Expected utility theory* with ”small worlds”

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Abstract

We formulate a new theory of expected utility in which risk and uncertainty is modelled by the usage of a so called event space which is a natural generalisation of a state space. The basic idea is that the decision maker for each group of related decisions creates a ”small world” (a local state space) from the events in the ”grand world” (the event space). We introduce a set of preference axioms similar in spirit to the Savage axioms, and show that they lead to a a more general expected utility theory, where in each ”small world” risk is described by an additive probability measure. All local risk measures appear as restrictions of a common integrated additive expectation functional defined on the ”grand world”. A benefit of the theory is that it allows for an intuitive distinction between risk and uncertainty. In particular, there is a numerical measure of the degree of uncertainty aversion associated with a given preference relation, which can be calculated. We illustrate the use of the theory for the Ellsberg paradox and for a no-trade portfolio decision problem, none of which can be captured using standard expected utility theory.

JEL classification: D8 and G12.

Key words: Expected utility, decision making under uncertainty, uncertainty aversion.

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1 Introduction

This paper develops a new theory of expected utility, which in a natural way may be considered as a generalisation of the standard expected utility theory, but is able to capture the difference between risk and uncertainty. The differentiation between risk and uncertainty is based on the idea that the decision maker for each group of similar or related decisions creates a "small world" consisting of only those events which are considered relevant in the given context. This may be interpreted as a cognitive process, where, before a decision is taken, it is grouped together with other decisions in a small and more manageable world.

The theory is based on two elements. One is the "global" event space which is introduced as a natural generalisation of a state space. The other is a set of axioms on preferences adapted to the more general theory of an event space. It is perhaps worth noting that our formulation preserves for "small worlds" the type of axioms introduced by Savage.

The notion of small worlds and their use in modelling decision making is described in Savage (1954)[1]:

> *In the sense under discussion a smaller world is derived from a larger by neglecting some distinctions between states, not by ignoring some states outright. The latter sort of contraction may be useful in case certain states are regarded by the person as virtually impossible so that they can be ignored.* *(page 9)*

However, Savage was also keenly aware that the theory he proposed could be seen as more appropriate when decision making takes place in a small world:

> *Any claim to realism made by this book - or indeed by almost any theory of personal decision of which I know - is predicated on the idea that some of the individual decision situations into which actual people tend to subdivide the simple grand decision do recapitulate in microcosm the mechanism of the idealized grand decision.* *(page 83)*

The ability to create "small worlds" from a set representing all possible "events in the world" is an integral part of Savage's decision theory, but he only considers "small worlds" which are created by partitioning a state space without considering alternative descriptions. Thus, despite his hesitations and the extensive discussion of the notion of "small worlds" and the link

[1] The page numbers refer to the 1972 edition of the book.

between these and the "grand world" Savage uses a framework in which there is no flexibility in the way "small worlds" are integrated into the "grand world".

In this paper we provide a way forward, which, in our view is a natural generalisation of Savage's work and contains Savage's results, as formulated in Anscombe and Aumann (1963), as a special case. It is obtained by using an event space which, contrary to a state space description, allows for a more sophisticated integration of different "small worlds" into the "grand world".

One advantage of the more general formulation is that it allows us to introduce a numerical measure between 0 and 1 of the degree of uncertainty aversion, where 0 corresponds to no uncertainty aversion and 1 to complete uncertainty aversion. In this sense, we are able to capture ambiguity or uncertainty by considering more carefully the grouping of events.

Following the terminology in Knight (1921) our approach thus allows us to distinguish between "risk" and "uncertainty". More precisely, uncertainty is related to the agent's perception, derived from his preferences, of events belonging to different "small worlds". Events belonging to a single "small world" are only risky. In fact, the standard axioms and the use of ordinary probability measures are maintained in each "small world".

In our view it is not only necessary but also desirable to replace the state space with a more general event space. Although an analysis based on counting naturally leads to a probabilistic description on a state space, the state space is an artifact of the counting process itself and counting may not be appropriate when agents are confronted with possibly irreversible choices about the uncertain future. This is particular so when agents are subject to events which they do not consider to be alternatives. Therefore, our use of a general event space amounts to a rejection of the idea that a probabilistic description is suitable for analysis of decision making.

In conclusion, to allow for the possibility that some events cannot be grouped together in the same "small world" we introduce the notion of an event space. We assume that the decision maker, depending on the context, prepares for a decision by creating a local state space, or "small world" in Savage's terminology, by grouping together suitable events taken from the "grand world". Local state spaces are thus created containing only events which by the decision maker are considered relevant (and true alternatives) for each decision to be made.

The paper is structured as follows: In the next section the general event space is introduced and discussed. In section 3 the main results are presented including the measure of uncertainty aversion. Section 4 contains two examples illustrating the use of the theory. One is the Ellsberg paradox, the other a portfolio decision problem under uncertainty. The last section discusses related literature.

2 The general event space

In this section we introduce a general event space which may be considered as a generalization of the notion of a sigma algebra on a state space. We choose projections to represent events simply because projections are equipped with exactly the properties which we naturally associate with the hierarchy and logical rules for the interplay of events. This should come as no surprise since, as we shall demonstrate, also a sigma algebra of events on a state space may be represented in this way. An event space is therefore an effectively more general way to represent an event structure than is the usual description by a sigma algebra on a state space. We demonstrate in a rigorous way how the notion of an event space is a natural generalisation of a state space. But first we introduce the "global" event space[2] and discuss the defining properties.

