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Capabilities and Equality of Health II: Capabilities as Options

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Capabilities and Equality of Health II: Capabilities as Options

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Abstract

The concept of capabilities, introduced originally by Sen, has inspired many researchers but has not found any simple formal representation which might be instrumental in the construction of a comprehensive theory of equality.

In a previous paper (Keiding, 2005), we investigated whether preferences over capabilities as sets of functionings can be rationalized by maximization of a suitable utility function over the set of functionings. Such a rationalization turned out to be possible only in cases which must be considered exceptional and which do not allow for interesting applications of the capability approach to questions of health or equality.

In the present paper we extend the notion of rationalizing orderings of capabilities to a dynamical context, in the sense that the utility function is not yet revealed to the individual at the time when the capabilities are ordered. It turns out that orderings which are in accordance with such probabilistic utility assignments can be characterized by a smaller set of the axioms previously considered.

Keywords: Capabilities, characteristics, equality of health.

JEL classification: D63, I10

1. Introduction

In recent years, the capability approach suggested by Sen (1980, 1985) has been applied in several different fields of economics, including research in poverty and inequality. The capability of an individual is defined as a set of functionings, each of which describes a way of transforming an intial given situation to a final outcome. A

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recent approach to the measurement of standard of living using functionings as a basic concept is Gaertner and Xu (2005).

Following the by now well-established approach to capabilities, we view the latter as *sets of functionings.* When it comes to specifying the nature of these functionings, the literature is less explicit, and this is of course unfortunate, as a more detailed explanation of what functionings are would be helpful in establishing possible properties of capabilities. In particular, the question of whether there is a sufficient supply of capabilities (that is whether capabilities, considered as sets, can take sufficiently many different forms) matters for the problem of rationalizing capabilities, finding out whether or not any well-behaved ordering of capabilities can be derived from an ordering of functionings. The answer to the rationalizability problem, in its turn, matters for deciding what is the primitive concept, functionings or capabilities. If functionings are the primitives, then an ordering of capabilities should have an explanation in terms of an underlying ordering of functionings (so that one capability is better than another if its best functioning in the first set is better than the best functioning in the other set). If orderings of capabilities cannot be explained in this way, then capabilities have intrinsic value over and above the constituent functionings; this may well be the case but it means that the theory of capabilities is left with a white spot still unexplored.

In a first paper on this problem (Keiding, 2005), we considered axioms on sets of capabilities and orderings on this set such that the ordering could be rationalized as maximization of a utility function defined on functionings. The results obtained were largely negative in the sense that many reasonable orderings could not be rationalized in this way, or, otherwise put, some of the axioms were rather more restrictive than what is desirable. In the present paper, we therefore modify the notion of rationalizing an ordering, allowing for situations where the utility on functionings becomes known only on a later stage, after the ranking of capabilities has taken place. This seems to fit well with (at least some of the) literature on capabilities, where the latter are seen as expressing the possibilities of later choice in life.

The paper is structured as follows. In the section 2, we introduce the notion of dynamic rationalization, understood as a ranking which is consistent with expected utility over the capability set for some given probability measure on the set of (linear) utility assignments on functionings. Section 3 gives the characterization result, and Section 4 contains a short discussion of the results obtained.

2. Capabilities as sets of functionings; the options approach

In this section, we develop the formalism to be used throughout the paper. A *functioning* is an element y of $\mathbb{R}^L$, L-dimensional Euclidean space. In accordance with the intuitive concept of functioning as a certain place to be filled in society, comprising productive as well as other activities, we shall allow for negative as well as positive coordinates in y. This makes functionings formally identical to net trade vectors in economic theory, but their interpretation may be different.

A *capability* C is a set of functionings, so C is a subset of $\mathbb{R}^L$. Keeping in mind the analogy with production sets, it seems reasonable to assume that capabilities are closed, convex, and comprehensive in the sense that if $x \in C$ and $y \in \mathbb{R}^L$ satisfies $y_h \leq x_h$ for $h = 1, \ldots, L$, then $y \in C$. Clearly, capabilities may have other formal properties as well, derived from the role which they play in application. To approach this problem, we study families $\mathcal{C}$, considered as possible sets of all potentially feasible capabilities.

