)}\}_{n=1}^{\infty}$ of step outcome-streams as described in Lemma A5, $\lim_{n\to\infty} \int_0^T a(t)v(\overline{\mathbf{x}}^{(n)}(t))dt = \lim_{n\to\infty} \int_0^T a(t)v(\underline{\mathbf{x}}^{(n)}(t))dt = \int_0^T a(t)v(\mathbf{x}(t))dt$.

(c) If $v(\mathbf{x}(t)) \le v(\mathbf{y}(t))$ a.e. on $[0,T]$, then exactly one of the following cases is true:

(i) $v(\mathbf{x}(t)) = v(\mathbf{y}(t))$ a.e. on $[0,T]$, and $\int_0^T a(t)v(\mathbf{x}(t))dt = \int_0^T a(t)v(\mathbf{y}(t))dt$.

(ii) $v(\mathbf{x}(t)) < v(\mathbf{y}(t))$ on a non-point interval, and $\int_0^T a(t)v(\mathbf{x}(t))dt < \int_0^T a(t)v(\mathbf{y}(t))dt$.

Proof. For (a), consider an outcome-stream $\mathbf{x}$ in R_T. Then, $\mathbf{x}$ is continuous a.e. and there are outcomes $\underline{x}$, $\overline{x}$ such that $\underline{x} \le \mathbf{x}(t) \le \overline{x}$ for t in $[0,T]$. Hence, the composite function $v(\mathbf{x}(t))$ is continuous a.e. (since the function $v(x)$ is continuous) and is bounded by $v(\underline{x})$ and $v(\overline{x})$ (since $v(x)$ is weakly increasing). Thus, $v(\mathbf{x}(t))$ is Riemann integrable. But $a(t)$ is Lebesgue integrable, and thus the product $a(t)v(\mathbf{x}(t))$ is Lebesgue integrable.

For (b), define $f_j^{(n)}(t) = \overline{\mathbf{x}}_j^{(n)}(t) - \underline{\mathbf{x}}_j^{(n)}(t)$, t in $[0,T]$, for each $j = 1,\ldots,N$. Then, the functions $f_j^{(n)}(t)$ are non-negative step functions and $f_j^{(1)}(t) \ge f_j^{(2)}(t) \ge \ldots$, t in $[0,T]$.

The sequence $\{f_j^{(n)}(t)\}_{n=1}^{\infty}$ converges for t in $[0,T]$. Define $f_j(t) = \lim_{n\to\infty} f_j^{(n)}(t)$. Then, $f_j(t)$ is Lebesgue integrable and $\int_0^T f_j(t)\,dt = \lim_{n\to\infty} \int_0^T f_j^{(n)}(t)\,dt$ by the Monotone Convergence Theorem. But $\lim_{n\to\infty} \int_0^T f_j^{(n)}(t)\,dt = 0$ since $\lim_{n\to\infty} \Delta(\overline{\mathbf{x}}^{(n)}, \underline{\mathbf{x}}^{(n)}) = 0$. Hence, $\int_0^T f_j(t)\,dt = 0$.

To show that $f_j(t) = 0$ a.e. on $[0,T]$, define $E = \{t \in [0, T] : f_j(t) > 0\}$ and $E_m = \{t \in [0, T] : f_j(t) > 1/m\}$, $m = 1, 2, \ldots$. Then, $E = \cup_{m=1}^{\infty} E_m$. If the measure $\lambda(E)$ of the union E is positive, then the measure $\lambda(E_m)$ of E_m is positive for some $m = 1, 2, \ldots$. But $\lambda(E_m) > 0$ implies that $\int_0^T f_j(t)\,dt \ge (1/m)\lambda(E_m) > 0$ which is a contradiction.

Thus, $\lim_{n\to\infty} \overline{\mathbf{x}}_j^{(n)}(t) = \lim_{n\to\infty} \underline{\mathbf{x}}_j^{(n)}(t) = \lim_{n\to\infty} \mathbf{x}_j^{(n)}(t)$ for all $j = 1,\ldots,N$ a.e. Since $v(x)$ is continuous, it follows that $\lim_{n\to\infty} v(\overline{\mathbf{x}}^{(n)}(t)) = \lim_{n\to\infty} v(\underline{\mathbf{x}}^{(n)}(t)) = \lim_{n\to\infty} v(\mathbf{x}^{(n)}(t))$ a.e. . Thus, $\lim_{n\to\infty} a(t)v(\overline{\mathbf{x}}^{(n)}(t)) = \lim_{n\to\infty} a(t)v(\underline{\mathbf{x}}^{(n)}(t)) = \lim_{n\to\infty} a(t)v(\mathbf{x}^{(n)}(t))$ a.e. .

Since $\{a(t)v(\overline{\mathbf{x}}^{(n)}(t))\}_{n=1}^{\infty}$ is a weakly decreasing sequence of Lebesgue integrable functions, the Monotone Convergence Theorem implies that $\lim_{n\to\infty} \int_0^T a(t)v(\overline{\mathbf{x}}^{(n)}(t))\,dt = \int_0^T a(t)v(\mathbf{x}(t))dt$. And by a similar argument, $\lim_{n\to\infty} \int_0^T a(t)v(\underline{\mathbf{x}}^{(n)}(t))\,dt = \int_0^T a(t)v(\mathbf{x}(t))dt$.