Definition 1 (Event space) An event space is a pair $(\mathcal{F}, H)$ consisting of a separable Hilbert space H and a family $\mathcal{F}$ of (self-adjoint) projections on H satisfying:

(i) The zero projection on H (denoted 0) and the identity projection on H (denoted 1) are both in $\mathcal{F}$.

(ii) $1 - P \in \mathcal{F}$ for arbitrary $P \in \mathcal{F}$.

(iii) The minorant projection $P \wedge Q \in \mathcal{F}$ for arbitrary $P, Q \in \mathcal{F}$.

(iv) $\sum_{i \in I} P_i \in \mathcal{F}$ for any family $(P_i)_{i \in I}$ of mutually orthogonal projections in $\mathcal{F}$.

We begin by listing some comments directly pertinent to the definition of an event space.

- The family $\mathcal{F}$ inherits the natural (partial) order relation $P \leq Q$ for projections on a Hilbert space. Notice that $0 \leq P \leq 1$ for arbitrary events $P \in \mathcal{F}$.

- We define a bijective mapping $P \to P^\perp$ of $\mathcal{F}$ onto itself by setting $P^\perp = 1 - P$. The event $P^\perp$ is called the event complementary to P.

- The minorant projection $P \wedge Q$ is the projection on the intersection of the ranges of P and Q. It has the property that $R \leq P \wedge Q$ for any event $R \in \mathcal{F}$ such that both $R \leq P$ and $R \leq Q$.

[2] The general event space to be used in the remainder of the paper is based on Hansen (2003).

The majorant projection $P \vee Q$ is the projection on the closure of the sum of the ranges of P and Q. It has the property that $P \vee Q \leq R$ for any event $R \in \mathcal{F}$ with $P \leq R$ and $Q \leq R$. Since

$$P \vee Q = 1 - (1 - P) \wedge (1 - Q)$$

it follows that $\mathcal{F}$ is closed also under majorant formation.

- Condition (iv) in the definition is a technical requirement[3] that ensures that $\mathcal{F}$ is closed under arbitrary formation of minorants or majorants. Thus to any family $(P_i)_{i \in I}$ of events in $\mathcal{F}$ there is a minorant event $\wedge_{i \in I} P_i$ and a majorant event $\vee_{i \in I} P_i$ both contained in $\mathcal{F}$.

An event space has a number of properties which are natural for the representation of events.

- An event space contains the projections 0 and 1 corresponding respectively to the vacuous (empty) event and the universal (sure) event.
- There is a partial order relation $\leq$ defined in $\mathcal{F}$ such that any event $P \in \mathcal{F}$ is placed between the vacuous and the universal events, that is $0 \leq P \leq 1$. More generally, for two events P and Q in $\mathcal{F}$ we consider Q to be a larger, more comprehensive event than P if $P \leq Q$.[4] The interpretation is that we know for sure that the event Q occurs (obtains) if P occurs.
- The joining of two events P and Q in $\mathcal{F}$ is represented by $P \wedge Q$ and the union is represented by $P \vee Q$, and these are both included in the event space $\mathcal{F}$.[5] It follows from (iv) that $\mathcal{F}$ is even closed under the joining or union of arbitrary families of events.

The bijective mapping $P \rightarrow P^{\perp} = 1 - P$ of $\mathcal{F}$ which associates an event with its complimentary event has the following natural properties:

- More comprehensive events have smaller complementary events, ie $P \leq Q \Rightarrow Q^{\perp} \leq P^{\perp} \quad$ for all $P, Q \in \mathcal{F}$.
- The joining between an event and its complementary event is the empty event, ie. $P \wedge P^{\perp} = 0 \quad$ for all $P \in \mathcal{F}$.

[3] The condition corresponds to the requirement that a sigma algebra is complete.

[4] It corresponds to the statement $A \subseteq B$ for measurable subsets A and B of a state space.

[5] We express this by saying that $\mathcal{F}$ is a lattice.

- The union between an event and its complementary event is the sure event, ie. $P \vee P^{\perp} = 1 \quad \text{for all } P \in \mathcal{F}$.

- The complementary event to the complementary event to an event is the event itself, ie. $P^{\perp\perp} = P \quad \text{for all } P \in \mathcal{F}$.

Suppose that the complementary event to a given event Q is more comprehensive than another event P, meaning that if P obtains then so does the complement to Q. If the events are represented by projections (here also denoted P and Q) on a Hilbert space H, then the condition is equivalent to the requirement $P \leq 1 - Q = Q^{\perp}$ which means that the ranges of P and Q are orthogonal subspaces of H. For this reason it becomes natural to denote such events as orthogonal events.

Definition 2 *We say that events P and Q in $\mathcal{F}$ are mutually exclusive if the minorant $P \wedge Q = 0$, and we say that P and Q are orthogonal*[6] *if $P \leq Q^{\perp}$.*

It follows readily from this definition that orthogonal events are mutually exclusive. However, it may happen that mutually exclusive events are not orthogonal. In fact, an event space not requiring mutually exclusive events to be orthogonal is the natural generalisation of a state space. Firstly, one can demonstrate (Hansen (2003)) that under very mild conditions every state space has an associated event space. Secondly, it can be shown that an event space is associated with a state space, if and only if each pair of mutually exclusive events are orthogonal. This associated state space then satisfies the same mild conditions as in the first result.