For our subsequent reasoning, it is convenient to assume that $\mathcal{C}$ is rich enough to contain some distinguished sets, in particular $\mathcal{C}$ contains the negative orthant $\mathbb{R}_-$ (corresponding to the functionings which can produce no net output of anything) and permits the operation of (Minkowski) weighted averages, i.e. if $C, C' \in \mathcal{C}$ and $\lambda \in [0,1]$, then the set

$$\lambda C + (1-\lambda)C' = \{y \in \mathbb{R}^L_+ \mid y = \lambda x + (1-\lambda)x',\, x \in C,\, x' \in C'\}$$

belongs to $\mathcal{C}$ as well. We shall say that a family $\mathcal{C}$ with these properties is regular.

In our present setup, the capabilities approach to QALY measurement would imply that there is an ordering (to be more precise, a complete preorder, that is a binary relation which is reflexive, transitive and complete) $\succsim$ on the sets $C \in \mathcal{C}$; we let $\succ$ and $\sim$ denote the associated strict order and indifference, respectively. For completeness of exposition, we state this as a first axiom.

In this paper, we shall be concerned with *rationalizability* of orderings $\succsim$ on a set of capabilities, whereby we understand that the ordering of capabilities can be derived from an underlying ordering the functionings. We distinguish between two different types of rationalizability: The ordering $\succsim$ of $\mathcal{C}$ is simply (or statically) rationalizable by the linear utility function $u : \mathbb{R}^L \to \mathbb{R}$ if

$$C \succsim C' \Leftrightarrow \max_C u > \max_{C'} u.$$

This notion of rationalizability was investigated in Keiding (2005), where it was shown that static rationalizability can be obtained only upon rather restrictive conditions on the family $\mathcal{C}$. Here, we extend the notion of rationalizability to allow for a situation which may be considered as closer to the intuitive content of capabilities and functionings: An ordering $\succsim$ on $\mathcal{C}$ is dynamically rationalizable if there is a *random* linear utility, that is a (measurable) map $\tilde{u} : X \to (\mathbb{R}^L)^*$, where $(X, \mathbb{X}, P)$ is a probability space, such that

$$C \succsim C' \Leftrightarrow \mathsf{E}_P\,[\max_C \tilde{u}] > \mathsf{E}_P\,[\max_{C'} \tilde{u}].$$

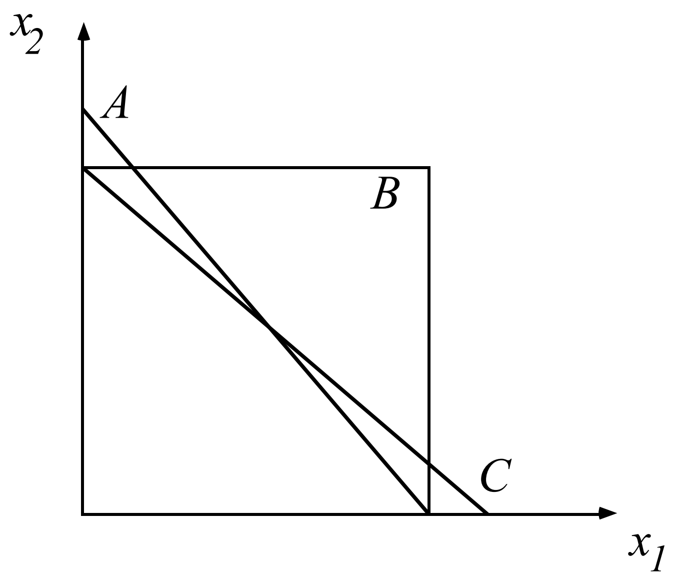

Figure 1

In the interpretation, we assume that the individual may have access to the functionings of given capability set, but that the preferences over functionings are revealed only at a later point of time. Thus, at the moment where the capability is evaluated, only the probabilities of the possible utility functions over functionings are known, not the actual utility functions. Therefore, evaluation is performed by taking averages over the value of the capability given the utility function drawn.

To see that the two forms of rationalizing orderings are effectively different, consider the simple example in Figure 1, where the family $\mathcal{C}$ of capabilities contain the triangles A and C and the rectangle B. Assume now that $A \sim C \succsim B$. This ordering cannot be rationalized in the static way, since a linear form on $\mathbb{R}^2$ which attains the same maximum on A and C must belong to $\{u \in \mathbb{R}^2 \mid u_1 \leq 0 \text{ or } u_2 \leq 0\}$, and for each such u, this maximum (which is $= 0$) is equal to the maximum on B.

On the other hand, it is not difficult to define a utility process $\tilde{u}$ such that the ordering is rationalized in the dynamic way; indeed, suppose that

$$\tilde{u} = \begin{cases} (1,0) & \text{with probability 1/2} \\ (0,1) & \text{with probability 1/2} \end{cases}.$$

Then $\mathsf{E}\left[\max_A \tilde{u}\right] = \mathsf{E}\left[\max_C \tilde{u}\right] > \mathsf{E}\left[\max_B \tilde{u}\right]$, so that the ordering $\succsim$ is indeed recovered by probabilistic utility maximization.

