

Discussion Papers
Department of Economics
University of Copenhagen

No. 07-01

Cardinal Scales for Public Health Evaluation

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ISSN: 1601-2461 (online)

Cardinal Scales for Public Health Evaluation*

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December 2006

Abstract

Policy studies often evaluate health for a population by summing the individuals' health as measured by a scale that is ordinal or that depends on risk attitudes. We develop a method using a different type of preferences, called preference intensity or cardinal preferences, to construct scales that measure changes in health. The method is based on a social welfare model that relates preferences between changes in an individual's health to preferences between changes in health for a population.

Keywords: public health evaluation, social welfare, preference intensity, health state

* We thank James K. Hammitt at the Harvard Center for Risk Analysis for making valuable suggestions.

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1. Introduction

Public health studies use a variety of scales and supporting procedures to measure health outcomes for an individual and to measure the effects of a public health policy for a population of individuals. In order to compare policy options, an analyst must decide how to aggregate the scale amounts for the individuals into a scale amount for the population. For example, should she use the sum or should she use a weighted sum, and should she use the scale amounts themselves or should she first transform the scale amounts to a different scale?

Moreover, the analyst may need to use the individual and population health scales to measure changes in health for an individual or for a population. For example, what procedures can she use to report a predicted improvement in the distribution of health over a population as an equivalent improvement that is hypothetical but easier to grasp?

This paper presents models and procedures that address these questions. The models involve a type of preferences usually referred to as preference intensity, cardinal preference, preference difference, or strength-of-preference. We will call such preferences an intensity relation. Unlike a preference relation that compares probability distributions of outcomes or that compares multiattribute outcomes, an intensity relation compares *changes* in outcomes. The next section argues that changes in outcomes can be compared in a public policy context and thus intensity relations have an operational meaning in such a context.

If an intensity relation satisfies certain conditions, it is represented by a function such that one change is preferred to another when the difference in function values for the first change is greater. Like an intensity relation, such a function has various names, e.g., a difference function, a measurable value function, and a cardinal utility function. We will call such a function either a difference function to emphasize that it has the above property or a cardinal scale to emphasize that (at least under certain conditions) it is unique up to a positive linear transformation.

Here, the outcomes are health outcomes. The general model contains: (i) for each member of a population, an intensity relation that compares changes in the person's health outcomes, and (ii) for the population, an intensity relation that compares changes in the health outcomes over the population. Each individual and population intensity relation is represented by a difference function. The model defines a Pareto condition that connects the individual and population

intensity relations, and it establishes that the intensity relations satisfy the condition if and only if a difference function for health over the population is a weighted sum of difference functions for the health outcomes of the population members. Thus, cardinal scale amounts for the individuals are aggregated by means of interpersonal weights into a cardinal scale amount for the population.

This model is an adaptation to the context of public health of social welfare models in Dyer and Sarin (1979) and Harvey (1999) that discuss preferences between changes in outcomes. These social welfare models are parallel to the well-known model due to Harsanyi (1955) that discusses preferences between probability distributions of outcomes.

The first part of this paper presents the health aggregation model outlined above. It also discusses when the interpersonal weights can or cannot be chosen as equal.

The second part presents cardinal models concerning health outcomes for an individual. The health outcomes are of two types: health states, and health-duration pairs. By a health state, we mean the health of a person, measured along one or more dimensions. Public health studies can make different assumptions as to the state's duration and what health states occur afterward. And by a health-duration pair, we mean a health state and as additional information the duration of the health state. Again, public health studies can make different assumptions as to what occurs afterward. Several options are death, optimal health, or a return to normal health.

This paper does not discuss models in which a health outcome contains more than one health state: e.g., a health outcome is a sequence of annual health states, a sequence of health-duration pairs, or a continuous stream of health states. For such health outcomes, Harvey and Østerdal (2006) present ordinal models that can serve as a basis for a cardinal health aggregation model.

In the third part of the paper, we assume that an analyst has used the models and procedures in the first two parts to obtain a difference function that compares changes in the distribution of health over a population. We discuss procedures by which the analyst can use such a function to evaluate a specified change in population health by calculating an indifferent change that can be simply stated. Hence, the models provide a basis and a method for evaluating changes in public health. For this reason, we will refer to the general model as a public health evaluation model.

Most of the results in this paper, as well as in Harvey (1999), are based on work by the Danish mathematician, J.L.W.V. Jensen (1905, 1906). Proofs of results are in the Appendix.

2. Is an Intensity Relation Meaningful?

Pareto (1896) and Fisher (1918) pointed out long ago that the term ‘utility’ had two different meanings: (i) an older, hedonic meaning in utilitarianism and material welfare theory as the degree of pleasure and pain (or welfare, wellbeing, etc.) that a person *experiences* as part of a specified consequence, and (ii) a newer, choice-oriented meaning in consumer theory as the degree to which a consequence satisfies a person’s *preferences*. Pareto gave an illustration in which the two meanings differ: a child may experience better health by taking medicine even though the child much prefers not to take the medicine.

An insufficient recognition of the distinction made by Pareto and Fisher led to a schism between welfare economists (using hedonic utility) and ordinalist economists (using choice-oriented utility); see, e.g., Cooter and Rappoport (1984). Later, a similar misunderstanding was caused by an insufficient recognition that while utility had a hedonic meaning in the expected-utility theory of Bernoulli (1738) it had a choice-oriented meaning in the expected-utility theory of von Neumann and Morgenstern (1944). In recent years, there has been a renewed interest in the hedonic meaning of utility; see, e.g., Broome (1991ab, 2004), Kahneman et al. (1997), and Kahneman (2000).

In this paper, we take the position that in a public health study one can assign either a hedonic meaning or a choice-oriented meaning to comparisons of changes in health.

Explicitly or implicitly, a public health study includes value judgments concerning individual and population health. Following what may be an abuse of language, we will refer to any such judgments as social values. Judgments that compare changes in a person’s health can be made by the person himself, or they can be made by someone else (an expert, an agency, or an idealized person). Usually, the social values will agree with the preferences of the affected individuals.

If an intensity relation defined on changes in an individual’s health is regarded as comparing differences in a hedonic scale of individual wellbeing, then the comparisons are possible in principle, no matter how difficult it may be to construct the scale. And if an intensity relation defined on changes in health over a population is regarded as comparing differences in a hedonic scale of population wellbeing, then comparisons are also possible.

But if an intensity relation defined on changes in the health of an individual or a population is regarded as the preferences of some entity, then an important question is whether the intensity relation is meaningful. Before presenting models that contain intensity relations, we need to address this question.

A classic technique of economic analysis is to specify a set of alternatives to be compared so that it includes not only possible alternatives but also alternatives that are similar but not possible. Here, we argue that in the context of public health, if this economic technique is reasonable, then comparisons of changes in a person's health and comparisons of changes in health distributions are meaningful—even though health changes have a different structure than health outcomes.

First, consider changes in health for an individual. When he faces a decision that affects only himself, his initial position is the same for any alternative, and thus his comparisons of options are comparisons of final outcomes, what are known as ordinal preferences. Comparisons of changes having different initial positions are not needed and in this sense are not meaningful.

But when a society faces a decision that affects many individuals, the initial positions of the individuals will most likely be different; e.g., the initial positions may be the different health outcomes of the individuals for a policy of non-intervention. The preferences of the society will include comparisons of changes in health from these different initial positions. Thus, preferences between changes in an individual's health have the same operational meaning for a society that preferences between final outcomes have for an individual decision maker.

Second, consider changes in health over a population. Since the policy options for the society have the same initial distribution of health, they can be described by their final distributions of health. In this situation, do preferences between changes in health have an operational meaning? We think so, for the following reason. The population can be divided into subpopulations, e.g., age groups, some of which have better distributions of health than others, and thus the society must compare changes for groups whose initial distributions of health are different. By the above argument, comparisons of changes for such a group are meaningful. If changes in distributions of health over the entire population are regarded as similar to changes in distributions of health over a subpopulation, then it follows that comparisons of changes in health for the entire population also are meaningful.

3. A Public Health Evaluation Model

In this section, we construct a public health evaluation model that is based on the social welfare model in Harvey (1999). Suppose that as part of a public health study one has specified a population of potentially affected individuals and a type of health outcomes for the individuals. The individuals will be indexed by $i = 1, \dots, N$; a health outcome for an i -th individual will be denoted by h_i ; and the set of health outcomes will be denoted by H_i . The health outcomes can be of any type: e.g., health states, health-duration pairs, or a type not discussed in this paper.

A distribution of health outcomes over the population will be called a health distribution and will be denoted by $\mathbf{h} = (h_1, \dots, h_N)$. Thus, we distinguish between a health outcome for an individual and a health distribution for a population. We assume that the set of health distributions is the product set $H_1 \times \dots \times H_N$.

A change from a health outcome h_i to a health outcome h'_i for an i -th individual will be denoted by $h_i \rightarrow h'_i$, and a change from a health distribution \mathbf{h} to a health distribution \mathbf{h}' for the population will be denoted by $\mathbf{h} \rightarrow \mathbf{h}'$. An intensity relation will compare changes in health outcomes, or it will compare changes in health distributions. We will refer to the comparisons as preferences even though they may equally well have a hedonic meaning.

As common notation for the individual and population cases, suppose that \succsim denotes an intensity relation for changes $h \rightarrow h'$ in a set H and that $w(h)$ denotes a difference function for \succsim , that is: $h \rightarrow h' \succsim \hat{h} \rightarrow \hat{h}'$ if and only if $w(h') - w(h) \geq w(\hat{h}') - w(\hat{h})$. Here, $h \rightarrow h' \succsim \hat{h} \rightarrow \hat{h}'$ means that the change $h \rightarrow h'$ is preferred to or indifferent to the change $\hat{h} \rightarrow \hat{h}'$.

The social welfare model assumes (as stated for a health context) that for each i -th individual there is an intensity relation that compares changes in health distributions over the population. The intensity relation may reflect social values that focus on the i -th individual or it may reflect the preferences or the hedonic experiences of the i -th individual.

Here, we assume that the intensity relation for an i -th individual depends only on the i -th components h_i in the health distributions. Thus, the intensity relation corresponds to another intensity relation, to be denoted by \succsim_i , that is defined on the set H_i and that compares changes in health outcomes for the i -th individual. We will call an individual intensity relation.

We expect that in most applications, the sets H_i of health outcomes will be chosen as a common set and the intensity relations \succsim_i will be chosen as a common intensity relation. These

assumptions permit the set of possible outcomes be a subset, even a small subset, of the common set and to differ from person to person; for example, younger people and older people may have different sets of possible outcomes.

The social welfare model also assumes that there is an intensity relation that compares changes in health distributions and that reflects social values regarding public health. We will denote such a relation by \succsim_P and call it a population intensity relation.

A difference function for an individual intensity relation \succsim_i will be called an individual difference function and will be denoted by $w_i(h_i)$. And a difference function for a population intensity relation \succsim_P will be called a population difference function and will be denoted by $W(\mathbf{h})$.

