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Rethinking the Concept of Long-Run Economic Growth [§]

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This paper argues that growth theory needs a more general "regularity" concept than that of exponential growth. This opens up for considering a richer set of parameter combinations than in standard growth models. Allowing zero population growth in the Jones (1995) model serves as our illustration of the usefulness of a general concept of "regular growth".

Keywords: Exponential growth; Arithmetic growth; Regular growth; Semi-endogenous growth; Knife-edge restrictions.

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1 Introduction

The concept of balanced growth, generally synonymous with exponential growth, has proved extremely useful in the theory of economic growth. This is not only because of the historical evidence (Kaldor's "stylized facts"), but also because of its convenient simplicity. Yet there may be a deceptive temptation to oversimplify and ignore other possible growth patterns. We argue there is a need to allow for a richer set of parameter constellations than in standard growth models and to look for a more general "regularity" concept than that of exponential growth. The motivation is the following:

First, when setting up growth models researchers place severe restrictions on preferences and technology such that the resulting model is compatible with balanced growth. For instance, models exhibiting balanced growth usually rely on some form of knife-edge restrictions, which drastically restrict the shape of preferences and production technology (Solow, 2000, Chapters 8-9). This paper demonstrates that regular long-run growth, in a sense specified below, can arise even if these restrictions are violated.

Second, standard R&D-based semi-endogenous growth models imply that the long-run growth rate is proportional to the growth rate of the labor force (Jones, 2005). This class of models is frequently used for positive and normative analysis since it appears to be empirically plausible in many respects. If we employ this type of model to evaluate the prospect of growth in the very long run, then we end up with the assertion that the growth rate converges to zero. This is simply due the fact that there must be limits to population growth. But then, what does this really imply for economic development in the very long run? This question has not received much attention so far and the answer is not that clear at first glance.

Third, everything less than exponential growth often seems interpreted as a fairly bad outcome and associated with economic stagnation. For instance, in the context of the Jones (1995) model with constant population, Young (1998, n. 10) states *"Thus, even if there are intertemporal spillovers, if they are not large enough to allow for constant growth, the development of the economy grinds to a halt."* However, to our knowledge, the case of zero population growth in the Jones model has not really been explored yet. We take the opportunity to let an analysis of this case serve as our illustration of the usefulness of the general concept of regular growth.

The paper is structured as follows. Section 2 introduces our regularity concept and shows how it is related to the cases of exponential and arithmetic growth. Section 3 illustrates that allowing a richer set of parameter combinations than in standard growth models indeed gives rise to other regularity patterns than exponential growth. Finally, Section 4 summarizes the findings.

2 Regular Growth

Growth theory explains long-run economic development as some pattern of regular growth. The most common regularity concept is that of exponential growth. Occasionally another regularity pattern turns up, namely that of arithmetic growth. Indeed, a Ramsey growth model with AK technology and CARA preferences features arithmetic GDP per capita growth (e.g., Blanchard and Fischer, 1989, pp. 44/45). Similarly, under Hartwick's rule, a model with essential non-renewable resources features arithmetic growth of capital (Solow, 1974; Hartwick, 1977). In similar settings with non-renewable resources Mitra (1983), Pezzey (2004) and Asheim et al. (2005) consider growth paths of the form $x(t) = x(0)(1+\mu t)^{\omega}$, $\mu, \omega > 0$, which, by the last-mentioned authors, is called "quasi-arithmetic growth". In these analyses the quasi-arithmetic growth pattern is associated with exogenous quasi-arithmetic growth in either population or technology. In this way results by Dasgupta and Heal (1979, pp. 303-308) on optimal growth within a classical utilitarian framework with non-renewable resources, constant population and constant technology are extended.

In our view there is a rationale for a concept of regular growth, subsuming exponential growth and arithmetic growth as well as the whole range between these two. Also some kind of less-than-arithmetic growth should be included. This general concept is labelled *regular growth*, for reasons that will become clear below. The example we consider in Section 3 shows how a quasi-arithmetic growth pattern may arise endogenously in a two-sector knowledge-driven growth model.