For (c), assume that $v(\mathbf{x}(t)) \le v(\mathbf{y}(t))$ a.e. Then, $\int_0^T a(t)v(\mathbf{x}(t))dt \le \int_0^T a(t)v(\mathbf{y}(t))dt$ since $a(t)$ is non-negative. But $\int_0^T a(t)v(\mathbf{x}(t))dt = \int_0^T a(t)v(\mathbf{y}(t))dt$ if $v(\mathbf{x}(t)) = v(\mathbf{y}(t))$ a.e.. Suppose $v(\mathbf{x}(t)) < v(\mathbf{y}(t))$ on a set E of positive measure. Since $\mathbf{x}, \mathbf{y}$ are continuous a.e., they are continuous at a time t_0 in E. Define $\varepsilon = v(\mathbf{y}(t_0)) - v(\mathbf{x}(t_0)) > 0$. There exists a non-point interval $\langle\alpha, \beta\rangle$ with $v(\mathbf{y}(t)) - v(\mathbf{x}(t)) > \varepsilon/2$ on $\langle\alpha, \beta\rangle$. Thus, $v(\mathbf{x}(t)) < v(\mathbf{y}(t))$ on $\langle\alpha, \beta\rangle$, and $\int_0^T a(t)v(\mathbf{y}(t))dt - \int_0^T a(t)v(\mathbf{x}(t))dt \geq (A(\beta) - A(\alpha))\,\varepsilon/2 > 0$.

Proof of Theorem 2. For the forward part of the proof, assume that an outcome-stream space $(R_T, \succsim)$ satisfies the stated conditions. Then, $\succsim$ restricted to the set S_T satisfies the conditions in Theorem 1. Thus, there exist functions $v(x)$, $a(t)$, $A(t)$, $V(\mathbf{x})$ with the properties (a)-(d) in Theorem 1 such that $V(\mathbf{x})$ is a value function for the space $(S_T, \succsim)$. Moreover, the function $v(x)$ is unique up to a positive linear transformation, and the function $A(t)$ is unique up to a positive multiple.

By Lemma A6(a), the function $a(t)v(\mathbf{x}(t))$ is Lebesgue integrable for any $\mathbf{x}$ in the set R_T and thus $V(\mathbf{x}) = \int_0^T a(t)v(\mathbf{x}(t))dt$ is well-defined on R_T. Our task is to show that $V(\mathbf{x})$ is a value function for the space $(R_T, \succsim)$. To do so, we establish the following properties.

(i) For any $\mathbf{x}$ in R_T and any $\varepsilon > 0$, there exists a $\mathbf{w}$ in S_T such that $\mathbf{w} \sim \mathbf{x}$ and $|V(\mathbf{w}) - V(\mathbf{x})| < \varepsilon$.

<u>Proof</u>: Consider $\mathbf{x}$ in R_T and $\varepsilon > 0$. By Lemmas A5, A6, there exist $\overline{\mathbf{x}}$, $\underline{\mathbf{x}}$ in S_T such that: (1) $\underline{\mathbf{x}}(t) \leq \mathbf{x}(t) \leq \overline{\mathbf{x}}(t)$, t in $[0,T]$, and (2) $|V(\mathbf{x}) - V(\underline{\mathbf{x}})| < \varepsilon$ and $|V(\overline{\mathbf{x}}) - V(\mathbf{x})| < \varepsilon$.

The inequalities (1) imply $\underline{\mathbf{x}}(t) \precsim_X \mathbf{x}(t) \precsim_X \overline{\mathbf{x}}(t)$ and $v(\underline{\mathbf{x}}(t)) \leq v(\mathbf{x}(t)) \leq v(\overline{\mathbf{x}}(t))$ for t in $[0,T]$ by Lemma 1. Hence, $\underline{\mathbf{x}} \precsim \mathbf{x} \precsim \overline{\mathbf{x}}$ by condition (A), and $V(\underline{\mathbf{x}}) \leq V(\mathbf{x}) \leq V(\overline{\mathbf{x}})$ by Lemma A6(c). If $\overline{\mathbf{x}} \sim \mathbf{x}$ or $\underline{\mathbf{x}} \sim \mathbf{x}$, then by the inequalities (2) we are through.

Assume the remaining case that $\underline{\mathbf{x}} \prec \mathbf{x}$ and $\mathbf{x} \prec \overline{\mathbf{x}}$. Then, $\underline{\mathbf{x}} \prec \overline{\mathbf{x}}$, and thus $\underline{\mathbf{x}}(t) \neq \overline{\mathbf{x}}(t)$ on a non-point interval $\langle a_{i-1}, a_i\rangle$. Hence, $\Delta(\underline{\mathbf{x}}, \overline{\mathbf{x}}) > 0$.

Define $\mathbf{x}_\lambda = \lambda\,\overline{\mathbf{x}} + (1-\lambda)\,\underline{\mathbf{x}}$ for $0 \leq \lambda \leq 1$. Then, (1) implies $\underline{\mathbf{x}}(t) \leq \mathbf{x}_\lambda(t) \leq \overline{\mathbf{x}}(t)$, t in $[0,T]$, and thus $V(\underline{\mathbf{x}}) \leq V(\mathbf{x}_\lambda) \leq V(\overline{\mathbf{x}})$ by Lemma A6. Hence, $|V(\mathbf{x}_\lambda) - V(\mathbf{x})| < \varepsilon$ by (2).

One can check that $\mathbf{x}_\mu - \mathbf{x}_\lambda = (\mu - \lambda)(\overline{\mathbf{x}} - \underline{\mathbf{x}})$ for λ, μ in [0, 1] and thus $\Delta(\mathbf{x}_\lambda, \mathbf{x}_\mu) = \sum_{i=1}^m (a_i - a_{i-1})\,|\,x_\lambda(i) - x_\mu(i)\,| = \sum_{i=1}^m (a_i - a_{i-1})\,|\lambda - \mu|\,|\,\overline{x}(i) - \underline{x}(i)\,| = |\lambda - \mu|\,\Delta(\underline{\mathbf{x}}, \overline{\mathbf{x}})$.