Various models have been constructed in which conditions on an intensity relation \succsim imply the existence of a difference function, but except for the model of Scott (1964) in which H is a finite set there is no model in which the conditions are equivalent to the existence of a difference function. The approach in the other models has been to add a non-necessary condition in order to obtain existence. And except for the model of Scott, the conditions that imply the existence of a difference function also imply that it is cardinally unique, that is: If $w(h)$ and $\hat{w}(h)$ are two difference functions, then there exist constants $a > 0$ and b such that $\hat{w}(h) = a w(h) + b$ for h in H . Here, we will use the following non-necessary condition.

Properness condition. The intensity relation \succsim defined on a set H is proper in the sense that it has a difference function $w(h)$ such that the range of $w(h)$ is a non-point interval.

As shown in the Appendix, the properness condition implies that $w(h)$ is cardinally unique. The reason for requiring a non-point range is to exclude the uninteresting case in which any changes $h \rightarrow h'$ in H are indifferent.

There are models in which conditions on a pair (H, \succsim) imply that \succsim is proper; see, e.g., Alt (1936), Debreu (1960), and Pfanzagl (1968). The social welfare model in Harvey (1999) assumes that each relation \succsim_i and \succsim_P satisfies one of these sets of conditions. The model in this section weakens this requirement by assuming that each relation and \succsim_P is proper. This change allows the use of any future conditions that imply that \succsim_i and \succsim_P are proper. In particular, we expect to present in later research a set of conditions that permit certain health scales in the description

of a health state to be categorical variables rather than continuous variables and that permit types of health outcomes other than health states and health-duration pairs.

Next, we consider how a population intensity relation \succsim_P might be connected to individual intensity relations \succsim_i . The condition below provides what seems a natural connection. We will call it a Pareto condition because of its similarity to Pareto conditions on preferences between probability distributions and on preferences between multivariable outcomes.

Pareto condition. For any changes $\mathbf{h} \rightarrow \mathbf{h}'$ and $\hat{\mathbf{h}} \rightarrow \hat{\mathbf{h}}'$ in health distributions, if there is an i -th individual such that $h_j \rightarrow h'_j$ is indifferent to $\hat{h}_j \rightarrow \hat{h}'_j$ for each $j \neq i$, then preferences between $h_i \rightarrow h'_i$ and $\hat{h}_i \rightarrow \hat{h}'_i$ according to the individual intensity relation \succsim_i imply preferences between $\mathbf{h} \rightarrow \mathbf{h}'$ and according to the population intensity relation \succsim_P .

In other words, when society does not need to make tradeoffs between changes in the health of different persons, then preferences between changes in public health agree with preferences between changes in health for the one person who matters. In this sense, the Pareto condition is a requirement of consumer sovereignty.

The result below is based on a result in Harvey (1999). Both results show that if individual and population intensity relations are connected in a simple fashion, then individual and population difference functions are connected in a simple fashion. Because of evaluation procedures discussed in the final section, we will refer to the result as a public health evaluation model.

Theorem 1. A set of proper intensity relations \succsim_i , $i = 1, \dots, N$, and \succsim_P satisfy the Pareto condition if and only if for any individual difference functions $w_i(h_i)$ there exist weights $a_i > 0$ such that the function:

$$W(\mathbf{h}) = a_1 w_1(h_1) + \dots + a_N w_N(h_N) \quad (1)$$

is a population difference function. The weights a_i in (1) are unique up to a positive multiple.

This model provides a foundation for evaluating changes in public health by a scale that is the sum or a weighted sum of scales that measure preferences regarding changes in health for individuals. Hence, it provides a foundation for the utilitarian principle that cardinal utility for a society is the sum (or a weighted sum) of cardinal utilities for the members of the society.

4. Interpersonal Weights

This section discusses two simplifications/restrictions in the public health evaluation model. The first is that the individual intensity relations \succsim_i are equal (and thus the sets H_i are equal). As remarked above, we think this simplification will be made in most applications of the model. It does not imply that the individuals have equal sets of possible health outcomes.

With this simplification, it is natural to choose individual difference functions $w_i(h_i)$ that are equal. Such a model will be called an equal preferences model. As notation that is concise (but that does overlap with our general notation for both the individual and population cases), suppose that: H_I denotes the common set of health outcomes, either h_i or h denotes a health outcome, \succsim_I denotes the common individual intensity relation, and $w(h)$ denotes an individual difference function. Theorem 1 states that for any individual difference function $w(h)$ there exists a population difference function of the form, $W(\mathbf{h}) = a_1 w(h_1) + \dots + a_N w(h_N)$, where the weights $a_i > 0$, $i = 1, \dots, N$, are unique up to a positive multiple.

In an equal preferences model, a ratio a_i / a_j of weights in a population difference function, $W(\mathbf{h}) = a_1 w(h_1) + \dots + a_N w(h_N)$, is the same as the ratio \hat{a}_i / \hat{a}_j of the corresponding weights in any other population difference function, $\hat{W}(\mathbf{h}) = \hat{a}_1 \hat{w}(h_1) + \dots + \hat{a}_N \hat{w}(h_N)$. This property of invariance is important for assessing the weights in terms of interpersonal tradeoffs, and for this reason, we will refer to the weights in an equal preferences model as interpersonal weights. In a general health evaluation model, the weights are not invariant in this sense.

The second simplification is to assume that the interpersonal weights in an equal preferences model are equal. By the invariance property, if some weights a_i are equal, then any weights \hat{a}_i are equal. An equal preferences model with equal interpersonal weights will be called an equal weights model. In such a model, any positive number can be chosen as a common interpersonal weight. In particular, we can choose $a_i = 1$ or $a_i = 1/N$. Thus, we have the following result.

Corollary 1. In an equal weights model, for any individual difference function $w(h)$ each of the following functions is a population difference function:

$$W(\mathbf{h}) = w(h_1) + \dots + w(h_N), \quad W(\mathbf{h}) = 1/N (w(h_1) + \dots + w(h_N)) \quad (2)$$

Next, we state conditions on a population intensity relation in an equal preferences model, each of which is satisfied if and only if the interpersonal weights are equal.

As what may be an awkward notation, suppose that for two health outcomes h, h' , \mathbf{h} denotes the health distribution in which $h_i = h$ for $i = 1, \dots, N$, and $(h_i = h'; \bar{h})$ denotes the health distribution in which $h_i = h'$ and $h_j = h$ for $j \neq i$. Suppose, moreover, that for three health outcomes h, h', h'' , $(h_i = h', h_j = h''; \bar{h})$ denotes the health distribution in which $h_i = h', h_j = h''$, and $h_k = h$ for $k \neq i, j$.

(a) For any health outcomes h, h' and for any two individuals i, j , society is indifferent between the change $\mathbf{h} \rightarrow (h_i = h'; \bar{h})$ in population health and the change $\mathbf{h} \rightarrow (h_j = h'; \bar{h})$ in population health. (For instance, if $h \rightarrow h'$ is an improvement in a person's health, then society is indifferent as to which person obtains the improvement.)

In terms of our general notation for both the individual and population cases, an intensity relation \succsim defined on a set H induces a so-called ordinal relation, to be denoted by \succsim^o , that compares the elements h in H . Often, the ordinal relation is defined by: $h \succsim^o h'$ if and only if $h \rightarrow h' \succsim h \rightarrow h$. A difference function for the intensity relation \succsim (i.e., a cardinal scale) also represents the ordinal relation \succsim^o , but a function that represents the ordinal relation \succsim^o (i.e., an ordinal scale) may or may not represent the intensity relation \succsim .

When an intensity relation \succsim has a difference function, then for a fixed initial element h : $h \rightarrow h' \succsim h \rightarrow h''$ if and only if $h' \succsim^o h''$ (as one can check). Thus, preferences between changes with the same initial element are described by the ordinal relation.

The following condition is based on the ordinal relations induced by \succsim_I and \succsim_P .

(b) For any health outcomes h, h', h'' and for any two individuals i, j , society is indifferent between the health distributions $(h_i = h', h_j = h''; \bar{h})$ and $(h_i = h', h_j = h''; \bar{h})$.

Theorem 2. In an equal preferences model, conditions (a) and (b) are equivalent to one another and to the condition that the interpersonal weights are equal.

When a population is divided into subpopulations (e.g., grouped according to age), each of the conditions (a), (b) can be restricted to the subpopulations.

Corollary 2. In an equal preferences model, suppose that the population is divided into subpopulations. Then, conditions (a) and (b) when weakened to hold within each subpopulation are equivalent to one another and to the condition that the interpersonal weights within each subpopulation are equal.

There are circumstances in which the relation \succsim_P satisfies conditions (a), (b) only when they are restricted to subpopulations and yet one can justifiably assign equal weights for the entire population. This is the case if the range of possible health distributions is limited such that the health outcomes in each health distribution have the same average amount in each subpopulation.

Theorem 3. Suppose that $w(h)$ is an individual difference function in an equal preferences model. Suppose also that the population is divided into subpopulations P_1, \dots, P_k of sizes n_1, \dots, n_k and that the population intensity relation \succsim_P satisfies conditions (a), (b) for each of the subpopulations. Then, for any health distribution $\mathbf{h} = (h_1, \dots, h_N)$ that has the same average amount of the cardinal scale $w(h)$ in each subpopulation, that is,

$$\frac{1}{n_1} \sum_{i \in P_1} w(h_i) = \dots = \frac{1}{n_k} \sum_{i \in P_k} w(h_i), \quad (3)$$

the evaluation of \mathbf{h} in (1) with the weights $a_1 + \dots + a_k = 1$ equals its evaluation in (2) with the equal weights $a_i = 1/N$, $i = 1, \dots, N$. Both evaluations are the common average in (3).

Moreover, the equalities (3) hold for any cardinal scale $w(h)$ or for no cardinal scale $w(h)$.

When are unequal interpersonal weights needed?

What differences among the members of a population should imply unequal importance for the same health outcomes for different individuals? One possibility is differences in age. Here, we discuss several reasons why different weights might be assigned to people of different ages.

First, age can be related to the social importance of individuals. Murray (1994) and Murray and Acharya (1997) argue that since working-aged adults make greater economic contributions than do children or seniors their health has greater social importance. The authors assign unequal weights to persons of different ages as part of a “Disability Adjusted Life Years” (DALY) scale for health time-streams. Often, in a DALY scale the weight assigned to a person aged 25 is about twice the weight assigned to someone aged 6 or 67. See Anand and Hanson (1997) for a critical

review. In our opinion, it is not justified to infer unequal social importance from unequal economic roles. Moreover, U.S. health policies (e.g., health programs for children and Medicare) implicitly assign greater weights to children and to seniors than to working-aged adults.

Second, age can be related to concerns for equality. Greater age tends to imply greater *lifetime* health and longevity. Williams (1997) argues for assigning weights that favor equality in people's lifetime quality-adjusted life years. The weights resulting from this 'fair-innings argument' will be greater for younger people than for older people.

Greater age also tends to imply lesser *future* health and longevity. One can argue for assigning weights that favor equality in people's future quality-adjusted life years. The weights resulting from this future-equality argument will be greater for older people than for younger people.