To describe our suggested concept of regular growth, a few definitions are needed. Let the variable $x(t)$ be a positively-valued differentiable function of time t. Then the growth rate of $x(t)$ at time t is:

$$g_1(t) := \frac{\dot{x}(t)}{x(t)},$$

where $\dot{x}(t) := dx(t)/dt$. We call $g_1(t)$ the first-order growth rate. Since we seek a more general concept of regular growth than exponential growth, we allow $g_1(t)$ to be time-variant. Indeed, the regularity we look for relates precisely to the way growth rates change over time. Presupposing $g_1(t)$ is strictly positive within the time range considered, let $g_2(t)$ denote the second-order growth rate of $x(t)$ at time t, i.e.,

$$g_2(t) := \frac{\dot{g}_1(t)}{g_1(t)}.$$

We suggest the following criterion as defining *regular growth*:

$$g_2(t) = -\beta g_1(t) \quad \forall t \geq 0, \tag{1}$$

where $\beta \geq 0$. That is, the second-order growth rate is proportional to the first-order growth rate with a non-positive factor of proportionality. The coefficient β is called the *damping coefficient*, since it indicates the rate of damping in the growth process.

Let x_0 and α denote the initial values $x(0) > 0$ and $g_1(0) > 0$, respectively. The unique solution of the second-order differential equation (1) may then be expressed as:

$$x(t) = x_0 \left(1 + \alpha\beta t\right)^{\frac{1}{\beta}}. \tag{2}$$

Note that this solution has at least one well-known special case, namely $x(t) = x_0 e^{\alpha t}$ for $\beta = 0$.[1] Moreover, it should be observed that, given x_0, (2) is also the unique solution of the first-order equation:

$$\dot{x}(t) = \alpha x_0^{\beta} x(t)^{1-\beta}, \qquad \alpha > 0, \beta \geq 0, \tag{3}$$

which is an autonomous Bernoulli equation. This gives an alternative and equivalent characterization of regular growth.

The simple formula (2) describes a family of growth paths, the members of which are indexed by the damping coefficient β. Figure 1 illustrates this family of regular growth paths.[2] There are three well-known special cases. For $\beta = 0$, we have $g_1(t) = \alpha$, a positive constant. This is the case of exponential growth. At the other extreme we have complete stagnation, i.e., the constant path $x(t) = x_0$. This can be interpreted as the limiting case $\beta \to \infty$.[3] Arithmetic growth, i.e., $\dot{x}(t) = \alpha$, $\forall t \geq 0$, is the special case $\beta = 1$.

[1] Indeed, $\lim_{\beta \to 0} x_0(1 + \alpha\beta t)^{\frac{1}{\beta}} = x_0 e^{\alpha t}$. To see this, use L'Hôpital's rule for "0/0" on $\ln(x(t)) = \ln(x_0) + \frac{1}{\beta}\ln(1 + \alpha\beta t)$.

[2] Figure 1 is based on $\alpha = 0.05$ and $x_0 = 1$. In this case, the time paths do not intersect. Intersections occur for $x_0 < 1$. However, for large t the picture always is as shown in Figure 1.

[3] Use L'Hôpital's rule for "∞/∞" on $\ln x(t)$. If we allow $g_1(0) = 0$, stagnation can of course also be seen as the case $\alpha = 0$.