Define $L = \{\lambda \in [0,1] : \mathbf{x}_\lambda \prec \mathbf{x}\}$ and $U = \{\lambda \in [0,1] : \mathbf{x}_\lambda \succ \mathbf{x}\}$. Then, L and U are disjoint, 0 is in L, and 1 is in U. Moreover, the sets L and U are open relative to $[0,1]$. For consider, e.g., an λ in L. Then, $\mathbf{x}_\lambda \prec \mathbf{x}$, and thus by condition (C), there is a $\delta > 0$

such that $\Delta(\mathbf{x}_\mu, \mathbf{x}_\lambda) < \delta$ implies $\mathbf{x}_\mu \prec \mathbf{x}$. Hence, $|\mu - \lambda| < \Delta(\underline{\mathbf{x}}, \overline{\mathbf{x}})^{-1}\delta$ implies that μ is in L. Since [0,1] is connected, there exists a ν in [0,1] that is not in L or U, and thus $\mathbf{x}_\nu \sim \mathbf{x}$ by the completeness of $\succsim$.

(ii) $V(\mathbf{x}) < V(\mathbf{y})$ implies $\mathbf{x} \prec \mathbf{y}$ for any $\mathbf{x}, \mathbf{y}$ in R_T.

Proof. Consider $\mathbf{x}, \mathbf{y}$ in R_T with $V(\mathbf{x}) < V(\mathbf{y})$. Define $\varepsilon = {}^1\!/_2\,(V(\mathbf{y}) - V(\mathbf{x}))$. By (i), there exist $\mathbf{w}, \mathbf{z}$ in S_T with $\mathbf{w} \sim \mathbf{x}$, $\mathbf{z} \sim \mathbf{y}$, $|V(\mathbf{w}) - V(\mathbf{x})| < \varepsilon$, and $|V(\mathbf{z}) - V(\mathbf{y})| < \varepsilon$. Hence, $V(\mathbf{w}) < V(\mathbf{z})$, and thus $\mathbf{w} \prec \mathbf{z}$ since the function $V(\mathbf{x})$ represents $(S_T, \succsim)$. Therefore, $\mathbf{x} \prec \mathbf{y}$ by the transitivity of $\succsim$.

(iii) $V(\mathbf{x}) = V(\mathbf{y})$ implies $\mathbf{x} \sim \mathbf{y}$ for any $\mathbf{x}, \mathbf{y}$ in R_T.

Proof. Consider $\mathbf{x}, \mathbf{y}$ in R_T with $V(\mathbf{x}) = V(\mathbf{y})$. By property (i) there exist $\mathbf{w}, \mathbf{z}$ in S_T such that $\mathbf{w} \sim \mathbf{x}$ and $\mathbf{z} \sim \mathbf{y}$. Then, $V(\mathbf{w}) = V(\mathbf{x})$ and $V(\mathbf{z}) = V(\mathbf{y})$ since otherwise, e.g., $V(\mathbf{w}) < V(\mathbf{x})$ which implies $\mathbf{w} \prec \mathbf{x}$ by (ii). Thus, $V(\mathbf{w}) = V(\mathbf{z})$. Since the function $V(\mathbf{x})$ represents $(S_T, \succsim)$, this equality implies that $\mathbf{w} \sim \mathbf{z}$. Therefore, $\mathbf{x} \sim \mathbf{y}$ by transitivity.

Properties (ii), (iii) imply that $V(\mathbf{x})$ is a value function for $(R_T, \succsim)$. For $\mathbf{x} \succsim \mathbf{y}$ implies not $\mathbf{x} \prec \mathbf{y}$ which implies $V(\mathbf{x}) \geq V(\mathbf{y})$ by (ii), and $V(\mathbf{x}) \geq V(\mathbf{y})$ implies $V(\mathbf{x}) > V(\mathbf{y})$ or $V(\mathbf{x}) = V(\mathbf{y})$ which implies $\mathbf{x} \succ \mathbf{y}$ or $\mathbf{x} \sim \mathbf{y}$ by (ii) and (iii) which implies $\mathbf{x} \succsim \mathbf{y}$.

For the converse part of the proof, assume that a function $V(\mathbf{x})$ of the form (2) is well-defined and is a value function for an outcome-stream space $(R_T, \succsim)$, and that the functions $v(x)$, $a(t)$, $A(t)$, and $V(\mathbf{x})$ satisfy the properties (a)-(d). Then, $\succsim$ satisfies condition (B) on R_T since $V(\mathbf{x})$ is a value function, and $\succsim$ satisfies conditions (D) and (E) on the set S_T by Theorem 1.

To show that $\succsim$ satisfies condition (A) on R_T, assume that $\mathbf{x}(t) \succsim_X \mathbf{y}(t)$ a.e. . Then, $v(\mathbf{x}(t)) \geq v(\mathbf{y}(t))$ a.e. by property (a). Hence, $V(\mathbf{x}) \geq V(\mathbf{y})$ by Lemma A6(c), and thus $\mathbf{x} \succsim \mathbf{y}$. If also $\mathbf{x}(t) \succ_X \mathbf{y}(t)$ a.e. on a non-point interval $\langle\alpha, \beta\rangle$, then $v(\mathbf{x}(t)) > v(\mathbf{y}(t))$ on $\langle\alpha, \beta\rangle$. Hence, $V(\mathbf{x}) > V(\mathbf{y})$ by Lemma A6(c), and thus $\mathbf{x} \succ \mathbf{y}$.

To show that $\succsim$ satisfies condition (C) on the pair of sets R_T, S_T, consider any $\mathbf{x}$ in R_T and $\mathbf{w}$ in S_T such that $\mathbf{w} \prec \mathbf{x}$. Then, $V(\mathbf{w}) < V(\mathbf{x})$. Define $\varepsilon = V(\mathbf{x}) - V(\mathbf{w}) > 0$. By property (c) in Theorem 1, there exists a $\delta > 0$ such that for any $\mathbf{z}$ in S_T $\Delta(\mathbf{z}, \mathbf{w}) < \delta$

implies $|V(\mathbf{z})-V(\mathbf{w})|<\varepsilon$. But $|V(\mathbf{z})-V(\mathbf{w})|<\varepsilon$ implies $V(\mathbf{z})<V(\mathbf{x})$ implies $\mathbf{z}\prec\mathbf{x}$. The argument when $\mathbf{w}\succ\mathbf{x}$ is similar, and thus condition (C) is satisfied.

