DISCUSSION PAPERS
Institute of Economics
University of Copenhagen

04-09

Voting in Assemblies of Shareholders and Incomplete Markets

Mich Tvede and Hervé Crès

Studiestræde 6, DK-1455 Copenhagen K., Denmark
Tel. +45 35 32 30 82 - Fax +45 35 32 30 00
http://www.econ.ku.dk

Voting in Assemblies of Shareholders and Incomplete Markets*

Mich Tvede[†]
University of Copenhagen

Hervé Crès
HEC School of Management

Abstract

An economy with two dates is considered, one state at the first date and a finite number of states at the last date. Shareholders determine production plans by voting – one share, one vote – and at ρ-majority stable stock market equilibria, alternative production plans are supported by at most $\rho \times 100$ percent of the shareholders. It is shown that a ρ-majority stable stock market equilibrium exists if

$$\rho \ \geq \ \frac{S-J}{S-J+1},$$

where S is the number of states at the last date and J is the number of firms. Moreover, an example shows that ρ-majority stable stock market equilibria need not exist for smaller ρ's.

Keywords: General Equilibrium, Incomplete Markets, Firms, Voting.

JEL-classification: D21, D52, D71, G39.

*The authors are grateful to an anonymous referee for helpful comments and suggestions. Financial support from the Danish Research Councils and hospitality of HEC is gratefully acknowledged by Mich Tvede.

[†] *Corresponding author:* Institute of Economics, University of Copenhagen, Studiestraede 6, DK-1455 Copenhagen K, Denmark.

1. Introduction

If markets are complete consumers have, at equilibrium, common shadow prices – namely the vector of market prices. Shareholders therefore agree that firms should maximize profits with respect to these common prices. However, if markets are incomplete, shadow prices need not be common. Thus, shareholders typically disagree on the production plans to be chosen. Therefore several suggestions have been put forward as reasonable objectives for firms.

It seems natural that production plans should satisfy the Pareto criterion: no alternative production plan results in some shareholders being better off and none being worse off. Unfortunately, the Pareto criterion is weak: production plans satisfy the Pareto criterion if and only if they maximize profits with respect to some price vector in the convex hull of the shadow prices of the shareholders.

Drèze (1974) and Grossman & Hart (1979) agree that production plans should satisfy the Pareto criterion and propose that side payments between shareholders should be allowed. Drèze (1974) (resp. Grossman & Hart (1979)) suggests that production plans should reflect the preferences of final (resp. initial) shareholders: this may be interpreted as production plans being determined after markets close (resp. before markets open).

Drèze (1985) suggests that production plans should be stable for simple majority voting between shareholders and unanimity between board members (without side payments): no alternative production plan results in all board members and a majority of shareholders being better off. As in Drèze (1974) production plans reflect preferences of final shareholders. It appears to be a drawback that unanimity between board members is essential for existence of equilibria. DeMarzo (1993) investigates some properties of equilibria at which production plans are stable for simple majority voting between shareholders. The largest shareholder typically determines the production plan at these equilibria. However, such equilibria need not exist, as known from the literature on aggregation of preferences in multi-dimensional settings (see

Plott (1967)).

The present paper addresses the problem of existence. Since the set of production plans is multi-dimensional, super majority voting rules are needed to ensure existence of equilibria (as, e.g., in Greenberg (1979) or Caplin & Nalebuff (1988, 1991)). The concept of ρ-majority stable stock market equilibrium (or ρ-MSSME) is introduced: At a ρ-MSSME, consumers do not want to change their portfolios, firms are not able to make more than $\rho \times 100$ percent of their shareholders better off by changing production plans and finally, markets clear. So ρ-MSSME are stable in respect to the joint operation of both a decentralized market mechanism and a centralized collective decision mechanism. It is shown that if portfolios are unbounded, then a ρ-MSSME exists provided that

$$\rho \geq \frac{S-J}{S-J+1}$$

where S is the number of states at the last date, J is the number of firms.

The latter result links the extent of the market failure (the degree of market incompleteness, $S-J$) with the coarseness of the aggregated preference: the more markets fail, the higher the ratio has to be, hence the coarser the aggregated preference. Indeed, the problem with super majority rules with high ratios is that they are conservative: the status quo tends to be protected. It has been a long standing idea that less heterogeneous individual preferences allow for less conservative voting rules. In the present model, the operation of a market mechanism partially reduces the heterogeneity of individual preferences over production plans. Indeed, market trading will lead to agreement amongst shareholders on the value of the J traded assets. This implies that shareholders will only disagree on the value of the $(S-J)$-dimensional set of 'non-marketed' assets. Thus the relevant 'disagreement space' is the projection of the production space onto this non-marketed space which has dimension $S-J$ at most. At the extreme, complete markets completely removes the heterogeneity of individual preferences over production plans:

shareholders are unanimous[1]. In case of the degree of market incompleteness being 1, a ρ-MSSME exists for simple majority voting, i.e. with $\rho = 1/2$, as argued by DeMarzo (1993). In case of a more severe degree of market incompleteness, additional assumptions on the primitive characteristics of the economy (shape of preferences, distribution of individual characteristics) are needed to get existence of ρ-MSSME for ratios smaller than the one provided in the present paper (see Crès and Tvede (2001) for existence results with ratios smaller than two thirds).

The paper is organized as follows. In Section 2 the economy and the notion of a ρ-majority stable *stock market* equilibrium, where firms are not able to make more than $\rho \times 100$ percent of their shareholders better off by changing production plans, are introduced. The timing is that production plans are determined *after* markets are closed as in Drèze (1974, 1985) and DeMarzo (1993). In Section 3 assumptions are introduced and the main result of the paper which is the existence of ρ-majority stable stock market equilibria, is stated. In Section 4 the notion of ρ-majority stable *no-arbitrage* equilibrium, where firms maximize profits with respect to state price vectors and are not able to make more than $\rho \times 100$ percent of their shareholders better off by changing state price vectors, are introduced. Moreover, it is shown that ρ-majority stable stock market equilibria exist because the two notions of equilibrium are equivalent and that ρ-majority stable no-arbitrage equilibria exist. In Section 5 the main result of the paper is extended to other timings: either production plans are determined *before* markets are open as in Grossman & Hart (1979) or *while* markets open. In both timings shareholders need to form expectations about how production plans influence prices and the notion of competitive price perceptions introduced in Grossman & Hart (1979) is considered. Finally Section 6 contains some concluding remarks.

[1] A ρ-MSSME with $\rho = 0$ then exists. Ekern & Wilson (1974) have shown that this result extends to the case of *partial spanning*, i.e. the sets of efficient production plans are subsets of the span of assets. Moreover, existence of ρ-MSSME for $\rho = 0$ holds in any model with incomplete markets where equilibrium allocations are Pareto optimal, e.g., under strong conditions for the CAPM (see Borch (1968) and Wilson (1968)).

The proof of existence of ρ-majority stable no-arbitrage equilibria is in an appendix.

2. The Economy

Consider an economy with 2 dates, $t \in \{0, 1\}$, 1 state at the first date $s = 0$, and S states at the second date $s \in \{1, \ldots, S\}$. There are: 1 commodity at every state, I consumers with $i \in \{1, \ldots, I\}$ and J firms with $j \in \{1, \ldots, J\}$. Consumers are characterized by their identical consumption sets $X = \mathbb{R}^{S+1}$, initial endowments $\omega_i \in \mathbb{R}^{S+1}$, utility functions $u_i : X \to \mathbb{R}$, and initial portfolio of shares in firms $\delta_i = (\delta_{i1}, \ldots, \delta_{iJ})$, where $\delta_{ij} \in \mathbb{R}$ and $\sum_{i=1}^{I} \delta_{ij} = 1$ for all j. Firms are characterized by their production sets $Y_j \subset \mathbb{R}^{S+1}$.

