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Abstract

This paper suggests a simple approximation procedure for the assessment of productivity scores with respect to fuzzy production plans. The procedure has a clear economic interpretation and all the necessary calculations can be performed in a spreadsheet making it highly operational.

Keywords: Fuzzy Production Plans, Productivity, Fuzzy DEA, Fuzzy Numbers, Fuzzy Index

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1 Introduction

Several authors have recognized the need to introduce some kind of data uncertainty into the linear programming models of Data Envelopment Analysis (DEA) since these non-parametric frontier models are extremely sensitive to measurement errors and outliers, see e.g. Färe, Grosskopf and Lovell [5]. A straightforward approach seems to be to represent the input and output data as intervals or, more generally, as fuzzy sets.

There are several ways to handle fuzzy data in connection with DEA. The main approach seems to be to transform fuzzy DEA into crisp DEA for given level sets, see e.g. Triantis and Girod [12], Kao and Liu [7], Entani et al. [4] and León et al. [9]. Another possibilistic approach treats the constraints of the DEA programs as fuzzy events, see e.g. Lertworasirikul et al. [10] while a third approach considers only pairwise dominance of fuzzy data, see e.g. Triantis and Vanden Eeckaut [14] and Triantis et al. [13].

In the present paper yet another approach is introduced. Based on a simple procedure for approximation of the DEA scores in the crisp case a generalized procedure for fuzzy production data is developed. The main advantage is the operationability since all calculations can be performed in a spreadsheet and, in any case, do not involve fuzzy programming. Moreover, the entire procedure (with its resulting crisp productivity scores) has an economic interpretation parallel to the original interpretation of the DEA scores.

The paper is organized as follows: In Section 2, the procedure is motivated by looking at the crisp case. Section 3, introduces the basic model. Section 4, defines a suitable fuzzy index and disclose some of its properties. The entire procedure is presented and discussed in Section 5. Section 6 closes with final remarks.

2 Motivation - the crisp case

Consider the standard DEA model as introduced by Charnes, Cooper and Rhodes [1] where the productivity score of production plan (or DMU) i is found by solving the program:

$$\max_{u,v \geq 0} \frac{u \cdot y_i}{v \cdot x_i}$$

s.t.
$$\frac{u \cdot y_r}{v \cdot x_r} \leq 1, \quad r = 1, \ldots, i, \ldots, k,$$

where x is a vector of inputs and y is a vector of outputs for the set of k production plans.

As noticed in [1] this program can be transformed into a linear programming problem by fixing either the input or the output side, i.e. $v \cdot x_i = 1$ or $u \cdot y_i = 1$. Clearly, the linear programming version is easier to solve and is known as the CCR model (implicitly assuming constant returns to scale in production). However, it is easy to see that a simple approximation to the solution of the above non-linear fractional programming problem can be found as follows:

Let, for (positive) output h and (positive) input j, weights u_h and v_j be such that

$$\frac{v_j}{u_h} = \max_r \frac{y_{hr}}{x_{jr}}.$$

Clearly, the weights $\hat{u} = (0, \ldots, 0, u_h, 0, \ldots, 0)$ and $\hat{v} = (0, \ldots, 0, v_j, 0, \ldots, 0)$ constitute a feasible solution since

$$\frac{\hat{u} \cdot y_r}{\hat{v} \cdot x_r} \leq 1, \quad \forall \ \ r = 1, \ldots, k.$$

Moreover, the value of the objective function

$$\frac{y_{hi}}{x_{ji}} / \max_r \frac{y_{hr}}{x_{jr}}$$

is obviously smaller than the optimal value of the fractional programming problem.

As such, the (CCR) productivity score can be approximated from below by considering the ratio matrix $[y_{hr}/x_{jr}]$, normalize (each column) with $\max_r \frac{y_{hr}}{x_{jr}}$ and then select the most favorable (maximal) normalized ratio for each production plan (DMU). The procedure therefore has a clear economic interpretation and all the necessary calculations can be performed in a spreadsheet making it highly operational.