To prove the uniqueness properties of the functions $v(x)$ and $A(t)$, consider two functions $V(\mathbf{x})=\int_0^T a(t)v(\mathbf{x}(t))dt$, $\hat{V}(\mathbf{x})=\int_0^T \hat{a}(t)\hat{v}(\mathbf{x}(t))dt$, and the associated functions, $A(t)=\int_0^t a(s)\,ds$, $\hat{A}(t)=\int_0^t \hat{a}(s)\,ds$. Assume that $V(\mathbf{x})$ and $\hat{V}(\mathbf{x})$ are value functions for the space $(R_T,\succsim)$ and that they satisfy the properties (a)-(d). Then, by Theorem 1 the function $\hat{v}(x)$ is a positive linear transformation of the function $v(x)$ and the function $\hat{A}(t)$ is a positive multiple of $A(t)$. It is straightforward to verify that, conversely, if $V(\mathbf{x})$ is a value function for $(R_T,\succsim)$, $\hat{v}(x)$ is a positive linear transformation of $v(x)$, and $\hat{A}(t)$ is a positive multiple of $A(t)$, then $\hat{V}(\mathbf{x})$ is a value function for $(R_T,\succsim)$.

Proof of Theorem 3. Since R_T is a subset of $R_{T'}$ for $T'>T$, R_f is the union of the sets R_T, $T\geq U$, for any horizon $U<\infty$. For our purposes, we choose $U=1$.

First, we show the forward implications. Lemma 1 implies that there exists an outcome $x^+\succ_X \mathrm{o}$ or an outcome $x^-\prec_X \mathrm{o}$. The arguments are the same in both cases, so it suffices to assume that there is an outcome $x^+\succ_X \mathrm{o}$.

The assumptions in Theorem 3 imply those in Theorem 2 for any horizon $T>0$. And Theorem 2 implies in particular that for any $T\geq 1$ there exist a value function $V_T(\mathbf{x})$ as described for the space $(R_T,\succsim)$. Moreover, we can assume that the associated functions $v_T(x)$, $A_T(t)$ are normalized such that $v_T(\mathrm{o})=0$, $v_T(x^+)=1$, $A_T(0)=0$, $A_T(1)=1$, and thus $v_T(x)$ and $A_T(t)$ are unique. Then, for any $T'>T\geq 1$, both $V_T(\mathbf{x})$ and $V_{T'}(\mathbf{x})$ are normalized value functions for $(R_T,\succsim)$, and thus $v_T(x)=v_{T'}(x)$ for x in X and $A_T(t)=A_{T'}(t)$ for $0\leq t\leq T$.

Hence, the following functions are well-defined: the function $v(x)$, x in X, defined by $v(x)=v_T(x)$ for x in X, and the function $A(t)$, $0\leq t<\infty$, defined by $A(t)=A_T(t)$ for $0\leq t\leq T$. We define a function $a(t)$, $0\leq t<\infty$, by $a(t)=A'(t)$ if the derivative $A'(t)$ exists and $a(t)=0$ otherwise. Hence, $a(t)=A'(t)=a_T(t)$ a.e. for $0\leq t\leq T$, and thus $V_T(\mathbf{x})=\int_0^T a_T(t)v_T(\mathbf{x}(t))\,dt=\int_0^T a(t)v(\mathbf{x}(t))\,dt$ for $\mathbf{x}$ in R_T. Finally, we define a function

$V(\mathbf{x})$, $\mathbf{x}$ in R_f, by $V(\mathbf{x}) = V_T(\mathbf{x})$ for $\mathbf{x}$ in R_T. Since $v(\mathrm{o}) = 0$, this definition implies that, $V(\mathbf{x}) = \lim_{T\to\infty} V_T(\mathbf{x}) = \lim_{T\to\infty} \int_0^T a(t)v(\mathbf{x}(t))\,dt$, for any $\mathbf{x}$ in R_f.

Theorem 2 implies that $V(\mathbf{x})$ is a value function for $(R_T, \succsim)$ for any $T \geq 1$ and that the functions $v(x)$, $A(t)$, $a(t)$, $V(\mathbf{x})$ have properties (a)-(d). Moreover, $V(\mathbf{x})$ is a value function for $(R_f, \succsim)$. For consider any $\mathbf{x}, \mathbf{y}$ in R_f. Then, $\mathbf{x}$ is in R_T and $\mathbf{y}$ is in $R_{T'}$ for some $T, T' \geq 1$. Assume that $T \leq T'$. Then, $\mathbf{x}$ and $\mathbf{y}$ are in $R_{T'}$ and thus can be compared by the normalized function $V_{T'}(\mathbf{x})$. Thus, $\mathbf{x} \succsim \mathbf{y}$ if and only if $V_{T'}(\mathbf{x}) \geq V_{T'}(\mathbf{y})$ if and only if $V(\mathbf{x}) \geq V(\mathbf{y})$ since $V(\mathbf{x}) = V_{T'}(\mathbf{x})$ and $V(\mathbf{y}) = V_{T'}(\mathbf{y})$.

To show the converse implications, assume that there exist functions $v(x)$, $A(t)$, $a(t)$, and $V(\mathbf{x})$ as described in Theorem 3. Then, Theorem 2 implies that for any $T > 0$ the preference relation $\succsim$ satisfies conditions (A)–(E) with regard to the sets R_T and S_T.