Let $q = (q_1, \ldots, q_J)$ where $q_j \in \mathbb{R}$ is the price of shares in firm j, be the price vector. Consumers choose consumption plans $x_i \in X$, and portfolios, $\theta_i \in \mathbb{R}^J$. Firms choose production plans, $y_j \in Y_j$.

The problem of consumer i given $(q, (y_j)_j)$ is

$$\begin{array}{ll} \max\limits_{x_i, \theta_i} & u_i(x_i) \\ \text{s.t.} & \left\{ \begin{array}{lcl} x_i^0 - \omega_i^0 & = & \sum_j q_j \delta_{ij} - \sum_j (q_j - y_j^0)\theta_{ij} \\ & & \\ x_i^s - \omega_i^s & = & \sum_j y_j^s \theta_{ij} \text{ for all } s \geq 1. \end{array} \right. \end{array} \tag{1}$$

There are no strategic considerations involved in the choices of portfolios.

The problem of firm j is more complicated to state because shareholders vote over production plans. First let $V_{ij} : X \times \mathbb{R} \times Y_j \to Y_j$ be a correspondence which associates a consumption bundle, a stock holding for consumer i and a production plan for firm j with the set of production plans for firm j that make consumer i better off, so

$$V_{ij}(x_i, \theta_{ij}, y_j) \; = \; \{y_j' \in Y_j | u_i(x_i + (y_j' - y_j)\theta_{ij}) > u_i(x_i)\}.$$

Next let $v_j : \prod_i (X \times \mathbb{R}) \times Y_j \times Y_j \to \{1, \ldots, I\}$ be the correspondence which associates a collection of individual consumption bundles and shares in firm

j and a pair of production plans with the set of consumers that are better off with the latter production plan than with the former production plan, so

$$v_j((x_i, \theta_{ij})_i, y_j, y'_j) \;=\; \{i \in \{1, \ldots, I\} | y'_j \in V_{ij}(x_i, \theta_{ij}, y_j)\}.$$

Finally, let $P^\rho_j : \prod_i (X \times \mathbb{R}) \times Y_j \to Y_j$ be a correspondence which associates a collection of individual consumption bundles and shares in firm j and a production plan for firm j with the set of production plans for firm j that makes more than $\rho \times 100$ percent of the shareholders better off, so

$$P^\rho_j((x_i, \theta_{ij})_i, y_j) \;=\; \begin{cases} \emptyset & \text{for } \sum_i \theta^+_{ij} = 0 \\ \{y'_j \in Y_j | \dfrac{\sum_{i \in v_j((x_i,\theta_{ij})_i, y_j, y'_j)} \theta^+_{ij}}{\sum_i \theta^+_{ij}} > \rho\} & \text{for } \sum_i \theta^+_{ij} > 0. \end{cases}$$

Then $P^\rho_j : \prod_i (X \times \mathbb{R}) \times Y_j \to Y_j$ is the preference of firm j and the problem of firm j given $(x_i, \theta_{ij})_i$ is to find y_j such that $P^\rho_j((x_i, \theta_{ij})_i, y_j) = \emptyset$.

An economic interpretation of the model is that shareholders vote over production plans after stock markets are closed, because changes in production plans are not perceived to influence prices.

Definition 1. *A* **ρ-majority stable stock market equilibrium** *is a price vector, a collection of individual consumption bundles and portfolios and a collection of individual production plans* $(\bar{q}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j)$ *such that:*

(C) $(\bar{x}_i, \bar{\theta}_i)$ *is a solution to the problem of consumer* i *given* $(\bar{q}, (\bar{y}_j)_j)$*;*

(F) $\bar{y}_j$ *is a solution to the problem of firm* j *given* $(\bar{x}_i, \bar{\theta}_{ij})_i$*, and;*

(E) $\sum_i \bar{x}_i \;=\; \sum_i \omega_i + \sum_j \bar{y}_j$ *and* $\sum_i \bar{\theta}_{ij} = 1$.

3. Existence of Equilibrium

In order to ensure the existence of a ρ-majority stable stock market equilibrium the assumptions below are imposed on the consumers, the firms and the production sector.

Consumer i is supposed to satisfy the following assumptions:

(A.1) $\omega_i \in X$;

(A.2) u_i is differentiable $u_i \in C^1(X, \mathbb{R})$;

(A.3) u_i has strictly positive derivatives $Du_i \in C(X, \mathbb{R}^{S+1}_{++})$;

(A.4) u_i is quasi-concave, so $u_i((1-t)x_i + tx_i') \geq \min\{u_i(x_i), u_i(x_i')\}$ for all $t \in [0, 1]$, and;

(A.5) $u_i^{-1}(a)$ is bounded from below for all $a \in \mathbb{R}$.

(A.1)-(A.5) are standard. However, the assumption that consumption sets are unbounded from below is not completely standard, but the assumption ensures that for all collections of individual production plans consumers are able to finance consumption plans in their consumption sets. Since firms aim at finding a production plans such that they are not able to increase the utility level of more than $\rho \times 100$ percent of the shareholders rather than maximize profits, the value of a firm may be negative. Therefore, if consumption sets are bounded from below, then consumers may not be able to finance any consumption plans in their consumption sets.

Let $Z_j \subset \mathbb{R}^{S+1}$ be the set of efficient production plans, so

$$Z_j \;=\; \{y_j \in Y_j | (\{y_j\} + \mathbb{R}^{S+1}_{+}) \cap Y_j = \{y_j\}\}$$

then firm j is supposed to satisfy the following assumptions:

(A.6) the production set Y_j is convex and closed, and;

(A.7) the set of efficient production plans Z_j is bounded.

Assumption (A.6) is standard, while assumption (A.7) includes "truncated" production sets such as

$$\{y \in \mathbb{R}^{S+1} | y^0 \in [\overline{y}, 0] \text{ and } y^s \leq (-y^0)^b \text{ for all } s \in \{1, \ldots, S\}\}$$

where $\overline{y} \leq 0$ and $b \in]0, 1]$.

Moreover, the production sector of the economy is supposed to satisfy the following assumption:

(A.8) for all collections of production plans $(y_j)_j$, where y_j is in the convex hull of the closure of the set of efficient production plans $y_j \in \text{co cl } Z_j$, production plans for date 1 are linearly independent.

Assumption (A.8) excludes that firms are able to replicate production plans of each other.

In the rest of the paper (A.1)-(A.8) are supposed to be satisfied.

Theorem 1. *All economies have a ρ-majority stable stock market equilibrium if and only if*

$$\rho \;\geq\; \frac{S-J}{S-J+1}.$$

Proof: In the next section, the notion of ρ-majority stable *no-arbitrage* equilibrium is introduced and it is shown (in Lemma 1 and Lemma 2) that a ρ-majority stable *no-arbitrage* equilibrium is a ρ-majority stable *stock market* equilibrium and vice versa and (in Proposion 1) that $\rho \geq (S-J)/(S-J+1)$ if and only if all economies have a ρ-majority stable *no-arbitrage* equilibrium. Therefore Theorem 1 follows from Lemma 1, Lemma 2 and Proposition 1 in the next section.