This type of approximation is analysed in detail in Despić [3] where it is demonstrated by several simulations that it is indeed a good approximation, in particular, in case of limited substitution possibilities among input and outputs. The aim of the present paper is therefore to generalize this simple and operational procedure to the assessment of productivity of fuzzy production plans. While [3] goes on to develop a comprehensive fuzzy linguistic framework, the present paper takes a more direct approach.

3 Fuzzy productivity ratios

Consider a finite set of *fuzzy production plans* (or DMU's) denoted by $\mathcal{F} = \{F_1, \ldots, F_k\}$ where each production plan transforms a finite number of fuzzy inputs $\{I_1, \ldots, I_m\}$ into a finite number of fuzzy outputs $\{O_1, \ldots, O_n\}$, i.e. $F_i = (I_i, O_i)$, for $i = 1, \ldots, k$. For the sake of simplicity each fuzzy input and output is given by a triangular fuzzy number.

In general, consider a triangular fuzzy number $A = (x; a, b, c)$ where $0 < a \leq b \leq c$, and let for $\alpha \in [0, 1]$, $A_\alpha = \{x \in \mathbf{R} | A(x) \geq \alpha\}$ be the α-level set associated with A. Since $A_\alpha = [a(\alpha), c(\alpha)]$ where $a(\alpha) = \inf A_\alpha$ and $c(\alpha) = \sup A_\alpha$, the triangular fuzzy number A may also be represented by intervals

$$A(\alpha) = [a(\alpha), c(\alpha)] = [(b-a)\alpha + a, -(c-b)\alpha + c] \quad \text{for all} \quad \alpha \in [0, 1].$$

In particular, $A(1) = b$ is called the *kernel* and $A(0) = [a, c]$ is called the *support* of the fuzzy number.

Now, let each *fuzzy input* $j = 1, \ldots, m$ be given by triangular fuzzy number $I_j = (x; a_j, b_j, c_j)$. Likewise, let each *fuzzy output* $h = 1, \ldots, n$ be given by a triangular fuzzy number $O_h = (y; d_h, e_h, f_h)$. Thus, the $k \times (m+n)$ data matrix is given by

$$(I_j^i, O_h^i), \; i = 1, \ldots, k, \; j = 1, \ldots, m, \; h = 1, \ldots, n.$$

Triangular fuzzy numbers seem to fit very well with practical applications since the kernel may be interpreted as the crisp value that has actually been measured and the support may be interpreted as a kind of confidence interval with the minimum value as the most pessimistic estimate and the maximum value as the most optimistic estimate of the 'true' value.

Now, invoking the assumption of constant returns to scale all fuzzy production plans can be compared through the $k \times mn$ fuzzy productivity ratio matrix given by

$$\begin{bmatrix} \frac{O_1^1}{I_1^1} & \cdots & \frac{O_1^1}{I_m^1} & \cdots & \frac{O_n^1}{I_1^1} & \cdots & \frac{O_n^1}{I_m^1} \\ \vdots & & & & & & \vdots \\ \frac{O_1^k}{I_1^k} & \cdots & \frac{O_1^k}{I_m^k} & \cdots & \frac{O_n^k}{I_1^k} & \cdots & \frac{O_n^k}{I_m^k} \end{bmatrix}.$$

Denote by

$$R_l^i = \frac{O_h^i}{I_j^i} = \frac{(y; d_h^i, e_h^i, f_h^i)}{(x; a_j^i, b_j^i, c_j^i)},$$

fuzzy productivity ratio $l = 1, \ldots, mn$ of DMU $i = 1, \ldots, k$.