Another criterion—one in the spirit of utilitarianism—is to assign weights that favor the sum of *improvements* in health and longevity. To examine this criterion, first assume that health outcomes are defined as health states—with the duration of a health state and any ensuing states unknown. A change in health states (e.g., from fatality to a state of no health problem) is likely to produce a greater improvement in a person's health and longevity for a younger person than for an older person. So when health outcomes are defined as health states the resulting weights may be greater for younger people. But when health outcomes are defined as health-duration pairs, a change in health outcomes entails the same improvement in health and longevity for an older person as for a younger person. So for health-duration pairs, knowing the ages of the population members does not provide a reason for assigning unequal weights.

Typically, the comparison of policy options in an equal weights model with health states will differ from a comparison of the same options in an equal weights model with health-duration pairs or health outcome-streams. In particular, fatalities will count the same regardless of age in an equal weights model with health states whereas a fatality for an older person will count less than a fatality for a younger person in an equal weights model with health-duration pairs or health outcome-streams. In such a context, one cannot assign equal weights both in a model with health states and in the corresponding model with health-duration pairs or health outcome-streams. Similar observations have been made for expected-utility models; see, e.g., Bordley (1994) and Hammitt (2002).

5. Health States

This section discusses the case in which a health outcome is a state of a person's health. For example, the health outcome may be a morbidity or an injury or it may be the condition of a person's health in general. A health outcome defined in this manner will be called a health state.

In the next section, a health state is part of a health outcome rather than the health outcome itself. In order to have a consistent notation, we will denote a health state by s_i and the set of health states by S_i . Hence, in this section h_i becomes s_i and H_i becomes S_i .

An important part of a model of health states is a scale that measures ordinal comparisons of health. The scale is often called 'health-related quality of life' or 'quality-of-life' (see, e.g., Gold et al., 1996, Dolan et al., 1996, and Dolan, 2000). We will call such a scale a quality scale, and we will denote it by $q(s_i)$.

The first part of this section discusses the possible linkage between a predetermined quality scale $q(s_i)$ and an individual intensity relation \succsim_i . In particular, we discuss when $q(s_i)$ is an individual difference function. The second part of this section broadens the discussion to include a population intensity relation \succsim_P . In particular, we discuss how the presence of \succsim_P makes possible certain procedures for assessing an individual difference function from a quality scale.

Quality scales have been developed both for morbidity and trauma. Most of the established scales are multiattribute expected-utility functions. As examples of this type of scale there are: the Functional Capacity Index, the Quality of Wellbeing scale (Kaplan and Anderson, 1996), the Health Utilities Indices (Torrance et al., 1982, 1995, 1996; Feeney et al., 1996; and Furlong et al., 1998), the Health and Activities Limitation Index (HALex), the SF-36 metric for health status measurement, and the EuroQoL quality of life scales (Kind, 1996; Dolan, 1997; and Richardson et al., 2001).

Procedures to assess a quality scale have been developed, either as part of a health study or as a separate undertaking. Examples of these assessment procedures (commonly called scales) include: time tradeoff scales, standard gamble scales, person tradeoff scales, and rating scales; see, e.g., Torrance (1976, 1986) and Dolan et al. (1996).

Here, we assume that a quality scale $q(s_i)$ has been chosen or assessed for the purpose of measuring the health of each individual in a population. This assumption implies that the sets S_i

of health states are a common set, which we will denote by S . The assumption does not imply any connection between $q(s_i)$ and the individual intensity relations \succsim_i . In order to examine when the scale $q(s_i)$ is connected to a relation \succsim_i , we will use the following condition.

Properness condition. The range of the quality scale $q(s_i)$ is a non-point interval.

We expect that in most if not all applications, the health states will be described by one or more so-called dimensions, each of which is measured by a scale. Then, a health state s_i is a vector of scale amounts, and the set S of health states is the product set of the ranges of the scales. If every scale is categorical, i.e., it has a finite set of values, then the set is finite, and thus the quality scale $q(s_i)$ cannot be proper. On the other hand, if at least one of the scales is continuous, i.e., it has an interval set of values, then the quality scale may or may not be proper. We intend to discuss in a later paper conditions which imply that a quality scale must be proper.

Conditions (c) and (d) below are weaker than the definitions of ordinal and cardinal scales, and hence easier to verify. Nevertheless, in the context of proper intensity relations and quality scales they suffice to imply that $q(s_i)$ is an individual difference function.

(c) For any health states, s_i and s'_i , $q(s_i) > q(s'_i)$ if and only if s_i is preferred to s'_i according to the ordinal preferences induced by the individual intensity relation \succsim_i .

(d) For any changes, $s_i \rightarrow s'_i$ and $\hat{s}_i \rightarrow \hat{s}'_i$, if $q(s'_i) - q(s_i) = q(\hat{s}'_i) - q(\hat{s}_i) > 0$, then the changes are indifferent according to the individual intensity relation \succsim_i .

Theorem 4. Suppose that $q(s_i)$ is a proper quality scale and that \succsim_i is a proper individual intensity relation. Then:

(i) $q(s_i)$ and \succsim_i satisfy condition (c) if and only if $q(s_i)$ is an ordinal scale for \succsim_i and any individual difference function has the form $w(s_i) = f(q(s_i))$ where $f(q)$ is a strictly increasing and continuous function.

(ii) $q(s_i)$ and \succsim_i satisfy conditions (c) and (d) if and only if $q(s_i)$ is an individual difference function for \succsim_i .

When $q(s_i)$ and \succsim_i satisfy conditions (c) and (d), the task of determining an individual difference function is completed. And when they satisfy the ordinal condition (c) but not the

cardinal condition (d), the task is reduced to that of assessing a function $f(q)$ having the properties in (i). Such a function will be called a conversion function since it converts the ‘ordinal’ units of the quality scale into the ‘cardinal’ units of an individual difference function.

The combination of conditions (c) and (d) for every $i = 1, \dots, N$ implies that the individual intensity relations \succsim_i are equal. However, condition (c) for every $i = 1, \dots, N$ does not have this implication since the conversion functions $f(q)$ can vary from one individual to another.

Next, we describe two types of procedures for assessing an individual difference function. Throughout this paper, we define the term ‘assessment’ to include any means of obtaining social values: e.g., elicitation from a decision maker, stated assumptions, or the use of survey results.

Equal-differences procedures

The following procedures for assessing an individual difference function will be called equal-differences procedures. First, we assume that a proper quality scale $q(s_i)$ that satisfies condition (c) has been determined, and second we omit this assumption.

As one equal-differences procedure, assess a sequence, $s^{(0)}, \dots, s^{(n)}$, of health states with increasing scale amounts $q(s_i)$ such that $s^{(k-1)} \rightarrow s^{(k)}$ is indifferent to $s^{(k)} \rightarrow s^{(k+1)}$ for each $k = 1, \dots, n-1$. A conversion function can be defined for these health states by $f(q(s^{(k)})) = k$, $k = 0, \dots, n$, and then can be extended to other health states by approximate interpolation.

As another equal-differences procedure, select a pair of extreme health states $s^{(0)}, s^{(1)}$ with $q(s^{(1)}) > q(s^{(0)})$. For example, $s^{(0)}$ and $s^{(1)}$ might be death and no injury. Assess a health state $s^{(1/2)}$ such that $s^{(0)} \rightarrow s^{(1/2)}$ and $s^{(1/2)} \rightarrow s^{(1)}$ are indifferent. Then, $f(q(s^{(1/2)})) - f(q(s^{(0)})) = f(q(s^{(1)})) - f(q(s^{(1/2)}))$, and thus $f(q(s^{(1/2)})) = 1/2 f(q(s^{(0)})) + 1/2 f(q(s^{(1)}))$. In a similar manner, assess a health state $s^{(1/4)}$ between $s^{(0)}$ and $s^{(1/2)}$ and a health state $s^{(3/4)}$ between $s^{(1/2)}$ and $s^{(1)}$. A conversion function can be defined for these health states by $f(q(s^{(k)})) = k$ for $k = 0, 1/4, \dots$ and then can be extended to other health states by approximate interpolation.

The above procedures can also be used to assess an individual difference function $w(s_i)$ when no quality scale is available. In the first procedure, define $w(s^{(k)}) = k$ for $k = 0, \dots, n$, and in the second procedure, define $w(s^{(k)}) = k$ for $k = 0, 1/4, \dots$.

In this situation, there are multiple health dimensions (since otherwise the single dimension provides a quality scale), and thus the procedure must be supplemented with other assessments.

For each health state $s^{(k)}$, one could for example use tradeoffs procedures to assess a number of health states that are indifferent to $s^{(k)}$.

At this point, we broaden the discussion to include a population intensity relation \succsim_P . Again, we first assume that a quality scale $q(s_i)$ that satisfies condition (c) has been determined. Then, there exists a conversion function $f(q)$ as described in Theorem 4 such that $w(s_i) = f(q(s_i))$ is an individual difference function and $W(s_1, \dots, s_N) = f(q(s_1)) + \dots + f(q(s_N))$ is a population difference function.

Person-tradeoffs procedures

The following procedures for assessing an individual difference function will be called person-tradeoffs procedures. Similar procedures for other types of scales have been proposed by several authors (see, e.g., Richardson 1994; Nord, 1995; and Green 2001). Some authors have observed that person-tradeoffs procedures are intuitively appealing while others have observed that they lack a foundation, that is, they lack stated conditions on preferences that justify the procedures. The public health evaluation model provides such a foundation.

Suppose that we have an equal weights model with a proper quality scale $q(s_i)$, and thus the task is to assess a conversion function. Select two extreme health states $s^{(0)}$, $s^{(1)}$ such that $q(s^{(1)}) > q(s^{(0)})$. Since by Theorem A1 an individual difference function is cardinally unique, there is a unique conversion function $f(q)$ such that $f(q(s^{(0)})) = 0$ and $f(q(s^{(1)})) = 1$. Choose a list consisting of $s^{(0)}$, $s^{(1)}$ and some intermediate health states, and for each health state s_i in the list assess a number $n(s_i)$ such that society is indifferent between an improvement from $s^{(0)}$ to s_i for the entire population and an improvement from $s^{(0)}$ to $s^{(1)}$ for $n(s_i)$ members of the population. Then, $N \times f(q(s_i)) = n(s_i) \times 1$, and thus $f(q(s_i)) = n(s_i) / N$. Hence, the conversion function can be defined by $f(q(s_i)) = n(s_i) / N$ for the health states s_i in the list, and then it can be extended to other health states by approximate interpolation.

The above procedure can also be used to assess an individual difference function $w(s_i)$ if no quality scale is available. In this case, define $w(s_i) = n(s_i) / N$ for the health states s_i in the list.

The health states in a person-tradeoffs procedure are selected (e.g., for simplicity) whereas those in an equal-differences procedure are assessed (and thus represent social values). Hence,

person-tradeoffs procedures but not equal-differences procedures can be used in those situations in which only a sparse set of health states can be visualized and thus are available to assess.

