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A GENERAL REPRESENTATION THEOREM FOR INTEGRATED VECTOR AUTOREGRESSIVE PROCESSES

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ABSTRACT. We study the algebraic structure of an $I(d)$ vector autoregressive process, where d is restricted to be an integer. This is useful to characterize its polynomial cointegrating relations and its moving average representation, that is to prove a version of the Granger representation theorem valid for $I(d)$ vector autoregressive processes.

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1. Introduction

Since Engle and Granger (1987) introduced the concept of cointegration many scholars have worked in this field either from the Wold representation of the process (Engle and Granger, 1987, Engle and Yoo, 1991, Phillips, 1991, Gregoire and Laroque, 1993, Stock and Watson, 1993, Gregoire and Laroque, 1994) or from the autoregressive formulation (Johansen, 1988, Ahn and Reinsel, 1990, Johansen, 1992, 1996, Paruolo, 1996) and have addressed and solved many issues regarding representation, estimation and inference.

More recently, we find a series of papers dealing exclusively with the algebraic properties of cointegrated systems which are relevant for understanding their order of integration and for deriving their different representations: Archontakis (1998) discusses the $I(1)$ case through the Jordan decomposition of the companion form, Haldrup and Salmon

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(1998) use the Smith-McMillan form to characterize $I(2)$ process, la Cour (1998) gives an algorithm for the $I(d)$ case, Neusser (2000) studies the $I(1)$ model by means of the Drazin inverse while Faliva and Zoia (2002) discuss $I(1)$ and $I(2)$ processes using a newly found result on the inversion of partitioned matrices. A general discussion of multiple frequency unit root processes can be found in Gregoire (1999) and in Bauer and Wagner (2003).

In this paper we extend the study of the algebraic structure and thus of the representation theory to $I(d)$ vector autoregressive process, where d can be any integer. The main difficulty in doing so resides in establishing the conditions under which the process it is integrated of the given order d; this is because the standard $I(1)$ and $I(2)$ rank conditions (see Johansen, 1996) are neither immediately available from the autoregressive nor from the error correction representation and are found only when the inversion of the characteristic polynomial is conducted explicitly (see Johansen, 2005, for an exhaustive survey of the mathematical results concerning the representation theory). When we want to consider higher order processes the conditions become more and more involved and it is very hard to be able to isolate them through direct inversion (an attempt to do so can be found in la Cour, 1998).

Here we proceed differently: starting from the fact that the order of integration of the process equals the difference between the multiplicity of the unit root in the characteristic equation and in the adjoint matrix polynomial (see Franchi, 2006), we use the adjoint matrix to give *i*) the Johansen's rank type condition, *ii*) the cointegration properties of the process, and *iii*) its polynomial cointegrating relations for any order of integration. This becomes feasible because the study of the adjoint allows to characterize the inverse without having to perform the inversion directly.

2. Basics

The definition of order of integration of a stochastic process is

Definition 2.1. *The linear process* $X_t = C(L)\epsilon_t$, $\epsilon_t \sim$ *i.i.d. is called* $I(0)$ *if* $C(z) = \sum_{i=0}^{\infty} C_i z^i$ *converges for* $|z| < 1 + \delta$ *for some* $\delta > 0$ *and* $C(1) \neq 0_p$*; if* $\Delta^d X_t$ *is* $I(0)$ *then* X_t *is* $I(d)$.

A stationary process is defined as the infinite moving average of an i.i.d. sequence with coefficients C_i such that its covariance structure

$cov(X_t,\ X_{t+h}) = \sum_{i=0}^{\infty} C_i \Omega C'_{i+h}$ is well defined. Note that by this definition any invertible $MA(\infty)$ is $I(0)$ but also those processes for which $|C(1)| = 0$ and $C(1) \neq 0$ are $I(0)$. In these cases there exist linear combinations of the original process which have a negative order of integration, the phenomenon called co-stationarity by Gregoire and Laroque (1993).

The definitions of cointegration and polynomial cointegration are

Definition 2.2. *The $I(d)$ process X_t is cointegrated if there exists β such that $\beta' X_t$ is $I(b)$ with $b < d$. It is polynomially cointegrated if there exists β_k for $k = 1, \cdots, d$, such that $\sum_{k=1}^{d} \beta'_k \Delta^{k-1} X_t$ is stationary.*

As usual we say that a process is cointegrated when it is integrated of a given order and there exist linear combinations of X_t having a lower order of integration; we say that it is polynomially cointegrated when it is possible to define a stationary process by combing linearly the process at different points in time. Obviously, the lowest order of integration for which polynomial cointegration arises is two.

We want to study the polynomial cointegration properties of the integrated process which is defined as the solution of the autoregressive equation

$$X_t = \Pi_1 X_{t-1} + \Pi_2 X_{t-2} + \cdots + \Pi_k X_{t-k} + \epsilon_t \tag{2.1}$$

where X_t is $p-$dimensional and ϵ_t is an i.i.d. sequence.

The solution of (2.1) is an integrated process when the roots of the characteristic equation are either one or lie outside the unit circle, that is

$$|\Pi(z)| = (z-1)^m g(z),\ g(1) \neq 0 \tag{2.2}$$

where $\Pi(z) = I - \sum_{i=1}^{k} \Pi_i z^i$ is the characteristic polynomial of (2.1), $|\Pi(z)| = det(\Pi(z))$ and the roots of $g(z)$ are all outside the unit circle. Note that $g(1) \neq 0$ implies that $m > 0$ is the multiplicity of the unit root in the characteristic equation. From $|\Pi(1)| = 0$ we have that $rank(\Pi(1)) = r_1 < p$ and thus that $\Pi(1)$ can be written as the product of two $p \times r_1$ matrices δ_1 and γ_1 of full rank r_1; that is $-\delta_1 \gamma'_1 = \Pi(1)$.