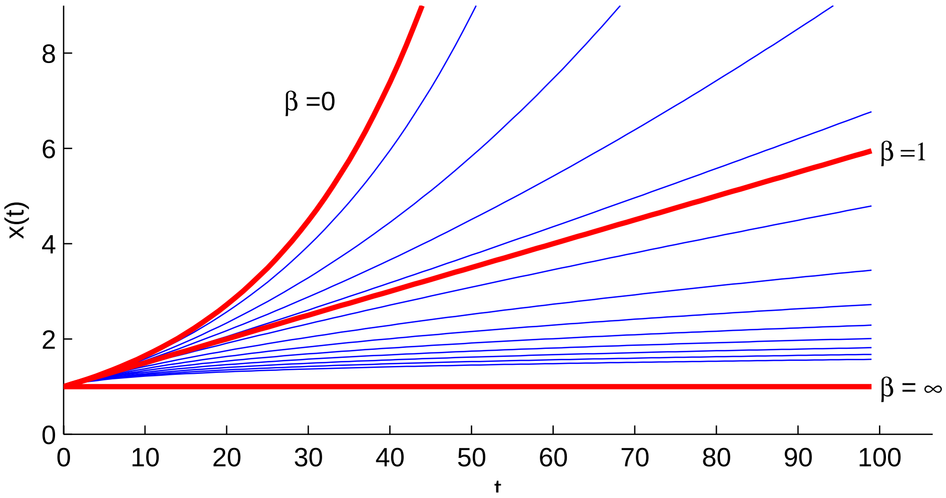

Figure 1: A family of growth paths indexed by β.

Table 1 lists these three cases and gives labels also to the intermediate ranges for the value of the damping coefficient β. Apart from being written in another (and perhaps less family-oriented) way, the "quasi-arithmetic growth" formula in Asheim et al. (2005) mentioned above, is subsumed under these intermediate ranges.

Table 1: Regular growth paths: $g_2(t) = -\beta g_1(t)\ \forall t \geq 0$, $\beta \geq 0$, $g_1(0) > 0$.

Label	Damping coefficient	Time path
Limiting case 1: exponential growth	$\beta = 0$	$x(t) = x_0 e^{\alpha t},\ \ \alpha > 0$
More-than-arithmetic growth	$0 < \beta < 1$	$x(t) = x_0(1 + \alpha\beta t)^{\frac{1}{\beta}},\ \ \alpha > 0$
Arithmetic growth	$\beta = 1$	$x(t) = x_0(1 + \alpha t),\ \ \alpha > 0$
Less-than-arithmetic growth	$1 < \beta < \infty$	$x(t) = x_0(1 + \alpha\beta t)^{\frac{1}{\beta}},\ \ \alpha > 0$
Limiting case 2: stagnation	$\beta = \infty$	$x(t) = x_0$

As to the case $\beta > 1$, notice that though the increase in x per time unit is falling over time, it remains positive; there is sustained growth in the sense that $x(t) \to \infty$ for $t \to \infty$.[4] Formally, also the case of $\beta < 0$ (more-than-exponential growth) could be included in the family of regular growth paths. However, this case should be considered as only relevant for a description of possible phases of *transitional* dynamics. A growth path (for, say, GDP per capita) with $\beta < 0$ is

[4]Empirical investigation of post-WWII GDP per-capita data of a sample of OECD countries yields non-negative damping factors between 0.17 (UK) and 1.43 (Germany). The associated initial (annual) growth rates in 1951 are 2.3% (UK) and 12.4% (Germany), respectively. The fit of the regular growth formula is remarkable.

explosive in a very dramatic sense: it leads to infinite output in finite time (Solow, 1994).

3 An Example

An optimal growth problem within the simple Jones (1995) framework is considered in order to illustrate how the regularities described above may arise. Population L is governed by $L = L(0)e^{nt}$, where $n \geq 0$ is constant. We include the case $n = 0$ not only for theoretical reasons, but also because it is of practical interest in view of the projected stationarity of the population of developed countries as a whole already from 2005 (United Nations, 2005).[5] Technologically the economy is described by:

$$\begin{aligned} Y &= A^{\sigma}K^{\alpha}(uL)^{1-\alpha}, \qquad \sigma > 0,\ 0 < \alpha < 1, && (4)\\ \dot{K} &= Y - cL, \qquad K(0) \text{ given}, && (5)\\ \dot{A} &= \gamma A^{\varphi}(1-u)L, \qquad \gamma > 0, \varphi \leq 1,\ \ A(0) \text{ given}, && (6) \end{aligned}$$