A quality scale $q(s_i)$ is an individual difference function if and only if it satisfies condition (d). One can verify (d) directly, or one can use an equal differences or person-tradeoffs procedure to determine whether the conversion function is linear. Thus, the public health evaluation model has the unusual feature that either intrapersonal equal-differences procedures or interpersonal person-tradeoffs procedures (or a combined procedure) can be used to assess its components.

Social Attitudes toward Inequality

Suppose that $\mathbf{s} = (s_1, \dots, s_N)$ denotes a health distribution in which the health outcomes are health states. A change $\mathbf{s} \rightarrow \mathbf{s}'$ from an initial health distribution \mathbf{s} to a final health distribution \mathbf{s}' consists of changes $s_i \rightarrow s'_i$ from initial health states s_i to final health states s'_i . What is society's attitude (as modeled in a public health evaluation model) toward inequality in the final health states? Here, we discuss two interpretations of this question.

(1) Society is neutral as to whether a change from one health distribution to another increases or decreases the degree of inequality in the final health states. To illustrate, suppose that s, s', s'' are three health states such that s is worst, s'' is best, and the changes, $s \rightarrow s'$ and $s' \rightarrow s''$, are indifferent according to the individual intensity relation. For example, the health states s, s', s'' might be death, severe injury, and no injury. Consider three health distributions $\mathbf{s}, \mathbf{s}', \mathbf{s}''$ in which an i -th individual has the health states s, s', s'' respectively, and everyone else has the intermediate health state s' . The Pareto condition implies that the changes, $\mathbf{s} \rightarrow \mathbf{s}'$ and $\mathbf{s}' \rightarrow \mathbf{s}''$, are indifferent according to the population intensity relation even though the change $\mathbf{s} \rightarrow \mathbf{s}'$ entails a decrease in inequality whereas the change $\mathbf{s}' \rightarrow \mathbf{s}''$ entails an increase in inequality.

(2) As we show below, in an equal weight model society may or may not be neutral as to whether a health distribution $\mathbf{s} = (s_1, \dots, s_N)$ has equal quality scale amounts $q(s_i)$. For a more extensive discussion of this issue of social preferences in the context of ordinal welfare functions rather than in the present context of cardinal welfare functions, see Harvey (1985).

Suppose that $\bar{\mathbf{s}} = (\bar{s}, \dots, \bar{s})$ is a health distribution with the same health state \bar{s} for each individual and that $\mathbf{s} = (s_1, \dots, s_N)$ is a health distribution with unequal quality scale amounts for at least two individuals.

(i) Social preferences will be called inequality neutral provided that for any health distributions \bar{s} , \mathbf{s} as described, $q(\bar{s}) = 1/N (q(s_1) + \dots + q(s_N))$ implies that \bar{s} is indifferent to \mathbf{s} .

(ii) Social preferences will be called inequality averse provided that for any health distributions \bar{s} , \mathbf{s} as described, $q(\bar{s}) = 1/N (q(s_1) + \dots + q(s_N))$ implies that \bar{s} is preferred to \mathbf{s} .

Theorem 5. For an equal weights model with a quality scale $q(s_i)$ that satisfies condition (c):

(i) Social preferences are inequality neutral if and only if the quality scale $q(s_i)$ is an individual difference function.

(ii) Social preferences are inequality averse if and only if any conversion function $f(q)$ is strictly concave. (Then, $q(s_i)$ is not an individual difference function.)

6. Health-Duration Pairs

A health outcome can be defined to include information on the timing of one or more states of health. This section discusses health outcomes that are defined as a constant health state and its duration. A health outcome defined in this manner will be called a health-duration pair. The duration of a health state can be defined as the time from a specified event, e.g., an accident or the onset of a disease, in which case different individuals will have different initial times, or it can be defined as the time from a common initial time, e.g., a specified present time.

A health state will again be denoted by s_i , and its duration in years will be denoted by t_i . Thus, a health-duration pair for an i -th individual will be denoted by $h_i = (s_i, t_i)$.

Most models of health-duration pairs are expected-utility models; see, e.g., Bleichrodt et al. (1997) for a discussion of the advantages of such models. The simplest utility functions are of the form $u(s_i, t_i) = t_i v(s_i)$ where the linear factor t_i represents risk neutrality toward duration. In these models, a health-duration pair (or a probability distribution of such pairs) is measured by a utility amount that represents an equivalent time in a state of optimal health; the amount is called ‘quality-adjusted life years’ (QALYs) or ‘healthy-years equivalents’ (HYEs).

We regard the various QALY models in Pliskin, Shepard, and Weinstein (1980) as the basic expected-utility models for health-duration pairs. Alternative conditions on preferences are

presented in Bleichrodt et al. (1997) and Miyamoto et al. (1998) for expected-utility models and in Doctor and Miyamoto (2003) and Østerdal (2005) for deterministic models. QALY models for health-duration pairs are also presented or discussed in Loomes and McKenzie (1989), Johannesson et al. (1994), Gold et al. (1996), and Dolan (2000).

A very different modeling approach is presented in Mehrez and Gafni (1989). They propose a procedure in which a health-duration pair is measured in “healthy-years equivalents” (HYEs) by assessing an indifferent health-duration pair in which HYE years of optimal health are followed by death. See also, e.g., Gafni et al. (1993), Loomes (1995), and Johannesson (1995).

Duration is an essential part of health outcomes modeled either as health states or as health-duration pairs. Simply stated, the reason is that health occurs over time. In the case of health states, duration is implicit. It might be constant—or at least independent of the health state and the individual. Or it might be unknown. In the case of health-duration pairs, duration is explicit. What is implicit is the person’s health afterward.

This section develops models to be applied when the health state in a health-duration pair is followed by a state of death. Either the person dies at the end of the time period, or the health state is a state of death and thus the person is dead from the initial time. We have this ‘death-afterward’ situation in mind when we assume, for instance, that longer durations are preferable.

As before, assume that the sets S_i of health states are a common set S and that a proper quality scale $q(s_i)$ is defined on the set S . Also, assume that S contains one or more states of death and that these states are the least preferred health states. Similar models can be developed in which any health state is better than death or in which there are health states worse than death. For expected-utility models having states worse than death, see Miyamoto et al. (1998).

Suppose that $q(s_i) \geq 0$ for any health state s_i and that $q(d) = 0$ where d denotes a specified state of death. If the predetermined scale $q(s_i)$ does not have these properties, then it can be replaced by the scale $q(s_i) - q(d)$. Thus, the range of the scale $q(s_i)$ is a non-point interval of non-negative numbers that contains zero as its lower endpoint.

We assume that the set of durations is the interval $T = (0, \infty)$. Thus, the set of health-duration pairs is defined as the product set $H = S \times (0, \infty)$. This choice of T avoids two issues: that of interpreting outcomes of zero duration, and that of assigning an upper bound on duration.

This section discusses the linkage between $q(s_i)$ and an individual intensity relation \succsim_i . We define conditions on $q(s_i)$ and \succsim_i that imply special forms of an individual difference function, and we discuss assessment procedures that use these forms. The resulting models are similar to the expected-utility models referenced above.

Whereas ordinal preferences between health states depend only on the quality scale $q(s_i)$, ordinal preferences between health-duration pairs also depend on the duration scale t_i . For this reason, the conditions on preferences in this section are somewhat more complicated than the corresponding conditions in the previous section.

An individual intensity relation \succsim_i will be called scale-proper provided that it has a difference function $w(s_i, t_i)$ such that for any health state s^* or duration t^* the range of the function $w(s_i, t^*)$ or $w(s^*, t_i)$ is an interval and at least one of these ranges is non-point. We show in the Appendix that if an individual intensity relation is scale-proper, then it is proper.

First, we define conditions on the ordinal preferences induced by an intensity relation \succsim_i .

- (e1)** For any pairs $(s_i, t_i), (s'_i, t_i)$ with a common duration t_i : $q(s_i) > q(s'_i)$ if and only if (s_i, t_i) is preferred to (s'_i, t_i) .
- (e2)** For any pairs $(s_i, t_i), (s_i, t'_i)$ with a common health state s_i : If $q(s_i) > 0$, then $t_i > t'_i$ if and only if (s_i, t_i) is preferred to (s_i, t'_i) . If $q(s_i) = 0$, then (s_i, t_i) is indifferent to (s_i, t'_i) .
- (e3)** For any health state s_i and any health-duration pair (s_i^*, t_i^*) with $q(s_i^*) > 0$, there exists a duration t_i such that (s_i, t_i) is less preferred than (s_i^*, t_i^*) .

Condition (e1) implies that for a fixed duration t_i , the quality scale $q(s_i)$ represents ordinal preferences between health states. Thus, (e1) is a direct analogue of condition (c).

Condition (e2) implies that for a fixed health state s_i , if s_i is preferred to death then longer durations are preferred, and if s_i is indifferent to death then all durations are indifferent.

Ethical arguments that social preferences should satisfy (e1) but not (e2) or vice versa have not been propounded. But arguments have been made for an 'equal value of life' principle that society should not compare health-duration pairs with non-death health states; see, e.g., Harris (1987) and Nord (2001). This principle appears to violate both (e1) and (e2). For reasons why

social preferences among health outcomes should depend on degrees of health and duration, see, e.g., Singer et al. (1995), Williams (1997), and Hasman and Østerdal (2004).

Roughly speaking, condition (e3) states that for any health state s_i there is a sufficiently brief duration t_i such that the outcome (s_i, t_i) is less preferred than the comparison outcome (s_i^*, t_i^*) . Whereas we define $T = (0, \infty)$ and introduce condition (e3), Bleichrodt et al. (1997) and Miyamoto et al. (1998) define $T = [0, \infty)$ and introduce a ‘zero-condition’ which states that any two outcomes $(s_i, 0)$ and $(s'_i, 0)$ are indifferent. This condition and the condition that ordinal preferences are continuous on the set $S \times [0, \infty)$ implies condition (e3) on the subset $S \times (0, \infty)$.

Conditions (e1)-(e3) on ordinal preferences imply that an individual intensity relation \succsim_i has a difference function of the form, $w(s_i, t_i) = g(q(s_i), t_i)$, as described in the Appendix. The two-variable function $g(q, t)$ has a general form and hence would be extremely difficult to assess.

Next, we introduce conditions on cardinal preferences which imply that $g(q, t)$ is a product of two single-variable functions. This reduction can greatly simplify the assessment task.

(f1) Preferences between two changes, $(s_i, t_i) \rightarrow (s_i, t'_i)$ and $(s_i, \hat{t}_i) \rightarrow (s_i, \hat{t}'_i)$, with a common health state s_i are the same for any s_i with $q(s_i) > 0$.

(f2) Preferences between two changes, $(s_i, t_i) \rightarrow (s'_i, t_i)$ and $(\hat{s}_i, t_i) \rightarrow (\hat{s}'_i, t_i)$, with a common duration t_i are the same for any t_i .