Before discussing when the solution of (2.1) is integrated of order d, note that the Taylor series representation of $\Pi(z)$

$$\Pi(z) = \sum_{v=0}^{d-1} \frac{\Pi^{(v)}(1)}{v!}(z-1)^v + (1-z)^d \Pi_d(z)$$

allows us to rewrite (2.1) as

$$\sum_{v=0}^{d-1}(-1)^v \frac{\Pi^{(v)}(1)}{v!}\Delta^v X_t + \Pi_d(L)\Delta^d X_t = \epsilon_t$$

from which we immediately see that since ϵ_t is i.i.d. and $\Pi_d(L)\Delta^d X_t$ is a finite moving average of an $I(0)$ process the linear combination $\sum_{v=0}^{d-1}(-1)^v \frac{\Pi^{(v)}(1)}{v!}\Delta^v X_t$ is stationary.

Hence the coefficients in

$$\gamma_1' X_t - \bar{\delta}_1' \sum_{v=1}^{d-1}(-1)^v \frac{\Pi^{(v)}(1)}{v!}\Delta^v X_t \tag{2.3}$$

provide the polynomial cointegrating relation in the γ_1 direction. As one can imagine γ_1 is one of the many cointegrating vectors of an $I(d)$ process; it will be clear that the others are derived exactly as this one, the only difference being that some more algebra is needed to reveal their coefficients.

3. Order of integration

Let $\Pi_a(z)$ be the adjoint matrix of $\Pi(z)$; then Franchi (2006) shows that

$$\Pi_a(z) = (z-1)^a H(z)$$

where $H(1) \neq 0$ and we call $a \geq 0$ the multiplicity[1] of the unit root in the adjoint matrix polynomial. The inverse is then equal to

$$\Pi(z)^{-1} = \frac{\Pi_a(z)}{|\Pi(z)|} = \frac{H(z)}{(z-1)^{m-a}g(z)},\ z \neq \{z : |\Pi(z)| = 0\},$$

where $H(1) \neq 0$, $g(1) \neq 0$ and $m - a > 0$.

It is interesting to see that the reason why the multiplicity of the unit root in the characteristic equation is not sufficient to determine the order of integration is simply that the factor $z-1$ appears both at the numerator and at the denominator of $\Pi(z)^{-1}$ and cancels. Exactly as it is in the univariate case the order of integration is equal to the order of the pole of the inverse function at the unit root. Then what is important is the difference between the two multiplicities and not the number of unit roots in the characteristic equation alone. That is, we have that

[1]Note that is in the matrix sense, that is the adjoint matrix is zero not only its determinant.

Theorem 3.1. *The process X_t in (2.1) is integrated of order*

$$d = m - a$$

where m is the multiplicity of the unit root in $|\Pi(z)|$ and a is the multiplicity of the unit root in $\Pi_a(z)$.

Proof. The roots of $g(z)$ being outside the unit circle imply that the function

$$C(z) = \frac{H(z)}{g(z)},\ z \neq \{z : g(z) = 0\}$$

is well defined for $|z| < 1 + \delta$ for some $\delta > 0$. Thus $C(z) = \sum_{i=0}^{\infty} C_i z^i$ converges on the same disc and it is such that $C(1) = \frac{H(1)}{g(1)} \neq 0$; this means that $\Delta^{m-a} X_t$ is $I(0)$ and thus that X_t is integrated of order $d = m - a$. ∎

Requiring $m - a = 1$ or $m - a = 2$ (see Franchi, 2006) is equivalent to assume the well known $I(1)$ and $I(2)$ rank conditions in Johansen (1996). So we replace a statement about the rank of matrix which is not immediately available from the autoregressive representation with one on that can be easily computed given $\Pi(z)$, as the following example shows.

Example: *The matrix polynomial*

$$\Pi(z) = \begin{pmatrix} 1 & 0 & -\frac{z}{2}(1-z)^2 \\ 0 & 1-z & 0 \\ -\frac{z}{2}(1-z) & 0 & (1-z)^3 \end{pmatrix}$$

has determinant

$$|\Pi(z)| = (1-z)^4(1 - \frac{z^2}{4})$$

so that $m = 4$ and $g(1) = \frac{3}{4}$. The adjoint matrix polynomial is

$$\Pi_a(z) = \begin{pmatrix} (1-z)^4 & 0 & \frac{z}{2}(1-z)^3 \\ 0 & (1-z)^3(1-\frac{z^2}{4}) & 0 \\ \frac{z}{2}(1-z)^2 & 0 & 1-z \end{pmatrix}$$

so that $a = 1$, $H(1) = diag(0, 0, 1) \neq 0$ and the process is $I(3)$.

4. Some interesting algebraic relations

It is the simplification in theorem 3.1 that makes the derivation of the general result in theorem 4.1 possible; in fact the sequence of rank conditions we are used to will appear very naturally without performing the inversion explicitly. The reason being that $\Pi(z)^{-1}$ is a scalar times

$H(z)$ and the relation between $\Pi(z)$ and $H(z)$ is incorporated in the identity $\Pi(z)\Pi_a(z) = \Pi_a(z)\Pi(z) = |\Pi(z)|I$, which is now written as

$$\Pi(z)H(z) = H(z)\Pi(z) = (z-1)^d g(z)I. \tag{4.1}$$

At the unit root this expression is $\delta_1\gamma_1' H(1) = H(1)\delta_1\gamma_1' = 0$ and since $H(1) \neq 0$ it implies

$$H(1) = \gamma_{1\perp}\phi_1\delta_{1\perp}' \tag{4.2}$$

for some $\phi_1 \neq 0$. Since $H(z) = H(1) + (z-1)H_1(z)$ for some finite matrix polynomial $H_1(z)$ we can write

$$\Pi(z)^{-1} = \frac{\gamma_{1\perp}\phi_1\delta_{1\perp}'}{(z-1)^d g(z)} + \frac{H_1(z)}{(z-1)^{d-1}g(z)},\ z \neq \{z : |\Pi(z)| = 0\},$$

from which we immediately see that γ_1 is a cointegrating vector since $\gamma_1'\Pi(z)^{-1}$ has at most a pole of order $d-1$ at the unit root. Obviously a vector β can be cointegrating if and only if it is such that $\beta' H(1) = 0$.