Since division performed at each α-level yields the correct form of the operation of division (see e.g. Kaufmann and Gupta [8]), the fuzzy productivity ratios are determined by intervals

$$R_l^i(\alpha) = \left[\hat{r}_l^i(\alpha), \check{r}_l^i(\alpha)\right] = \left[\frac{(e_h^i - d_h^i)\alpha + d_h^i}{-(c_j^i - b_j^i)\alpha + c_j^i}, \frac{-(f_h^i - e_h^i)\alpha + f_h^i}{(b_j^i - a_j^i)\alpha + a_j^i}\right] \;\forall\; \alpha \in [0,1].$$

The economic interpretation of such ratios is straightforward: The kernel ($\alpha = 1$) can simply be interpreted as the usual crisp productivity ratio (e_h^i/b_j^i) while the support ($\alpha = 0$) is given by the interval from the worst possible ratio (when output has its lowest value and input has its highest value, i.e. d_h^i/c_j^i) to the best possible ratio (when output has its highest value and input its lowest value, i.e. f_h^i/a_j^i).

The fuzzy ratios R_l^i may also be approximated linearly as triangular fuzzy numbers

$$R_l^i \approx (z; \frac{d_h^i}{c_j^i}, \frac{e_h^i}{b_j^i}, \frac{f_h^i}{a_j^i}), \quad i = 1, \ldots, k, \quad l = 1, \ldots, mn.$$

However, as demonstrated in Giachetti and Young [6] this approximation may yield substantial errors, see also Example 1 below.

As mentioned in Section 2, the crisp scores of DEA (using the CCR model) can be approximated from below by a procedure where, for each productivity ratio, a maximal (benchmark) ratio is found across the set of production plans (that is, for each column in the ratio matrix) and by dividing each column with its respective maximal ratio a normalized productivity ratio matrix is obtained.

Mimicking this procedure for fuzzy production data we may define the maximal (benchmark) ratio for each column of the fuzzy productivity ratio matrix by

$$R_l^{max}(\alpha) = [\hat{r}_l^{max}(\alpha), \check{r}_l^{max}(\alpha)] = \left[\max_{i=1,\dots,k}\left\{\frac{(e_h^i - d_h^i)\alpha + d_h^i}{-(c_j^i - b_j^i)\alpha + c_j^i}\right\}, \max_{i=1,\dots,k}\left\{\frac{-(f_h^i - e_h^i)\alpha + f_h^i}{(b_j^i - a_j^i)\alpha + a_j^i}\right\}\right],$$

for all $\alpha \in [0,1]$ and $l = 1, \dots, mn$.

In terms of production economics, such maximal benchmarks (or pseudo units) represent a combination of the best performance that has been observed with respect to kernel and support for each possible fuzzy productivity ratio. Consequently, such benchmarks may not represent data from an existing production unit (cf. Example 1 below).[1]

Unfortunately, normalizing each column in the fuzzy productivity ratio matrix by dividing with the respective maximal ratios (for example defined like R_l^{max} above) may result in fuzzy efficiency scores without a sound economic interpretation, as demonstrated by the following example:

Example 1: Consider the following fuzzy data matrix (Table 1) where each fuzzy input and output is given by triangular fuzzy numbers.

Table 1: *Fuzzy data*

DMU no.	Input 1	Input 2	Output 1	Output 2
1	$(x; 2, 3, 6)$	$(x; 5, 7, 8)$	$(y; 4, 6, 7)$	$(y; 5, 6, 7)$
2	$(x; 2, 4, 5)$	$(x; 1, 5, 6)$	$(y; 6, 8, 9)$	$(y; 6, 7, 7)$
3	$(x; 5, 6, 7)$	$(x; 4, 5, 5)$	$(y; 8, 9, 9)$	$(y; 10, 10, 11)$
4	$(x; 3, 5, 6)$	$(x; 3, 6, 7)$	$(y; 5, 7, 8)$	$(y; 8, 9, 10)$
5	$(x; 1, 2, 2)$	$(x; 3, 4, 4)$	$(y; 4, 6, 7)$	$(y; 5, 6, 6)$
6	$(x; 3, 4, 4)$	$(x; 1, 2, 2)$	$(y; 3, 4, 5)$	$(y; 4, 5, 6)$

[1] To compare observed production data with extreme performance of pseudo units is in the spirit of non-parametric efficiency analysis (like DEA) where the efficient frontier of the production function is estimated from observed production data by adding assumptions of convexity and returns to scale. Thereby observed production data are often compared to efficient combinations of existing productions when assessing the DEA efficiency score, see e.g. [5].