Theorem 6. Suppose that $q(s_i)$ is a proper quality scale and that \succsim_i is a scale-proper individual intensity relation. Then, the following are equivalent: (i) $q(s_i)$ and \succsim_i satisfy conditions (e1)-(e3) and (f1); (ii) $q(s_i)$ and \succsim_i satisfy conditions (e1)-(e3) and (f2); and (iii) \succsim_i has an individual difference function with the product form:

$$w(s_i, t_i) = D(t_i) f(q(s_i)) \quad (4)$$

where $f(q)$ and $D(t)$ are continuous, strictly increasing functions such that $f(q(d)) = 0$ and $\lim_{t \rightarrow 0} D(t) = 0$. Moreover, each of the functions $D(t)$, $f(q)$ is unique up to a positive multiple.

The function $f(q)$ in (4) can be interpreted as a conversion function, and the function $D(t)$ in (4) can be interpreted as a cumulative discounting function, that is, $D(t) = \int_0^t d(u) du$ where

$d(u)$ is a discounting function. For preferences that are timing neutral (non-discounting), one can choose $d(u) = 1$ for $u > 0$ and thus $D(t) = t$ for $t > 0$.

Now, we define conditions that strengthen conditions (f1) and (f2) respectively.

(g1) For any changes $(s_i, t_i) \rightarrow (s_i, t'_i)$ and $(s_i, \hat{t}_i) \rightarrow (s_i, \hat{t}'_i)$ with a common health state s_i , if $t'_i - t_i = \hat{t}'_i - \hat{t}_i$, then the changes are indifferent.

(g2) For any changes $(s_i, t_i) \rightarrow (s'_i, t_i)$ and $(\hat{s}_i, t_i) \rightarrow (\hat{s}'_i, t_i)$ with a common duration t_i , if $q(s'_i) - q(s_i) = q(\hat{s}'_i) - q(\hat{s}_i)$, then the changes are indifferent.

Condition (g1) requires that for a common health state society is indifferent between a change in a short duration (thus in the near future) and the same change in a long duration (thus in the distant future). There are ethical arguments in favor of such non-discounting, but most people place far more importance on the near future than on the distant future; for example, they would prefer an increase in a short duration to the same increase in a long duration.

Condition (g2) requires that for a common duration society is indifferent between a change from a bad health state and a change from a good health state that has an equal difference in quality scale amounts. This condition seems appropriate for many quality scales, and thus there may be many situations in which (g2) but not (g1) is satisfied.

Theorem 7. Suppose that $q(s_i)$ is a proper quality scale and that \succsim_i is a scale-proper individual intensity relation. Then:

(i) $q(s_i)$ and \succsim_i satisfy conditions (e1)-(e3) and (g1) if and only if \succsim_i has an individual difference function that is linear in the variable t_i , that is:

$$w(s_i, t_i) = t_i f(q(s_i)) \quad (5a)$$

where $f(q)$ is a continuous, strictly increasing function with $f(q(d)) = 0$.

(ii) $q(s_i)$ and \succsim_i satisfy conditions (e1)-(e3) and (g2) if and only if \succsim_i has an individual difference function that is linear in the variable $q(s_i)$, that is:

$$w(s_i, t_i) = D(t_i) q(s_i) \quad (5b)$$

where $D(t)$ is a continuous, strictly increasing function with $\lim_{t \rightarrow 0} D(t) = 0$.

(iii) $q(s_i)$ and \succsim_i satisfy conditions (e1)-(e3) and (g1)-(g2) if and only if \succsim_i has an individual difference function of the form:

$$w(s_i, t_i) = t_i q(s_i) \quad (5c)$$

Moreover, each of the functions $f(q)$, $D(t)$ in (5a), (5b) is unique up to a positive multiple.

Expected-utility functions of the form (5a) represent risk neutrality toward future duration, and expected-utility functions of the form (5b) represent risk neutrality toward quality scale amounts. See, e.g., Pliskin et al. (1980) and Broome (1993) for discussions of such functions.

7. QALY and YALQ Scales

A health-duration pair can be evaluated by selecting a ‘standard amount’ of s_i or t_i and finding an amount of the other such that the resulting health-duration pair is indifferent to (s_i, t_i) . The well-known method is to select a state s^* of optimal health as the standard health state, and for a given health-duration pair (s_i, t_i) assess a duration \hat{t} such that (s^*, \hat{t}) is indifferent to (s_i, t_i) . As mentioned in Section 6, \hat{t} is called the ‘quality-adjusted life years’ for the pair (s_i, t_i) and is denoted by QALY. If the health state s_i in a pair (s_i, t_i) is less preferred than s^* , then the quality-adjusted life years \hat{t} for the pair (s_i, t_i) will be less than t_i .

The alternative method is to select a standard duration t^* . Perhaps, t^* is an extremely long duration (what might be called a Methuselah duration). For a given health-duration pair (s_i, t_i) , assess a health state \hat{s} such that (\hat{s}, t^*) is indifferent to (s_i, t_i) . We will refer to the amount $\hat{q} = q(\hat{s})$ as the years-adjusted life quality of the pair (s_i, t_i) , and we will denote \hat{q} by YALQ. If the duration t_i in a pair (s_i, t_i) is less than t^* , then the years-adjusted life quality \hat{q} for the pair (s_i, t_i) will be less than $q(s_i)$.

The QALY scale is well-known, while the YALQ scale appears to be new. For technical reasons, we will modify the above definitions so that each scale is a function of pairs $(q(s_i), t_i)$ rather than a function of health-duration pairs (s_i, t_i) . As shown in the Appendix, conditions (e1)-(e3) imply that each scale represents ordinal preferences between health-duration pairs.

Suppose that $\hat{t} = \text{QALY}(q(s_i), t_i)$ denotes a QALY scale and that $\hat{q} = \text{YALQ}(q(s_i), t_i)$ denotes a YALQ scale. A QALY scale is defined on pairs $(q(s_i), t_i)$ such that $0 < q(s_i) \leq q^*$

(where $q^* = q(s^*)$) and $t_i > 0$, and a YALQ scale is defined on pairs $(q(s_i), t_i)$ such that $0 < t_i \leq t^*$ and $q(s_i) > 0$. The inequalities $t_i > 0$ are not a restriction since zero durations have been excluded while the inequalities $q(s_i) > 0$ exclude health states that are indifferent to death.

In general, a QALY scale or a YALQ scale would be difficult to assess since it is a function of two variables, namely $q(s_i)$ and t_i . By choosing $t_i = t^*$ or $q(s_i) = q^*$, the scales induce the single-variable scales: $Q^*(q(s_i)) = \text{QALY}(q(s_i), t^*)$ and $Y^*(t_i) = \text{YALQ}(q^*, t_i)$. As shown in the Appendix, the functions $Q^*(q(s_i))$ and $Y^*(t_i)$ are inverses of one another.

Theorem 8. Suppose that a quality scale $q(s_i)$ and an individual intensity relation \succsim_i satisfy the assumptions in Theorem 6 and that a health-duration pair (s^*, t^*) with $q^* = q(s^*) > 0$ has been specified. Then, there exists a unique individual difference function of the product form, $w^*(s_i, t_i) = D(t_i) f(q(s_i))$, where $f(q)$ and $D(t)$ are continuous, strictly increasing functions such that $f(q(d)) = 0$, $f(q(s^*)) = 1$ and $\lim_{t \rightarrow 0} D(t) = 0$, $D(t^*) = 1$. Moreover:

$$w^*(s_i, t_i) = D(\text{QALY}(q(s_i), t_i)) = f(\text{YALQ}(q(s_i), t_i)) \quad (6)$$

and

$$w^*(s_i, t_i) = D(t_i) D(Q^*(q(s_i))) = f(q(s_i)) f(Y^*(q(s_i))) \quad (7)$$

for any health-duration pair (s_i, t_i) with $0 < q(s_i) \leq q^*$ and $0 < t_i \leq t^*$.

Suppose that condition (g1) is satisfied, that is, society is neutral toward differences in duration (non-discounting). Then, $D(t) = t/t^*$, and formulas (6), (7) imply that $w^*(s_i, t_i) = \text{QALY}(q(s_i), t_i)(t^*)^{-1} = t_i Q^*(q(s_i))(t^*)^{-2}$. As another special case, suppose that condition (g2) is satisfied, that is, society is neutral toward differences in quality scale amounts. Then, $f(q) = q/q^*$, and thus $w^*(s_i, t_i) = \text{YALQ}(q(s_i), t_i)(q^*)^{-1} = q(s_i) Y^*(t_i)(q^*)^{-2}$. In particular, a YALQ scale $Y^*(t_i)$ is more appropriate than a QALY scale $Q^*(q(s_i))$ when: (i) Discounting is used (rather than timing neutrality), and (ii) The quality scale is defined as or is converted to a scale such that equal differences in quality scale amounts have the same social importance.

Imagine a situation in which neither of the conditions (g1), (g1) is satisfied. The product form of a difference function $w^*(s_i, t_i)$ and the formulas (6), (7) suggest the following methods for assessing $w^*(s_i, t_i)$: (i) Assess a cumulative discounting function $D(t)$ and a conversion

function $f(q)$, (ii) Assess a cumulative discounting function $D(t)$ and a QALY scale $Q^*(q(s_i))$, and (iii) Assess a conversion function $f(q)$ and a YALQ scale $Y^*(t_i)$.

The single-variable functions and scales can be assessed by equal-differences procedures, person-tradeoffs procedures, or other types of procedures. This paper does not undertake a comprehensive discussion of assessment procedures—either those that like the equal-differences and person-tradeoffs procedures consist of direct assessments or those that like constant discounting depend on assuming that the function or scale belongs to a parametric family.

8. Evaluating Policy Options

This section discusses how a population difference function that has been constructed as part of a public health study can be used to evaluate the policy options in the study. In contrast to procedures to obtain a population difference function, these procedures use such a function. The health outcomes can be of any type: e.g., health states, health-duration pairs, or health streams.

Since a population difference function is a cardinal scale, a change in public health can be legitimately measured by a difference in function values. By contrast, an expected-utility function or an ordinal scale cannot be used in this manner unless it is also a difference function.

Assume that each policy option in a public health study is predicted to produce a change from one health distribution to another. The initial distribution may be, e.g., the predicted public health for the current policy of intervention or for a policy of non-intervention, and the final distribution may be, e.g., the predicted public health for the proposed policy option.

It would be more realistic to assume that each policy option leads to a probability distribution of changes. However, this paper does not discuss such models nor does it discuss when the extra realism of using probability distributions of changes rather than predicted changes is needed.

By the evaluation of a policy option, we mean the following process: (1) Calculate a change in public health that is indifferent to the change produced by the policy option. The hypothetical change may be simpler and more easily understood than the predicted change. (2) Report the calculated change as an evaluation of the policy option.

Below, we discuss two methods for this process. Each method assumes that a public health evaluation model (as described in Section 3) has been constructed. In particular, suppose that

$\mathbf{h} = (h_1, \dots, h_N)$ denotes a health distribution and that $W(\mathbf{h}) = a_1 w_1(h_1) + \dots + a_N w_N(h_N)$ denotes a population difference function.