In theorem 4.1 we will show (see (4.6), (4.8) and (4.9)) that for an $I(d)$ process, $H(1)$ has the following nested structure

$$H(1) = \gamma_{1\perp}\cdots\gamma_{d\perp}\phi_d\delta_{d\perp}'\cdots\delta_{1\perp}',$$

where ϕ_d is the full rank matrix which provides the Johansen's rank condition for the order of integration and δ_s and γ_s are defined by the reduced rank nature of specific matrices (see (4.5) and (4.7)). So we are basically extending what we already know for $I(1)$ and $I(2)$ processes, the main difficulty being that we need to keep track of the evolution of the reduced rank matrices that define the sequence of δ_s and γ_s.

Once we understand $H(1)$ we understand cointegration; in fact the cointegrating vectors are simply given by $\beta_s = \bar{\gamma}_{1\perp}\cdots\bar{\gamma}_{s-1\perp}\gamma_s$. The polynomial cointegration properties of the process will instead be understood by studying the term $\beta_s' H_1(z)$ (see (4.4)).

Since $(z-1)^d g(z)I$ needs to be differentiated at least d times to be different from zero at $z=1$, the derivative of order n of (4.1) at $z=1$ immediately gives the following relations[2]

(4.3)
$$-\delta_1\gamma_1' H^{(n)} + \sum_{v=1}^{n} \binom{n}{v} \Pi^{(v)} H^{(n-v)} = \begin{cases} 0 & \text{if} \quad n = 1, \cdots, d-1 \\ d!gI & \text{if} \quad n = d. \end{cases}$$

[2]For notational convenience we write Π, H and g instead of $\Pi(1)$, $H(1)$ and $g(1)$; similarly, we also drop the reference to one in the derivatives, that is we let $\Pi^{(n)} = \frac{d^n}{dz^n}\Pi(z)\big|_{z=1}$ and $H^{(n)} = \frac{d^n}{dz^n}H(z)\big|_{z=1}$. When convenient we write $\dot{\Pi}$ and $\ddot{\Pi}$ for first and second derivatives.

The manipulation of this expression yields *i*) the Johansen's rank type condition for the order of integration (see (4.8)), *ii*) the cointegration properties of the process (see (4.9)), and *iii*) its polynomial cointegration properties (see (4.4)). The main algebraic results are collected in the following theorem.

Theorem 4.1. *Let* $m - a = d$, $n = 1, \cdots, d-1$, $s = 1, \cdots, n$ *and* $n_s = n + 1 - s$*; then*

$$\beta_s' H^{(n_s)} = \bar{\alpha}_s' \sum_{v=1}^{n_s} \binom{n_s}{v} \theta_v^s H^{(n_s - v)} \tag{4.4}$$

where $\beta_s = \bar{\zeta}_{s-1}\gamma_s$, $\zeta_s = \zeta_{s-1}\gamma_{s\perp}$, $\alpha_s = \bar{\eta}_{s-1}\delta_s$, $\eta_s = \eta_{s-1}\delta_{s\perp}$ *and* $\eta_0 = \zeta_0 = I$*;* ϕ_1 *in (4.2) and the coefficients in (4.4) are defined by the recursion*

$$-\delta_{s+1}\gamma_{s+1}' = \eta_s'\theta_1^s\zeta_s \tag{4.5}$$

and

$$\gamma_{s+1\perp}\phi_{s+1}\delta_{s+1\perp}' = \phi_s \tag{4.6}$$

where

$$\theta_v^1 = \Pi^{(v)} \text{ and } \theta_v^s = \theta_1^{s-1} \sum_{j=1}^{s-1} \bar{\beta}_j \bar{\alpha}_j' \theta_v^j + \frac{\theta_{v+1}^{s-1}}{v+1} \text{ for } s \neq 1. \tag{4.7}$$

The recursion ends with

$$\phi_d = g(\eta_d'\theta_1^d\zeta_d)^{-1} \tag{4.8}$$

being full rank and then

$$H = \zeta_d\phi_d\eta_d'. \tag{4.9}$$

Proof. See Appendix. ■

Some interesting features of the result should be considered: first of all note that as long as $s < d$, $\delta_{s\perp}' \cdots \delta_{1\perp}'\theta_1^s\gamma_{1\perp} \cdots \gamma_{s\perp}$ in (4.5) has reduced rank and thus it can be written as the product of two non square matrices δ_{s+1} and γ_{s+1} that span the same lower dimensional space generated by the matrix. When $s = d$ (see (4.8)) the matrix $\eta_d'\theta_1^d\zeta_d$ is full rank, it spans the full space in which it lives and no additional smaller base can be defined.

The main difficulty in getting these rank properties correctly resides in the fact that it requires to look at the right matrices (the θ_1^s) in the right coordinates (the η_s and ζ_s) and the evolution of θ_1^s is not trivial (see (4.7)). These rank conditions are very important because they define the sequence of coefficients that will be used for cointegration and polynomial cointegration.

In theorem 5.2 we will show that the polynomial cointegration properties can be fully understood from (4.4). Even more simply, from (4.9) we immediately see that $\beta_s = \bar{\zeta}_{s-1}\gamma_s$ is a cointegrating vector (see theorem 5.1): the nested structure of $H = \gamma_{1\perp}\gamma_{2\perp}\cdots\gamma_{d\perp}\phi_d\eta_d'$ defines the d directions in which linear combinations of the process have lower order of integration. The first ones lie in the space which is orthogonal to $sp(\gamma_{1\perp})$, the second ones lie in that part of $sp(\gamma_{1\perp})$ which is orthogonal to $sp(\gamma_{2\perp})$, the third ones lie in that part of $sp(\gamma_{2\perp})$ which is orthogonal to $sp(\gamma_{3\perp})$, and so on. At any round we split the space spanned by $sp(\gamma_{s-1\perp})$ into the two orthogonal subspaces given by $sp(\gamma_s)$ and $sp(\gamma_{s\perp})$ and use the first direction for β_s and part of the second one for β_{s+1}. We keep on splitting smaller and smaller spaces until we reach the full rank matrix ϕ_d which fills up all the space in which it lives and no other cointegrating vector can be defined.