where Y is aggregate manufacturing output (net of capital depreciation), A society's stock of "knowledge", K society's capital, u the fraction of the labour force (= population) employed in manufacturing and c per-capita consumption; σ, α, γ and φ are constant parameters. The criterion functional of the social planner is:

$$U_0 = \int_0^{\infty} \frac{c^{1-\theta} - 1}{1-\theta} Le^{-\rho t} dt,$$

where $\theta > 0$ and $\rho \geq 0$, both constant. In the spirit of Ramsey (1928) we include the case $\rho = 0$, since giving less weight to future generations than to current might be deemed "ethically indefensible". When $\rho = 0$, there exist feasible paths for which the integral U_0 does not converge. In that case our optimality criterion is the catching-up criterion, see Case 4 below. The social planner chooses a plan $(c, u)_{t=0}^{\infty}$, where $c > 0$ and $u \in [0, 1]$, to optimize U_0 under the constraints (4), (5) and (6) as well as $K \geq 0$ and $A \geq 0$, $\forall t \geq 0$.

Case 1: $\varphi = 1, \rho > n = 0$. This is the fully-endogenous growth case considered by Romer (1990).[6] An interior optimal solution converges to exponential growth with growth rate $g_c = (1/\theta)\left[\sigma\gamma L/(1-\alpha) - \rho\right)]$ and $u = 1 - (1-\alpha)g_c/(\sigma\gamma L)$.

Case 2: $\varphi < 1, \rho > n > 0$. This is the semi-endogenous growth case considered by Jones (1995). An interior optimal solution converges to exponential growth

[5] From now, the explicit timing of the variables is suppressed when not needed for clarity.

[6] Contrary to Romer (1990), though, we allow $\sigma \neq 1-\alpha$ for reasons explained in Alvarez-Pelaez and Groth (2005).

with growth rate $g_c = n/(1-\varphi)$ and $u = \frac{(\sigma/(1-\alpha))(\theta-1)n+(1-\varphi)\rho}{(\sigma/(1-\alpha))\theta n+(1-\varphi)\rho}$.[7]

Case 3: $\varphi < 1$, $\rho > n = 0$. In this case the economy can be shown to end up in stagnation (constant c), as is indicated by putting $n = 0$ in the formula for u in Case 2. The explanation is the combination of a) no population growth to countervail the diminishing marginal returns to knowledge ($\partial \dot{A}/\partial A \to 0$ for $A \to \infty$), and b) a positive constant rate of time preference.

Case 4: $\varphi < 1$, $\rho = n = 0$. Depending on the values of φ, σ, α and θ, a continuum of dynamic processes emerges which fill the whole range between stagnation and exponential growth.[8] Since this case does not seem investigated in the literature, we shall spell it out here. The optimality criterion is the *catching-up criterion:* a feasible path $(\hat{K}, \hat{A}, \hat{c}, \hat{u})_{t=0}^{\infty}$ is catching-up optimal if

$$\lim_{t\to\infty} \inf \left(\int_0^t \frac{\hat{c}^{1-\theta}-1}{1-\theta} d\tau - \int_0^t \frac{c^{1-\theta}-1}{1-\theta} d\tau \right) \geq 0$$

for all feasible paths $(K, A, c, u)_{t=0}^{\infty}$.