Proposition 1 (Event space associated with a state space)
Suppose that $(\Omega, \mathcal{S}, \mu)$ is a probability space such that Ω is a locally compact, second countable Hausdorff space, and μ is the completion of the Riesz representation of a Radon measure. Then the set $\mathcal{F}$ of projections in $L^{\infty}(\Omega, \mathcal{S}, \mu)$ acting as multiplication operators on the Hilbert space $H = L^2(\Omega, \mathcal{S}, \mu)$ is an event space $(\mathcal{F}, H)$ with the special property that each pair of mutually exclusive events are orthogonal.

Note that a typical projection in $L^{\infty}(\Omega, \mathcal{S}, \mu)$ is the indicator function of a measurable set in $\mathcal{S}$. The proof of the proposition is straightforward and is left as an exercise to the reader[7].

At present the authors are not aware of any economic applications using a state space formulation which do not satisfy the conditions in the above

[6] Notice that the definition is symmetric in P and Q, ie. $P \leq Q^{\perp}$ if and only if $Q \leq P^{\perp}$.

[7] The only complication is condition (iv) in Definition 1 which is proved by appealing to Lebesque's theorem of dominated convergence.

proposition. We have thus demonstrated that any state space, under the very mild conditions of the proposition, possesses an associated event space. The following result is found in Hansen (2003).

Theorem 1 *Let* $(\mathcal{F}, H)$ *be an event space, cf. Definition 1, satisfying the following additional condition:*

(#) *Each pair of mutually exclusive events in* $\mathcal{F}$ *are orthogonal.*

Then the event space $(\mathcal{F}, H)$ *is associated, as specified in the proposition, with a uniquely defined state space* $(\Omega, \mathcal{S}, \mu)$ *.*

One can easily check if a given event space is associated with a state space or not. This can be done by relying on the insight that the projections in an event space satisfying (#) necessarily commute, ie. if an event space only contains commuting projections then it can associated with a state space. One the other hand, if an event space contains non-commuting projections then it cannot be associated with a state space.[8]

3 Small worlds and expected utility

In this section we first introduce the notions of "small worlds", acts and preferences. Then we introduce the set of axioms which we require preferences to satisfy. After introducing these "building blocks" we state and prove the main result in the paper. In the last part of the section we introduce a measure of uncertainty aversion.

To avoid technical difficulties, we assume in the remainder of the paper that the Hilbert space H is of finite dimension. This corresponds to a finite state space in the standard model.

3.1 Small worlds

Given an event space we introduce the notion of a "small world" or preparation as a subdivision of the sure event into those constituent parts (risky events) which are pertinent for a particular set of acts. The notion of a "small world" or preparation thus fits neatly into Savage's concept of "neglecting some distinctions between states".

Definition 3 *A preparation or "small world" in* $(\mathcal{F}, H)$ *is a set* $\{P_1, \ldots, P_n\}$ *of projections in* $\mathcal{F}$ *with sum* $P_1 + \cdots + P_n = 1$ *(note that such projections*

[8]Two projections P and Q commute if $PQ = QP$. Note that the multiplicative structure plays no direct role in the theory.

automatically are mutually orthogonal). The set of preparations of $(\mathcal{F}, H)$ *is denoted by* $P(H)$.

The events in a "small world" are thus mutually exclusive and their majorant event is the sure event. Therefore exactly one of the events obtains. This is why a preparation acts like a local state space. The events function as (local) states and the obtaining event as the "true state of nature". The set of preparations or "small worlds" $P(H)$ thus becomes a set of state spaces, each describing a certain part of the "grand world" as specified by the event space $(\mathcal{F}, H)$.

3.2 Acts and consequences

Definition 4 *An act is a pair* (α, f) *consisting of a preparation* $\alpha \in P(H)$ *and a mapping* $f : \alpha \to C$*, where* C *is the set of consequences.*

The set of consequences C is assumed to have an affine structure. This implies that we may define the convex combination (α, h) of acts (α, f) and (α, g) with the same preparation $\alpha \in P(H)$ by setting

$$h(P) = tf(P) + (1 - t)g(P) \qquad t \in [0, 1]$$

for each event $P \in \alpha$.

To a given event space $(\mathcal{F}, H)$ and set of consequences C, we consider a set L of actions. For each preparation $\alpha \in P(H)$ we assume that the subset $L_\alpha \subseteq L$, consisting of the acts in L with preparation α, is convex and includes the set of constant acts.

3.3 Preferences

Preferences are specified by a binary preference relation $\succeq$ over the set L.

For any consequence $c \in C$ and preparation $\alpha \in P(H)$ consider the constant action $(\alpha, c) \in L_\alpha$ defined by setting $c(P) = c$ for every $P \in \alpha$. Note that for each preparation $\alpha \in P(H)$, the preference relation on L induces a preference relation $\succeq_\alpha$ on C by setting

$$c \succeq_\alpha d \qquad \text{if} \quad (\alpha, c) \succeq (\alpha, d)$$

for consequences c and d in C.

We are now ready to introduce the set of axioms which we will require the preference relation $\succeq$ over L to satisfy.

(i) **Totality**: For all (α, f) and (β, g) in L we have either $(\alpha, f) \succeq (\beta, g)$ or $(\beta, g) \succeq (\alpha, f)$.