Before discussing the stochastic counterpart of these algebraic results, let us see how they specialize in the $I(1)$ and $I(2)$ cases; from (4.7) we have that $\theta_1^1 = \dot{\Pi}$ and $\theta_1^2 = \dot{\Pi}\bar{\gamma}_1\bar{\delta}_1'\dot{\Pi} + \frac{\ddot{\Pi}}{2}$, thus when $d = 1$ (4.8) gives

$$|\delta_{1\perp}'\dot{\Pi}\gamma_{1\perp}| \neq 0$$

which is the well known $I(1)$ condition; when $d = 2$ (4.5) defines

$$-\delta_2\gamma_2' = \delta_{1\perp}'\dot{\Pi}\gamma_{1\perp}$$

and from (4.8) we know that

$$\eta_2'\theta_1^2\zeta_2 = \delta_{2\perp}'\delta_{1\perp}'\{\dot{\Pi}\bar{\gamma}_1\bar{\delta}_1'\dot{\Pi} + \frac{\ddot{\Pi}}{2}\}\gamma_{1\perp}\gamma_{2\perp} \tag{4.10}$$

is full rank, which is the usual $I(2)$ condition. Then (4.9) gives

$$H = \gamma_{1\perp}\phi_1\delta_{1\perp}'$$

with $|\phi_1| \neq 0$ in the first case and

$$H = \gamma_{1\perp}\gamma_{2\perp}\phi_2\delta_{2\perp}'\delta_{1\perp}'$$

with $|\phi_2| \neq 0$ in the second.

5. The stochastic counterpart

Theorem 5.1 (Cointegration). *The vectors β_s, $s = 1, \cdots, d$, in theorem 4.1 (and no others) are the cointegrating vectors.*

Proof. From $\zeta_d = \zeta_{s-1}\gamma_{s\perp}\cdots\gamma_{d\perp}$, $\beta_s = \bar{\zeta}_{s-1}\gamma_s$ and $H = \zeta_d \phi_d \eta_d'$ it follows that $\beta_s' H = 0$; then $\beta_s'\Pi(z)^{-1}$ has a pole at most of order $d-1$ at $z = 1$ and the process $\beta_s' X_t$ is integrated of order $b < d$. ∎

An $I(d)$ process is such that there are d directions in which linear combinations of X_t have a lower order of integration and these are given by $\beta_1, \beta_2, \cdots, \beta_d$ as defined in theorem 4.1. When $d = 1$ the only cointegrating relation is given by $\gamma_1' X_t$ and the process is directly reduced to stationary; when $d = 2$ both $\gamma_1' X_t$ and $\gamma_2'\bar{\gamma}_{1\perp}' X_t$ are $I(1)$ processes. The important difference among the β_s can be appreciated only when we consider the polynomial cointegrating relations. The reason being that depending on which direction we choose the order of integration can be reduced differently. Think about the well known $I(2)$ case: in the $\beta_1 = \gamma_1$ direction we can combine the two $I(1)$ processes $\gamma_1' X_t$ and ΔX_t in such a way that their linear combination is stationary but no such way exists in the $\beta_2 = \bar{\gamma}_{1\perp}\gamma_2$ direction where the only way of reducing $\gamma_2'\bar{\gamma}_{1\perp}' X_t$ to stationarity is by differentiation.

This is exactly what happens in the general case (see theorem 5.2): in the β_1 direction we can go from $I(d)$ to stationarity by taking linear combinations, in the β_2 direction from $I(d)$ to $I(1)$ by linear combinations and then to stationarity by first differences, in the β_3 direction from $I(d)$ to $I(2)$ and then use Δ^2 to achieve stationarity, and so on until the β_d direction in which no linear combination can reduce the order of integration and we must use Δ^{d-1} to achieve stationarity.

As theorem 5.2 makes clear this is the consequence of the algebraic relations among the derivatives of $H(z)$ which are described in (4.4).

Theorem 5.2 (Polynomial cointegration). *Let α_s, β_s and θ_v^s be as in theorem 4.1 and $\psi_v^s = \frac{(-1)^v}{v!}\theta_v^s$; then the process*

$$\beta_s'\Delta^{s-1}X_t - \bar{\alpha}_s' \sum_{v=1}^{d-s} \psi_v^s \Delta^{v+s-1} X_t, \; s = 1, \cdots, d$$

is stationary.

Proof. See Appendix. ∎

Note that for $s = 1$ the polynomial cointegrating relation $\beta_1' X_t - \sum_{v=1}^{d-1} \psi_v^1 \Delta^v X_t$ is nothing but

$$\gamma_1' X_t - \bar{\delta}_1' \sum_{v=1}^{d-1} (-1)^v \frac{\Pi^{(v)}}{v!} \Delta^v X_t$$

which was derived in (2.3) applying the balancing argument. While these coefficients were immediately available from the outset of the analysis, α_s, β_s and ψ_v^s have required some manipulations to be revealed.

Now we state the general representation theorem.

Theorem 5.3 (Representation of integrated processes)**.** *Let X_t in (2.1) be $I(d)$, the coefficients as in theorem 4.1, $\psi_v^s = \frac{(-1)^v}{v!}\theta_v^s$ and $s = 1, \cdots, d$; then X_t has the representation*

$$X_t = C_d S_t^d + \cdots + C_1 S_t^1 + C_d(L)\epsilon_t + A \tag{5.1}$$

where $S_t^b \sim I(b)$ is obtained by cumulating b times ϵ_t, $C_d(L)\epsilon_t$ is stationary, A depends on initial values, and

$$C_d = (-1)^d \frac{H}{g} = (-1)^d \zeta_d (\eta_d' \theta_1^d \zeta_d)^{-1} \eta_d'. \tag{5.2}$$

Moreover, β_s is a cointegrating vector and

$$\beta_s' \Delta^{s-1} X_t - \bar{\alpha}_s' \sum_{v=1}^{d-s} \psi_v^s \Delta^{v+s-1} X_t \tag{5.3}$$

is a polynomial cointegrating relation.