We have a characterization of dynamically rationalized orderings of families of capabilities which uses support functions of convex sets. For $C \subset R^L$ a capability, the support function of C is the map $\delta^*(\cdot \mid C) : \triangle \to \mathbb{R}_+$ defined by

$$\delta^*(\cdot \mid C) = \sup \{u \cdot x \mid x \in C\};$$

the support function is convex, and it characterizes the capability C fully, in the sense that if $g : \triangle \to \mathbb{R}_+$ is a convex function, then $g(u) = \delta^*(u \mid C^g)$, all $u \in \triangle$, where

$$C^g = \cap_{u \in \triangle} \{x \in \mathbb{R} \mid u \cdot x \leq g(u)\}.$$

THEOREM 1. *Let $\mathcal{C}$ be a family of capabilities ordered by $\precsim$, and suppose that $\precsim$ is dynamically rationalized. Then there is a probability measure P on $\triangle$ such that*

$$C \precsim C' \Leftrightarrow \int_\triangle \delta^*(u \mid C)\, dP(u) \geq \int_\triangle \delta^*(u \mid C)\, dP(u).$$

PROOF: This follows immediately from the definition of dynamic rationalizability, since the random utility $\tilde{u} : X \to \triangle$ gives rise to a probability measure P on $\triangle$ defined by

$$P(A) = \tilde{P}\left(\tilde{u}^{-1}(A)\right)$$

for all Borel sets U of $\triangle$, and clearly $\mathsf{E}_{\tilde{P}}\left[\max_C \tilde{u}\right] = \int_\triangle \delta^*(u \mid C)\, dP(u)$. □

If $\mathcal{C}$ is a set of capabilities, then an ordering $\precsim$ of capabilities induces an ordering $\precsim^*$ of their support functions in the obvious way. It is seen from Theorem 1 that $(\mathcal{C}, \precsim)$ is dynamically rationalizable if and only if $\precsim^*$ is the ordering derived from integration with respect to a suitable probability measure on $\triangle$.

3. Properties of rationalizable orderings of capabilities

In this section, we present a characterization of dynamically rationalizable orderings of families of capabilities. The axioms to be used are the Axioms 1 – 3 from Keiding (2005), which for completeness are listed below:

AXIOM 1. *The preference relation $\precsim$ on the family $\mathcal{C}$ is a complete preorder, and it is continuous in the sense that $\{C' \mid C \precsim C'\}$ and $\{C' \mid C' \precsim C\}$ are closed (in the topology on $\mathcal{C}$ induced by the Hausdorff distance) for all $C \in \mathcal{C}$.*

We assume that the indifference relation $\sim$ is stable under averages:

AXIOM 2. *Let (C_1, C_2) and (C_1', C_2') be pairs of elements of $\mathcal{C}$ with $C_1 \sim C_2$, $C_1' \sim C_2'$, and let $\lambda \in [0, 1]$. Then*

$$\lambda C_1 + (1 - \lambda) C_1' \sim \lambda C_2 + (1 - \lambda) C_2'.$$

AXIOM 3. *If $C_1 \subset \operatorname{int} C_2$, then $C_2 \succ C_1$.*

As in our previous discussion of these axioms, we fell reasonably comfortable with the Axioms 1 and 3, whereas Axiom 2, dealing with mixtures of capabilities, is less intuitive. We return to the axioms at the end of the paper.

The following is straightforward:

THEOREM 2. *Let $\mathcal{C}$ be a rich family of capabilities ordered bu $\precsim$, and suppose that $\precsim$ is dynamically rationalized. Then $\precsim$ satisfies Axioms 1 – 3.*

PROOF: To show that Axiom 1 holds, let $(C_n)_{n\in\mathbb{N}}$ be a sequence of capabilities with $C_n \succsim C$ converging to C_0 in the Hausdorff topology. Using Theorem 1 we have that there îs P such that

$$\int_{\triangle} \delta^*(u \mid C_n)\, dP(u) \geq \int_{\triangle} \delta^*(u \mid C_n)\, dP(u)$$

for all $n \in \mathbb{N}$. Since the map $C' \mapsto \delta^*(\cdot \mid C')$ is continuous, we get that

$$\int_{\triangle} \delta^*(u \mid C_0)\, dP(u) \geq \int_{\triangle} \delta^*(u \mid C_0)\, dP(u)$$

or $C_0 \succsim C$, and we conclude that the set $\{C' \mid C' \succsim C\}$ is closed for every $C \in \mathcal{C}$. Closedness of $\{C' \mid C \succsim C'\}$ is established in similar way.

