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05-19

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Merging and splitting in cooperative games: some (im-)possibility results

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October 2005

Abstract

Solutions for cooperative games with side-payments can be manipulated by merging a coalition of players into a single player, or, conversely, splitting a player into a number of smaller players. This paper establishes some (im-)possibility results concerning merging- or splitting-proofness of core solutions of balanced and convex games.

JEL classification: C71, D23, D71.

Keywords: Cooperative games, Manipulation, Dutta-Ray solution.

1 Introduction

A *cooperative game with side-payments* specifies a set of players and a *worth,* in monetary units, for each coalition. This worth can be interpreted as

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the income, or surplus, that a coalition can obtain if it stands alone, i.e. if it chooses not to cooperate with the other players. An *allocation* is a distribution of the worth of the grand coalition, and a *core allocation* is an allocation for which the worth of each coalition does not exceed its aggregate income. A *solution* is a rule that, for each game, specifies an allocation.

A cooperative game with side-payments is a very summary representation of an underlying game of conflict. It is therefore essential for an analyst to understand to what extend it matters how the player set itself is specified from the data of the situation. In many applications of cooperative game theory, players may represent groups of persons, for example labor unions or nations, or they may be other economic variables of the situation, for example factors of production or objectives of an economic project (Peleg and Sudhölter, 2003), and there may be more than one way of fixing the variables of the game.

In other applications, players are persons who can exit (entry) the game by handing over (receiving) their assets to (from) other players, or groups of persons can merge and then jointly act as one decision unit, e.g. as a household or a firm. Depending on the specifics of the game and solution, players may have incentives to merge, or to split themselves into smaller units, i.e. the game itself may be subject to strategic manipulation.

Manipulation of solutions for cooperative game situations has been a recurrent theme in the literature. In the context of *bargaining problems*, Harsanyi (1977) discusses the *joint-bargaining paradox* of the Nash bargaining solution. Harsanyi points out that if two players merge into a single bargaining unit, they tend to weaken their bargaining position. In *rationing problems*, interpreted for example as bankruptcy- or simple cost-sharing situations, conditions similar to the joint properties of merging- and splitting-proofness have been used to characterize proportional allocation, see, e.g., Moulin (2002, Section 1.2) for a survey.

In the context of cooperative games with side-payments, Lehrer (1988) investigates bilateral mergers, called *amalgamations*, where two players merge into one player. Lehrer shows that for the Banzhaf value it is always profitable to merge and he uses this condition for an axiomatic characterization. Haviv (1995) uses some consistency with respect to consecutive mergers for a characterization of the Shapley value. Derks and Tijs (2000) consider a partition of the player set and study the game that evolves when the players in each compartment of the partition merge into one player, and formulate a set of conditions implying that a merger in a given compartment is profitable. Derks and Tijs assume that players are rewarded according to the Shapley value.

Haller (1994) investigates collusion properties of the Shapley value, the Banzhaf value and other probabilistic values for bilateral *proxy*- and *association agreements*. Proxy agreements are similar to mergers (if disregarding dummy players). An association agreement modifies the games such that if just one of the players in the association enters some coalition, then the player's contribution to its worth is as if all the players in the association were entering. Segal (2003) also considers probabilistic values and gives conditions for profitable agreements under different types of integration.

The present paper examines whether (core) solutions can be *merging-proof* or, conversely, *splitting-proof*, and provides some impossibility and possibility results in this direction. Section 2.1 considers balanced games, i.e. games with a non-empty set of core allocations. An anonymous solution cannot simultaneously be merging- and splitting-proof. Anonymous solutions can be merging-proof *or* splitting-proof, but we show that then they cannot be core solutions. Section 2.2 considers convex games, i.e. games where the incentives for joining a coalition increase as the coalition grows (Shapley, 1971). We show that the Dutta-Ray solution is, in fact, merging-proof, and we find a core solution which is splitting-proof on the class of strictly monotonic convex games.

1.1 Definitions

A *cooperative game with side-payments* is a pair (N, v), where N is a finite set and v is a real-valued function defined on the subsets of N and $v(\emptyset) = 0$. The elements of N are called *players*. To save on notation, we write $v(i)$ instead of $v(\{i\})$, $v(i, j)$ instead of $v(\{i, j\})$, and so on.