Calculating a common change for everyone

In the first type of evaluation, a health outcome $h^{(0)}$ has been specified and a health outcome \hat{h} is to be calculated. Suppose that $\mathbf{h}^{(0)}$ and $\hat{\mathbf{h}}$ denote the health distributions in which everyone experiences the health outcome $h^{(0)}$ or \hat{h} respectively.

For a policy option with a predicted change $\mathbf{h} \rightarrow \mathbf{h}'$ in public health, calculate a health distribution $\hat{\mathbf{h}}$ such that $W(\hat{\mathbf{h}}) - W(\mathbf{h}^{(0)}) = W(\mathbf{h}') - W(\mathbf{h})$. Then, the change $\mathbf{h}^{(0)} \rightarrow \hat{\mathbf{h}}$ is indifferent to the change $\mathbf{h} \rightarrow \mathbf{h}'$, and in that sense it is equivalent to $\mathbf{h} \rightarrow \mathbf{h}'$. Hence, one can evaluate the policy option as the hypothetical change $h^{(0)} \rightarrow \hat{h}$ in health outcomes for everyone.

This method of evaluation will be especially simple in two situations. First, it may be appropriate to choose the same initial distribution $\mathbf{h}^{(0)}$ to evaluate each of the policy options. In this case, assume that the individual difference functions $w_i(h_i)$ have been partially normalized such that $w_i(h^{(0)}) = 0$ for $i = 1, \dots, N$, and thus $W(\mathbf{h}^{(0)}) = 0$. Then, $W(\hat{\mathbf{h}}) - W(\mathbf{h}^{(0)}) = W(\hat{\mathbf{h}})$, and thus a policy option can be evaluated as a common hypothetical health outcome \hat{h} .

Second, an equal preferences model may be appropriate. In this case, suppose that the common individual difference function is denoted by $w(h)$ and that the interpersonal weights have been normalized so that they sum to unity. Then, $W(\hat{\mathbf{h}}) - W(\mathbf{h}^{(0)}) = w(\hat{h}) - w(h^{(0)})$, and thus a policy option can be evaluated as a difference $\Delta w = w(\hat{h}) - w(h^{(0)})$ in the cardinal scale $w(h)$. The difference Δw is independent of the health outcomes $h^{(0)}$ and \hat{h} .

Suppose that the health outcomes in an equal preferences model are health states, $h_i = s_i$. Then, $w(\hat{h}) - w(h^{(0)}) = f(q(\hat{s})) - f(q(s^{(0)}))$. In the case that equal differences in health quality are equally important (i.e., condition (d) is satisfied), and thus $q(s_i)$ is a cardinal scale, it follows that $w(\hat{h}) - w(h^{(0)}) = q(\hat{s}) - q(s^{(0)})$. Then, a policy option can be evaluated as a difference $\Delta q = q(\hat{s}) - q(s^{(0)})$ in the quality scale $q(s_i)$.

Next, suppose that the health outcomes in an equal preferences model are health-duration pairs, $h_i = (s_i, t_i)$. In this case, one may restrict the health-duration pair $\hat{h} = (\hat{s}, \hat{t})$ to one of the forms, $\hat{h} = (s^{(0)}, \hat{t})$ or $\hat{h} = (\hat{s}, t^{(0)})$. In the first case, a policy option can be evaluated as a change

from $t^{(0)}$ to \hat{t} in duration conditional on the health state $s^{(0)}$, and in the second case a policy option can be evaluated as a change from $s^{(0)}$ to \hat{s} in health conditional on the duration $t^{(0)}$.

If the individual difference function $w(h)$ has a product form, then $w(\hat{s}, \hat{t}) - w(s^{(0)}, t^{(0)}) = D(\hat{t})f(q(\hat{s})) - D(t^{(0)})f(q(s^{(0)}))$. When timing neutrality is appropriate (i.e., condition (g1) is satisfied), it follows that $w(s^{(0)}, \hat{t}) - w(s^{(0)}, t^{(0)}) = (\hat{t} - t^{(0)})f(q(s^{(0)}))$. Then, one can evaluate a policy option as a change $\Delta t = \hat{t} - t^{(0)}$ in duration conditional on the health state $s^{(0)}$. And when equal differences in health quality are equally important (i.e., condition (g2) is satisfied), it follows that $w(\hat{s}, t^{(0)}) - w(s^{(0)}, t^{(0)}) = D(t^{(0)})(f(q(\hat{s})) - f(q(s^{(0)})))$. Then, one can evaluate a policy option as a change $\Delta q = q(\hat{s}) - q(s^{(0)})$ in health quality conditional on the duration $t^{(0)}$.

Calculating a subpopulation with a specified change

Again, assume a public health evaluation model as described in Section 3. Assume that two health outcomes h_* , h^* have been specified, and suppose that \mathbf{h}_* and \mathbf{h}^* denote the health distributions in which everyone experiences the health outcome h_* or h^* respectively. For $n = 0, 1, \dots, N$, suppose that $\mathbf{h}^{(n)}$ denotes the health distribution in which the first n members of the population experience the health outcome h_* and the remaining $N - n$ members of the population experience the health outcome h^* . Then, in particular $\mathbf{h}^{(0)} = \mathbf{h}_*$ and $\mathbf{h}^{(N)} = \mathbf{h}^*$.

Consider a policy option with a predicted change $\mathbf{h} \rightarrow \mathbf{h}'$ in public health. To focus the discussion, assume that both the change $\mathbf{h} \rightarrow \mathbf{h}'$ and the change $\mathbf{h}_* \rightarrow \mathbf{h}^*$ are improvements in public health and that $\mathbf{h} \rightarrow \mathbf{h}'$ is less preferred than $\mathbf{h}_* \rightarrow \mathbf{h}^*$.

One method for evaluating such a policy option is to calculate a number n between 0 and N such that $W(\mathbf{h}^{(n)}) - W(\mathbf{h}_*)$ is approximately equal to $W(\mathbf{h}') - W(\mathbf{h})$. In effect, one is calculating a subpopulation consisting of the first n members of the population. Then, the change $\mathbf{h}_* \rightarrow \mathbf{h}^{(n)}$ is indifferent to the change $\mathbf{h} \rightarrow \mathbf{h}'$. Thus, one can evaluate the policy option as the hypothetical change $h_* \rightarrow h^*$ in health outcomes for the subpopulation defined as the first n members of the population. In an equal weights model, one can report either the number n or the fraction n/N of population members with the hypothetical change $h_* \rightarrow h^*$.

Appendix: Proofs of Results

First, we establish the following result which strengthens the condition of ‘properness.’

Theorem A1. If an intensity relation \succsim is proper, i.e., it has a difference function $w(h)$ with a non-point interval range I , then any difference function for \succsim is a positive linear transformation of $w(h)$ and has a non-point interval range.

Proof. Suppose that a function $z(h)$ with a range J is a difference function for \succsim . Then, $w(h)$ and $z(h)$ are both ordinal scales for \succsim , and thus $z(h) = f(w(h))$ for some strictly increasing function $f(w)$. The domain of $f(w)$ is the non-point interval I , and its range is the set J .

For $w < w'$ in I , define $\bar{w} = 1/2 w + 1/2 w'$. There exist h, \bar{h}, h' in H such that $w(h) = w$, $w(\bar{h}) = \bar{w}$, and $w(h') = w'$. Thus, $\bar{w} - w = w' - \bar{w}$ implies that $h \rightarrow \bar{h} \sim \bar{h} \rightarrow h'$ which implies $z(\bar{h}) - z(h) = z(\bar{h}) - z(h')$ which implies $f(\bar{w}) = 1/2 f(w) + 1/2 f(w')$. Jensen (1905, 1906) proved that any continuous function that satisfies the above equation is linear (see, e.g., Aczél, 1966, p. 43), and essentially the same argument can be used to show that any increasing function that satisfies the equation is linear. The further arguments are straightforward.

Proof of Theorem 1. We will show that the health evaluation model in Section 3 is a corollary of the social welfare (SW) model in Harvey (1999). The set C of alternative consequences for n individuals in the SW model corresponds to the set $H_1 \times \dots \times H_N$ of health distributions for N individuals in the health evaluation model, and the intensity relations \succsim_i and \succsim in the SW model correspond to the intensity relations \succsim_i and \succsim_P here.

We have made significant changes in the assumptions concerning each intensity relation. The SW model assumes that the set C mentioned above is topologically connected and that each of the intensity relations \succsim_i and \succsim satisfies a set of conditions that implies that it has a continuous difference function. It follows (as can be shown) that the range of any difference function is an interval. As the primary part of the condition of ‘properness,’ we merely assume that there exists a difference function with an interval range. This assumption provides a more general model since it does not imply the assumptions mentioned above that are used in the SW model.

The ‘properness’ condition does require, however, that the interval ranges mentioned above are non-point (which is equivalent to the condition that not all changes are indifferent). In this sense, the model here is a restriction of the SW model. Our motive for the requirement of a non-point range is to exclude a situation that we believe to be unimportant and a distraction.

The health evaluation model assumes that an intensity relation \succsim_i depends only on changes in the i -th component of the set $H_1 \times \dots \times H_N$, that is, it depends only on changes in health for the i -th individual. This assumption implies condition (D) in the SW model which states (in our terminology) that for any health distributions $\mathbf{h}^{(i)}$, $i = 1, \dots, N$, there is a health distribution \mathbf{h} such that \mathbf{h} is indifferent to $\mathbf{h}^{(i)}$ according to \succsim_i for each $i = 1, \dots, N$. The reason is that for the given distributions $\mathbf{h}^{(i)}$ we can define a health distribution $\mathbf{h} = (h_1^{(1)}, \dots, h_N^{(N)})$ where $h_i^{(i)}$ is the i -th component of $\mathbf{h}^{(i)}$. Then, for each $i = 1, \dots, N$ the distribution \mathbf{h} has the same i -th component as the distribution $\mathbf{h}^{(i)}$, and thus it is indifferent to $\mathbf{h}^{(i)}$ according to \succsim_i .

As one can verify, the Pareto condition here is equivalent to the Pareto conditions (A), (B) in the SW model. Theorem A1 implies that any individual difference function has an interval range. Hence, the Pareto condition implies by the proof of the SW model that for any given individual difference functions $w_i(h_i)$, $i = 1, \dots, N$, there exists a population difference function of the weighted-sum form (1). Theorem A1 also implies that any individual difference function has a non-point range, and it follows that the interpersonal weights must be positive. It is straightforward to verify that the weights are unique up to a positive multiple and to verify the converse implication, namely that the weighted-sum form (1) implies the Pareto condition.

Proof of Corollary 1. In an equal weights model, for any individual difference function $w(h)$ one can choose the interpersonal weights as any single positive number.

Proof of Theorem 2. Suppose that $w(h)$ is a common individual difference function and that $W(\mathbf{h}) = a_1 w(h_1) + \dots + a_N w(h_N)$ is an associated population difference function.