(ii) **Transitivity**: If $(\alpha, f) \succeq (\beta, g)$ and $(\beta, g) \succeq (\gamma, h)$ for actions (α, f), (β, g) and (γ, h) in L, then $(\alpha, f) \succeq (\gamma, h)$.

A total and transitive order relation is also called a weak ordering. If $(L, \succeq)$ satisfies the axioms (i) and (ii), then it follows that the induced order relation $\succeq_\alpha$ on C, for each preparation $\alpha \in P(H)$, enjoys the same properties.

(iii) **Independence**: Let (α, f), (α, g) and (α, h) be acts in L_α for a preparation $\alpha \in P(H)$. Then

$$(\alpha, f) \succ (\alpha, g) \quad \text{implies} \quad (\alpha, tf + (1-t)h) \succ (\alpha, tg + (1-t)h)$$

for each $t \in (0, 1]$.

(iv) **Continuity**: Let (α, f), (α, g) and (α, h) be acts in L_α for a preparation $\alpha \in P(H)$. If $(\alpha, f) \succ (\alpha, g)$ and $(\alpha, g) \succ (\alpha, h)$, then

$$(\alpha, tf + (1-t)h) \succ (\alpha, g) \quad \text{and} \quad (\alpha, g) \succ (\alpha, sf + (1-s)h)$$

for some numbers $t, s \in (0, 1)$.

(v) **Monotonicity**: Let (α, f) and (α, g) be acts in L_α for a preparation $\alpha \in P(H)$. If $f(P) \succeq_\alpha g(P)$ for each event $P \in \alpha$, then $(\alpha, f) \succeq (\alpha, g)$.

(vi) **Non-degeneracy**: To each preparation $\alpha \in P(H)$ there exist acts (α, f) and (α, g) in L_α such that $(\alpha, f) \succ (\alpha, g)$.

Note that the axioms (iii) through (vi) apply to each small world at a time. As each small world can be viewed as a context dependent state space, and the said axioms coincide with the axioms considered in the version of Savage's theory as presented in Anscombe and Aumann (1963), we immediately obtain the following result:

Theorem 2 (Anscombe-Aumann) *Assume that the preference relation $\succeq$ satisfies the axioms (i) through (vi). Then there exists for each preparation $\alpha \in P(H)$ a map $u_\alpha \colon C \to \mathbf{R}$, unique up to an affine transformation, such that*

$$c \succ_\alpha d \quad \textit{if and only if} \quad u_\alpha(c) > u_\alpha(d)$$

for consequences $c, d \in C$. Furthermore, there exists a unique subjective probability distribution E_α over α such that

$$(\alpha, f) \succ (\alpha, g) \quad \textit{if and only if} \quad U_\alpha(f) > U_\alpha(g)$$

for arbitrary acts (α, f) *and* (α, g) *in* L_α*, where the expected utility function* U_α *is defined by setting*

$$U_\alpha(\alpha, f) = \sum_{i=1}^{n} E_\alpha(P_i) u_\alpha(f(P_i))$$

for any act $(\alpha, f) \in L_\alpha$ *with preparation* $\alpha = \{P_1, \ldots, P_n\} \in P(H)$*, where* n *is some natural number depending on* α*.*

For a proof see Anscombe and Aumann (1963), Mas-Colell and Whinston (1995, Chapter 6) or the discussion in Schmeidler (1989, page 578). The proof is facilitated by the fact that each local state space or preparation $\alpha \in P(H)$, in the present version of the theory, is a finite set.

Corollary 1 *Any act* (α, f) *in* L *with preparation* $\alpha \in P(H)$ *is under the conditions of Theorem 2 equivalent to a constant act* (α, c)*.*

Proof: Set $c = E_\alpha(P_1)f(P_1) + \cdots + E_\alpha(P_n)f(P_n) \in C$ where the preparation $\alpha = \{P_1, \ldots, P_n\}$, and let (α, c) be the act with constant consequence c. Since $(\alpha, c) \in L_\alpha$ and u_α is unique up to an affine transformation we obtain $U_\alpha(\alpha, c) = u_\alpha(c) = U_\alpha(\alpha, f)$. **QED**

Having introduced the "small world" axioms we now introduce the "grand world" axioms. They are essentially the only new axioms that we introduce.

(vii) **Indifference**: To each consequence $c \in C$ and to any preparations $\alpha, \beta \in P(H)$ the constant acts (α, c) and (β, c) are equivalent.

The axiom states that constant acts with the same consequence are equivalent across all "small worlds". This can be interpreted as the requirement that a sure bet should be equally attractive, independent of the context in which it is available. With the indifference axiom in place (which we henceforth will assume), it is clear that the induced order relations $\succeq_\alpha$ on C are equivalent for all preparations $\alpha \in P(H)$. We may therefore suppress the subscript in $\succeq_\alpha$ and just write

$$c \succeq d \qquad \text{if} \quad (\alpha, c) \succeq (\beta, d)$$

for consequences c and d in C, and preparations $\alpha, \beta \in P(H)$.