Q.E.D

If $(\bar{q}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j)$ satisfies (E), then gradients $(Du_i(\bar{x}_i))_i$ are orthogonal to the dividend vectors $(\bar{d}_j)_j$, where $\bar{d}_j \in \mathbb{R}^{S+1}$ is defined by

$$\bar{d}_j \;=\; \begin{pmatrix} \bar{y}_j^0 - \bar{q}_j \\ \bar{y}_j^1 \\ \vdots \\ \bar{y}_j^S \end{pmatrix}.$$

Therefore gradients are in a $S+1-J$ dimensional subspace of $\mathbb{R}^{S+1}$. Let $\Delta \subset \mathbb{R}^{S+1}_+$ denote the simplex $\Delta = \{\mu \in \mathbb{R}^{S+1}_+ | \sum \mu_s = 1\}$ and let $\pi_i : X \to \Delta$ denote the normalized gradient for consumer i

$$\pi_i(x_i) \;=\; \frac{1}{\sum_s D_{x_i^s} u_i(x_i)} Du_i(x_i).$$

Then normalized gradients are in a $S-J$ dimensional subset of the simplex.

In the next section it is shown that the problem of the firm can be decomposed into two: (1) maximizing profits with respect to state price vectors, and (2) problems of finding state price vectors which reflect the interests of their shareholders. All shareholders want their own gradients to be the state price vector for which firms maximize profits, so, intuitively, the relevant set of state price vectors is the convex hull of the set of normalized gradients. Therefore, again intuitively, the relevant set of state price vectors has dimension $S-J$ at most. In Greenberg (1979), where a society consists of a set of alternatives B and a finite set of agents $\{1, \ldots, L\}$, who are described by their preference correspondences $R_\ell : B \to B$, it is shown that there exists a ρ-majority stable voting equilibrium if and only if $\rho \geq (\dim B)/(\dim B + 1)$. The result of Greenberg is applied to the problems of determining state price vectors for which firms maximize profits.

4. No-Arbitrage Equilibria

In order to provide a proof of Theorem 1 and to explore how firms determine production plans another notion of equilibrium is introduced.

Let $W_i : X \times \Delta \to \Delta$ be a correspondence which associates a consumption bundle for consumer i and a state price vector with the set of state price vectors that are closer to the normalized gradient, so

$$W_i(x_i, \mu) \;=\; \{\mu' \in \Delta | \|\mu' - \pi_i(x_i)\| < \|\mu - \pi_i(x_i)\|\}.$$

Next let $w_j : \prod_i X \times \Delta \times \Delta \to \{1, \ldots, I\}$ be a correspondence which associates a collection of individual consumption bundles and a pair of state price vectors with the set of consumers whose normalized gradients are closer to the latter state price vector than to the former state price vector, so

$$w((x_i)_i, \mu, \mu') \;=\; \{i \in \{1, \ldots, I\} | \mu' \in W_i(x_i, \mu)\}.$$

Finally, let $Q^\rho : \prod_i (X \times \mathbb{R}) \times \Delta$ be a correspondence which associates a collection of individual consumption bundles and portfolios and a state price vector with the set of state price vectors are closer to the normalized gradients of more than $\rho \times 100$ percent of the shareholders, so

$$Q^{\rho}((x_i,\theta_i)_i,\mu) = \begin{cases} \emptyset & \text{for } \sum \theta_i^+ = 0 \\ \{\mu' \in \Delta | \frac{\sum_{i \in w((x_i)_i,\mu,\mu')} \theta_i^+}{\sum_i \theta_i^+} > \rho\} & \text{for } \sum \theta_i^+ > 0. \end{cases}$$

Obviously W_i, w and Q^{ρ} are purely artificial constructions in the sense that the only information W_i, w and Q^{ρ} convey about shareholders is their normalized gradients and portfolios.

Definition 2. *A* **ρ-majority stable no-arbitrage equilibrium** *is a state price vector, a collection of individual consumption bundles and portfolios and a collection of individual production plans and state price vectors*

$$(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$$

such that:

(C') $(\bar{x}_i, \bar{\theta}_i)$ maximizes the utility of consumer i given $(\bar{\lambda}, (\bar{y}_j)_j)$, so $(\bar{x}_i, \bar{\theta}_i)$ is a solution to

$$\max_{x_i,\theta_i} \quad u_i(x_i)$$
$$s.t. \quad \begin{cases} \bar{\lambda} \cdot x_i = \bar{\lambda} \cdot \omega_i + \bar{\lambda} \cdot \sum_j \bar{y}_j \delta_{ij} \\ x_i^s - \omega_i^s = \sum_j y_j^s \theta_{ij} \text{ for all } s \geq 1; \end{cases} \tag{2}$$

(F') $\bar{y}_j$ maximizes the profit of firm j given $\bar{\mu}_j$, so $\bar{y}_j$ is a solution to

$$\max_{y_j} \quad \bar{\mu}_j \cdot y_j$$
$$s.t. \quad y_j \in Y_j;$$

(F") $Q^{\rho}((\bar{x}_i, \bar{\theta}_{ij})_i, \bar{\mu}_j) = \emptyset$, and;

(E) $\sum_i \bar{x}_i = \sum_i \omega_i + \sum_j \bar{y}_j$ and $\sum_j \bar{\theta}_{ij} = 1$.

If there exists a coalition $C \subset \{1, \ldots, I\}$, where $\sum_{i \in C} \bar{\theta}_{ij} > \rho \sum_i \bar{\theta}_{ij}$, such that $\bar{\mu}_j$ is not in the convex hull of the gradients of the shareholders in C, then there exists μ_j such that $\|\mu_j - \pi_i(\bar{x}_i)\| < \|\bar{\mu}_j - \pi_i(\bar{x}_i)\|$ for all $i \in C$, so $\mu_j \in Q^\rho((\bar{x}_i, \bar{\theta}_i)_i, \bar{\mu}_j)$. Therefore $\bar{\mu}_j$ has to be in the intersection of the convex hulls of gradients for coalitions of shareholders with more than $\rho \times 100$ percent of the shares.

Lemma 1 and Lemma 2 below show the equivalence of stock market equilibria and no-arbitrage equilibria.

Lemma 1. *If $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ is a ρ-majority stable no-arbitrage equilibrium, then $(\bar{q}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j)$, where $\bar{q}_j = (1/\bar{\lambda}_0)\bar{\lambda} \cdot \bar{y}_j$, is a ρ-majority stable stock market equilibrium.*

Proof: Suppose that $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ is a ρ-majority stable no-arbitrage equilibrium, then (E) in Definition 1 is satisfied. Therefore in order to show that $(\bar{q}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j)$, where $\bar{q}_j = (1/\bar{\lambda}_0)\bar{\lambda} \cdot \bar{y}_j$, is a ρ-majority stable stock market equilibrium it suffices to show that (C) and (F) in Definition 1 are satisfied.

"(C)" Clearly, $\bar{\lambda}_0 > 0$ because $(\bar{x}_i, \bar{\theta}_i)$ solves problem (2) given $(\bar{\lambda}, (\bar{y}_j)_j)$ and, according to (A.3), gradients are positive vectors. Therefore, according to Magill & Quinzii (1996), $(\bar{x}_i, \bar{\theta}_i)$ is a solution to Problem (1) given $(\bar{q}, (\bar{y}_j)_j)$, where $\bar{q}_j = (1/\bar{\lambda}_0)\bar{\lambda} \cdot \bar{y}_j$, if and only if $(\bar{x}_i, \bar{\theta}_i)$ is a solution to Problem (2) given $(\bar{\lambda}, (\bar{y}_j)_j)$.