An element x of $\mathbb{R}^N$ is called a payoff vector. For $x \in \mathbb{R}^N$ and $S \subseteq N$ we define $x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$. If $x(N) = v(N)$ then x is called an *allocation*. The *core* of a game (N, v) is the set $C(N, v) = \{x \in \mathbb{R}^N | x(S) \geq v(S)$ for all $S \subset N$ and $x(N) = v(N)\}$.

A game (N, v) is *convex* if $v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$ for all $T \subset S \subseteq N$, $i \notin S$, it is *superadditive* if $v(S \cup T) \geq v(T) + v(S)$ for all $S, T \subseteq N$, $S \cap T = \emptyset$, and (strictly) monotonic if $v(S \cup \{i\}) \geq (>) v(S)$ for all i and $S \subseteq N$, $i \notin S$ (see, e.g., Peleg and Sudhölter, 2003).

A *solution* is a function ϕ that for each game (N, v) assigns an allocation in $\mathbb{R}^N$. We restrict attention to *anonymous* solutions. To be precise, let (N, v) be a game with $|N| = n$, let σ be a bijective correspondence from N to $\{1, ..., n\}$, and define a game v' by $v'(\sigma(S)) = v(S)$ for all $S \subseteq N$. Then we have $\phi_i(N, v) = \phi_{\sigma(i)}(\{1, ..., n\}, v')$ for all $i \in N$.

For a game (N, v) and $T \subset N$ we define the *T-merger game* (N^T, v^T) as follows: $N^T = \left\{T, \{i\}_{i \in N \setminus T}\right\}$ and $v^T(S) = v(\overline{S})$ for all $S \subseteq N^T$, where $\overline{S} = \left\{\{i\}_{i \in T}, \{i\}_{i \in S \setminus T}\right\}$ if $T \in S$ and $\overline{S} = S$ otherwise. Note that T is a coalition in (N, v) and a player in the T-merger game.

A solution ϕ is *merging-proof (splitting-proof)* if for any game (N, v) and any *T-merger game* (N^T, v^T), $T \subseteq N$, that $\phi_T(N^T, v^T) \leq (\geq) \sum_{i \in T} \phi_i(N, v)$. In words, a solution is merging-proof if the players in a coalition never have incentives to merge and act as one player. Splitting-proofness says that regardless of how a player can be split up into smaller units, the player will never have an incentive to do that.[1]

[1] Put differently, a solution is merging-proof if regardless of how a player is able to divide herself into a group of smaller players, it is always profitable. And a solution is

2 Results

2.1 Balanced games

It is easily verified that if (N, v) is a balanced game and $T \subset N$, then (N^T, v^T) is balanced. Even if restricting attention to balanced games, the combination of merging- and splitting-proofness is inconsistent with anonymity.

Proposition 1 *On the class of balanced games, there exists no anonymous merging- and splitting-proof solution.*

Proof: By contradiction. Assume that ϕ is an anonymous, merging- and splitting proof solution. Let (N, v) be a game with $N = \{1, 2, 3\}$, $v(1) = v(2) = 1$, $v(3) = 2, v(2, 3) = v(1, 3) = 3, v(1, 2) = 4, v(1, 2, 3) = 6$. We claim that $\phi_1(N, v) = \phi_2(N, v) = \phi_3(N, v) = 2$. For this, notice that by anonymity $\phi_1(N, v) = \phi_2(N, v)$. Furthermore, consider a merger $S = \{1, 2\}$. The resulting game $(\{S, 3\}, v^S)$ is then defined by $v^S(S) = 4$, $v^S(3) = 2, v^S(S, 3) = 6$. By merging- and splitting-proofness $\phi_S(\{S, 3\}, v^S) = \phi_1(N, v) + \phi_2(N, v)$. By merging- and splitting-proofness and anonymity, for the game $(\{1, 2, 3\}, w)$ with $w(1) = w(2) = w(3) = 2$, $w(i, j) = 4, w(1, 2, 3) = 6$ we must have $\phi_1(\{1, 2, 3\}, w) + \phi_2(\{1, 2, 3\}, w) = \phi_S(\{S, 3\}, v^S) = 4$ proving the claim.