Proof. Since $\Pi(z)^{-1}$ has a pole of order d at $z = 1$ its Laurent expansion is given by

$$\Pi(z)^{-1} = \sum_{v=0}^{d-1} \frac{C_{d-v}}{(1-z)^{d-v}} + L(z)$$

where

$$C_{d-v} = (-1)^{d-v} \left. \frac{d^v}{dz^v} \frac{H(z)}{g(z)} \right|_{z=1}$$

and $L(z)$ converges for $|z| < 1 + \delta$ for some $\delta > 0$; thus (5.1) and (5.2) follow. The cointegration and polynomial cointegration properties were proved in theorems 5.1 and 5.2. ∎

From the moving average representation in (5.1) we see that X_t is composed of $I(1)$ up to $I(d)$ processes which are generated by cumulating ϵ_t plus a stationary infinite moving average part given by $C_d(L)\epsilon_t$.

Each of the non stationary components is loaded into X_t through the corresponding C coefficient and in (5.2) we give the explicit expression of C_d, which defines the cointegrating relations β_s. The other C coefficients are more complicated and not very interesting in themselves; what is important is to understand which linear combinations of the process are stationary. These are the polynomial cointegrating relations described in (5.3) which state that the processes

$$\begin{gathered}
\beta_1' X_t - \bar{\alpha}_1' \textstyle\sum_{v=1}^{d-1} \psi_v^1 \Delta^v X_t, \\
\beta_2' \Delta X_t - \bar{\alpha}_2' \textstyle\sum_{v=1}^{d-2} \psi_v^2 \Delta^{v-1} X_t, \\
\vdots \\
\beta_{d-1}' \Delta^{d-2} X_t - \bar{\alpha}_{d-1}' \psi_1^{d-1} \Delta^{d-1} X_t,
\end{gathered}$$

and

$$\beta_d' \Delta^{d-1} X_t$$

are stationary. So we see that in the β_1 direction we can combine the process in such a way that we go from $I(d)$ to stationarity, in the β_2 direction from $I(d)$ to $I(1)$ by polynomial cointegration and then to stationarity by first differences, in the β_3 direction from $I(d)$ to $I(2)$ and then to stationarity by Δ^2, and so on up to the β_d direction in which no polynomial cointegration is present we must use Δ^{d-1} to achieve stationarity.

Note that the result specializes for $d = 1$ into

$$X_t = C_1 \sum_{i=1}^{t} \epsilon_i + C_1(L)\epsilon_t + A$$

where

$$C_1 = -\gamma_{1\perp}(\delta_{1\perp}' \dot{\Pi} \gamma_{1\perp})^{-1} \delta_{1\perp}'$$

and

$$\gamma_1' X_t$$

being stationary and for $d = 2$ into

$$X_t = C_2 \sum_{j=1}^{t} \sum_{i=1}^{j} \epsilon_i + C_1 \sum_{i=1}^{t} \epsilon_i + C_2(L)\epsilon_t + A$$

where

$$C_2 = \gamma_{1\perp}\gamma_{2\perp}(\eta_2' \theta_1^2 \zeta_2)^{-1} \delta_{2\perp}' \delta_{1\perp}'$$

and

$$\gamma_1' X_t + \bar{\delta}_1' \dot{\Pi} \Delta X_t \text{ and } \gamma_2' \bar{\gamma}_{1\perp}' \Delta X_t$$

being stationary. These are the well known results for $I(1)$ and $I(2)$ processes (see Johansen, 1996).

6. An example: the representation of $I(3)$

As a Corollary to Theorem 4.1 we have that

Corollary 6.1. *Let*

$$\Pi = -\alpha\beta',$$

$$\alpha'_\perp \dot{\Pi} \beta_\perp = \xi\eta', \text{ and}$$

$$\alpha'_2 \theta \beta_2 = \gamma\lambda'$$

where γ *and* λ *are* $p-r-s \times u$ *matrices of full rank* $u < p-r-s$, $\beta_2 = \beta_\perp \eta_\perp$, $\alpha_2 = \alpha_\perp \xi_\perp$, *and* $\theta = \frac{\ddot{\Pi}}{2} + \dot{\Pi}\bar{\beta}\bar{\alpha}'\dot{\Pi}$.

The $I(3)$ *rank condition is*

$$|\alpha'_3 \theta_1 \beta_3| \neq 0$$

where

$$\alpha_3 = \alpha_\perp \xi_\perp \gamma_\perp,$$

$$\beta_3 = \beta_\perp \eta_\perp \lambda_\perp,$$

$$\theta_1 = \frac{\dddot{\Pi}}{6} + \dot{\Pi}\bar{\beta}\bar{\alpha}'\frac{\ddot{\Pi}}{2} + \theta\bar{\beta}\bar{\alpha}'\dot{\Pi} - \theta\bar{\beta}_1\bar{\alpha}'_1\theta, \text{ and}$$

$$\beta_1 = \bar{\beta}_\perp \eta, \ \alpha_1 = \bar{\alpha}_\perp \xi.$$

Moreover,

$$\beta' \dot{H} = \bar{\alpha}' \dot{\Pi} H, \tag{6.1}$$

$$\beta' \ddot{H} = \bar{\alpha}' \ddot{\Pi} H + 2\bar{\alpha}' \dot{\Pi} \dot{H} \tag{6.2}$$

$$\beta'_1 \dot{H} = -\bar{\alpha}'_1 \theta H, \text{ and} \tag{6.3}$$

$$H = \beta_\perp \eta_\perp \lambda_\perp \phi \gamma'_\perp \xi'_\perp \alpha'_\perp, \ \phi = (\alpha'_3 \theta_1 \beta_3)^{-1} g. \tag{6.4}$$