"(F)" The line of proof is to show that if $P_j^\rho((\bar{x}_i, \bar{\theta}_{ij})_i, \bar{y}_j) \neq \emptyset$, then $Q^\rho((\bar{x}_i, \bar{\theta}_{ij})_i, \bar{\mu}_j) \neq \emptyset$. Suppose that $P_j^\rho((\bar{x}_i, \bar{\theta}_{ij})_i, \bar{y}_j) \neq \emptyset$, then there exists $y_j \in Y_j$, such that

$$\frac{\sum_{i \in v_j((\bar{x}_i, \bar{\theta}_{ij})_i, \bar{y}_j, y_j)} \bar{\theta}_{ij}^+}{\sum_i \bar{\theta}_{ij}^+} \ > \ \rho$$

For $i \in v_j((\bar{x}_i, \bar{\theta}_{ij})_i, \bar{y}_j, y_j)$ and $\bar{\theta}_{ij} > 0$, then

$$\pi_i(\bar{x}_i) \cdot (y_j - \bar{y}_j) \ > \ 0.$$

Let z_j be the orthogonal projection of $y_j - \bar{y}_j$ on $\langle \bar{\mu}_j \rangle^{\perp}$, where $\langle \bar{\mu}_j \rangle^{\perp}$ is the linear subspace orthogonal to $\bar{\mu}_j$, so

$$z_j \;=\; y_j - \bar{y}_j + \frac{\bar{\mu}_j \cdot (\bar{y}_j - y_j)}{\bar{\mu}_j \cdot \bar{\mu}_j} \bar{\mu}_j,$$

then $\bar{\mu}_j \cdot z_j = 0$ and $\pi_i(\bar{x}_i) \cdot z_j \geq \pi_i(\bar{x}_i) \cdot (y_j - \bar{y}_j)$. (Indeed, $\bar{\mu}_j \cdot (\bar{y}_j - y_j) \geq 0$, because $\bar{y}_j$ maximizes profit given $\bar{\mu}_j$ and, since both $\pi_i(\bar{x}_i)$ and $\bar{\mu}_j$ are vectors with non-negative coordinates, $\pi_i(\bar{x}_i) \cdot \bar{\mu}_j \geq 0$.)

Clearly, $\mu_j \in W_i(\bar{x}_i, \bar{\mu}_j)$ if and only if

$$\left(\pi_i(\bar{x}_i) - \frac{1}{2}(\mu_j + \bar{\mu}_j)\right) \cdot (\mu_j - \bar{\mu}_j) \;>\; 0.$$

Let $\mu_j \in \mathbb{R}^{S+1}$ be defined by

$$\mu_j^s \;=\; \bar{\mu}_j^s + z_j^s - \frac{1}{S+1} \sum_s z_j^s,$$

then $\sum_s \mu_j^s = 1$. Next, define $(\mu_{jn})_n$ by $\mu_{jn} = (1/n)\mu_j + ((n-1)/n)\bar{\mu}_j$. Obviously, if there exists an n such that $\mu_{jn} \in Q^{\rho}((\bar{x}_i, \bar{\theta}_{ij})_i, \bar{\mu}_j)$ for some n, then the proof is finished, because $Q^{\rho}((\bar{x}_i, \bar{\theta}_{ij})_i, \bar{\mu}_j) \neq \emptyset$.

If the sequence $(\mu_{jn})_n$ converges to $\bar{\mu}_j$, then there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $\mu_{jn} \in \Delta$, because $\bar{\mu}_j$ is in the interior of the simplex. (Indeed, $\bar{\mu}_j$ is in the convex hull of the normalized gradients of the consumers, because otherwise $Q^{\rho}((\bar{x}_i, \bar{\theta}_{ij})_i, \bar{\mu}_j) \neq \emptyset$, and, according to (A.3), the normalized gradients of the consumers are in the interior of the simplex.)

Moreover, there exists an $N' \in \mathbb{N}$ such that if $n \geq N'$ and $\pi_i(\bar{x}_i) \cdot z_j > 0$, then

$$\left(\pi_i(\bar{x}_i) - \frac{1}{2}(\mu_{jn} + \bar{\mu}_j)\right) \cdot (\mu_{jn} - \bar{\mu}_j) \;>\; 0,$$

because $\pi_i(\bar{x}_i) \cdot z_j > 0$ if and only if $(\pi_i(\bar{x}_i) - \bar{\mu}_j) \cdot (\mu_j - \bar{\mu}_j) > 0$. Indeed, easy computations yield that

$$\begin{aligned}\pi_i(\bar{x}_i)\cdot z_j &= \pi_i(\bar{x}_i)\cdot(\mu_j-\bar{\mu}_j)+\sum_s z_j^s \\ &= \pi_i(\bar{x}_i)\cdot(\mu_j-\bar{\mu}_j)-\bar{\mu}_j\cdot z_j+\sum_s z_j^s \\ &= (\pi_i(\bar{x}_i)-\bar{\mu}_j)\cdot(\mu_j-\bar{\mu}_j),\end{aligned}$$

because $\bar{\mu}_j\cdot z_j=0$. Thus $\mu_{jn}\in Q^{\rho}((\bar{x}_i,\bar{\theta}_{ij})_i,\bar{\mu}_j)$ for $n\geq\max\{N,N'\}$.

Q.E.D

The next lemma requires stronger assumptions than Lemma 1: In addition to (A.1)-(A.8) production sets are supposed to be smooth manifolds with boundary of dimension $S+1$.

Lemma 2. *Suppose that production sets Y_j are smooth manifolds with boundary of dimension $S+1$. If $(\bar{q},(\bar{x}_i,\bar{\theta}_i)_i,(\bar{y}_j)_j)$ is a ρ-majority stable stock market equilibrium, then there exist $\bar{\lambda}$ and $(\bar{\mu}_j)_j$ such that $(\bar{\lambda},(\bar{x}_i,\bar{\theta}_i)_i,(\bar{y}_j,\bar{\mu}_j)_j)$ is a ρ-majority stable no-arbitrage equilibrium.*

Proof: Suppose that $(\bar{q},(\bar{x}_i,\bar{\theta}_i)_i,(\bar{y}_j)_j)$ is a ρ-majority stable stock market equilibrium for an economy, where production sets are smooth manifolds with boundary of dimension $S+1$, then (E) in Definition 1 is satisfied. Therefore in order to show that there exists $\bar{\lambda}$ where $\bar{q}_j=(1/\bar{\lambda}_0)\bar{\lambda}\cdot\bar{y}_j$, and $(\bar{\mu}_j)_j$ such that $(\bar{\lambda},(\bar{x}_i,\bar{\theta}_i)_i,(\bar{y}_j,\bar{\mu}_j)_j)$ is a ρ-majority stable no-arbitrage equilibrium it suffices to show that (C'), (F') and (F") in Definition 2 are satisfied.

"(C)" According to Magill & Quinzii (1996), $(\bar{x}_i,\bar{\theta}_i)$ is a solution to Problem (1) given $(\bar{q},(\bar{y}_j)_j)$ if and only if $(\bar{x}_i,\bar{\theta}_i)$ is a solution to Problem (2) given $(\bar{\lambda},(\bar{y}_j)_j)$, where $\bar{\lambda}=\pi_i(\bar{x}_i)$ for some i.