To show that each function $A(t)$ and $v(x)$ is unique up to a positive multiple, consider two value functions $V(\mathbf{x}) = \lim_{T\to\infty} \int_0^T a(t)v(\mathbf{x}(t))\,dt$ and $\hat{V}(\mathbf{x}) = \lim_{T\to\infty} \int_0^T \hat{a}(t)\hat{v}(\mathbf{x}(t))\,dt$ as described in Theorem 3. In particular, $v(\mathrm{o}) = \hat{v}(\mathrm{o}) = 0$ and $A(0) = \hat{A}(0) = 0$.

For any $T > 0$, the functions $V(\mathbf{x})$, $\hat{V}(\mathbf{x})$ restricted to R_T are $V(\mathbf{x}) = \int_0^T a(t)v(\mathbf{x}(t))dt$ and $\hat{V}(\mathbf{x}) = \int_0^T \hat{a}(t)\hat{v}(\mathbf{x}(t))\,dt$. Thus, Theorem 2 implies that there exist constants $\alpha_T > 0$, β_T, and $\gamma_T > 0$ such that $\hat{v}(x) = \alpha_T v(x) + \beta_T$, x in X, and $\hat{A}(t) = \gamma_T A(t)$, $0 \leq t \leq T$. Then, $\beta_T = 0$ since $v(\mathrm{o}) = \hat{v}(\mathrm{o}) = 0$. Moreover, α_T and γ_T are independent of $T > 0$. The reason is that for $T \leq T'$: $\alpha_T v(x) = \hat{v}(x) = \alpha_{T'} v(x)$, x in X and $\gamma_T A(t) = \hat{A}(t) = \gamma_{T'} A(t)$, $0 \leq t \leq T$. Since $v(x^+) \neq 0$, $A(T) \neq 0$, it follows that $\alpha_T = \alpha_{T'}$ and $\gamma_T = \gamma_{T'}$.

Proof of Lemma 2. To show (a), consider an $\mathbf{x}$ in R_f. Then, there is a $T \geq 1$ such that $\mathbf{x}_{(t,\infty)} = \mathbf{o}_{(t,\infty)}$ for $t \geq T$. But $a \prec_X b \prec_X c$ implies $(\mathbf{a}_{[0,1]}, \mathbf{o}) \prec (\mathbf{b}_{[0,1]}, \mathbf{o}) \prec (\mathbf{c}_{[0,1]}, \mathbf{o})$ by the definition of $\succsim_X$, and thus $\mathbf{x}$ satisfies Definition 5.

For (b), consider two outcome-streams $\mathbf{x}$, $\mathbf{y}$ in R_∞ such that $\mathbf{y}$ is in $R_\succsim$. If there exists a time $U > 0$ such that $\mathbf{x}(t) = \mathbf{y}(t)$ for any $t > U$, then $\mathbf{x}_{(t,\infty)} = \mathbf{y}_{(t,\infty)}$ for $t \geq U$, and thus $\mathbf{y}$ satisfies Definition 5 with the horizon T replaced by the horizon $\max\{T, U\}$.

Lemma A7. Suppose that the preference relation $\succsim$ in an outcome-stream space $(R_\infty, \succsim)$ satisfies conditions (B), (D) on the set $R_\succsim$ and satisfies conditions (A), (C) on each set

S_T, $T > 0$. Then, for any outcome-stream $\mathbf{x}$ in $R_{\succsim}$, any non-point, bounded interval $\langle\alpha, \beta\rangle$, and any outcomes $a \prec_X b \prec_X c$, there exists a time $T \geq \beta$ such that:

$$(\mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{o}) \prec (\mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o}, \mathbf{x}_{(t,\infty)}) \prec (\mathbf{c}_{\langle\alpha,\beta\rangle}, \mathbf{o}), \; t \geq T$$

Proof. It suffices to show that for any non-point, bounded, disjoint intervals $\langle\alpha, \beta\rangle$, $\langle\alpha', \beta'\rangle$, if an outcome-stream in $R_{\succsim}$ satisfies the comparability condition with respect to $\langle\alpha', \beta'\rangle$, then it satisfies the comparability condition with respect to $\langle\alpha, \beta\rangle$. This result implies the lemma for the case that [0, 1] and $\langle\alpha, \beta\rangle$ are not disjoint since we can introduce a third interval that is disjoint from [0, 1] and from $\langle\alpha, \beta\rangle$ and then use the result twice. We will consider only the left-hand strict preferences since the argument for the right-hand strict preferences is similar.

Consider an $\mathbf{x}$ in $R_{\succsim}$ that satisfies the comparability condition with respect to a non-point, bounded interval $\langle\alpha', \beta'\rangle$; for example, for any outcomes $b^- \prec_X b$ there exists a $T \geq \beta'$ such that: (i) $(\mathbf{b}^-_{\langle\alpha',\beta'\rangle}, \mathbf{o}) \prec (\mathbf{b}_{\langle\alpha',\beta'\rangle}, \mathbf{o}, \mathbf{x}_{(t,\infty)})$ for $t \geq T$. Lemma 2 states that any outcome-stream in R_∞ that is in R_f or that equals $\mathbf{x}$ after a finite time is in $R_{\succsim}$.

Suppose that $\langle\alpha, \beta\rangle$ is a non-point, bounded interval that is disjoint from $\langle\alpha', \beta'\rangle$ and that a is an outcome such that $a \prec_X b$. Lemma A3 implies that there exists an outcome $b^- \prec_X b$ such that: (ii) $(\mathbf{b}_{\langle\alpha',\beta'\rangle}, \mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{o}) \prec (\mathbf{b}^-_{\langle\alpha',\beta'\rangle}, \mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o})$.

By condition (D), (i) implies: (iii) $(\mathbf{b}^-_{\langle\alpha',\beta'\rangle}, \mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o}) \prec (\mathbf{b}_{\langle\alpha',\beta'\rangle}, \mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o}, \mathbf{x}_{(t,\infty)})$, $t \geq \max\{T, \beta\}$. Then, (ii) and (iii) imply $(\mathbf{b}_{\langle\alpha',\beta'\rangle}, \mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{o}) \prec (\mathbf{b}_{\langle\alpha',\beta'\rangle}, \mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o}, \mathbf{x}_{(t,\infty)})$ by transitivity, and condition (D) implies $(\mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{o}) \prec (\mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o}, \mathbf{x}_{(t,\infty)})$, $t \geq \max\{T, \beta\}$.