The moving average representation is

$$X_t = C_3 S_t^3 + C_2 S_t^2 + C_1 S_t^1 + C_3(L)\epsilon_t + A$$

where

$$C_3 = \frac{H}{g}, \ C_2 = \frac{\dot{H}}{g} - \frac{\dot{g}}{g} C_3, \text{ and } C_1 = \frac{\ddot{H}}{2g} - \frac{\dot{g}}{g} C_2 - \frac{\ddot{g}}{2g} C_3.$$

By (6.4) we have

$$\beta' C_3 = 0, \ \eta' \bar{\beta}'_\perp C_3 = 0, \text{ and } \lambda' \bar{\eta}'_\perp \bar{\beta}'_\perp C_3 = 0$$

and by (6.1), (6.2), and (6.3) that

$$\beta' C_2 = \bar{\alpha}' \dot{\Pi} C_3, \tag{6.5}$$

$$\beta' C_1 = \bar{\alpha}' \frac{\ddot{\Pi}}{2} C_3 + \bar{\alpha}' \dot{\Pi} C_2, \text{ and} \tag{6.6}$$

$$\beta_1' C_2 = -\bar{\alpha}_1' \theta C_3. \tag{6.7}$$

Thus cointegration and polynomial cointegration occur in the following way:

Corollary 6.2 (Polynomial cointegration in $I(3)$ systems)**.** *Let X_t be $I(3)$ and $\lambda_1' = \lambda' \bar{\eta}_\perp' \bar{\beta}_\perp'$; then*

i) $\beta' X_t$, $\beta_1' X_t$ and $\lambda_1' X_t$ are $I(2)$,
ii) $\beta_1' X_t + \bar{\alpha}_1' \theta \Delta X_t$ is $I(1)$, and
iii) $\beta' X_t - \bar{\alpha}' \dot{\Pi} \Delta X_t - \frac{1}{2} \bar{\alpha}' \ddot{\Pi} \Delta^2 X_t$ is $I(0)$.

7. Conclusion

We prove the Granger representation theorem for $I(d)$ vector autoregressive processes and characterize the cointegration and polynomial cointegration properties of such processes.

Appendix

Proof of Theorem 4.1. The proof of (4.4) - (4.7) is by induction on s, so we begin by proving that the result holds for $s = 1$.

For convenience here we write (4.3)

$$-\delta_1\gamma_1' H^{(n)} + \sum_{v=1}^{n} \begin{pmatrix} n \\ v \end{pmatrix} \Pi^{(v)} H^{(n-v)} = 0,\ n = 1, \cdots, d-1. \tag{7.1}$$

Pre-multiply it by $\bar{\delta}_1'$, let $\beta_1 = \gamma_1$, $\alpha_1 = \delta_1$, $n_1 = n$ and $\theta_v^1 = \Pi^{(v)}$; then we have

$$\beta_1' H^{(n_1)} = \bar{\alpha}_1' \sum_{v=1}^{n_1} \begin{pmatrix} n_1 \\ v \end{pmatrix} \theta_v^1 H^{(n_1-v)} \tag{7.2}$$

which shows that (4.4) and (4.7) hold for $s = 1$.

To see that also (4.5) and (4.6) hold, write (7.1) for $n = 1$ and use $H = \gamma_{1\perp}\phi_1\delta_{1\perp}'$, $\phi_1 \neq 0$ to have

$$-\delta_1\gamma_1'\dot{H} + \theta_1^1\gamma_{1\perp}\phi_1\delta_{1\perp}' = 0,$$

pre and post-multiply it by $\delta_{1\perp}'$ and $\bar{\delta}_{1\perp}$, let $\eta_1 = \delta_{1\perp}$ and $\zeta_1 = \gamma_{1\perp}$ to get

$$\eta_1'\theta_1^1\zeta_1\phi_1 = 0.$$

Since $|\eta_1'\theta_1^1\zeta_1| \neq 0$ contradicts $\phi_1 \neq 0$, $\eta_1'\theta_1^1\zeta_1$ must be of reduced rank and thus it can be written as the product of two non square matrices

$$-\delta_2\gamma_2' = \eta_1'\theta_1^1\zeta_1. \tag{7.3}$$

Then $\delta_2\gamma_2'\phi_1 = \phi_1\delta_2\gamma_2' = 0$ follow from the identities in (4.1) and imply

$$\gamma_{2\perp}\phi_2\delta_{2\perp}' = \phi_1$$

for some $\phi_2 \neq 0$ and $H = \gamma_{1\perp}\gamma_{2\perp}\phi_2\delta_{2\perp}'\delta_{1\perp}' = \zeta_2\phi_2\eta_2'$. This completes the proof of the statement for $s = 1$.

To see how the proof works in general we now discuss the case $s = 2$; the second derivative of (4.1) at $z = 1$ is

$$\Pi\ddot{H} + 2\dot{\Pi}\dot{H} + \ddot{\Pi}H = 0$$

that is (see (7.10) below)

$$-\delta_1\gamma_1'\ddot{H} + 2\theta_1^1\dot{H} + \theta_2^1 H = 0.$$

Pre-multiplying it by $\eta_1' = \delta_{1\perp}'$ we have that (see (7.11) below)

$$2\eta_1'\theta_1^1\dot{H} + \eta_1'\theta_2^1 H = 0 \tag{7.4}$$

and by the identity $I = \gamma_{1\perp}\bar{\gamma}'_{1\perp} + \bar{\gamma}_1\gamma'_1 = \zeta_1\bar{\zeta}'_1 + \bar{\beta}_1\beta'_1$ we write

$$\eta'_1\theta_1^1\dot{H} = \eta'_1\theta_1^1\zeta_1\bar{\zeta}'_1\dot{H} + \eta'_1\theta_1^1\bar{\beta}_1\beta'_1\dot{H}.$$

By (7.3) we have that

$$\eta'_1\theta_1^1\zeta_1\bar{\zeta}'_1\dot{H} = -\delta_2\gamma'_2\bar{\zeta}'_1\dot{H}$$

and by (7.2) for $n_1 = 1$ that

$$\beta'_1\dot{H} = \bar{\alpha}'_1\theta_1^1 H$$

which means that

$$\eta'_1\theta_1^1\bar{\beta}_1\beta'_1\dot{H} = \eta'_1\theta_1^1\bar{\beta}_1\bar{\alpha}'_1\theta_1^1 H.$$