"(F')" Clearly if $(\bar{q},(\bar{x}_i,\bar{\theta}_i)_i,(\bar{y}_j)_j)$ is a ρ-majority stable stock market equilibrium, then $\bar{y}_j$ is in the set of efficient production plans Z_j. Therefore, there exists $\bar{\mu}_j\in\Delta$ such that $\bar{y}_j$ maximizes the profit of firm j given $\bar{\mu}_j$.

"(F")" The line of proof is to show that if $Q^{\rho}((\bar{x}_i,\bar{\theta}_{ij})_i,\bar{\mu}_j)\neq\emptyset$, then $P_j^{\rho}((\bar{x}_i,\bar{\theta}_{ij})_i,\bar{y}_j)\neq\emptyset$. Suppose that $Q^{\rho}((\bar{x}_i,\bar{\theta}_{ij})_i,\bar{\mu}_j)\neq\emptyset$, then there exists $\bar{\mu}_j$

such that

$$\frac{\sum_{i \in w((\bar{x}_i)_i, \bar{\mu}_j, \mu_j)} \bar{\theta}_{ij}^{+}}{\sum_i \bar{\theta}_{ij}^{+}} \;>\; \rho$$

Clearly, if $i \in w((\bar{x}_i)_i, \bar{\mu}_j, \mu_j)$, then

$$\left(\pi_i(\bar{x}_i) - \frac{1}{2}(\mu_j + \bar{\mu}_j)\right) \cdot (\mu_j - \bar{\mu}_j) \;>\; 0.$$

Therefore,

$$(\pi_i(\bar{x}_i) - \bar{\mu}_j) \cdot (\mu_j - \bar{\mu}_j) \;>\; 0,$$

because

$$(\pi_i(\bar{x}_i) - \bar{\mu}_j) \cdot (\mu_j - \bar{\mu}_j) - (\pi_i(\bar{x}_i) - \frac{1}{2}(\mu_j + \bar{\mu}_j)) \cdot (\mu_j - \bar{\mu}_j) \;\geq\; 0$$

if and only if $(\mu_j - \bar{\mu}_j) \cdot (\mu_j - \bar{\mu}_j) \geq 0$. Let $z_j \in \langle \bar{\mu}_j \rangle^{\perp}$ be defined by

$$z_j^s \;=\; (\mu_j - \bar{\mu}_j) - \bar{\mu}_j \cdot (\mu_j - \bar{\mu}_j),$$

then $\pi_i(\bar{x}_i) \cdot z_j \;>\; 0$, because

$$\pi_i(\bar{x}_i) \cdot z_j \;=\; (\pi_i(x_i) - \bar{\mu}_j) \cdot (\mu_j - \bar{\mu}_j).$$

According to Milnor (1965), the boundary of the production set bd Y_j is a smooth manifold of dimension S, because the production set Y_j is a smooth manifold with boundary of dimension $S+1$. Therefore there exist an open subset A of $\mathbb{R}^S$, a neighborhood B of $\bar{y}_j$ and a diffeomorphism $\phi_j : A \to B \cap \text{bd } Y_j$. Let $T_j(\bar{y}_j)$ be the tangent space of the boundary of the production set bd Y_j at $\bar{y}_j$, then $T_j(\bar{y}_j) = \langle \bar{\mu}_j \rangle^{\perp}$. Let $\bar{a}_j = \phi_j^{-1}(\bar{y}_j)$ and $a_j = D\phi_j(\bar{y}_j)^{-1} z_j$, then there exists $n \in N$ such that if $n \geq N$, then $\bar{a} + (1/n)a_j \in A$. Let $(y_{jn})_{n \geq N}$ be defined by $y_{jn} = \phi_j(\bar{a}_j + (1/n)a_j)$, then

$$\lim_{n \to \infty} \frac{\|y_{jn} - \bar{y}_j - (1/n)z_j\|}{\|(1/n)a\|} \;=\; 0,$$

so

$$\lim_{n \to \infty} n(y_{jn} - \bar{y}_j) \;=\; z_j.$$

Hence, there exists $N' \in \mathbb{N}$, such that if $n \geq N'$, $\pi_i(\bar{x}_i) \cdot z_j > 0$ and $\bar{\theta}_{ij} > 0$, then $u_i(\bar{x}_i + (y_{jn} - \bar{y}_j)\bar{\theta}_{ij}) > u_i(\bar{x}_i)$. Thus, $y_{jn} \in P_j^\rho((\bar{x}_i, \bar{\theta}_{ij})_i, \bar{y}_j)$ for $n \geq \max\{N, N'\}$.

Q.E.D

Drèze (1974) suggests that firms in stock market equilibria should maximize profits with respects to the average gradients of the shareholders $\bar{\mu}_j^A$, where

$$\bar{\mu}_j^A = \sum_i \frac{\bar{\theta}_{ij}^+}{\sum_i \bar{\theta}_{ij}^+} \pi_i(\bar{x}_i).$$

However, it follows from Lemma 2 that at ρ-majority stable stock market equilibria for economies where the production sets are smooth manifold with boundary of dimension $S+1$, production plans maximize profit with respect to some state price vector in the intersection of the convex hulls of gradients for coalitions of shareholders with more than $\rho \times 100$ percent of the shares.

For an economy with $I = 4$, $S - J = 2$, so normalized gradients are in a 2-dimensional subset of Δ, and $\rho = 2/3$, suppose that $\bar{\theta}_{1j} = \ldots = \bar{\theta}_{4j} = 0.25$ and that gradients are distributed as shown in Figure 1 below. Coalitions of

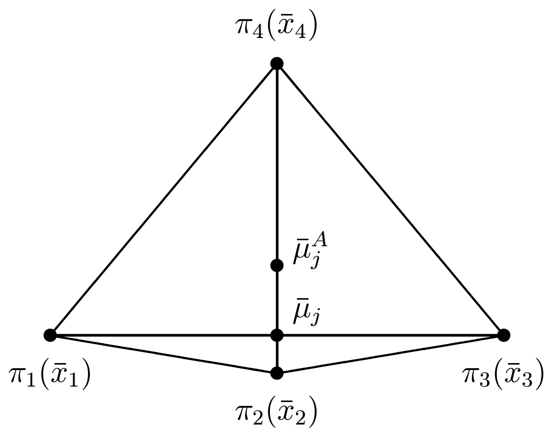

Figure 1: *Distribution of gradients.*

two or less shareholders are not able to change the outcome, because these coalitions control less than $\rho \times 100$ percent of the shares. All coalitions of three or more shareholders are able to change the outcome, because these coalitions control more than $\rho \times 100$ percent of the shares. Therefore $\bar{\mu}_j$ has to be in the intersection of the convex hulls of gradients for three or more shareholders and $\bar{\mu}_j$ is the only point in the intersection. Obviously $\bar{\mu}_j$ is different from the average gradient of the shareholders $\bar{\mu}_j^A$.

Proposition 1. *All economies have a ρ-majority stable no-arbitrage equilibrium if and only if*

$$\rho \;\geq\; \frac{S-J}{S-J+1}.$$

Remark: Since the proof of Proposition 1 is rather long and not too complicated, it is delegated to the appendix. The proof of the "if" of the assertion is based on the theorem on existence of equilibria in abstract economies in Shafer & Sonnenschein (1975) and Theorem 2 on existence of super majority stable equilibria in societies in Greenberg (1979). The proof of the "only if" of the assertion is an example of a family of economies in which for all $\rho < (S-J)/(S-J+1)$ there exist economies which do not have a ρ-majority stable stock market equilibrium.