Lemma 1 *Assume that the preference relation* $\succeq$ *satisfies the axioms* (*i*) *through* (*vii*)*. Then there exists a common utility function* $u : C \to \mathbf{R}$*, unique up to an affine transformation, such that*

$$c \succ d \quad \textit{if and only if} \quad u(c) > u(d)$$

for consequences $c, d \in C$*.*

Proof: The statement follows since all the preference relations $\succeq_\alpha$, for $\alpha \in P(H)$, are equivalent. **QED**

(viii) **Separation**: Let $\alpha, \beta \in P(H)$ be preparations with a common event $P \in \alpha \cap \beta$. There exist equivalent actions (α, f) and (β, g) in L and non-equivalent consequences $a, b \in C$ such that

$$f(P) \sim g(P) \sim a \quad \text{and} \quad f(Q) \sim g(R) \sim b$$

for every $Q \in \alpha \backslash \{P\}$ and $R \in \beta \backslash \{P\}$.

The axiom is interpreted as the requirement that if two small "worlds" share a common event, it must be possible to make equivalent and non-trivial bets contingent on the event in both worlds.

Lemma 2 *Assume that the preference relation $\succeq$ satisfies the axioms (i) through (viii). If two preparations $\alpha, \beta \in P(H)$ have a common event $P \in \alpha \cap \beta$, then $E_\alpha(P) = E_\beta(P)$.*

Proof: Consider two preparations $\alpha, \beta \in P(H)$ with a common event $P \in \alpha \cap \beta$. By the separation axiom there exist equivalent actions (α, f) and (β, g) in L and non-equivalent consequences $a, b \in C$ such that

$$f(P) \sim g(P) \sim a \quad \text{and} \quad f(Q) \sim g(R) \sim b$$

for every $Q \in \alpha \backslash \{P\}$ and $R \in \beta \backslash \{P\}$. Since a and b are non-equivalent we may assume $u(a) < u(b)$. We set

$$d = E_\alpha(P)a + (1 - E_\alpha(P))b \in C$$

and calculate the α-utility

$$E_\alpha(\alpha, d) = u(d) = E_\alpha(P)u(a) + (1 - E_\alpha(P))u(b) = E_\alpha(\alpha, f)$$

of the constant action (α, d), and observe that (α, f) is equivalent to (α, d). Since the constant actions (α, d) and (β, d) are equivalent by the indifference axiom, we conclude that (β, g) and (β, d) are equivalent. Therefore,

$$u(d) = E_\beta(\beta, d) = E_\beta(\beta, g) = E_\beta(P)u(a) + (1 - E_\beta(P))u(b).$$

We have thus written $u(d)$ as two convex combinations of $u(a)$ and $u(b)$. Since $u(a) < u(b)$ we conclude that $E_\alpha(P) = E_\beta(P)$. **QED**

3.4 The main result

Lemma 2 ensures that we unambiguously can define a function

$$E : \mathcal{F} \to [0, 1]$$

by setting $E(P) = E_\alpha(P)$ for any preparation $\alpha \in P(H)$ containing P. This function has the property

$$E(P_1) + \cdots + E(P_n) = 1$$

for any sequence $P_1, \ldots, P_n$ of projections in $\mathcal{F}$ with sum $P_1 + \cdots + P_n = 1$. A function with this property is called a frame function, and such functions were studied by Mackey (1957), Gleason (1957), Varadarajan (1968), Piron (1976) and others. The following remarkable result was conjectured by Mackey and proved by Gleason.

Gleasons' theorem *Let $\mathcal{F}$ be the (lattice of self-adjoint) projections on a (real or complex) separable Hilbert space H of dimension greater than or equal to three, and let $F : \mathcal{F} \to [0, 1]$ be a frame*[9] *function. Then there exists a positive semi-definite trace class operator h on H with unit trace such that*

$$F(P) = \operatorname{Tr}(hP)$$

for any $P \in \mathcal{F}$.

Given this, we can now prove our main result:

Theorem 3 *Let $(\mathcal{F}, H)$ be the event-lattice consisting of all (self-adjoint) projections on a (real or complex) Hilbert space of finite dimension greater than or equal to three, let C be a set of consequences equipped with an affine structure, and let L be a set of actions. For each preparation $\alpha \in P(H)$ we assume that the subset $L_\alpha \subseteq L$, consisting of the acts in L with preparation α, is convex and includes the set of constant acts. The primitive datum of the utility theory is a binary preference relation $\succeq$ over the set L satisfying the axioms (i) through (ix). There exists then a map $u \colon C \to \mathbf{R}$, unique up to an affine transformation, and a positive semi-definite operator h on H with unit trace such that*

$$(\alpha, f) \succ (\beta, g) \quad \textit{if and only if} \quad U(\alpha, f) > U(\beta, g)$$

[9]Note that such a frame function automatically is continuous by Gleason's theorem.

for arbitrary acts (α, f) and (β, g) in L, where the expected utility function U is defined by setting

$$U(\alpha, f) = \sum_{i=1}^{n} \mathrm{Tr}\,(hP_i)u(f(P_i))$$

for any act $(\alpha, f) \in L$ with preparation $\alpha = \{P_1, \ldots, P_n\} \in P(H)$, where n is some natural number depending on α.

Proof: The two acts (α, f) and (β, g) in L are by Corollary 1 equivalent to constant acts (α, c) and (β, d) respectively, and since $U_\gamma = U$ for any preparation $\gamma \in P(H)$ we obtain

$$U(\alpha, f) = U(\alpha, c) = u(c) \quad \text{and} \quad U(\beta, g) = U(\beta, d) = u(d).$$

But since constant acts are ordered by u the statement follows. **QED**

Note that the statement in the main result implies that the indifference axiom and the separation axiom for preferences across "small worlds" also are necessary conditions. The implication is that these two axioms must be satisfied in any expected utility formulation of the given form.