By the fuzzy data in Table 1, the fuzzy productivity ratios can be obtained. In Table 2 below the linear approximation is used for simplicity:

Table 2: *Approximated fuzzy productivity ratios*

DMU no.	O_1/I_1	O_1/I_2	O_2/I_1	O_2/I_2
1	$(z; 0.66, 2, 3.5)$	$(z; 0.5, 0.86, 1.4)$	$(z; 0.83, 2, 3.5)$	$(z; 0.63, 0.86, 1.4)$
2	$(z; 1.2, 2, 4.5)$	$(z; 1, 1.6, 9)$	$(z; 1.2, 1.75, 3.5)$	$(z; 1, 1.4, 7)$
3	$(z; 1.14, 1.5, 1.8)$	$(z; 1.6, 1.8, 2.25)$	$(z; 1.43, 1.66, 2.2)$	$(z; 2, 2, 2.75)$
4	$(z; 0.83, 1.4, 2.66)$	$(z; 0.71, 1.17, 2.66)$	$(z; 1.33, 1.8, 3.33)$	$(z; 1.14, 1.5, 3.33)$
5	$(z; 2, 3, 7)$	$(z; 1, 1.5, 2.33)$	$(z; 2.5, 3, 6)$	$(z; 1.25, 1.5, 2)$
6	$(z; 0.75, 1, 1.66)$	$(z; 1.5, 2, 5)$	$(z; 1, 1.25, 2)$	$(z; 2, 2.5, 6)$

The approximated maximal ratio for each column, R_l^{max}, is respectively:

$$(z; 2, 3, 7),\ (z; 1.6, 2, 9),\ (z; 2.5, 3, 6) \text{ and } (z; 2, 2.5, 7).$$

Hence, dividing each column with the associated maximal ratio yields the normalized productivity ratio matrix $[R_l^i/R_l^{max}]$. For each fuzzy production plan a fuzzy productivity score may now be defined by selecting, from the normalized ratio matrix, the largest kernel and support across the normalized ratios, i.e. the score of production plan i is the triangular fuzzy number

$$P(F_i) = (z; \max_l\{\frac{d_h^i/c_j^i}{\max_i\{d_h^i/c_j^i\}}\}, \max_l\{\frac{e_h^i/b_j^i}{\max_i\{e_h^i/b_j^i\}}\}, \max_l\{\frac{f_h^i/a_j^i}{\max_i\{f_h^i/a_j^i\}}\}),$$

yielding:

Table 3: *Fuzzy productivity scores*

DMU no.	Triangular fuzzy scores
1	$(z; 0.13, 0.66, 1.75)$
2	$(z; 0.2, 0.8, 5.63)$
3	$(z; 0.29, 0.9, 1.41)$
4	$(z; 0.22, 0.6, 1.66)$
5	$(z; 0.42, 1, 3.5)$
6	$(z; 0.29, 1, 3.13)$

Now, the kernels $(0.66, 0.8, 0.9, 0.6, 1, 1)$ can be directly compared to the Farrell productivity score that results from DEA using the CCR model on the crisp data set given by the kernels of the fuzzy data matrix. The DEA scores are,

$$(0.66, 0.98, 1, 0.86, 1, 1).$$

As mentioned in Section 2, the procedure above yields a lower estimate.

However, notice that the fuzzy scores, as such, have no clear economic interpretation because the normalization may result in meaningless fuzzy ratios. For example, consider DMU no. 5, where the ratio $(z; 2, 3, 7)$ is divided by itself (as maximal ratio) yielding ratio $(z; 0.29, 1, 3.5)$. Clearly, this new ratio is meaningless: For example, the value $(2/7 =)$ 0.29 is interpreted as the normalized ratio related to the scenario where the ratio R_l^i is at its lowest value while the maximal ratio R_l^{max} is at its highest value (the worst possible case) - but this is *not* a possible scenario since we speak about the same production plan.