Now, consider a merger $T = \{2, 3\}$. The resulting game $(\{1, T\}, v^T)$ is then defined by $v^T(T) = 3, v^T(1) = 1$, and $v^T(1, T) = 6$. We claim that $\phi_T(\{1, T\}, v^T) = \frac{18}{4}$ and $\phi_1(\{1, T\}, v^T) = \frac{6}{4}$. For this, consider the game $(\{1, 2, 3, 4\}, q)$ with $q(i) = 1$, $q(i, j) = 2$, $q(i, j, k) = 3$ and $q(\{1, 2, 3, 4\}) = 6$. By anonymity we have $\phi_i(\{1, 2, 3, 4\}, q) = \frac{6}{4}$ for all i, and from merging- and splitting-proofness our claim follows.

We have now obtained a contradiction, since for the game (N, v) the merger $T = \{2, 3\}$ strictly increases aggregate payoff for coalition members. $\square$

splitting-proof if it is always profitable for any given coalition to merge.

It it noteworthy that the proof of Proposition 1 involves only monotonic convex games (and the impossibility applies therefore to this subfamily of games).

There exist anonymous and merging-proof solutions (for example, the *equal split* solution that divides $v(N)$ equally among the players), and on the class of superadditive balanced games we can find anonymous and splitting-proof solutions (for example, the solution that for a game (N, v) divides $v(N)$ between the players who have the highest single-player worth $v(i)$). However such merging-proof or splitting-proof solutions cannot be core solutions.

Proposition 2 *On the class of balanced games, an anonymous core solution can neither be merging-proof nor splitting-proof.*

Suppose that ϕ is an anonymous merging-proof core solution. For $N = \{1, 2, 3\}$ define v as follows. $v(i) = 0$ for all i, $v(2, 3) = 0$, $v(1, 2) = v(1, 3) = 1$ and $v(1, 2, 3) = 1$. Then $C(N, v) = (1, 0, 0)$ and since ϕ is a core solution we accordingly have $\phi_1(N, v) = 1$ and $\phi_2(N, v) = \phi_3(N, v) = 0$.

Now, for $T = \{2, 3\}$ consider the T-merger game which is defined as follows: $N^T = \{1, T\}$, $v^T(1) = v^T(T) = 0$, and $v(1, T) = 1$. By anonymity we have $\phi_1(N^T, v^T) = \phi_T(N^T, v^T) = \frac{1}{2}$, contradicting that ϕ is merging-proof.

Next, suppose that ϕ is an anonymous splitting-proof core solution. Let $N^S = \{S, 4\}$ and define the game v^S as follows. $v^S(S) = v^S(4) = 2$ and $v^S(N) = 5$. By anonymity $\phi_S(v^S, N^S) = \phi_4(v^S, N^S) = \frac{5}{2}$.

Suppose that by splitting the coalition S into three individual players $\{1, 2, 3\}$, the game (N, v) is obtained with $N = \{1, 2, 3, 4\}$, $v(1) = v(2) = v(3) = 0$, $v(4) = 2 = v(1, 2) = v(2, 3) = v(1, 3) = 2$, $v(1, 4) = v(2, 4) = v(3, 4) = 3$, $v(1, 2, 3) = 2$, $v\,(i, j, k) = 3$ for any other three-player coalition, and $v(N) = 5$. Since ϕ is a core solution we must have $\phi_i(N, v) = 1$ for $i = 1, 2, 3$, and $\phi_4(N, v) = 2$ contradicting that the solution is splitting-proof. $\square$

We notice that the proof of Propositions 2 only involves monotonic superadditive games.

2.2 Convex games

For the family of probabilistic values, Haller (1994, Corollary 3.3) gives sufficient conditions for which bilateral proxy agreements are always (un)profitable. The Shapley value does not satisfy these conditions, and core compatibility was not addressed. Indeed, the Shapley value is neither merging-proof, nor splitting-proof, not even on the class of on convex games, as showed in Example 1 below. Note that bilateral merging-proofness (splitting-proofness) does not necessarily imply merging-proofness (splitting-proofness).