For Axiom 2, let (C_1, C_2) and (C_1', C_2') be pairs of elements of $\mathcal{C}$ with $C_1 \sim C_2$, $C_1' \sim C_2'$, and let $\lambda \in [0, 1]$. Using the definition of support functions, we have that

$$\delta^*(u \mid \lambda C_j + (1-\lambda) \mid C_j') = \lambda\delta^*(u \mid C_j) + (1-\lambda)\delta^*(u \mid C_j'),$$

$j = 1, 2$, for each $u \in \triangle$, so that

$$\int_{\triangle} \delta^*(u \mid \lambda C_j + (1-\lambda) \mid C_j')\, dP(u) = \lambda \int_{\triangle} \delta^*(u \mid C_1)\, dP(u) + (1-\lambda)\int_{\triangle} \delta^*(u \mid C_1')\, dP(u),$$

$j = 1, 2$, we get from repeated use of Theorem 1 that $\lambda C_1 + (1-\lambda)C_1' \sim \lambda C_2 + (1-\lambda)C_2'$.

Axiom 3 follows easily from the fact that $\delta^*(u, C_1) < \delta^*(u, C_2)$ for all $u \in \triangle$. □

To proceed we need a general result about topological vector spaces.

THEOREM 3. *Let $(V, \geq)$ be an ordered topological vector space with positive cone V_+ having nonempty interior, let A be a convex cone contained in V_+, and let $\precsim$ be a continuous total preorder on A. Then the following are equivalent:*

(i) $\precsim$ is monotonic (in the sense that $x, x' \in A$, $x \geq x'$ implies $x \succsim x'$) and satisfies the following condition: If $x, x', y, y' \in A$ with , and $\lambda, \mu \in \mathbb{R}_+$, then

$$[x \succsim y,\ x' \succsim y',\ \lambda x + \mu x' \in A,\ \lambda y + \mu y' \in A] \Rightarrow \lambda x + \mu x' \succsim \lambda y + \mu y'. \qquad (1)$$

(ii) $\precsim$ has a utility representation u, where $u = v_{|A} : A \to \mathbb{R}_+$ is the restriction to A of a positive linear form u on V.

PROOF: (ii)⇒(i): Since for $x, x' \in A$, $x \precsim x'$ if and only if $u(x) \geq u(x')$, and u is the restriction of a positive linear form on V, we have immediately that $\precsim$ is monotonic. The second condition in (ii) follows similarly: If $u(x) \geq u(y)$ and $u(x') \geq u(y')$, then by linearity of u we get that

$$u(\lambda x + \mu x') = \lambda u(x) + \mu u(x'),\ u(\lambda y + \mu y') = \lambda u(y) + \mu u(y'),$$

and the conclusion follows.

(i)⇒(ii): Choose an element x of A with $x \succ 0$ (such an element exists since 0 is minimal for $\precsim$ by monotonicity). Clearly, $\lambda x \succ x$ for $\lambda > 1$ and $x \succ \lambda x$ for $\lambda < 1$. Let $I(x) = \{x' \in A \mid x' \sim x\}$ be the set of elements of A which are equivalent to x in $\precsim$ (that is, such that $x' \precsim x$ and $x \precsim x'$). Using (1) with $y = y' = x$ we get that $I(x)$ is the intersection of A with an affine subset in V; we write this set as $K + \{x\}$, where K is a closed subspace of V.

By monotonicity, the set K does not intersect int V_+, and by the Hahn-Banach theorem (see, e.g., Rudin (1973), Theorem 3.4), there is a continuous linear form v on V such that $v(x) = 0$ for $x \in K$ and $v(x) > 0$ for $x \in$ int V_+. It is easily seen that $u = v_{|A}$ satisfies the conditions in (i). □

Now we may prove a converse of Theorem 2, giving us the desired characterization of dynamically rationalized orderings of capabilities.

THEOREM 4. *Let $\mathcal{C}$ be a family of closed, convex, and comprehensive subsets of $\mathbb{R}^L$, and assume that if $\mathcal{C}$ contains sets C and C', then it contains also $\lambda C + \mu C'$ for all $\lambda, \mu \in \mathbb{R}_+$.*

If $\precsim$ is a an ordering of $\mathcal{C}$ which satisfies Axioms 1 – 3, then there exists a probability space and a random utility on this probability space rationalizing $\precsim$.