Setting $\beta_2 = \bar{\zeta}_1\gamma_2$, we then have that

$$\eta'_1\theta_1^1\dot{H} = -\delta_2\beta'_2\dot{H} + \eta'_1\theta_1^1\bar{\beta}_1\bar{\alpha}'_1\theta_1^1 H$$

which implies that (7.4) can be written as (see (7.12) below)

$$-\delta_2\beta'_2\dot{H} + \eta'_1\theta_1^2 H = 0 \tag{7.5}$$

where

$$\theta_1^2 = \theta_1^1\bar{\beta}_1\bar{\alpha}'_1\theta_1^1 + \frac{\theta_2^1}{2}.$$

Pre-multiplying (7.5) by $\bar{\delta}'_2$ and setting $\bar{\alpha}_2 = \eta_1\bar{\delta}_2$, we see that (see (7.13) below)

$$\beta'_2\dot{H} = \bar{\alpha}'_2\theta_1^2 H.$$

Now pre and post-multiply (7.5) by $\delta'_{2\perp}$ and $\bar{\eta}_2$ and use $H = \zeta_2\phi_2\eta'_2$ to get

$$\eta'_2\theta_1^2\zeta_2\phi_2 = 0.$$

Since $|\eta'_2\theta_1^2\zeta_2| \neq 0$ contradicts $\phi_2 \neq 0$, $\eta'_2\theta_1^2\zeta_2$ must have reduced rank and thus it can be written as

$$-\delta_3\gamma'_3 = \eta'_2\theta_1^2\zeta_2.$$

Then $\delta_3\gamma'_3\phi_2 = \phi_2\delta_3\gamma'_3 = 0$ follow from the two versions of the identity (4.1) and imply

$$\gamma_{3\perp}\phi_3\delta'_{3\perp} = \phi_2$$

for some $\phi_3 \neq 0$. This completes the proof of the statement for $s = 2$.

Now we show that if the statement holds for $s = 1, \cdots, k$ then it holds for $s = k + 1$ for any k. Let $\beta_k = \bar{\zeta}_{k-1}\gamma_k$, $\zeta_k = \zeta_{k-1}\gamma_{k\perp}$,

$\bar{\alpha}_k = \eta_{k-1}\bar{\delta}_k$, $\eta_k = \eta_{k-1}\delta_{k\perp}$, $n_k = n + 1 - k$ and write (4.4) - (4.7) for $s = k$; that is

$$\beta_k' H^{(n_k)} = \bar{\alpha}_k' \sum_{v=1}^{n_k} \binom{n_k}{v} \theta_v^k H^{(n_k - v)}, \tag{7.6}$$

$$-\delta_{k+1}\gamma_{k+1}' = \eta_k' \theta_1^k \zeta_k, \tag{7.7}$$

$$\gamma_{k+1\perp}\phi_{k+1}\delta_{k+1\perp}' = \phi_k, \tag{7.8}$$

and

$$\theta_v^k = \theta_1^{k-1} \sum_{j=1}^{k-1} \bar{\beta}_j \bar{\alpha}_j' \theta_v^j + \frac{\theta_{v+1}^{k-1}}{v+1}. \tag{7.9}$$

Substituting $\bar{\alpha}_k = \eta_{k-1}\bar{\delta}_k$ into (7.6) we see that

$$-\delta_k \beta_k' H^{(n_k)} + \eta_{k-1}' \sum_{v=1}^{n_k} \binom{n_k}{v} \theta_v^k H^{(n_k - v)} = 0 \tag{7.10}$$

and by pre-multiplying it by $\delta_{k\perp}'$, letting $\eta_k = \eta_{k-1}\delta_{k\perp}$, changing index in the summation, and using $n_{k+1} = n_k - 1$ that

$$\binom{n_k}{1} \eta_k' \theta_1^k H^{(n_{k+1})} + \eta_k' \sum_{v=1}^{n_{k+1}} \binom{n_k}{v+1} \theta_{v+1}^k H^{(n_{k+1} - v)} = 0. \tag{7.11}$$

By the identity $I = \zeta_k \bar{\zeta}_k' + \sum_{j=1}^k \bar{\beta}_j \beta_j'$ we write

$$\eta_k' \theta_1^k H^{(n_{k+1})} = \eta_k' \theta_1^k \zeta_k \bar{\zeta}_k' H^{(n_{k+1})} + \eta_k' \theta_1^k \sum_{j=1}^k \bar{\beta}_j \beta_j' H^{(n_{k+1})};$$

by (7.7) we have that

$$\eta_k' \theta_1^k \zeta_k \bar{\zeta}_k' H^{(n_{k+1})} = -\delta_{k+1} \gamma_{k+1}' \bar{\zeta}_k' H^{(n_{k+1})}$$

and by (4.4) for $j = 1, \cdots, k$ that

$$\beta_j' H^{(n_{k+1})} = \bar{\alpha}_j' \sum_{v=1}^{n_{k+1}} \binom{n_{k+1}}{v} \theta_v^j H^{(n_{k+1} - v)}$$

which means that

$$\eta_k' \theta_1^k \sum_{j=1}^k \bar{\beta}_j \beta_j' H^{(n_{k+1})} = \eta_k' \theta_1^k \sum_{j=1}^k \bar{\beta}_j \bar{\alpha}_j' \sum_{v=1}^{n_{k+1}} \binom{n_{k+1}}{v} \theta_v^j H^{(n_{k+1} - v)}.$$

Rearranging terms and setting $\beta_{k+1} = \bar{\zeta}_k \gamma_{k+1}$, we then have that

$$\eta_k' \theta_1^k H^{(n_{k+1})} = -\delta_{k+1} \beta_{k+1}' H^{(n_{k+1})} + \eta_k' \sum_{v=1}^{n_{k+1}} \binom{n_{k+1}}{v} \theta_1^k \sum_{j=1}^k \bar{\beta}_j \bar{\alpha}_j' \theta_v^j H^{(n_{k+1} - v)}$$

which implies that (7.11) can be written as

$$(7.12) \qquad -\delta_{k+1}\beta'_{k+1}H^{(n_{k+1})} + \eta'_k \sum_{v=1}^{n_{k+1}} \binom{n_{k+1}}{v} \theta_v^{k+1} H^{(n_{k+1}-v)} = 0$$

where

$$\theta_v^{k+1} = \theta_1^k \sum_{j=1}^{k} \bar{\beta}_j \bar{\alpha}'_j \theta_v^j + \frac{\theta_{v+1}^k}{v+1}$$

and hence (4.7) holds for $s = k+1$.