Lemma A8. Suppose that a non-negative function $a(t)$ on $[0, \infty)$ is Lebesgue integrable on each interval $[0,T]$, $T > 0$, and that the indefinite integral $A(t) = \int_0^t a(s)\,ds$ is strictly increasing on $[0, \infty)$. Suppose also that a function $v(x)$ defined on X is continuous and weakly increasing. Then, for any $\mathbf{x}, \mathbf{y}$ in R_∞ and any $T > 0$:

(a) If $\int_0^T a(t)v(\mathbf{x}(t))dt < \int_0^T a(t)v(\mathbf{y}(t))dt$, then $v(\mathbf{x}(t)) < v(\mathbf{y}(t))$ on a non-point interval in $[0,T]$.

(b) If $v(\mathbf{x}(t)) < v(\mathbf{y}(t))$ on a set of positive measure in $[0,T]$, then for any $\varepsilon > 0$ there exists a non-point interval $\langle\alpha, \beta\rangle$ in $[0,T]$ and outcomes a, b such that:

(i) $v(\mathbf{x}(t)) \le v(a) < v(b) \le v(\mathbf{y}(t))$ for t in $\langle\alpha, \beta\rangle$.

(ii) $0 \le \int_\alpha^\beta a(t)v(b)dt - \int_\alpha^\beta a(t)v(\mathbf{x}(t))dt < \varepsilon$ and $0 \le \int_\alpha^\beta a(t)v(\mathbf{y}(t))dt - \int_\alpha^\beta a(t)v(a)dt < \varepsilon$.

Proof. Assume that $\int_0^T a(t)v(\mathbf{x}(t))dt < \int_0^T a(t)v(\mathbf{y}(t))dt$. Then, $v(\mathbf{x}(t)) < v(\mathbf{y}(t))$ on a set E in $[0,T]$ of positive measure since otherwise $v(\mathbf{x}(t)) \ge v(\mathbf{y}(t))$ a.e. in $[0,T]$ which implies $\int_0^T a(t)v(\mathbf{x}(t))dt \ge \int_0^T a(t)v(\mathbf{y}(t))dt$ by Lemma A6. Since $\mathbf{x}, \mathbf{y}$ are continuous a.e., there is a t_0 in E such that the function $v(\mathbf{y}(t)) - v(\mathbf{x}(t))$ is continuous at t_0. Thus, there is a non-point interval $\langle\alpha, \beta\rangle$ in $[0,T]$ such that $v(\mathbf{y}(t)) - v(\mathbf{x}(t)) > 0$ for t in $\langle\alpha, \beta\rangle$.

For (b), assume that $v(\mathbf{x}(t)) < v(\mathbf{y}(t))$ on a set in $[0,T]$ of positive measure. By a slight extension of the above argument, there is an amount $\delta > 0$ and a non-point interval $\langle\alpha, \beta\rangle$ in $[0,T]$ such that $v(\mathbf{y}(t)) - v(\mathbf{x}(t)) > \delta$ for t in $\langle\alpha, \beta\rangle$. Define $v_1 = \inf\{v(\mathbf{x}(t)) : t \in \langle\alpha,\beta\rangle\}$ $v_2 = \sup\{v(\mathbf{x}(t)) : t \in \langle\alpha,\beta\rangle\}$, $v_3 = \inf\{v(\mathbf{y}(t)) : t \in \langle\alpha,\beta\rangle\}$, and $v_1 = \sup\{v(\mathbf{y}(t)) : t \in \langle\alpha,\beta\rangle\}$. Then, $v_1 \le v_2 < v_3 \le v_4$ where $v_3 - v_2 \ge \delta$.

The range of the function $v(x)$ is an interval I by Lemma 1. The functions $v(\mathbf{x}(t))$, $v(\mathbf{y}(t))$ restricted to $[0,T]$ have bounds in I since $\mathbf{x}, \mathbf{y}$ are in R_∞ and the function $v(x)$ is weakly increasing. Thus, the amounts $v_1, \dots, v_4$ are in I, and thus there are outcomes $a_1, \dots, a_4$ in X such that $v_1 = v(a_1)$, etc. Hence, part (i) is established.

For part (ii), we construct a subinterval $\langle\gamma, \delta\rangle$ of $\langle\alpha, \beta\rangle$ that is sufficiently small. But $\int_\gamma^\delta a(t)v(a_3)dt - \int_\gamma^\delta a(t)v(\mathbf{x}(t))dt \le \left(A(\delta) - A(\gamma)\right)\left(v(a_3) - v(a_1)\right)$ for any subinterval $\langle\gamma, \delta\rangle$, and a similar inequality is true for $\mathbf{y}$. The function $A(t)$ is strictly increasing and continuous, and thus we can choose $\gamma < \delta$ such that the inequalities in (ii) are true.

Proof of Theorem 4. To show the forward implications, assume the stated conditions. Then, by Lemma 2, R_f is a subset of $R_\succsim$, and thus the space $(R_f, \succsim)$ satisfies the conditions of Theorem 3. Hence, there exist functions $v(x)$, $a(t)$, $A(t)$, and $V(\mathbf{x})$ (as in (3)) that have the properties (a)-(d) in Theorem 3 and such that $V(\mathbf{x})$ represents $(R_f, \succsim)$.