3.5 Uncertainty aversion

We are now ready to introduce a measure of uncertainty aversion. With this purpose in mind, consider two events P and Q in an event space $(\mathcal{F}, H)$ and a decision maker with preferences as given in Theorem 3. Let us to keep matters simple assume that the events are mutually exclusive, ie. $P \wedge Q = 0$. If the number

$$\nu(P, Q) = E(P \vee Q) - (E(P) + E(Q))$$

is positive, this is interpreted as reflecting the decision maker's uncertainty aversion. We may think of an experiment in which a ball is drawn from an urne with an unknown distribution of red and black balls. The event P represents the drawing of a red ball while the event Q represents the drawing of a black ball. The union (majorant) of the two events $P \vee Q$ is the sure event so $E(P \vee Q) = 1$. The decision maker assigns so low probabilities to the individual events that their sum is less than the probability of the union. Therefore the decision maker is uncertainty averse.

Let now $P_1, \ldots, P_N$ be events in $\mathcal{F}$ with no further assumptions and consider the number

$$\nu(P_1, \ldots, P_N) = E(P_1 \vee \cdots \vee P_N) - \sum_{i=1}^{N} E(P_i).$$

This number is obviously less or equal to one and it may be negative, for example if the events are identical and $N \geq 2$. But note that if the events form (or are just part of) a preparation, then $P_1 \vee \cdots \vee P_N = P_1 + \cdots + P_N$ and $\nu(P_1, \ldots, P_N) = 0$.

Definition 5 The number

$$\nu = \sup\{\nu(P_1, \ldots, P_n) \mid P_1, \ldots, P_n \in \mathcal{F},\ N = 1, 2, \ldots\}.$$

is defined as the decision maker's uncertainty aversion.

Thus, a decision maker's degree of uncertainty aversion is determined as the largest possible difference between the weight attached to the union and the sum of the weights of the individual events. Note that by focusing on the "worst possible" situation the introduced measure of uncertainty aversion is linked to that of Schmeidler (1989). It is clear from the preceding remarks that the decision maker's uncertainty aversion ν is a real number in the interval $[0, 1]$.

Proposition 2 *Let h be the positive semi-definite operator (matrix) on H with unit trace such that $E(P) = \mathrm{Tr}\,(hP)$ for any event $P \in \mathcal{F}$. Then*

$$\nu = 1 - \lambda_{min} \cdot \dim H,$$

where $\dim H$ is the finite dimension of the Hilbert space H and λ_{min} is the minimal eigenvalue of the operator h.

Proof: Consider the expression $\nu(P_1, \ldots, P_N)$ for some events $P_1, \ldots, P_N$. Since E is additive we may without loss of generality assume the majorant event $P_1 \vee \cdots \vee P_N = 1$ and that all the constituent projections are one-dimensional. We may then discart events until all remaining events are needed to maintain the sure event as majorant. In this situation $N = \dim H$ and the remaining events are necessarily (self-adjoint) projections on a set of basis vectors in H. The supremum is then obtained by choosing a sequence of bases of H with each basis vector converging to an eigenvector for the minimal eigenvalue of h. **QED**

If the decision maker's uncertainty aversion $\nu = 0$, then the proposition entails that h is the identity operator on H (the identity matrix) divided by $\dim H$, hence

$$E(P) = \frac{\dim R(P)}{\dim H} \qquad P \in \mathcal{F},$$

where $R(P)$ denotes the range of P. An uncertainty neutral ($\nu = 0$) decision maker is thus assigning weight to an event solely according to the dimension of the representing projection.

4 Examples

4.1 Ellsbergs' paradox

As in the example by Ellsberg (1961) a decision maker is presented with an urn containing 90 balls. He is told that 30 of the balls are red and that the remaining 60 balls are either black or yellow, but he is given no information about the distribution of the black and yellow balls. The decision maker is first asked to state his preferences between three bets, each on the exact color of a single drawn ball. The decision maker is then asked to state his preferences between three bets in which he is given a choice between two colors of a single drawn ball. All six bets pay out the same amounts, conditional on the outcome of the draw.

Since the decision maker has exact information about the fraction of the red balls, he considers a bet on the red ball to be a simple lottery described by a probability distribution given the weight $1/3$ to the event "the ball is red" and the weight $2/3$ to the event "the ball is not red", and this last event is recognized to be the same event as "the ball is either black or yellow".

This may be modelled by letting the event "red ball" be represented by the projection R and the event "not red ball" or "black or yellow ball" be represented by the projection $1-R$ where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 1 - R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The decision maker is in the absence of further information not able to subdivide the "black or yellow ball" event into two orthogonal single color events with a probability distribution. They are in Knight's words "ambiguous events for which ordinary probabilities are not defined". The two single color events "black ball" and "yellow ball" belong to different "small worlds" and may be represented by the projections B and Y given by

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

Note that the three single color events have minorant 0, that is

$$R \wedge B = 0, \quad B \wedge Y = 0, \quad R \wedge Y = 0,$$

and as required the majorant event $B \vee Y = 1 - R$. The two other majorant events are then easily calculated to be

$$R \vee B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R \vee Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