End of remark

5. Timing and Price Perceptions

In Section 2 and Section 3, voting is supposed to take place after markets are closed as in Drèze (1985) and DeMarzo (1993). However, voting may also take place before markets open as in Grossman & Hart (1979), so the voting weights in firm j are $(\delta_{ij})_i$, or while markets are open, so the voting weights in firm j are $(\theta_{ij})_i$. In both cases, shareholders are able to adjust their portfolios after the outcome of the voting is known, so they need to form expectations about how the outcome of voting influences prices. Grossman & Hart (1979) introduced the notion of *competitive price perceptions*, where consumers perceive that income vectors are valued by their normalized gradients. Indeed if $(\bar{q}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j)$ satisfies (C), then consumer i perceives that a change of production plan from $\bar{y}_j$ is to y_j for firm j changes the price from $\bar{q}_j$ to q_j, where $q_j = (1)/(\pi_i^0(\bar{x}_i))\pi_i(\bar{x}_i) \cdot y_j$.

In general, changes of production plans influence trading opportunities through two channels: they change the value of portfolios as well as the span of assets. From this perspective, competitive price perceptions represent an extreme: if a consumer perceives that a change of a production plan of a firm is going to make the consumer better off, then the consumer perceives that the change is going to increase the value of the firm.

If $(\bar{q}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j)$ satisfies (C) and voting takes place before markets open, then a change for firm k from $\bar{y}_k$ to y_k is perceived by consumer i to change the price of firm k from $\bar{q}_k$ to $q_k = (1/(\pi_i^0(\bar{x}_i))\pi_i(\bar{x}_i) \cdot y_j$. Therefore the change of production plan is perceived by consumer i to result in consumption bundle x_i, where:

$$x_i^s \;=\; \begin{cases} \omega_i^0 + \sum_j q_j \delta_{ij} + \sum_j (y_j^0 - q_j)\theta_{ij} \text{ for } s = 0 \\ \\ \omega_i^s + \sum_j y_j^s \theta_{ij} \text{ for all } s \geq 1, \end{cases}$$

where $q_j = \bar{q}_j$ and $y_j = \bar{y}_j$ for $j \neq k$. Therefore

$$x_i^s - \bar{x}_i^s \;=\; \begin{cases} \sum_j (q_j - \bar{q}_j)\delta_{ij} - \sum_j (\bar{y}_j^0 - \bar{q}_j)\bar{\theta}_{ij} + \sum_j (y_j^0 - q_j)\theta_{ij} \text{ for } s = 0 \\ \\ -\sum_j \bar{y}_j^s \bar{\theta}_{ij} + \sum_j y_j^s \theta_{ij} \text{ for all } s \geq 1, \end{cases}$$

and

$$\pi_i(\bar{x}_i) \cdot (x_i - \bar{x}_i) \;=\; \pi_i(\bar{x}_i) \cdot (y_k - \bar{y}_k)\delta_{ik}.$$

Hence, if the voting weights in firm j are changed from $(\bar{\theta}_{ij})_i$ to $(\delta_{ij})_i$ in Section 2-4, then it follows that all economies have a ρ-majority stable stock market equilibrium if and only if $\rho \geq (S - J)/(S - J + 1)$. Since voting weights (in case voting takes place after markets close and before markets open) differ, the set of equilibria for the two different voting schemes probably differ.

If $(\bar{q}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j)$ satisfies (C) and voting takes while markets are open, then a change for firm k from $\bar{y}_k$ to y_k is perceived by consumer i to change the price of firm k from $\bar{q}_k$ to $q_k = (1/(\pi_i^0(\bar{x}_i))\pi_i(\bar{x}_i) \cdot y_j$. Therefore the change of production plan is perceived by consumer i to result in consumption bundle x_i, where:

$$x_i^s \;=\; \begin{cases} \omega_i^0 + \sum_j \bar{q}_j \delta_{ij} + \sum_j (q_j - \bar{q}_j)\bar{\theta}_{ij} + \sum_j (y_j^0 - q_j)\theta_{ij} \text{ for } s = 0 \\ \\ \omega_i^s + \sum_j y_j^s \theta_{ij} \text{ for all } s \geq 1, \end{cases}$$

where $q_j = \bar{q}_j$ and and $y_j = \bar{y}_j$ for $j \neq k$. Therefore

$$x_i^s - \bar{x}_i^s \;=\; \begin{cases} \sum_j (q_j - \bar{q}_j)\bar{\theta}_{ij} - \sum_j (\bar{y}_j^0 - \bar{q}_j)\bar{\theta}_{ij} + \sum_j (y_j^0 - q_j)\theta_{ij} \text{ for } s = 0 \\ \\ - \sum_j \bar{y}_j^s \bar{\theta}_{ij} + \sum_j y_j^s \theta_{ij} \text{ for all } s \geq 1, \end{cases}$$

and

$$\pi_i(\bar{x}_i) \cdot (x_i - \bar{x}_i) \;=\; \pi_i(\bar{x}_i) \cdot (y_k - \bar{y}_k)\bar{\theta}_{ik}.$$

Hence, the model in Section 2-4 may reflect that either voting takes while markets are open or after markets are closed. Thus in case voting takes place while markets are open, all economies satisfying (A.1)-(A.8) have a ρ-majority stable stock market equilibrium if and only if $\rho \geq (S-J)/(S-J+1)$.

6. Final remarks

The present paper shows existence of equilibria which are stable with respect to the joint operation of a market mechanism and a voting mechanism within firms. Since the set of production plans is multi-dimensional, super majority voting rules are needed to ensure existence of equilibria. The ratio proposed here is the upper bound on the lowest io necessary to guarantee existence; an upper bound obtained by relaxing the assumptions on the primitive characteristics of the economy as much as the usual standards of general equilibrium theory allow. The literature on social choice yields many ways to improve this type of results, through additional assumptions on the shape of preferences and on the distribution of primitive characteristics (see Crès and Tvede (2001) for existence results with ratios smaller than two thirds).

The fact that the equilibrium concept at stake is based on stability, at the same time, with respect to both a (decentralized) market mechanism and a (centralized) collective decision making mechanism is an interesting feature of the model. Indeed, the society of shareholders studied here is a coherent laboratory of our societies where both types of resource allocation mechanisms are intertwined. And the result obtained links the extent of the market failure with the conservativeness of the voting rule necessary to ensure existence of equilibria. The present paper reinforces the idea that the impossibility results that the theory of social choice has proposed over the last three decades (in the logic of the present study, the necessity of the super majority rule to be close to unanimity) can be partially resolved by the operation of a decentralized mechanism for resource allocation that endogenizes individual preferences over political choices. The proposed ratio gives, for this laboratory, a measure of this partial resolution.

References

Borch, K., General Equilibrium in the Economics of Uncertainty, in K. Borch & J. Mossin eds., *Risk and Uncertainty*, Macmillan (1968).

Border, K., *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press (1985).

Caplin, A. & B. Nalebuff, On 64%-Majority Rule, Econometrica **56** (1988), 787-814.

Caplin, A. & B. Nalebuff, Aggregation and Social Choice: A Mean Voter Theorem, Econometrica **59** (1991), 1-23.

Crès, H. & M. Tvede, Proxy Fights in Incomplete Markets: When Majority Voting and Sidepayments are Equivalent, Cahier de Recherche du

Groupe HEC (2001),

DeMarzo, P., Majority Voting and Corporate Control: The Rule of the Dominant Shareholder, Review of Economic Studies **60** (1993), 713-734.