To show that $V(\mathbf{x})$ converges for any $\mathbf{x}$ in $R_\succsim$, it suffices to show that for any $\varepsilon > 0$ there exists a $T > 0$ such that $V((\mathbf{x}_{[0,t]}, \mathbf{o})) - V((\mathbf{x}_{[0,s]}, \mathbf{o})) < \varepsilon$ for $s,\ t \ge T$. Then, also $V((\mathbf{x}_{[0,s]}, \mathbf{o})) - V((\mathbf{x}_{[0,t]}, \mathbf{o})) < \varepsilon$ for $s, t \ge T$. Hence, $\{V((\mathbf{x}_{[0,n]}, \mathbf{o}))\}_{n=1}^\infty$ is a Cauchy sequence and thus has a limit V. The inequalities then imply that $\lim_{t\to\infty} V((\mathbf{x}_{[0,t]}, \mathbf{o})) = V$.

For $\mathbf{x}$ in $R_{\succsim}$ and $\varepsilon > 0$, choose outcomes $a \prec_X b \prec_X c$ with $A(1)(v(c)-v(a)) < \varepsilon$. By Lemma A7, there exists a time $T \geq 1$ such that $(\mathbf{a}_{[0,1]}, \mathbf{o}) \prec (\mathbf{b}_{[0,1]}, \mathbf{o}, \mathbf{x}_{(s,\infty)})$ and $(\mathbf{b}_{[0,1]}, \mathbf{o}, \mathbf{x}_{(t,\infty)}) \prec (\mathbf{c}_{[0,1]}, \mathbf{o})$ for $s, t \geq T$. Therefore, $(\mathbf{a}_{[0,1]}, \mathbf{x}_{(1,s]}, \mathbf{o}) \prec (\mathbf{b}_{[0,1]}, \mathbf{x}_{(1,\infty)})$ and $(\mathbf{b}_{[0,1]}, \mathbf{x}_{(1,\infty)}) \prec (\mathbf{c}_{[0,1]}, \mathbf{x}_{(1,t]}, \mathbf{o})$ by condition (D). Thus, transitivity implies that $(\mathbf{a}_{[0,1]}, \mathbf{x}_{(1,s]}, \mathbf{o}) \prec (\mathbf{c}_{[0,1]}, \mathbf{x}_{(1,t]}, \mathbf{o})$. Hence, $V((\mathbf{a}_{[0,1]}, \mathbf{x}_{(1,s]}, \mathbf{o})) < V((\mathbf{c}_{[0,1]}, \mathbf{x}_{(1,t]}, \mathbf{o}))$ since the function $V(\mathbf{x})$ represents $\succsim$ on R_f. Adding $V(\mathbf{x}_{[0,1]})$ to both sides, we obtain, $A(1)v(a) + V((\mathbf{x}_{[0,s]}, \mathbf{o})) < A(1)v(c) + V((\mathbf{x}_{[0,t]}, \mathbf{o}))$. Thus, $V((\mathbf{x}_{[0,s]}, \mathbf{o})) - V((\mathbf{x}_{[0,t]}, \mathbf{o})) <$ $A(1)\,(v(c)-v(a)) < \varepsilon$ for any $s, t \geq T$.

Next, we show that $V(\mathbf{x})$ represents the space $(R_{\succsim}, \succsim)$. First, suppose that an $\mathbf{x}$ in $R_{\succsim}$ is upper extremal in the sense that $\mathbf{x} \succsim \mathbf{y}$ for any $\mathbf{y}$ in $R_{\succsim}$. Then, $V(\mathbf{x}) \geq V(\mathbf{y})$ for any $\mathbf{y}$ in $R_{\succsim}$. For assume that $V(\mathbf{y}) > V(\mathbf{x})$. Then, $V((\mathbf{y}_{[0,T]}, \mathbf{o})) > V((\mathbf{x}_{[0,T]}, \mathbf{o}))$ for some $T > 0$, and thus $v(\mathbf{y}(t)) > v(\mathbf{x}(t))$ on a nonpoint interval $\langle\alpha, \beta\rangle$ in $[0,T]$ by Lemma A8. Hence, $\mathbf{y}(t) \succ_X \mathbf{x}(t)$ for t in $\langle\alpha, \beta\rangle$. But the outcome-stream $(\mathbf{y}_{\langle\alpha,\beta\rangle}, \mathbf{x})$ is in $R_{\succsim}$ by Lemma 2, and $(\mathbf{y}_{\langle\alpha,\beta\rangle}, \mathbf{x}) \succ \mathbf{x}$ by condition (A). A similar argument can be given for the case in which an outcome-stream $\mathbf{x}$ in $R_{\succsim}$ is lower extremal.

Next, suppose that an $\mathbf{x}$ in $R_{\succsim}$ is non-extremal, i.e., there exist $\mathbf{y}^-, \mathbf{y}^+$ in $R_{\succsim}$ such that $\mathbf{y}^- \prec \mathbf{x} \prec \mathbf{y}^+$. In this case, we will show that for some $T > 0$ there exists an outcome-stream in S_T that is indifferent to $\mathbf{x}$. The argument is parallel to that in Theorem 3; in particular, the properties (i)–(iii) below correspond to (i)–(iii) in the proof of Theorem 3.

(i) For any non-extremal outcome-stream $\mathbf{x}$ in $R_{\succsim}$ and any $\varepsilon > 0$, there exist a time $T > 0$ and an outcome-stream $\mathbf{w}$ in S_T such that $\mathbf{w} \sim \mathbf{x}$ and $|V(\mathbf{w}) - V(\mathbf{x})| < \varepsilon$.

To prove (i), it suffices to show that for any $\varepsilon > 0$ there exists a time $T > 0$ and step outcome-streams $\mathbf{w}^-, \mathbf{w}^+$ in S_T such that $\mathbf{w}^- \precsim \mathbf{x} \precsim \mathbf{w}^+$ and $|V(\mathbf{w}^-) - V(\mathbf{x})| < \varepsilon$, $|V(\mathbf{w}^+) - V(\mathbf{x})| < \varepsilon$. For we can then use the proof of (i) in Theorem 3 to obtain an outcome-stream of the form, $\mathbf{w} = \lambda\,\mathbf{w}^+ + (1-\lambda)\,\mathbf{w}^-$, $0 \leq \lambda \leq 1$, that satisfies (i).