Example 1 Let (N, v) be a convex game, where $N = \{1, 2, 3, 4\}$ and v is given by $v(S) = 1$ if $|S| = 1$, $v(S) = 3$ if $|S| = 2$, $v(S) = 6$ if $|S| = 3$ and $v(N) = 9$. The Shapley value is $\phi^{Sh}(N, v) = \frac{1}{4}(9, 9, 9, 9)$.[2] Now for $T = \{3, 4\}$ consider the T-merger game where $N^T = \{1, 2, T\}$ and v^T takes the following values: $v^T(1) = v^T(2) = 1$, $v^T(T) = 3$, $v^T(1, 2) = 3$, $v^T(1, T) = v^T(2, T) = 6$ and $v^T(N^T) = 9$. Then $\phi^{Sh}_T(N^T, v^T) = \frac{14}{3} > \frac{9}{2}$. Thus the merger of players 3 and 4 is profitable.

Next, consider the game (N, w) where $N = \{1, 2, 3\}$, $w(S) = |S|$ if $|S| < 3$, $w(N) = a$ where $a > 3$. Then $\phi^{Sh}_i(N, w) = \frac{a}{3}$ for all i. For $T = \{1, 2\}$, the T-merger game w^T is defined by $N^T = \{T, 3\}$, $w^T(T) = 2$, $w^T(3) = 1$ and $w^T(N^T) = a$. Then $\phi^{Sh}_T(N^T, w^T) = \frac{a+1}{2} < \frac{2a}{3}$, i.e. splitting T is profitable. $\square$

Proposition 3 *Let (N, v) be a convex game and $T \subseteq N$. Then the T-merger game (N^T, v^T) is convex.*

[2] The Shapley value can be defined as

$$\phi^{Sh}_i(N, v) = \sum_{S \subseteq N, S \ni i} \frac{(|S|-1)!(|N|-|S|)!}{|N|!}(v(S) - v(S \backslash \{i\})).$$

Proof: Let $S, S^1 \subseteq N^T = \{T, \{i\}_{i \in N \setminus T}\}$. First, we claim that $\overline{S \cap S^1} = \overline{S} \cap \overline{S^1}$. For this, consider a player $i \in N \setminus T$. Then $i \in \overline{S \cap S^1}$ if and only if $i \in S$ and $i \in S^1$ if and only if $i \in \overline{S}$ and $i \in \overline{S^1}$. Further, consider the player T in N^T. Then $\{i\}_{i \in T} \in \overline{S \cap S^1}$ if and only if $T \in S$ and $T \in S^1$ if and only if $\{i\}_{i \in T} \in \overline{S}$ and $\{i\}_{i \in T} \in \overline{S^1}$, which proves the claim.

Second, we claim that $\overline{S \cup S^1} = \overline{S} \cup \overline{S^1}$, which is proved in a similar way: Consider a player $i \in N \setminus T$. Then $i \in \overline{S \cup S^1}$ if and only if $i \in S$ or $i \in S^1$ if and only if $i \in \overline{S}$ or $i \in \overline{S^1}$. Further consider the player T in N^T. Then $\{i\}_{i \in T} \in \overline{S \cap S^1}$ if and only if $T \in S$ or $T \in S^1$ if and only if $\{i\}_{i \in T} \in \overline{S}$ or $\{i\}_{i \in T} \in \overline{S^1}$, which proves the claim.

The game (N^T, v^T) is convex if

$$v^T(S \cap S^1) + v^T(S \cup S^1) \geq v^T(S) + v^T(S^1) \text{ for all } S, S^1 \subseteq N^T,$$

i.e. if

$$v(\overline{S \cap S^1}) + v(\overline{S \cup S^1}) \geq v(\overline{S}) + v(\overline{S^1}) \text{ for all } S, S^1 \subseteq N^T. \tag{1}$$