Drèze, J., Investment under Private Ownership: Optimality, Equilibrium and Stability, in J. Drèze, ed., *Allocation under Uncertainty: Equilibrium and Optimality*, Macmillan (1974), 129-166.

Drèze, J., (Uncertainty and) The Firm in General Equilibrium Theory, Economic Journal **95** (1985), 1-20.

Ekern, S. & R. Wilson, On the Theory of the Firm in an Economy with Incomplete Markets, Bell Journal of Economics **5** (1974), 171-80.

Greenberg, J., Consistent Majority Rules over Compact Sets of Alternatives, Econometrica **47** (1979), 627-636.

Grossman, S. & O. Hart, A Theory of Competitive Equilibrium in Stock Market Economies, Econometrica **47** (1979), 293-330.

Magill, M. & M. Quinzii, *The Theory of Incomplete Markets*, The MIT Press (1996).

Milnor, J., *Topology from the Differentiable Viewpoint*, University Press of Virginia (1965).

Plott, C., A Notion of Equilibrium and its Possibility under the Majority Rule, American Economic Review **52** (1967), 787-806.

Shafer, W. & H. Sonnenschein, Equilibrium in Abstract Economies without Ordered Preferences, Journal of Mathematical Economics **2** (1975), 345-348.

Appendix: Proof of Proposition 1

The proof consists of 3 parts: In Part 1, the economy is transformed into an abstract economy; in Part 2, the abstract economy is shown to have an equilibrium and equilibria of abstract economies are shown to be ρ-majority stable no-arbitrage equilibria of the original economies, and; in Part 3, an example is provided to show that for all $\rho < (S-J)/(S-J+1)$ there exists an economy and does not have a ρ-majority stable no-arbitrage equilibrium. Hence, Part 1 and Part 2 are the *"if"* part of the proof and Part 3 is the *"only if"* part of the proof.

Part 1: *Transformation into an abstract economy*

In an abstract economy or generalized game every agent is described by a strategy set A_k, a constraint correspondence $C_k : A \to A_k$, where $A = \prod_k A_k$, and a preference correspondence $Q_k : A \to A_k$ and an equilibrium is a vector $a = (a_k)_k$ such that $a_k \in C_k(a)$ and $Q_k(a) \cap C_k(a) = \emptyset$, for all k.

There are four categories of agents: an auctioneer, who takes care of (E) in Definition 2, I consumers, who take care of (C') in Definition 2, J firms which takes care of (F') in Definition 2, and groups of shareholders which take care of (F") in Definition 2. Indeed, the auctioneer (agent 0) determines a state price vector in order to maximize the value of excess demand. The consumers (agent $k \in \{1, \ldots, I\}$) determine maximal consumption bundles and portfolios for their preferences. The firms (agent $k \in \{I+1, \ldots, I+J\}$) determine production plans that maximize profits with respect to a state price vector which reflect the interests of their shareholders. The groups of shareholders (agent $k \in \{I+J+1, \ldots, I+2J\}$, one group per firm) determine a state price vector for which firms maximize profits.

"Auctioneer" For agent k, where $k = 0$, let the strategy set $A_k \subset \mathbb{R}^{S+1}$ be defined by

$$A_k \ = \ \Delta,$$

let the constraint correspondence $C_k : A \to A_k$ be defined by

$$C_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j) \ = \ A_k$$

and let the preference correspondence $Q_k : A \to A_k$ be defined by

$$\begin{aligned} Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j) \;&=\; \{\lambda' \in A_k | \\ &\qquad (\lambda' - \lambda) \cdot (\textstyle\sum_i x_i - \sum_i \omega_i - \sum_j y_j) > 0\}. \end{aligned}$$

Clearly, A_k is compact and convex, C_k is continuous and the graph of Q_k is open and λ is not in the convex hull of $Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j)$, because $Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j)$ is convex and, by construction,

$$\lambda \;\notin\; Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j).$$

"Consumers" There exists a compact and convex set $A_k \subset X \times \mathbb{R}^J$ such that if $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ satisfies (C'), (F') and (E) in Definition 2, then $(\bar{x}_i, \bar{\theta}_i)$ is in the interior of A_k because of (A.5), (A.7) and (A.8). For agent $k \in \{1, \ldots, I\}$, where $k = i$, let the strategy set $A_k \subset X \times \mathbb{R}^J$ be defined by

$$A_k \;=\; X \times \mathbb{R}^J,$$

let the constraint correspondence $C_k : A \to A_k$ be defined by

$$\begin{aligned} C_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j) \;&=\; \{(x'_i, \theta'_i) \in A_k | \lambda \cdot x'_i \le \lambda \cdot \omega_i + \textstyle\sum_j \lambda \cdot y_j \delta_{ij} \\ &\qquad \text{and } x'^s_i - \omega^s_i \le \textstyle\sum_j y^s_j \theta'_{ij} \text{ for all } s \ge 1\} \end{aligned}$$

and let the preference correspondence, $Q_k : A \to A_k$ be defined by

$$Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j) \;=\; \{(x'_i, \theta') \in A_k | u_i(x'_i) > u_i(x_i)\}.$$

Clearly, A_k is compact and convex, C_k is continuous and the graph of Q_k is open with (x_i, θ_i) not in the convex hull of $Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j)$ because, according to (A.4), $Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j)$ is convex and, by construction,

$$(x_i, \theta_i) \;\notin\; Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j).$$

"Firms" For agent $k \in \{I+1, \ldots, I+J\}$, where $k = I+j$, let the strategy set $A_k \subset Y_j$ be defined by

$$A_k \;=\; \text{co cl } Z_j,$$

let the constraint correspondence $C_k : A \to A_k$ be defined by

$$C_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j) \;=\; A_k$$

and the preference correspondence $Q_k : A \to A_k$ be defined by

$$Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j) \;=\; \{y'_j \in A_k | \mu_j \cdot (y'_j - y_j) > 0\}.$$

Clearly, A_k is compact and convex, C_k is continuous and the graph of Q_k is open with y_j not in the convex hull of $Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j)$, because $Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j)$ is convex and, by construction,

$$y_j \;\notin\; Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j).$$

"Shareholders" For agent $k \in \{I+J+1, \ldots, I+2J\}$, where $k = I+J+j$, let the strategy set $A_k \subset \mathbb{R}^{S+1}$ be defined by

$$A_k \;=\; \Delta,$$

let the constraint correspondence $C_k : A \to A_k$ be defined by

$$C_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j) \;=\; \text{co}\,\{\pi_1(x_1), \ldots, \pi_I(x_I)\},$$

and let the preference correspondence $Q_k : A \to A_k$ be defined by

$$Q_k(\lambda, (x_i, \theta_i)_i, (y_j)_j, (\mu_j)_j) \;=\; Q^\rho((x_i, \theta_i)_i, \mu_j)$$

Clearly, A_k is compact and convex, C_k is continuous and the graph of Q_k is open.

Part 2: *Existence of a ρ-majority stable no-arbitrage equilibrium*

According to the theorem in Shafer & Sonneschein (1975) (Theorem 19.8 in Border (1985)), there exists $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j, (\bar{\mu}_j)) \in A$ such that for all $k \in \{0, \ldots, I+J\}$

$$Q_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j, (\bar{\mu}_j)) \cap C_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j, (\bar{\mu}_j)) \;=\; \emptyset$$

which implies that $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j, (\bar{\mu}_j))$ satisfies (C'), (F') and (E) in Definition 2, and for all $k \in \{I+J+1, \ldots, I+2J\}$ either

$$Q_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j, (\bar{\mu}_j)) \cap C_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j, (\bar{\mu}_j)) \; = \; \emptyset$$

or

$$\bar{\mu}_j \;\in\; (\text{co } Q_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j, (\bar{\mu}_j))) \cap C_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j, (\bar{\mu}_j)).$$

Below in Corollary 1 which is a corollary to Theorem 2 in Greenberg (1979), it is shown that if $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ satisfies (C'), (F') and (E) in Definition 2, then $\bar{\mu}_j$ is not in the convex hull of $Q_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ for $k \in \{I + J+1, \ldots, I+2J\}$.