Pre-multiplying (7.12) by $\bar{\delta}'_{k+1}$ and setting $\bar{\alpha}_{k+1} = \eta_k \bar{\delta}_{k+1}$, we see that

$$(7.13) \qquad \beta'_{k+1}H^{(n_{k+1})} = \bar{\alpha}'_{k+1} \sum_{v=1}^{n_{k+1}} \binom{n_{k+1}}{v} \theta_v^{k+1} H^{(n_{k+1}-v)}$$

which is (4.4) for $s = k+1$.

To see that also (4.5) and (4.6) hold for $s = k+1$, note that the repeated application of (7.8) implies $H = \zeta_{k+1}\phi_{k+1}\eta'_{k+1}$; now let $n_{k+1} = 1$ in (7.12), pre and post-multiply it by $\delta'_{k+1\perp}$ and $\bar{\eta}_{k+1}$ to get

$$\eta'_{k+1}\theta_1^{k+1}\zeta_{k+1}\phi_{k+1} = 0.$$

Since $|\eta'_{k+1}\theta_1^{k+1}\zeta_{k+1}| \neq 0$ contradicts $\phi_{k+1} \neq 0$, $\eta'_{k+1}\theta_1^{k+1}\zeta_{k+1}$ must have reduced rank and thus it can be written as

$$-\delta_{k+2}\gamma'_{k+2} = \eta'_{k+1}\theta_1^{k+1}\zeta_{k+1}$$

proving that (4.5) holds for $s = k+1$. Then $\delta_{k+2}\gamma'_{k+2}\phi_{k+1} = \phi_{k+1}\delta_{k+2}\gamma'_{k+2} = 0$ follow from the two versions of the identity (4.1) and imply

$$\gamma_{k+2\perp}\phi_{k+2}\delta'_{k+2\perp} = \phi_{k+1}$$

for some $\phi_{k+2} \neq 0$, which is (4.6) for $s = k+1$. Then (4.4) - (4.7) hold for $s = k+1$ and the induction part of the proof is complete.

To see that (4.8) is true, note that using the previous recursion the derivative of order d can be written as

$$-\delta_d\gamma'_d\dot{H} + \eta'_{d-1}\theta_1^d H = gI.$$

Pre-multiplying by $\delta'_{d\perp}$ and using $\eta_d = \eta_{d-1}\delta_{d\perp}$, we have $\eta'_d\theta_1^d H = g\delta'_{d\perp}$. Since $H = \zeta_d\phi_d\eta'_d$ and $\delta'_{d\perp}\bar{\eta}_d = I$, post-multiplication by $\bar{\eta}_d$ turns it into

$$\eta'_d\theta_1^d\zeta_d\phi_d = gI$$

and the proof is complete. ■

Proof of Theorem 5.2. The Taylor expansion of $H(z)$ at $z = 1$ is written as

$$H(z) = \sum_{v=0}^{d-s} \frac{H^{(v)}}{v!}(z-1)^v + (z-1)^{d-s+1}A(z)$$

and then

$$\beta_s' H(z) = \beta_s' \sum_{v=1}^{d-s} \frac{H^{(v)}}{v!}(z-1)^v + (z-1)^{d-s+1}\beta_s' A(z) \tag{7.14}$$

follows from $\beta_s' H = 0$.

Using (4.4) and rearranging terms, we have that

$$\beta_s' \sum_{v=1}^{d-s} \frac{H^{(v)}}{v!}(z-1)^v = \bar{\alpha}_s' \sum_{v=1}^{d-s} \frac{\theta_v^s}{v!} \sum_{k=v}^{d-s} \frac{H^{(k-v)}}{(k-v)!}(z-1)^k$$

and since

$$(z-1)^v H(z) = \sum_{k=v}^{d-s} \frac{H^{(k-v)}}{(k-v)!}(z-1)^k + (z-1)^{d-s+1}B(z)$$

we have that

$$\beta_s' \sum_{v=1}^{d-s} \frac{H^{(v)}}{v!}(z-1)^v = \bar{\alpha}_s' \sum_{v=1}^{d-s} \frac{\theta_v^s}{v!}(z-1)^v H(z) + (z-1)^{d-s+1}C(z).$$

Then (7.14) is rewritten as

$$\{\beta_s' - \bar{\alpha}_s' \sum_{v=1}^{d-s} \frac{\theta_v^s}{v!}(z-1)^v\}H(z) = (z-1)^{d-s+1}D(z). \tag{7.15}$$

Dividing both sides of (7.15) by $(z-1)^d g(z)$ we have that

$$\{\beta_s' - \bar{\alpha}_s' \sum_{v=1}^{d-s} \frac{\theta_v^s}{v!}(z-1)^v\}\Pi(z)^{-1} = \frac{D(z)}{(z-1)^{s-1}g(z)}$$

which means that

$$(z-1)^{s-1}\{\beta_s' - \bar{\alpha}_s' \sum_{v=1}^{d-s} \frac{\theta_v^s}{v!}(z-1)^v\}\Pi(z)^{-1}$$

has no pole at $z = 1$. Since the difference operator is defined as $\Delta = 1 - L$ we use (-1) to turn $z-1$ into $1-z$ and the proof is complete. ■

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