But since $v(\overline{S \cap S^1}) = v(\overline{S} \cap \overline{S^1})$ and $v(\overline{S \cup S^1}) = v(\overline{S} \cup \overline{S^1})$, (1) is equivalent to

$$v(\overline{S} \cap \overline{S^1}) + v(\overline{S} \cup \overline{S^1}) \geq v(\overline{S}) + v(\overline{S^1}) \text{ for all } S, S^1 \subseteq N^T,$$

which is satisfied since (N, v) is convex and $\overline{S}, \overline{S^1} \subseteq N$. $\square$

For the core $C(N, v)$ of a convex game (N, v), the set of Lorenz-maximal elements $L(N, v) \subseteq C(N, v)$ is, in fact, a singleton (Dutta and Ray, 1989). Let $\phi^{DR}(N, v) = L(N, v)$ denote the *Dutta-Ray solution*, which we define on the class of convex games. By Hardy et al. (1934, Theorem 108) if $f : \mathbb{R} \to \mathbb{R}$ is strictly concave, then $\phi^{DR}(N, v)$ is the maximizer of the additive symmetric social welfare function $\sum_{i \in N} f(x_i)$ subject to the constraint $x \in C(N, v)$.

We shall prove that the Dutta-Ray solution is merging-proof. We make use of a lemma which says that we can always go from one core allocation to another by a sequence of *bilateral transfers* for which the allocation following each step is also in the core.[3]

Lemma 1 *Let (N, v) be a convex game and $x, y \in C(N, v)$. Then there is a transfer matrix $\Gamma = \{\gamma_{ij}\}_{i,j\in N}$ of bilateral transfers leading from x to y, and an ordering $k(ij)$ of bilateral transfers in Γ such that after each bilateral transfer the resulting allocation is in $C(N, v)$. In fact, for any ordering $r(i)$ of receivers (payers) i, there is a transfer matrix Γ and a sequence of core compatible bilateral transfers such that all payments to (from) the receivers follow the sequence $r(i)$, i.e. if $r(i) < r(j)$ then all transfers to (from) player i will be carried out before there are any payments to (from) player j.*

Proof: Let $x', y' \in C(N, v)$. Let $P = \{i|x'_i > y'_i\}$, $R = \{i|x'_i < y'_i\}$ and $U = \{i|x'_i = y'_i\}$.

First, we claim that for an arbitrary player i in P, we can always find some player j in R such that the transfer of some amount $0 < \varepsilon \leq \min\{x'_i - y'_i, y'_j - x'_j\}$ leads to a new allocation which is also in $C(N,v)$.

For this, consider a player $i \in P$, and suppose to the contrary that there is no player j in R for which there can be transferred some amount $0 < \varepsilon_{ij} \leq \min\{x'_i - y'_i, y'_j - x'_j\}$ from i to j (upholding the core constraints). This means that for any $j \in R$, there must be a zero-excess coalition S^j at x' (i.e. $x'(S^j) = v(S^j)$) for which $i \in S^j$ and $j \notin S^j$. By Shapley (1971), the set of zero-excess coalitions is a ring (i.e. closed under union and intersection). In particular, $\cap_{j\in R} S^j$ is a zero–excess coalition. Since $i \in \cap_{j\in R} S^j$ and since $\cap_{j\in R} S^j$ has empty intersection with R, it contradicts that y' is a core allocation.

[3] Suppose that $x, y \in \mathbb{R}^N$, and for some $\gamma \geq 0$ and some $i, j \in N$ we have $y_i - \gamma = x_i$, $y_j + \gamma = x_j$ and $x_k = y_k$ for $k \neq i, j$. We then say that y is reached from x after a bilateral transfer γ from player i to j. A *transfer matrix* is a matrix $\Gamma = \left[\gamma_{ij}\right]_{i,j\in N}$, where is γ_{ij} a bilateral transfer from i to j, satisfying the following conditions: if $\gamma_{ij} > 0$ then $\gamma_{ji} = 0$, if there exists j such that $\gamma_{ij} > 0$ then there exist no j' such that $\gamma_{j'i} > 0$, and $\gamma_{ii} = 0$ for all i. A player that is neither a *payer* nor a *receiver* is called *unaffected*, so a transfer matrix induces a tri-partition of N in *payers*, *receivers*, and *unaffected* players. See also Hougaard and Østerdal (2005).