First, assume condition (a). If two changes, $\mathbf{h} \rightarrow (h_i = h'; \bar{h})$ and $\mathbf{h} \rightarrow (h_j = h'; \bar{h})$, are indifferent, then $a_i w(h') - a_i w(h) = a_j w(h') - a_j w(h)$ by cancelling common terms. Since we can choose h, h' such that $w(h) \neq w(h')$, it follows that $a_i = a_j$. Next, assume condition (b). If

two health distributions $(h_i = h', h_j = h''; \bar{h})$ and $(h_i = h', h_j = h''; \bar{h})$ are indifferent, then $a_i w(h') + a_j w(h) = a_i w(h) + a_j w(h')$ by cancelling common terms, and it follows that $a_i = a_j$.

One can verify that, conversely, if the weights are equal then the intensity relations \succsim_I and \succsim_P satisfy conditions (a) and (b).

Proof of Corollary 2. Apply Theorem 2 to each subpopulation.

Proof of Theorem 3. For a given individual difference function $w(h)$, Theorem 1 states that there exist weights a_1, \dots, a_N such that $W(\mathbf{h}) = a_1 w(h_1) + \dots + a_N w(h_N)$ is a population difference function, and Corollary 2 states that there exist weights $\hat{a}_1, \dots, \hat{a}_k$ such that $\hat{W}(\mathbf{h}) = \hat{a}_1 \sum_{i \in P_1} w(h_i) + \dots + \hat{a}_k \sum_{i \in P_k} w(h_i)$ is a population difference function. We can assume that the weights are normalized such that $a_1 + \dots + a_N = 1$ and $n_1 \hat{a}_1 + \dots + n_k \hat{a}_k = 1$. Then, $a_i = \hat{a}_1$ for i in P_1 and so forth. Hence, $W(\mathbf{h}) = \sum_{i \in P_1} a_i w(h_i) + \dots + \sum_{i \in P_k} a_i w(h_i) = \hat{a}_1 \sum_{i \in P_1} w(h_i) + \dots + \hat{a}_k \sum_{i \in P_k} w(h_i) = \hat{W}(\mathbf{h})$ for any health distribution \mathbf{h} .

We wish to compare the population difference function $W(\mathbf{h})$ with the function $\bar{W}(\mathbf{h}) = 1/N (w(h_1) + \dots + w(h_N))$ where $\bar{W}(\mathbf{h})$ may or may not be a population difference function. Suppose that \mathbf{h} is a health distribution which has equal scale averages as in (3), and denote the common average by \bar{w} . Then, $W(\mathbf{h}) = \hat{W}(\mathbf{h}) = n_1 \hat{a}_1 \bar{w} + \dots + n_k \hat{a}_k \bar{w} = \bar{w}$. Moreover, $\bar{W}(\mathbf{h}) = 1/N \sum_{i \in P_1} w(h_i) + \dots + 1/N \sum_{i \in P_k} w(h_i) = 1/N (n_1 \bar{w} + \dots + n_k \bar{w}) = \bar{w}$, and thus $W(\mathbf{h}) = \bar{W}(\mathbf{h}) = \bar{w}$ as was to be shown.

Proof of Theorem 4. Since the quality scale $q(s_i)$ and the individual intensity relation \succsim_i are proper, $q(s_i)$ has a non-point interval range and \succsim_i has an individual difference function $w(s_i)$ with a non-point interval range. Suppose that I_q and I_w denote the ranges of $q(s_i)$ and $w(s_i)$.

The function $w(s_i)$ represents the ordinal relation \succsim_i^o induced by \succsim_i as well as the intensity relation \succsim_i itself. It follows in particular that the ordinal relation \succsim_i^o is complete, i.e., for any health states s_i and s'_i , either s_i and s'_i are indifferent or one is preferred to the other.

For part (i), first assume condition (c). Then, for any health states s_i and s'_i , s_i is preferred to s'_i if and only if $q(s_i) > q(s'_i)$. To show that $q(s_i)$ is an ordinal scale, it remains to show that s_i is indifferent to s'_i iff $q(s_i) = q(s'_i)$. If s_i is not indifferent to s'_i , then by completeness either

s_i is preferred to s'_i or s'_i is preferred to s_i . In either case, it follows by (c) that $q(s_i) \neq q(s'_i)$. And if $q(s_i) \neq q(s'_i)$, it follows by (c) that either s_i is preferred to s'_i or s'_i is preferred to s_i .

Hence, both $q(s_i)$ and $w(s_i)$ are ordinal scales for \succsim_i^0 . By the uniqueness property of ordinal scales, there is a strictly increasing function $f(q)$ defined on the interval I_q such that $w(s_i) = f(q(s_i))$ for s_i in S . The range of the function $f(q)$ is the range I_w of the function $w(s_i)$. Since I_q and I_w are intervals, it follows that the function $f(q)$ is continuous.

For part (ii). First assume conditions (c) and (d). By part (i) there is a continuous, strictly increasing function $f(q)$ such that $w(s_i) = f(q(s_i))$ is an individual difference function. First, we show that the function $f(q)$ is a linear. For any $q < q'$ in the interval I_q , there are health states s_i, s'_i and \hat{s}_i in S such that $q(s_i) = q$, $q(s'_i) = q'$, and $q(\hat{s}_i) = 1/2 q + 1/2 q'$. Then, $q(s'_i) - q(\hat{s}_i) = q(\hat{s}_i) - q(s_i) > 0$ which implies by condition (d) that the changes, $s_i \rightarrow \hat{s}_i$ and $\hat{s}_i \rightarrow s'_i$, are indifferent which implies that $w(s'_i) - w(\hat{s}_i) = w(\hat{s}_i) - w(s_i)$ which implies that $f(q(s'_i)) - f(q(\hat{s}_i)) = f(q(\hat{s}_i)) - f(q(s_i))$. Hence, $f(q(\hat{s}_i)) = 1/2 f(q(s_i)) + 1/2 f(q(s'_i))$, and thus $f(1/2 q + 1/2 q') = 1/2 f(q) + 1/2 f(q')$. Since the function $f(q)$ is continuous and strictly increasing, this Jensen's equation implies that $f(q) = aq + b$ for some constants $a > 0$ and b . But an individual difference function is cardinally unique by Theorem A1, and thus $q(s_i)$ is an individual difference function.

Proofs of the converse implications are straightforward.

Proof of Theorem 5. By Theorem 4, there exists a conversion function $f(q)$ that is continuous, strictly increasing, and whose domain I_q is a non-point interval. By Theorem A1, an individual difference function is cardinally unique, and thus any conversion function has these properties.

Consider any points $q < q'$ and $\bar{q} = 1/2 q + 1/2 q'$ in I_q . There exist health distributions $\bar{\mathbf{s}} = (\bar{s}_1, \dots, \bar{s}_N)$ and $\mathbf{s} = (s_1, \dots, s_N)$ such that: $q(\bar{s}) = \bar{q}$, $q(s_1) = q$, $q(s_2) = q'$, and $q(s_i) = \bar{q}$ for $i > 2$. Hence, \mathbf{s} has unequal health scale amounts, and $1/N (q(s_1) + \dots + q(s_N)) = q(\bar{s}) = \bar{q}$.

If $\bar{\mathbf{s}}$ is indifferent to \mathbf{s} , then $N \times f(q(\bar{s})) = f(q(s_1)) + \dots + f(q(s_N))$ which implies that $f(\bar{q}) = 1/2 f(q) + 1/2 f(q')$. Since $q < q'$ are any points in I_q and the function $f(q)$ is continuous, this Jensen's equation implies that $f(q)$ is linear on I_q . It is also strictly increasing, and thus by cardinal uniqueness the quality scale $q(s_i)$ is an individual difference function.

If \bar{s} is preferred to s , then $N \times f(q(\bar{s})) > f(q(s_1)) + \dots + f(q(s_N))$ which implies that $f(\bar{q}) > 1/2 f(q) + 1/2 f(q')$. Since $q < q'$ are any points in the interval I_q and the function $f(q)$ is continuous, it follows that $f(q)$ is strictly concave on I_q .

Proofs of the converse implications are straightforward.

Theorem A2. If an individual intensity relation \succsim_i is scale-proper, then it is proper.

Proof. Suppose that $w(s_i, t_i)$ is a scale-proper difference function for \succsim_i and that $w^{(1)}, w^{(2)}$ are two amounts in the range of $w(s_i, t_i)$. Then, there exists health-duration pairs $(s^{(1)}, t^{(1)})$, $(s^{(2)}, t^{(2)})$ such that $w(s^{(1)}, t^{(1)}) = w^{(1)}$ and $w(s^{(2)}, t^{(2)}) = w^{(2)}$. Define $s^* = s^{(1)}$, $t^* = t^{(2)}$, and $w^* = w(s^*, t^*)$. The interval range I of the function $w(s_i, t^*)$ contains w^* and $w^{(2)}$, and the interval range J of the function $w(s^*, t_i)$ contains $w^{(1)}$ and w^* . Thus, the union of I and J is an interval, and it contains $w^{(1)}$ and $w^{(2)}$. Hence, any amount between $w^{(1)}$ and $w^{(2)}$ is in the range of the function $w(s_i, t_i)$. It follows that the range of $w(s_i, t_i)$ is an interval. It is non-point since for some range of $w(s_i, t_i)$ restricted to a single variable is non-point.

Theorem A3. Suppose that $q(s_i)$ is a proper quality scale and that \succsim_i is a scale-proper individual intensity relation. Then, $q(s_i)$ and \succsim_i satisfy conditions (e1)-(e3) if and only if \succsim_i has a continuous difference function of the form $w(s_i, t_i) = g(q(s_i), t_i)$ such that:

- (i) For any duration t , $g(q, t)$ is a strictly increasing function of q .
- (ii) For any health quality $q > 0$, $g(q, t)$ is a strictly increasing function of t .
- (iii) $g(0, t) = 0$ for any duration t , and $\lim_{t \rightarrow 0} g(q, t) = 0$ for any health quality q .

In this case, the ordinal relation \succsim_i^o defined on health-duration pairs (s_i, t_i) induces an ordinal relation \succsim_q^o defined on pairs (q, t) , and the function $g(q, t)$ is an ordinal scale for \succsim_q^o .

Proof. The properness assumptions imply that the range Q of $q(s_i)$ is a non-point interval and that \succsim_i has a difference function $w(s_i, t_i)$. The function $w(s_i, t_i)$ also represents the ordinal relation \succsim_i^o defined by \succsim_i . and thus \succsim_i^o is complete.

Assume conditions (e1)-(e3). Since \succsim_i^o is complete, condition (e1) implies that for any s_i , s'_i , and t_i : $q(s_i) = q(s'_i)$ iff (s_i, t_i) is indifferent to (s'_i, t_i) . But, (s_i, t_i) is indifferent to (s'_i, t_i) iff $w(s_i, t_i) = w(s'_i, t_i)$. Therefore, $w(s_i, t_i) = g(q(s_i), t_i)$ where $g(q, t)$ is a function of quality scale amounts $q = q(s_i)$ in the interval Q and durations $t = t_i$ in the interval T .