The three two color events $B \vee Y$, $R \vee B$ and $R \vee Y$ are thus endogenously given by the lattice operations once the one color events R, B and Y are specified. Assume that the decision maker prefers a bet on a "red ball" to a bet on a "black ball" and is indifferent between a bet on a "black ball" and a bet on a "yellow ball", that is

$$\text{Bet}(R) \succ \text{Bet}(B) \sim \text{Bet}(Y).$$

Assume furthermore, following Ellsberg (1961) and others, that the decision maker belongs to the group of people (appearing in numerous empirical experiments) which prefer a bet on a "black or yellow ball" to a bet on either a "red or yellow ball" or a bet on a "black or red ball", and is indifferent between these two last bets, that is

$$\text{Bet}(B \vee Y) \succ \text{Bet}(R \vee Y) \sim \text{Bet}(B \vee R).$$

We will show that these preferences may be expressed in terms of the expected utility theory developed in the last section. The set C of consequences is the unit interval $[0, 1]$ and each of the six bets is an act with a two-element preparation of projections on the Hilbert space $\mathbf{R}^3$ equipped with the scalar product (this is the simplest case covered by Gleason's theorem). The bet on a "red ball" is thus the act

$$\big(\alpha_R, \text{Bet}(R)\big) \quad \text{with preparation} \quad \alpha_R = \{R, 1 - R\}$$

such that $\text{Bet}(R)(R) = 1$ and $\text{Bet}(R)(1 - R) = 0$. The five other bets are in the same way defined as acts such that the consequence is 1 in the projection (the event) associated with the bet, and 0 in the orthogonal complement.

Preferences satisfying axioms (i) through (ix) are by Theorem 3 represented by an expected utility function U constructed from a linear functional E on the form

$$E(A) = \text{Tr}\,(hA) \qquad A \in B(H),$$

where h is a positive semi-definite unit trace operator. We may calculate the expected utility of the bet on a "red ball",

$$U(\alpha_R, \text{Bet}(R)) = E(R) \cdot 1 + E(1 - R) \cdot 0 = E(R),$$

and find that it equals the expected value $E(R)$ of the event R associated with the red ball. The same result applies, mutatis mutandis, to the five other bets. If we choose

$$h = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/6 & -1/6 \\ 0 & -1/6 & 1/2 \end{pmatrix}$$

and calculate the expectations

$$\begin{aligned} E(R) &= \frac{1}{3} & \qquad E(B) &= \frac{1}{6} & \qquad E(Y) &= \frac{1}{6} \\ E(B \vee Y) &= \frac{2}{3} & \qquad E(R \vee B) &= \frac{1}{2} & \qquad E(R \vee Y) &= \frac{1}{2} \end{aligned}$$

we find that

$$E(R) > E(B) = E(Y) \quad \text{and} \quad E(B \vee Y) > E(R \vee Y) = E(B \vee R).$$

This shows that the decision maker's preferences may be represented by an expected utility function also when he is unable to fit the ambiguous events "black ball" and "yellow ball" into the same "small world" and assign ordinary probabilities.

Given the weights attached to the different events, the behaviour may be interpreted as reflecting uncertainty aversion. By Proposition 2 we find the degree of uncertainty aversion to be:

$$\nu = 1 - \lambda_{min} \dim H = 1 - \frac{1}{6}(2 - \sqrt{2}) \cdot 3 = \frac{\sqrt{2}}{2}.$$

The "non red ball" and the "black or yellow ball" events are well defined and identical, while the ambiguous events "non black ball" and "red or yellow ball" are represented by different projections. This may seem meaningless from the set up of the experiment, but one should remember that we are modelling the decision maker's perception of the given situation (as reflected in his decisions) and not the physical properties of the system.

Note that if the decision maker is uncertainty neutral, then he assigns weight to the various events only according to the dimension of the representing projections, and this cannot lead to a representation of the stated preferences even with an event space description. On the other hand, we may retain the preferences given by h but investigate other representations of the events. But if the ambiguous "non black ball" and "red or yellow ball" events are perceived by the decision maker as identical, that is $1-B = R \vee Y$

then B and Y are orthogonal and thus $1 = R+B\vee Y = R+B+Y$. This would correspond to the decision maker using a 3 point state space, in which case the linear expectation functional E becomes an ordinary probability measure on that space. However, as is well known, this description is not compatible with the decision maker's preferences.

4.2 Portfolio decisions

In this section we consider an example presented in Dow and Werlang (1992) which illustrates the portfolio decisions of an agent who faces uncertainty. The example has a single investor with wealth $W > 0$, a risk free asset and a single risky asset in a one-period model. The price of the risky asset is p and the present value of an investment in the risky asset is either H (high) or L (low). To avoid arbitrage possibilities we assume $L < p < H$.

The investor may choose between going long (strategy 1) or going short (strategy 2) in the risky asset at the given price, or he may choose to invest his wealth in the risk free asset (strategy 3). The investor considers the two risky investment strategies to be qualitatively different and belonging to different "small worlds". In the language of the theory developed in the previous sections we say that the two acts have different preparations.