Second, we claim that for an arbitrary player j in R, we can always find some player i in P such that the transfer of some amount $0 < \varepsilon_{ij} \leq \min\{x'_i - y'_i, y'_j - x'_j\}$ is possible (upholding the core constraints).

For this, consider a player $j \in R$, and suppose to the contrary that there is no player i in P for which there can be transferred some amount $0 < \varepsilon_{ij} \leq \min\{x_i - y_i, y_j - x_j\}$ from i to j. This means that for any $i \in P$, there must be a zero-excess coalition S^i at x' for which $i \in S^i$ and $j \notin S^i$. Since $\cup_{i \in P} S^i$ is then also a zero-excess coalition, $P \subseteq \cup_{i \in P} S^i$ and $j \notin \cup_{i \in P} S^i$ it contradicts that y is a core allocation.

To complete the proof, we must show we can actually obtain y' from x' by a finite number of any such bilateral transfers. For this, we show that for any $x', y' \in C(N, v)$ and sets P and R as described, any player $i \in P$ can transfer a total amount $x'_i - y'_i$ to players in R in at most $|R|$ steps (upholding the core constraint in each step). The argument for that any player $j \in R$ can obtain a total amount of $y'_i - x'_i$ from players in P in at most $|P|$ steps (upholding the core constraint in each step) is similar and omitted.

Consider therefore an arbitrary player $i \in P$, and let $0 < m_i \leq x'_i - y'_i$ denote the supremum of the total amounts of payoff that can be transferred from player i to (a subset of) players in R by an ordered (finite or countable infinite) sequence of core compatible bilateral transfers. Denote the final allocation obtained in the limit of such a sequence of bilateral transfers with y''. First, we notice that the same final allocation y'' can be obtained by an ordered sequence of at most $|R|$ transfers: Let $0 \leq m_{ij} \leq m_i$ denote the supremum of the total amount transferred from i to j. Since $C(N, v)$ is a closed set, the allocation y'' is in the core. Further, we can transfer the entire amounts m_{ij} from i to j in a arbitrary sequence of bilateral transfers involving at most $|R|$ step. Indeed, if the core constraint for a coalition S, $i \in S$, is violated after some step, then the final allocation would also violate this constraint for coalition S - a contradiction. Second, we notice that we cannot have $m_i < x'_i - y'_i$ since y'' is in $C(N, v)$, hence there must be an additional core-compatible bilateral transfer from i to some j player in R for which $y''_j < y'_j$ - a contradiction. $\square$

Proposition 4 *On the class of convex games, there exists a merging-proof core solution: The Dutta-Ray solution is merging-proof.*

Proof: Let (N, v) be a convex game and $x = \phi^{DR}(N, v)$. Let $T \subset N$ and consider the T-merger game (N^T, v^T), and $y = \phi^{DR}(N^T, v^T)$. We want to show that $x(T) \geq y_T$. For this shall argue that if $x(T) < y_T$, then x cannot be the Dutta-Ray solution for the game (N, v) - a contradiction.

From x define the following allocation $\widetilde{x}$ in $\mathbb{R}^{N^T}$: $\widetilde{x}_T = x(T)$ and $\widetilde{x}_i = x_i$ for $i \in N^T \backslash T$. Then $\widetilde{x} \in C(N^T, v^T)$: Indeed, for any coalition $S \subseteq N^T$ we have $\widetilde{x}(S) = x(\overline{S}) \geq v(\overline{S}) = v^T(S)$.

We define the following two sets of players in N^T: $P = \{i \in N^T \backslash T | y_i < x_i\}$ and $R = T \cup \{i \in N^T \backslash T | y_i > x_i\}$. Hence in $C(N^T, v^T)$ we can obtain y from $\widetilde{x}$ by bilateral transfers from players in P to players in R. By Lemma 1, there exists a sequence of these bilateral transfers, such that after each step in this sequence, the allocation obtained is in $C(N^T, v^T)$ and the player T first begins to receive payoff from a subset P' of the players P when all other players in R have obtained all their payoff (i.e. each player $i \in R \backslash T$ has received $y_i - x_i$). Further, by Lemma 1, these bilateral transfers to T can be made in an arbitrary sequence (upholding the core constraints). Hence each of these transfer from players in P to T must increase social welfare measured by $\sum_{i \in N^T} f$.