Moreover, as demonstrated in [6], the linear approximation of non-linear operations like product and division may generally lead to substantial errors. In the present case, with fuzzy data as given in Table 1, the errors related to the approximated fuzzy productivity ratios also prove to be significant. To pick an extreme case; for DMU 2, ratio R_2, the upper bound of the 0.5 α-level is 87 % off the true value. □

Consequently, instead of dividing each column with the maximal ratio (which may be one of the existing ratios), a fuzzy index (to be defined in Section 4) will be used to compare each ratio to the maximal ratio yielding a (crisp) productivity score larger than or equal to 1. This score can be interpreted as the relative improvement potential of the given fuzzy production plan along the lines of the standard interpretation of the DEA scores in the crisp case.

4 An index ranking fuzzy numbers

There is a large literature on indexes ranking fuzzy numbers, see e.g. Wang and Kerre [15] for a recent survey. For this particular application, however, it is chosen to define an index in line with the 'spirit' of DEA in the sense that the index compares each fuzzy number relatively to the maximal fuzzy number associated with the set of alternatives. As such, the index is related to indices defined in Liou and Wang [11] and Choobineh and Li [2].

In general, consider a finite set of normalized fuzzy numbers

$$\mathcal{A} = \{A_1, \ldots, A_n\},$$

and let A_i be represented by the α-levels $A_i(\alpha) = [a_i(\alpha), c_i(\alpha)]$ for all $\alpha \in$

$[0,1]$. Further, let

$$A^{max} = [a^{max}(\alpha), c^{max}(\alpha)] = [\max_{i=1,\ldots,n}\{a_i(\alpha)\}, \max_{i=1,\ldots,n}\{c_i(\alpha)\}]$$

for all $\alpha \in [0,1]$.

Now, define the following index:

$$J(A_i) = \lambda \check{t}_i + (1-\lambda)\hat{t}_i, \quad \lambda \in [0,1]$$

where

$$\check{t}_i = \int_0^1 \frac{c^{max}(\alpha)}{c_i(\alpha)} d\alpha, \quad \hat{t}_i = \int_0^1 \frac{a^{max}(\alpha)}{a_i(\alpha)} d\alpha.$$

Clearly, $J(A^{max}) = 1$ and $J(A_i) \geq 1, \ \forall i$, with strict inequality if $A_i \neq A^{max}$.

As such, the fuzzy index $J(\cdot)$ can be interpreted as follows: For each level set of a given fuzzy number the factor necessary to scale the smallest (largest) value to the smallest (largest) value of the corresponding level set of the maximal fuzzy number is found and $\hat{t}_i$ ($\check{t}_i$) is the average of those factors over all level sets. Now, the index value is found as a weighted average of $\check{t}_i$ and $\hat{t}_i$. The weight λ can be considered as exogeneously chosen by the analyst and indicates whether the overall focus is pessimistic or optimistic, i.e. relates to the left ($\lambda = 0$) or right spred ($\lambda = 1$) of the triangular fuzzy productivity ratios.

The index $J(\cdot)$ induces a natural ordering of the set $\mathcal{A}$ where $A_i \succeq A_j \ \Leftrightarrow J(A_i) \leq J(A_j)$. The relations $\succ$ and $\sim$ are defined as $A_i \succeq A_j \ \wedge \ \neg A_j \succeq A_i$ and $A_i \succeq A_j \ \wedge \ A_j \succeq A_i$, respectively.

4.1 Properties of $J(\cdot)$

The first obvious property of index $J(\cdot)$ is the following:

Proposition 1: *Consider two fuzzy numbers $A_i, A_j \neq A^{max}$ where for all $\alpha \in [0,1]$, $A_j = [\beta a_i(\alpha), \beta c_i(\alpha)]$, then*

$$J(A_j) = \frac{1}{\beta} J(A_i).$$

Proof: Clearly, $\hat{t}_j = \hat{t}_i/\beta$ and $\check{t}_j = \check{t}_i/\beta$. $\square$

Moreover, the following properties will turn out to be convenient.