Let p be the shadow price of knowledge in terms of the capital good. Then, the value ratio $x \equiv pA/K$ is capable of being stationary in the long run. Indeed, as shown in the appendix, the first order conditions of the problem lead to:

$$\dot{x} = \frac{\gamma L A^{\varphi-1}}{1-\alpha} \left\{ (\alpha - s)xu - [\sigma + (1-\alpha)(1-\varphi)]\, u + (1-\alpha)(1-\varphi) \right\} x, \qquad (7)$$

where s is the saving rate $= 1 - cL/Y$; further,

$$\dot{u} = \frac{\gamma L A^{\varphi-1}}{1-\alpha} \left[-(1-s)xu + \sigma u + \frac{1-\alpha}{\alpha}\sigma \right] u, \quad \text{and} \qquad (8)$$

$$\dot{s} = \frac{\gamma L A^{\varphi-1}}{1-\alpha} \left[-(\frac{1}{\theta} - s)\alpha x u - (1-\alpha)\sigma u + (1-\alpha)\sigma + \frac{(1-\alpha)^2 \dot{u}/u}{\gamma L A^{\varphi-1}} \right] (1-s). \qquad (9)$$

Provided $\theta > 1$, this dynamic system has a unique stationary state:

$$x^* = \frac{\sigma\theta}{\alpha(\theta-1)} > 0, u^* = \frac{\sigma + \alpha(1-\varphi)}{\frac{\theta}{\theta-1}\sigma + \alpha(1-\varphi)} \in (0,1), s^* = \frac{\sigma + 1 - \varphi}{\theta\left[\frac{\sigma}{\alpha} + 1 - \varphi\right]} \in (0, \frac{1}{\theta}). \qquad (10)$$

The resulting paths for A, K, Y and c feature regular growth with positive damping coefficient:

$$A(t) = \left[A(0)^{1-\varphi} + (1-\varphi)\gamma(1-u^*)Lt\right]^{\frac{1}{1-\varphi}} = A(0)\left(1 + \mu t\right)^{\frac{1}{1-\varphi}},$$

[7] The Jones (1995) model also includes a negative duplication externality in R&D, which is not relevant for our discussion. Convergence of this model is shown in Arnold (2006).

[8] The entire spectrum of regular growth patterns can alternatively be obtained in an elementary version of the Jones (1995) model with no capital, but two types of (immobile) labor, i.e., unskilled labour in final goods production and skilled labour in R&D.

where $\mu \equiv (1-\varphi)\gamma(1-u^*)LA(0)^{\varphi-1} > 0$;

$$\begin{aligned} K(t) &= \left(\frac{1-\alpha}{\gamma x^*(u^*L)^\alpha}\right)^{\frac{1}{1-\alpha}} A(0)^{\frac{\sigma+1-\varphi}{1-\alpha}} (1+\mu t)^{\frac{\sigma+1-\varphi}{(1-\alpha)(1-\varphi)}}, \\ Y(t) &= (u^*L)^{\frac{1-2\alpha}{1-\alpha}} \left(\frac{1-\alpha}{\gamma x^*}\right)^{\frac{\alpha}{1-\alpha}} A(0)^{\frac{\sigma+\alpha(1-\varphi)}{1-\alpha}} (1+\mu t)^{\frac{\sigma+\alpha(1-\varphi)}{(1-\alpha)(1-\varphi)}}. \end{aligned}$$

Finally, $c(t) = (1-s^*)Y(t)/L$.[9]

When $0 < \varphi < 1$ (the "standing on the shoulders" case), the damping coefficient $\beta = 1-\varphi < 1$, i.e., knowledge features more-than-arithmetic growth. When $\varphi < 0$ (the "fishing out" case), the damping coefficient is $1-\varphi > 1$, and knowledge features less-than-arithmetic growth. In the intermediate case, $\varphi = 0$, knowledge features arithmetic growth.[10] More interesting is perhaps the path of Y to which the path of c is proportional. We see that Y features more-than-arithmetic growth if and only if $\sigma > (1-2\alpha)(1-\varphi)$. A sufficient condition for this is that $\frac{1}{2} \leq \alpha < 1$; it is interesting that $\varphi > 0$ is not needed. Notice also that the capital-output ratio features arithmetic growth *always*, i.e., independently of the size relation between the parameters. Indeed, $K/Y = [K(0)/Y(0)]\,(1+\mu t)$. This is like in Hartwick's rule (Hartwick, 1977). A mirror image of this is that the marginal product of capital always approaches zero for $t \to \infty$, a property not surprising in view of $\rho = 0$.