Consider now the game (N, v) and $C(N, v)$. Since $\widetilde{x}_T \geq x_i$ for all $i \in T$, for any player i in P' it follows that there is a (sufficiently small) amount of payoff p_i such that a bilateral transfer of p_i from i to any player in T increases social welfare measured by $\sum_{i \in N} f$. Since $x = \phi^{DR}(N, v)$ any such transfer must violate a core constraint. In particular, for an arbitrary player $i \in P'$ and any player j in T there must be a zero-excess coalition S^j at x such that $i \in S^j$ and $j \notin S^j$. Hence $i \in \cap_{j \in T} S^j \subset N \backslash T$ and

$\cap_{j \in T} S^j$ is a zero-excess coalition, contradicting that y is in $C(N^T, v^T)$ since $v^T(\cap_{j \in T} S^j) = v(\cap_{j \in T} S^j) = x(\cap_{j \in T} S^j) > y(\cap_{j \in T} S^j)$. □

For the family of strictly monotonic convex games, splitting-proof core solutions exist. Note that a convex game (N, v) is strictly monotonic if and only if $v(i) > 0$ for all $i \in N$.

Proposition 5 *On the class of strictly monotonic convex games, there exists an anonymous splitting-proof core solution.*

Proof: We define a core solution, called ϕ^*, and show that a merger is always profitable; that is, for any (N, v) and any $T \subset N$ then $\phi^*_T(N^T, v^T) \geq \sum_{i \in T} \phi^*_i(N, v)$.

For any game (N, v), there is $1 \leq k \leq |N|$ and a partition $P_1, ..., P_k$ of N, classifying players according to increasing contribution to the grand coalition, i.e. for any $1 \leq m \leq n \leq k$, if $i \in P_m$ and $j \in P_n$ then $v(N) - v(N \backslash \{i\}) \leq v(N) - v(N \backslash \{j\})$.

Let $\sigma = (i, j, k, ...)$ be an ordering of the players such that i' is listed before j' if there is $m < n$ such that $i' \in P_m$ and $j' \in P_n$. Let

$$p(\sigma) = (v(i), v(ij) - v(i), v(ijk) - v(ij), ...)$$

be the partial marginal associated with σ. We then define ϕ^* to be the center of gravity of the $|P_1|!|P_2|! \cdots |P_k|!$ partial marginals that can be generated by all such orderings σ, i.e.

$$\phi^*(N, v) = \frac{p(\sigma^1) + p(\sigma^2) + ...}{|P_1|!|P_2|! \cdots |P_k|!},$$

where $\sigma^1, \sigma^2, ..$ are all possible orderings satisfying the condition described above.

We claim that for any $T \subseteq N$, a T-merger is always profitable for the players in T. For this, note that by convexity and strict monotonicity, $v(S) -$

$v(S\backslash\{i\}) \geq v(i) > 0$ for all $S \ni i$. Now consider a coalition $T \subset N$, and let λ_i be the highest possible partial marginal in (N, v) to player i taken over all orderings σ (i.e. the partial marginal when player i has the last possible position in σ). We then have

$$\sum_{i \in T} \phi_i^*(N, v) \leq \sum_{i \in T} \lambda_i.$$

However, for the T-merger game (N^T, v^T) we have

$$\phi_T^*(N^T, v^T) \geq \sum_{i \in T} \lambda_i,$$

because every partial marginal for player T in (N^T, v^T), for which the players are ordered according to increasing contributions to the grand coalition, is greater than or equal to $\sum_{i \in T} \lambda_i$, since $v^T(N^T) - v(N^T \backslash T) > v(N) - v(N \backslash \{i\})$ for all $i \in T$. □

3 Concluding remarks

It remains an open question whether there exists a splitting-proof core solution on the class of (not necessarily monotonic) convex games.

Merging-proofness of the Dutta-Ray solution appeared to be closely connected to the defining property of this solution of selecting the most equal allocation in the core. We conjecture that the Dutta-Ray solution is the *only* merging-proof core solution on the class of convex games.

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