P1: (Dominance) If for any pair A_i, A_j that $a_i(\alpha) \geq a_j(\alpha)$ for all $\alpha \in [0,1]$ and $c_i(\alpha) \geq c_j(\alpha)$ for all $\alpha \in [0,1]$ with strict inequality for at least one $\alpha-$level, then $A_i \succ A_j$.

Loosely speaking, dominance ensures that if one fuzzy number is located to right of another fuzzy number then this fuzzy number should be preferred.

P2: (Pairwise symmetric spread) Let $\mathcal{A} = \{A_i, A_j\}$ where $A_i = (x; a^i, b^i, c^i)$ and $A_j = (x; a^j, b^j, c^j)$ are two triangular fuzzy numbers with $b^i = b^j$, $a^j = a^i - \epsilon$ and $c^j = c^i + \epsilon$, $\epsilon > 0$. Then $A_i \succ A_j$.

Loosely speaking, pairwise symmetric spread ensures that increased uncertainty leads to lower preference. In this sense, production plans are not rewarded for being associated with uncertain data measurement.

Proposition 2: *The ordering induced by index $J(\cdot)$ satisfies P1 and P2.*

Proof: P1 follows from $\check{t}_i \leq \check{t}_j$ and $\hat{t}_i \leq \hat{t}_j$, where at least one of the inequalities is strict. P2 follows from $\check{t}_i < \hat{t}_j$ and $\hat{t}_i = \check{t}_j = 1$. $\square$

Remark: Note that if $|\mathcal{A}| > 2$ then P2 is not necessarily satisfied. Thus, the index $J(\cdot)$ does *not* satisfy $[A_i \succeq A_j \text{ on } \mathcal{A} \;\Leftrightarrow\; A_i \succeq A_j \text{ on } \hat{\mathcal{A}}]$ where $A_i, A_j \in \mathcal{A} \cap \hat{\mathcal{A}}$ for $\mathcal{A}$ and $\hat{\mathcal{A}}$ being two arbitrary sets of fuzzy numbers.[2]

5 Using the fuzzy index $J(\cdot)$ to obtain productivity scores

Recall that the fuzzy productivity ratio matrix is given by

$$R_l^i(\alpha) = \left[\hat{r}_l^i(\alpha), \check{r}_l^i(\alpha)\right] = \left[\frac{(e_h^i - d_h^i)\alpha + d_h^i}{-(c_j^i - b_j^i)\alpha + c_j^i}, \frac{-(f_h^i - e_h^i)\alpha + f_h^i}{(b_j^i - a_j^i)\alpha + a_j^i}\right] \;\forall\; \alpha \in [0,1],$$

[2]Note also, that since the ordering is induced by an index it satisfies central properties like completeness (for all $A_i, A_j \in \mathcal{A},\; A_i \succeq A_j$ or $A_j \succeq A_i$) and transitivity (for all $A_i, A_j, A_h \in \mathcal{A},\; [A_i \succeq A_j,\; A_j \succeq A_h] \;\Rightarrow\; A_i \succeq A_h$).

for $i = 1, \ldots, k$ and $l = 1, \ldots, mn$.

Moreover, the maximal (benchmark) ratio for each column of the fuzzy productivity ratio matrix is given by

$$\begin{aligned} R_l^{max}(\alpha) = \ & [\hat{r}_l^{max}(\alpha), \check{r}_l^{max}(\alpha)] = \\ & \left[\max_{i=1,\ldots,k}\left\{\frac{(e_h^i - d_h^i)\alpha + d_h^i}{-(c_j^i - b_j^i)\alpha + c_j^i}\right\}, \max_{i=1,\ldots,k}\left\{\frac{-(f_h^i - e_h^i)\alpha + f_h^i}{(b_j^i - a_j^i)\alpha + a_j^i}\right\}\right] , \end{aligned}$$

for all $\alpha \in [0, 1]$ and $l = 1, \ldots, mn$.