4 Summary and Conclusion

Our proposed concept of regular growth has the following advantages: (1) The concept allows researchers to get rid of the largely arbitrary knife-edge restriction, which underlies both standard neoclassical and endogenous growth models. (2) Since the resulting dynamic process has one more degree of freedom compared to exponential growth, it is at least as plausible in empirical terms. (3) The concept covers a continuum of dynamic processes which fill the whole range between exponential growth and complete stagnation, a range which may deserve more attention in view of the likely future demographic development in the world. (4) Finally, as our analysis of zero population growth in the Jones (1995) model shows, falling growth rates need not mean that economic development grinds to a halt.

[9] The usual transversality conditions require $\theta > (\sigma+1-\phi)/\left[\sigma+\alpha(1-\phi)\right]$, which we assume satisfied (see the appendix). This condition is slightly stronger than the requirement $\theta > 1$.

[10] The coefficient μ could be called the *growth momentum*.

5 Appendix

This appendix derives the results reported for Case 4 in Section 3. The Hamiltonian for the control problem in Case 4 is:

$$H = \frac{c^{1-\theta} - 1}{1-\theta} L + \lambda_1 (Y - cL) + \lambda_2 \gamma A^{\varphi} (1-u) L,$$

where $Y = A^{\sigma} K^{\alpha} (uL)^{1-\alpha}$, and λ_1 and λ_2 are the co-state variables associated with physical capital and knowledge, respectively. Necessary first order conditions (see Seierstad and Sydsaeter, 1987, p. 234) for an interior solution are:

$$\frac{\partial H}{\partial c} = c^{-\theta} L - \lambda_1 L = 0, \tag{11}$$

$$\frac{\partial H}{\partial u} = \lambda_1 (1-\alpha) \frac{Y}{u} - \lambda_2 \gamma A^{\varphi} L = 0, \tag{12}$$

$$\frac{\partial H}{\partial K} = \lambda_1 \alpha \frac{Y}{K} = -\dot{\lambda}_1, \tag{13}$$

$$\frac{\partial H}{\partial A} = \lambda_1 \sigma \frac{Y}{A} + \lambda_2 \varphi \gamma A^{\varphi-1} (1-u) L = -\dot{\lambda}_2. \tag{14}$$

Combining (11) and (13) gives the Keynes-Ramsey rule

$$\frac{\dot{c}}{c} = \frac{1}{\theta} \alpha A^{\sigma} K^{\alpha-1} (ul)^{1-\alpha}. \tag{15}$$

Given the definition $p = \lambda_2/\lambda_1$, (12), (13) and (14) yield

$$\frac{\dot{p}}{p} = \alpha A^{\sigma} K^{\alpha-1} (ul)^{1-\alpha} - \frac{\sigma \gamma A^{\varphi-1} uL}{1-\alpha} - \varphi \gamma A^{\varphi-1} (1-u) L. \tag{16}$$

Let $x \equiv pA/K$. Log-differentiating x w.r.t. time and using (12), (6), (5) and (4) give (7). Log-differentiating (12) w.r.t. time, using (16), (5), (4) and (6), gives (8). Finally, log-differentiating $1 - s \equiv cL/Y$, using (15), (4), (6) and (5), gives (9).

Due to non-concavity of the maximized Hamiltonian, not all the Arrow sufficiency conditions (Seierstad and Sydsaeter 1987, p. 236) hold, and so far we have found no alternative set of sufficient conditions satisfied. Yet, at least the transversality conditions, $\lim_{t\to\infty} \lambda_1(t) K(t) = 0$ and $\lim_{t\to\infty} \lambda_2(t) A(t) = 0$, can be shown to hold along the unique regular growth path if (and only if) $\theta > (\sigma + 1 - \varphi)/\left[\sigma + \alpha(1-\varphi)\right]$.

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