PROOF: Let $V = C(\triangle)$ be the set of continuous real functions on $\triangle$, endowed with the topology of uniform convergence. Then V is a topological vector space, and its positve cone (the set of all nonnegative functions on $\triangle$) has nonempty interior (consisting of the strictly positive functions).

We define the set $A = \{\delta^*(\cdot \mid C) \mid C \in \mathcal{C}\}$. Since $\lambda C + \mu C' \in \mathcal{C}$ for all $C, C' \in \mathcal{C}$ and $\lambda, \mu \in \mathbb{R}_+$, and $\delta^*(\cdot \mid \lambda C) = \lambda\delta^*(\cdot \mid C)$, $\delta^*(\cdot \mid \mu C') = \mu\delta^*(\cdot \mid C')$ we have that A is a convex cone.

Define the preorder $\precsim$ on A in the obvious way, that is by

$$\delta^*(\cdot \mid C) \precsim \delta^*(\cdot \mid C') \Leftrightarrow C \precsim C'$$

(since this is basically the "same" ordering, only transferred from sets to their support functions, we have kept the same notation). We leave it to the reader to check that $\precsim$ is a continuous preorder on A (the continuity part was shown in the proof of Theorem 2).

Monotonicity of $\precsim$ defined on A is a straightforward consequence of Axiom 3. Similarly, it is easily seen that (1) follows from Axiom 2. We thus have that all the conditions in (i) of Theorem 3 are fulfilled.

Using now Theorem 3, we get that $\precsim$ has a utility representation U, which is the restriction to A of a positive linear form on $V = C(\triangle)$. By the Riesz representation theorem (see e.g., Rudin (1966), Thm.6.19), a positive linear form on V can be identified with a probability measure P on $\triangle$, so that

$$U(\delta^*(\cdot \mid C) = \int_\triangle \delta^*(u \mid C)\, dP(u),$$

which by Theorem 1 is exactly the expression of dynamic rationalizability. □

4. Concluding remarks

In this second paper on the foundations of capabilities, we have investigated another interpretation of capabilities, namely the options approach, according to which the capability of an individual expresses the possibilities of choosing between several different ways of functioning in society, whereby the emphasis is on possibility of choice, to be exercised when more is known about the advantages and disadvantages of different functionings. In our formalization the idea of choosing after having obtain additional information is reduced to a choice to be performed after the utility function has been revealed. This means that for our initial ordering of capabilities, the utility function on functionings is known only probabilistically, with a given density of utility assignments.

Parallel to what we did in the first paper, we have investigated properties of orderings of capabilities which arise from the above type of probabilistic utility maximization. It turns out that we can characterize such orderings by the first three of the four axioms already considered, combined with the assumption that the set of capabilities is large enough to contain particular sets which we need in order to apply the axioms.As we noticed above, among the three axioms characterizing the dynamically rationalized orderings of capabilities, Axiom 2 on mixtures of capabilities stands out as less intuitive than the two others. First of all, it relies on the notion of a mixture of two capabilities, defined as the totality of all mixtures of functionings from the two capabilities (with fixed coefficients). Once the idea of scalar multiples of functionings and capabilities have been accepted, it might not be overly restrictive to assume that orderings are robust with respect to taking mixtures. On the other hand, the demand that scalar multiples of capabilities should be meaningful does of course put some restrictions on the possible applications.

To interpret the result of this second paper, it is useful to go back to the starting point for the axiomatic approach to ordering capabilities: Why are we at all interested in such orderings? The point here is that in order to use capabilities for measuring inequality, we would like capabilities to be ordered in a more or less objective way, making it possible to decide whether or not an individual is better or worse off getting one capability instead of another; in the absence of such a common scale of preferences we would have to be satisfied with weak measures of inequality such as non-envy or egalitarian-equivalence, which clearly are much less appealing that standard equality measures. Our result shows that the members of a reasonably broad class of orderings on capabilities are uniquely determined by a probability distribution over utility assingments. Assuming this probability distribution given in society, a unique ranking of capabilities will emerge. This is a much better result than what we obtained in the first paper, where orderings related directly to utilities, that is to subjective properties of the individual, and therefore could not reasonably be taken as valid for all individuals in society. With the options approach to capabilities, it does make sense to speak of objective ranking of capabilities as better or worse. In this sense, the results of the paper may be seen as a possibility result opening up for measuring inequality through capabilities. Needless to say, this measurement faces many other problems than the conceptual one considered here, but at least we have moved one step towards the goal.

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