Thus, for for each DMU ($i = 1, \ldots, k$) and each fuzzy productivity ratio ($l = 1, \ldots, mn$) we get index value

$$J(R_l^i) = \lambda \int_0^1 \frac{\check{r}_l^{max}(\alpha)}{\check{r}_l^i(\alpha)} d\alpha \ + (1 - \lambda) \int_0^1 \frac{\hat{r}_l^{max}}{\hat{r}_l^i} d\alpha,$$

and the *productivity score* $P(F_i)$ of fuzzy production plan F_i is consequently defined as

$$P(F_i) = \min_l \{J(R_l^i)\}.$$

These scores can be interpreted along the lines of the DEA score in the crisp case: For each production plan we select the ratio for which the utilization is most favorable compared relatively to the best performer with respect to this particular ratio (which may be a pseudo unit just like reference units may be pseudo units in DEA). In other words, a score on 1.2 can be interpreted as a need to increase the utilization with 20 pct in order to be as productive as the benchmark.

Example 1 continued: Recall the fuzzy data matrix in Table 1 and assume that the analyst is neither optimistic nor pessimistic when considering the fuzzy productivity ratios, i.e. $\lambda = 0.5$ when using the index $J(\cdot)$ of Section 4.

Thus, for example, the index value of the first fuzzy ratio for the first DMU, R_1^1, is found as

$$J(R_1^1) = \frac{1}{2}\left(\int_0^1 \frac{(7 - \alpha)/(\alpha + 1)}{(7 - \alpha)/(\alpha + 2)} d\alpha + \int_0^1 \frac{(2\alpha + 4)/2}{(2\alpha + 4)/(6 - 3\alpha)} d\alpha\right) = 1.97.$$

Similarly, we obtain the index values in Table 4 below:

Table 4: *Index values for each ratio*

DMU no.	$J(R_1)$	$J(R_2)$	$J(R_3)$	$J(R_4)$
1	1.97	2.94	1.91	3.27
2	1.56	1.24	1.81	1.70
3	2.32	1.41	1.97	1.38
4	2.31	1.97	1.74	1.73
5	1.00	1.65	1.00	1.93
6	3.15	1.06	2.52	1.00

Consequently, the productivity scores for each fuzzy production plan, $P(\cdot)$, are given in Table 5, where also the scores arising from the crisp data set given by the kernels are presented and compared.

Table 5: *Comparison of productivity scores for fuzzy and crisp production data*

DMU no.	Scores, fuzzy data	Scores, crisp data	Δ in %-points
1	1.91	1.50	41
2	1.24	1.25	-1
3	1.38	1.11	27
4	1.73	1.67	6
5	1.00	1.00	0
6	1.00	1.00	0

For DMU's 1-4 the adding of uncertain data in form of triangular fuzzy numbers (with the crisp data as kernels) has resulted in adjustments of the productivity score. The adjustments may go in both directions: For DMU 2 the score has improved slightly while for DMU 1,3 and 4 the uncertainty has lead to an increased improvement potential. In particular, it can be noted that DMU 1 and 3, are affected by the fuzziness introduced since the improvement potential has increased with 41 and 27 percentage points respectively.

For example, consider DMU no. 1. In this case the large difference between the crisp and the fuzzy score relates to a large left-spread of R_3^1 compared to the left-spread of R_3^{max}. Considering crisp data, the best performance of DMU 1 is found w.r.t. ratio R_1 and R_3 (since for example $R_3^1 = 2$ and $R_3^{max} = 3$ making $3/2 = 1.50 = \min_l\{R_l^1/R_l^{max}\}$, cf. kernel values in Table 5). Now, introducing fuzziness, the fuzzy score is related to ratio R_3. Since $R_3^1 = [(\alpha+5)/(6-3\alpha), (7-\alpha)/(2+\alpha)]$ has a relatively large left-spread compared to $R_3^{max}(\alpha) = [0.5\alpha + 2.5, 6/(\alpha+1)]$ the value of $\hat{t}_1$ becomes quite large ($\hat{t}_1 = 2.25$) while the value of $\check{t}_1 = 1.56$ is close to the crisp value. As we assume that $\lambda = 0.5$ we obtain (as the average of these two values) fuzzy score 1.91 - see also Table 6 below.