Assume that strategy 1 is given by (α, l) with preparation $\alpha = \{P, 1-P\}$ and strategy 2 is given by (β, s) with preparation $\beta = \{Q, 1-Q\}$ such that

$$l(P) = H - p \quad \text{and} \quad l(1-P) = L - p$$

together with

$$s(Q) = -H + p \quad \text{and} \quad s(1-Q) = -L + p.$$

Note that the investor does not question the outcomes of the two different strategies, but only considers the known consequences to be triggered by different events. The two preparations may be specified by setting

$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that P and Q are (self-adjoint) projections on the real Hilbert space $\mathbf{R}^3$ equipped with the scalar product (this is the simplest case covered by Gleason's theorem). Preferences satisfying axioms (i) through (ix) are by Theorem 3 represented by an expected utility function U constructed from a linear functional E on the form

$$E(A) = \mathrm{Tr}\,(hA) \qquad A \in B(H),$$

where h is a positive semi-definite unit trace operator. If for example

$$h = \begin{pmatrix} 1/3 & -1/6 & 1/12 \\ -1/6 & 1/6 & -1/6 \\ 1/12 & -1/6 & 1/2 \end{pmatrix}$$

we find

$$E(P) = \frac{1}{6} \quad \text{and} \quad E(Q) = \frac{1}{3}$$

and may calculate the expected utilities

$$U(\alpha, l) = E(P)(H - p) + E(1 - P)(L - p) = \frac{5}{6}L + \frac{1}{6}H - p$$

and

$$U(\beta, s) = E(Q)(-H + p) + E(1 - Q)(-L + p) = -\frac{2}{3}L - \frac{1}{3}H + p$$

of going either long or short in one unit of the risky asset. The investor has a third possible strategy which is to invest in the risk free asset, and this constant act has a utility level of 0. If the price p of the risky asset is confined to the smaller interval (of length $(H - L)/6$) given by

$$L < \frac{5}{6}L + \frac{1}{6}H < p < \frac{2}{3}L + \frac{1}{3}H < H,$$

we obtain both $U(\alpha, l) < 0$ and $U(\beta, s) < 0$. In this case the investor prefers to invest in the risk free asset to going either long or short in the risky asset.

As in the Ellsberg example, these preferences may be interpreted as reflecting uncertainty aversion, which is modelled by choosing different preparations for the two risky investment strategies. Given the weights attached to the different events and using Proposition 2, we find that the degree of uncertainty aversion ν is approximately 0.8815.

5 Related literature

Since Ellsberg (1961) a substantial empirical literature has documented that subtle differences between sources of risk or uncertainty can lead a decision maker to treat them differently. Reflecting this, there has been a growing theoretical literature which has focused on modelling decision making under uncertainty, while at the same time allowing for a clear distinction between risk and uncertainty in the spirit of Knight (1921). Karni and Schmeidler

(1991) and more recently Wakker (2004) provide comprehensive surveys of this literature.[10] In general this literature has been focused on weakening the Savage/Anscombe-Aumann axioms. Some authors, including in particular Vind (2003), has chosen to abandon the Savage axioms - in the case of Vind (2003) the totality of preferences - to construct more flexible expected utility models.

Our paper is clearly related to this literature. In particular our approach is related to that taken by Schmeidler (1989) in terms of the modelling of uncertainty and uncertainty aversion. The approach taken by Schmeidler (1989) is, in general terms, to maintain the use of a "grand world" state space and instead assume that decision makers assign non-additive probabilities as a reflection of uncertainty aversion. It is then shown, that imposing slightly weaker versions of the Anscombe-Aumann axioms on the preferences it is possible to capture preferences towards uncertainty and risk aversion in an expected utility formulation. In this framework subjective probabilities that sum to less than one are interpreted as reflecting uncertainty aversion. Clearly, this work demonstrates that it is possible to formulate expected utility theories that capture a notion of uncertainty-aversion while still relying on the use of a state space. A small but growing group of researchers has begun to apply this type of framework to analyse economic situations.[11]

To our knowledge, there are only a few papers on decision theory that do not rely on the explicit presence of state space. Gilboa and Schmeidler (2001) models subjective distributions without relying on a state space by modelling preferences over acts conditional on bets. Assuming the existence of an outcome-independent linear utility on bets subjective probabilities are derived on the outcome that is consistent with expected value maximizing behavior. Karni (2004) develops an axiomatic theory of decision making under uncertainty that dispenses with the Savage state space. A subjective expected utility theory that does not invoke the notion of states of the world to resolve uncertainty is formulated. Importantly, this approach does not rule out that decision makers may mentally construct a state space to help organize their thoughts - but it does not require that they do. Thus, "traditional" theory can, as in ours, be embedded in this framework.

Chew and Sagi (2003) assumes a Savage state space, but the authors provide a set of axioms which allow for domains of events that arise endogenously according to the preferences of the decision maker ("small worlds") and the manner in which sources of uncertainty are treated. The authors also show, given weak assumptions, that preferences restricted to a domain

[10]Early contributions include Fellner (1961) and Quiggin (1982).

[11]See Mukerji and Tallon (2004) for a survey of this literature.

exhibit probabilistic sophistication. This allows for an endogenous formulation of the two-stage approach and a distinction between risk and uncertainty in a setting with a Savage state space. However, as opposed to Savage's formulation, the approach taken is to model decisions as generally taking place at the "small world" level, hence leaving the question of consistent extension of decision making across distinct "small worlds" unanswered.

Finally, our work also has links to discussions of the foundation of quantum physics, in particular quantum-mechanical derivations of probability which is closely modelled on the classical theory. See Wallace (2003a) and Wallace (2003b) for a discussion of how decision theory may be applied in quantum-mechanics.

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