To show the existence of a step outcome-stream $\mathbf{w}^+$ as described, we will construct a sequence of the form: $\mathbf{x}$, $(\mathbf{x}_{[0,T]}, \mathbf{o})$, $(\mathbf{x}_{[0,T]-\langle\alpha,\beta\rangle}, \mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o})$, $\mathbf{w}^+$. The arguments to show the existence of $\mathbf{w}^-$ are similar. Assume that an amount $\varepsilon > 0$ is given.

For the first step, observe that since the integral $V(\mathbf{x})$ converges there exists a time $T_1 > 0$ such that $|V(\mathbf{x}) - V((\mathbf{x}_{[0,t]}, \mathbf{o}))| < \varepsilon/3$ for any $t \geq T_1$.

For the second step, observe that since $\mathbf{x}$ is non-extremal there exists an outcome-stream $\mathbf{y} \succ \mathbf{x}$ in $R_{\succsim}$. Condition (A) implies that $\mathbf{y}(t) \succ_X \mathbf{x}(t)$ on a set of positive measure. Thus, there is a time $T_2 > 0$ such that $\mathbf{y}(t) \succ_X \mathbf{x}(t)$ on a set E of positive measure in the interval $[0, T_2]$. Thus, $v(\mathbf{y}(t)) > v(\mathbf{x}(t))$ on E. Then, Lemma A8b implies that there is a non-point interval $\langle\alpha, \beta\rangle$ in $[0, T_2]$ and outcomes a, b such that: (1) $v(\mathbf{x}(t)) \leq v(a) < v(b)$ for t in $\langle\alpha, \beta\rangle$, and (2) $0 \leq \int_\alpha^\beta a(t)v(b)dt - \int_\alpha^\beta a(t)v(\mathbf{x}(t))dt < \varepsilon/3$.

The inequalities (1) imply by condition (A) that $\mathbf{x} \precsim (\mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{x})$, and they imply by Lemma A7 that there exists a $T_3 \geq \beta$ such that $(\mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{o}, \mathbf{x}_{(t,\infty)}) \prec (\mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o})$ for $t \geq T_3$. Condition (D) then implies that $(\mathbf{a}_{\langle\alpha,\beta\rangle}, \mathbf{x}) \prec (\mathbf{x}_{[0,t]-\langle\alpha,\beta\rangle}, \mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o})$ for $t \geq T_3$. Hence, $\mathbf{x} \prec (\mathbf{x}_{[0,t]-\langle\alpha,\beta\rangle}, \mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o})$ for $t \geq T_3$ by transitivity.

Next, the inequalities (2) imply that $|V((\mathbf{x}_{[0,t]-\langle\alpha,\beta\rangle}, \mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o})) - V((\mathbf{x}_{[0,t]}, \mathbf{o}))| = |\int_\alpha^\beta a(t)v(b)dt - \int_\alpha^\beta a(t)v(\mathbf{x}(t))dt| < \varepsilon/3$ for $t \geq T_3$.

For the third step, define $T = \max\{T_1, T_2, T_3\}$. By property (i) in the proof of Theorem 3, there exists a step outcome-stream $\mathbf{w}^+$ in S_T such that $\mathbf{w}^+ \sim (\mathbf{x}_{[0,t]-\langle\alpha,\beta\rangle}, \mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o})$ and $|V(\mathbf{w}^+) - V((\mathbf{x}_{[0,t]-\langle\alpha,\beta\rangle}, \mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o}))| < \varepsilon/3$.

To conclude, $\mathbf{x} \prec (\mathbf{x}_{[0,t]-\langle\alpha,\beta\rangle}, \mathbf{b}_{\langle\alpha,\beta\rangle}, \mathbf{o}) \sim \mathbf{w}^+$ implies $\mathbf{x} \prec \mathbf{w}^+$ by transitivity. And by adding the above three inequalities, it follows that $|V(\mathbf{w}^+) - V(\mathbf{x})| < \varepsilon$.

The arguments for (ii) and (iii) below are the same as those for (ii), (iii) in Theorem 3. Moreover, the argument that (ii), (iii) suffice to show that $V(\mathbf{x})$ is a value function for the non-extremal outcome-streams in $(R_{\succsim}, \succsim)$ is the same as in the proof of Theorem 3.

(ii) $V(\mathbf{x}) < V(\mathbf{y})$ implies $\mathbf{x} \prec \mathbf{y}$ for any non-extremal $\mathbf{x}$, $\mathbf{y}$ in $R_{\succsim}$.

(iii) $V(\mathbf{x}) = V(\mathbf{y})$ implies $\mathbf{x} \sim \mathbf{y}$ for any non-extremal $\mathbf{x}$, $\mathbf{y}$ in $R_{\succsim}$.

To show the converse implications, assume that $(R_\infty, \succsim)$ is an outcome-stream space, that $V(\mathbf{x})$ is a function of the form (4) that is well-defined on the set $R_{\succsim}$ and is a value function for the space $(R_{\succsim}, \succsim)$, and that the functions $a(t)$, $A(t)$, $v(x)$, and $V(\mathbf{x})$ satisfy the properties in (b). Then, it is straightforward to verify that $\succsim$ satisfies the conditions (A), (B), and (D) on the set $R_{\succsim}$. By Theorem 3, $\succsim$ satisfies condition (E) on each set S_T, $T > 0$, and one can use an argument similar to that in Theorem 3 to show that $\succsim$ satisfies condition (C) on each pair of sets $R_{\succsim}$, S_T, $T > 0$.

By Theorem 3, each of the functions $A(t)$, $v(x)$ is unique up to a positive multiple.

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