Now, in order to get an impression of the role of parameter λ we consider the index values for DMU 1, for extreme values of λ.

Table 6: *Values $J(\cdot)$ related to DMU 1, for extreme values of λ*

DMU no. 1	Scores, $\lambda = 1$	Scores, $\lambda = 0$
$J(R_1)$	1.69	2.25
$J(R_2)$	3.21	2.67
$J(R_3)$	1.56	2.25
$J(R_4)$	3.48	3.06

As it appears in Table 6, focussing on the right-spread ($\lambda = 1$), the score of DMU 1 will be 1.56 which is better than the score 1.91 obtained for $\lambda = 0.5$ (cf. Table 5). Note that the score is related to ratio R_3 in both cases. However, if focus is set on the left-spread ($\lambda = 0$), then the score is 2.25, but this time it is also related to ratio R_1. In other words, focussing on the right-spread the ratio 'closest' to the maximal ratio is R_3 while focussing on the left-spread the ratios R_1 and R_3 are equally 'close' to their maximal ratio. $\square$

5.1 Indexes for fuzzy numbers and the operation of division

It is important to notice that using indices like $J(\cdot)$, or indeed any non-trivial index where the left and right spread of the membership function plays a role, rankings are not necessarily preserved by the operation of division. To be more specific, dividing all outputs with the same input yields some ranking of the productivity ratios according to the index values but dividing all outputs with another input may yield a different ranking according to the (new) index values. At first sight, this seems to make the present fuzzy approach incompatible with the economics of productivity analysis. However, the dominance property, as shown in Section 4.1, is crucial in this connection: Knowing that the index respects dominance relations we know that the ranking will only change slightly and not violate dominance. Loosely speaking, the problem occurs because division change the spread of the membership functions and since the index relates to all level sets the rankings may change when division by different fuzzy numbers change the spread of the ratios differently

- clearly, this is not the case in a crisp framework where division is order preserving. Consider Example 2 below.

Example 2: For simplicity we focus of the right spread of the membership function, i.e. $\lambda = 1$ using index $J(\cdot)$. When $b^{max} = c^{max} = 20$ we have that $F_i \sim F_j$ when $b_i = 3,\ c_i = 5$ and $b_j = c_j = 3.737$. Now, dividing these data by fuzzy number $\hat{F}$ where $\hat{a} = 2$ and $\hat{b} = 5$, we get that $J(F_i) = 18.54 < 18.73 = J(F_j)$, i.e. preferences are now strict, $F_i \succ F_j$. □

6 Final remarks

The present approach provides a simple approximation of (crisp) productivity scores for fuzzy production data. What is needed in addition to the usual crisp production data of DEA is basically a confidence interval related to the measurement of each input and output. Ideally, specific membership functions can be used to describe the relation between the actually observed data and their confidence intervals but this relation may be approximated by triangular fuzzy numbers (as in the present exposition - and indeed, in the fuzzy DEA literature in general).

Contrary to other approaches to fuzzy DEA (as cited in the Section 1), the present approach does not use fuzzy programming techniques (or fuzzy dominance relations) but only the well defined operation of division and a specific index for the ranking of fuzzy numbers. Since an index is used the procedure results in crisp productivity scores directly comparable with the usual DEA scores. As illustrated by Example 1 above, this comparison between scores related to fuzzy data and scores related to crisp data, provides the analyst with an immediate impression of the impact of data uncertainty on the improvement potential of specific DMU's.

At first sight, it seems that the implicit aggregation process of the index greatly simplifies the performance evaluation compared to the often overwhelming amount of information contained in sets of fuzzy scores for certain parameter values or pairwise degrees of fuzzy dominance for the entire set of DMU's, that are the results of the known approaches. However, in order to further understand the role played by fuzziness for particular DMU's the analyst will have to relate the index values to the original fuzzy productivity ratios and the maximal ratios. Hence, complexity is to a certain extent still inherent in the evaluation process.

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