Condition (e1) implies that for any t^* in T , the function $g(q, t^*)$ is strictly increasing. Condition (e2) implies that for any $q^* > 0$ in Q , the function $g(q^*, t)$ is strictly increasing and that for $q^* = 0$ the function $g(0, t)$ is constant. Since the domains and ranges of these functions are intervals, it follows that they are continuous. Since they are strictly increasing or constant, it then follows the function $g(q, t)$ is continuous.

Since the difference function $w(s_i, t_i)$ is cardinally unique, we can subtract the constant $g(0, t)$ from it and from $g(q, t)$ to obtain functions, which we also denote by $w(s_i, t_i)$ and $g(q, t)$ such that $g(0, t) = 0$ for t in the interval T .

To show that $\lim_{t \rightarrow 0} g(q, t) = 0$ for any health quality q , it suffices to show that for any $\varepsilon > 0$, there exists a duration t such that $g(q, t) < \varepsilon$. Since $g(0, t) = 0$ for any t , there is nothing to show when $q = 0$. Consider $q > 0$ and $\varepsilon > 0$. First, choose $t^* = 1$. Since the function $g(q, 1)$ is continuous and strictly increasing, there is a $q^* > 0$ such that $g(q^*, 1) < \varepsilon$. Choose an s^* with $q(s^*) = q^*$ and an s_i with $q(s_i) = q$. Condition (e4) implies that there is a duration t such that $w(s_i, t) < w(s^*, 1)$. Hence, $g(q, t) < g(q^*, 1) < \varepsilon$.

The converse implications are straightforward. Moreover, the ordinal relation \succsim_i^o on pairs (s_i, t_i) induces an ordinal relation \succsim_q^o on pairs (q, t) since for any fixed duration $t = t_i$ the quality scale $q(s_i)$ is an ordinal scale for \succsim_i^o . And the function $g(q, t)$ is an ordinal scale for \succsim_q^o since the function $g(q(s_i), t_i)$ is an ordinal scale for \succsim_i^o .

Proof of Theorem 6. Assume conditions (e1)-(e3) and (f2). Then, there exists a function $w(s_i, t_i) = g(q(s_i), t_i)$ as described in Theorem A3. The intensity relation \succsim_i on pairs (s_i, t_i) induces an intensity relation \succsim_g on the pairs (q, t) , and $g(q, t)$ is a difference function for \succsim_g . Condition (f1) implies that for any amount $q > 0$, the relation \succsim_g induces the same intensity relation on changes in duration. Each function $g(q, t)$ with fixed $q > 0$ is a difference function with an interval range for this common intensity relation. Select an amount $q^* > 0$. Cardinal uniqueness implies that $g(q, t) = a(q)g(q^*, t) + b(q)$ for any $q > 0$ where $a(q) > 0$. Thus, $a(q)$ and $b(q)$ are functions of $q > 0$. But, $\lim_{t \rightarrow 0} g(q, t) = 0$ for any amount q , and thus $b(q) = 0$ for $q > 0$. Hence, $g(q, t) = a(q)g(q^*, t)$ for $q > 0$. Define $a(0) = 0$. Then, $g(q, t) = a(q)g(q^*, t)$ for any amount q . Theorem A3 implies that the function $f(q)$ defined as $a(q)$ and the function

$D(t) = g(q^*, t)$ are continuous and strictly increasing with $f(0) = 0$ and $\lim_{t \rightarrow 0} D(t) = 0$. Hence, the difference function $w(s_i, t_i) = D(t_i) f(q(s_i))$ has the product form as described.

Conditions (e1)-(e3) and (f2) imply the form (4) as described by similar arguments, and it is straightforward to verify the converse implications.

If $w(s_i, t_i) = D(t_i) f(q(s_i))$ and $\hat{w}(s_i, t_i) = \hat{D}(t_i) \hat{f}(q(s_i))$ are two difference functions as described, then $\hat{w} = a w + b$ for some constants $a > 0$ and b . But $b = 0$ since $f(0) = \hat{f}(0) = 0$. Hence, $\hat{D}(t_i) \hat{f}(q(s_i)) = a D(t_i) f(q(s_i))$ where $a > 0$. It is straightforward to show that this implies that $\hat{D}(t_i) = a_1 D(t_i)$ and $\hat{f}(q(s_i)) = a_2 f(q(s_i))$ for some constants $a_1, a_2 > 0$.

Proof of Theorem 7. For part (i), assume conditions (e1)-(e3) and (g1). For any durations $t < t'$, define $\hat{t} = 1/2 t + 1/2 t'$. Condition (g1) implies that $g(q, t') - g(q, \hat{t}) = g(q, \hat{t}) - g(q, t)$ for any amount q . Hence, $g(q, 1/2 t + 1/2 t') = 1/2 g(q, t) + 1/2 g(q, t')$. For a fixed q , $g(q, t)$ is a strictly increasing or constant function, and thus this Jensen's equation implies that $g(q, t) = a(q) t + b(q)$ where $a(q) > 0$ for $q > 0$. By the same arguments as in the previous theorem, it follows that the difference function $w(s_i, t_i) = g(q(s_i), t_i)$ is as described in (5a).

The remaining arguments are omitted. The reason is that the proof of the forward implications in (ii) and (iii) are similar to the above argument and the proof of the converse implications and of the uniqueness statement are straightforward verifications.

Theorem A4. Suppose that a quality scale $q(s_i)$ and an individual intensity relation \succsim_i satisfy the assumptions in Theorem A3 and that a pair (q^*, t^*) with $q^* > 0$ has been specified. Then (using the notation in Theorem A3):

(i) For any pair $(q(s_i), t_i)$ with $0 < q(s_i) \leq q^*$ there exists a unique amount \hat{t} such that (q^*, \hat{t}) is indifferent to $(q(s_i), t_i)$. Moreover, $0 < \hat{t} \leq t_i$. The scale $\hat{t} = \text{QALY}(q(s_i), t_i)$ defined in this manner is a continuous ordinal scale for pairs $(q(s_i), t_i)$ with $0 < q(s_i) \leq q^*$.

(ii) For any pair $(q(s_i), t_i)$ with $0 < \hat{t} \leq t_i$ there exists a unique amount \hat{q} such that (\hat{q}, t^*) is indifferent to $(q(s_i), t_i)$. Moreover, $0 < \hat{q} \leq q(s_i)$. The scale $\hat{q} = \text{YALQ}(q(s_i), t_i)$ defined in this manner is a continuous ordinal scale for pairs $(q(s_i), t_i)$ with $0 < \hat{t} \leq t_i$.

(iii) The single-variable functions $Q^*(q) = \text{QALY}(q, t^*)$ and $Y^*(t) = \text{YALQ}(q^*, t)$ defined on $0 < q \leq q^*$ and $0 < t \leq t^*$ are inverses of one another.

Proof. The proofs of parts (i) and (ii) are similar, and thus we omit the proof of (ii).

To show part (i), suppose that $g(q(s_i), t_i)$ is an ordinal scale as described in Theorem A3, and define $g_*(t) = g(q^*, t)$. Then, $g_*(t)$ is a continuous and strictly increasing function whose domain is the interval $T = (0, \infty)$ and whose range is a non-point interval of the form $(0, b]$ or $(0, b)$ where the upper endpoint b in $(0, b)$ may be infinite.

Consider a pair $(q(s_i), t_i)$ with $0 < q(s_i) \leq q^*$. Then, $0 < g(q(s_i), t_i) \leq g(q^*, t_i) = g_*(t_i)$. Hence, there exists a unique \hat{t} such that $g_*(\hat{t}) = g(q(s_i), t_i)$, and thus (q^*, \hat{t}) is indifferent to $(q(s_i), t_i)$. Moreover, $0 < \hat{t} \leq t^*$. The function $w = g_*(t)$ has a continuous, strictly increasing inverse function $g_*^{-1}(w)$, and thus $\hat{t} = g_*^{-1}(g(q(s_i), t_i))$.

Define $\text{QALY}(q(s_i), t_i)$ as \hat{t} for $0 < q(s_i) \leq q^*$. Then, $\text{QALY}(q(s_i), t_i) = g_*^{-1}(g(q(s_i), t_i))$. Since $g(q(s_i), t_i)$ is an ordinal scale and the function $g_*^{-1}(w)$ is strictly increasing, it follows that $\text{QALY}(q(s_i), t_i)$ is an ordinal scale. And the function $\text{QALY}(q(s_i), t_i)$ is continuous since the functions $g(q(s_i), t_i)$ and $g_*^{-1}(w)$ are continuous.

For part (iii), suppose that $t = Q^*(q) = \text{QALY}(q, t^*)$ where $0 < q \leq q^*$ and thus $0 < t \leq t^*$. Then, (q^*, t) is indifferent to (q, t^*) , and thus $Y^*(t) = \text{YALQ}(q^*, t) = q$. In a similar manner, $q = Y^*(t) = \text{YALQ}(q^*, t)$ implies that $t = Q^*(q) = \text{QALY}(q, t^*)$. Hence the functions $Q^*(q)$ and $Y^*(t)$ are inverses of one another.

Proof of Theorem 8. Consider a pair $(q(s_i), t_i)$ with $0 < q(s_i) \leq q^*$ and $0 < t_i \leq t^*$. Then, there exist unique amounts \hat{q} and \hat{t} such that the pairs (\hat{q}, t^*) , $(q(s_i), t_i)$, and (q, t^*) are indifferent. Moreover, $0 < \hat{q} \leq q(s_i)$, $0 < \hat{t} \leq t_i$ and $\hat{t} = \text{QALY}(q(s_i), t_i)$, $\hat{q} = \text{YALQ}(q(s_i), t_i)$. By Theorem A3, the individual difference function $w^*(s_i, t_i)$ can be written as $w^*(s_i, t_i) = g^*(q(s_i), t_i)$.

Consider formula (6). The normalization $f(q^*) = 1$ implies that $w^*(s_i, t_i) = g^*(q^*, \hat{t}) = D(\hat{t})f(q^*) = D(\text{QALY}(q(s_i), t_i))$, and the normalization $D(t^*) = 1$ implies that $w^*(s_i, t_i) = g^*(\hat{q}, t^*) = D(t^*)f(\hat{q}) = f(\text{YALQ}(q(s_i), t_i))$.

Next, consider formula (7). The definition $Q^*(q(s_i)) = \text{QALY}(q(s_i), t^*)$ implies that the pairs $(q(s_i), t^*)$ and $(q^*, Q^*(q(s_i)))$ are indifferent. Hence, $g^*(q(s_i), t^*) = g^*(q^*, Q^*(q(s_i)))$, and thus $D(t^*)f(q(s_i)) = D(Q^*(q(s_i)))f(q^*)$ which implies that $f(q(s_i)) = D(Q^*(q(s_i)))$. Hence, $w^*(s_i, t_i) = D(t_i)f(q(s_i)) = D(t_i)D(Q^*(q(s_i)))$. The proof that $w^*(s_i, t_i) = f(q(s_i))f(Y^*(q(s_i)))$ is similar.

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