Corollary 1. *Suppose $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ satisfies (C'), (F') and (E) in Definition 2. Then for $k = I + J + j$, $\bar{\mu}_j$ is not in the convex hull of $Q_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$.*

Proof: Every group of shareholders is transformed into a society a la Greenberg (1979), where a society consists of a set of alternatives B and a finite set of agents $\{1, \ldots, L\}$, who are described by their preference correspondences $R_\ell : B \to B$. Let the correspondence $r : B \times B \to \{0, \ldots, L\}$ be defined by

$$r(\mu, \mu') \;=\; \text{the cardinality of } \{\ell \in \{1, \ldots, L\} | \mu' \in R_\ell(\mu)\}$$

and let the correspondence $R^\rho : B \to B$ be defined by

$$R^\rho(\mu) \;=\; \{\mu' \in B | r(\mu, \mu') > \rho L\},$$

then in the proof of Theorem 2 in Greenberg (1979) it is shown that if $\rho \geq (\text{dim } B)/(\text{dim } B + 1)$, then $\mu \notin \text{co } R^\rho(\mu)$. However, in the proof of Theorem 2 in Greenberg, all agents have identical voting weights, while in the present setup the voting weights of shareholders depend on their portfolios. Therefore, every shareholder is transformed into a number of agents, where the number of agents depends on the portfolio of the shareholder in order to apply the proof of Theorem 2 in Greenberg (1979).

For $k = I + J + j$, let $B \subset \Delta$ be defined by

$$B \;=\; \text{co}\,\{\pi_1(\bar{x}_1), \ldots, \pi_I(\bar{x}_I)\}.$$

Let $[r] \in \mathbf{Z}$ be the integer part of $r \in \mathbb{R}$ and let n satisfy the following conditions: if $\bar{\theta}_{ij} > 0$, then $[\bar{\theta}_{ij} n] \geq 1$, and; for all subsets of consumers $C \subset \{1, \ldots, I\}$ if

$$\frac{\sum_{i \in C} \bar{\theta}^+_{ij}}{\sum_i \bar{\theta}^+_{ij}} \;>\; \rho,$$

then

$$\frac{\sum_{i \in C} [\bar{\theta}^+_{ij} n]}{\sum_i [\bar{\theta}^+_{ij} n]} \;>\; \rho.$$

Let $L = \sum_i [\bar{\theta}_{ij} n] \leq n \sum_i \bar{\theta}_{ij}(\lambda, x, y)^+$ and let the preference correspondence of agent ℓ, where $\ell \in L$ such that $\sum_{a<i} [\bar{\theta}_{aj} n] + 1 \leq \ell \leq \sum_{a \leq i} [\bar{\theta}_{aj} n]$, be defined by

$$R_\ell(\mu) \;=\; \{\mu' \in B | \|\mu' - \pi_i(\bar{x}_i)\| < \|\mu - \pi_i(\bar{x}_i)\|\},$$

so $R_\ell(\mu) = W_i(\bar{x}_i, \mu) \cap B$ and

$$Q_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j, (\bar{\mu}_j)_j) \cap C_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j, (\bar{\mu}_j)) \;\subset\; R^\rho(\bar{\mu}_j).$$

Therefore, according to the proof of Theorem 2 in Greenberg (1979), if $\rho \geq (\dim B)/(\dim B + 1)$, then $\mu \notin \text{co}\; R^\rho(\mu)$.

If $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ satisfies (C') in Definition 2, then $\bar{\lambda}_0 > 0$ because, according to (A.3), gradients are positive vectors. Therefore $\pi_i(\bar{x}_i)$ is orthogonal to the dividend vectors $(\bar{d}_j)_j$, where $\bar{d}_j \in \mathbb{R}^{S+1}$ is defined by

$$\bar{d}_j \;=\; \begin{pmatrix} \bar{y}^0_j - \bar{q}_j \\ \bar{y}^1_j \\ \vdots \\ \bar{y}^S_j \end{pmatrix},$$

because $(\bar{x}_i, \bar{\theta}_i)$ is a solution to Problem (1) given $(\bar{q}, (\bar{y}_j)_j)$ where $\bar{q}_j = (1/\bar{\lambda}_0)\bar{\lambda} \cdot \bar{y}_j$ if and only if $(\bar{x}_i, \bar{\theta}_i)$ is a solution to Problem (1) given $(\bar{\lambda}, (\bar{y}_j)_j)$.

Moreover, if $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ satisfies (F') in Definition 2, then, according to (A.8), the dividend vectors are linearly independent. Hence, $\dim B \leq S - J$, so if $\rho \geq (S-J)/(S-J+1)$, then $\mu \notin \text{co } R^\rho(\mu)$. Thus, $\bar{\mu}_j$ is not in the convex hull of $Q_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$.

Q.E.D

According to Corollary 1, $\bar{\mu}_j$ is not in the convex hull of $Q_k(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ for $k = I + J + j$, so $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j)_j, (\bar{\mu}_j))$ satisfies (F") in Definition 2. Therefore, Corollary 1 implies that the abstract economy has an equilibrium and, by construction, an equilibrium of the abstract economy is a ρ-majority stable no-arbitrage equilibrium of the economy.

Part 3: *An example showing that the bound is binding*

Consider a family of economies parametrized by $\gamma = (\alpha, \beta)$ where $\alpha, \beta \geq 0$, with $I = S - J + 1$ consumers. The consumers have identical utility functions $u_i : \mathbb{R}^{S+1} \to \mathbb{R}$ defined by

$$u_i(x_i) \;=\; \{t \in \mathbb{R} | x_i^s - t > 0 \text{ for all } s \text{ and} \textstyle\sum_s \ln(x_i^s - t) \;=\; 0\},$$

and identical initial portfolios of shares $\delta_i = (1/I)(1, \ldots, 1)$.

In order to define initial endowments and productions sets, suppose that the vectors $(\tilde{y}_j)_j$, where $\tilde{y}_j \in \mathbb{R}^{S+1}$, are linearly independent and let $\tilde{\omega}_i \in \mathbb{R}^{S+1}$ be defined by

$$\tilde{\omega}_i^s + \sum_j \tilde{y}_j^s \delta_{ij} \;=\; 1$$

for all s. Let initial endowments $(\omega_i^\alpha)_i$ be defined such that $\|\omega_i^\alpha - \tilde{\omega}_i\| \leq \alpha$ and $\sum_s \ln(\tilde{x}_i^\alpha) = 0$, where $\tilde{x}_i^\alpha = \omega_i^\alpha + \sum_s \tilde{y}_j \delta_{ij}$. Moreover, if $\alpha > 0$, then the normalized gradients $(\pi(\tilde{x}_i^\alpha))_i$ are supposed to be orthogonal to the dividend vectors $(\tilde{d}_j)_j$ where $\tilde{d}_j \in \mathbb{R}^{S+1}$ is defined by

$$\tilde{d}_j \;=\; \begin{pmatrix} \tilde{y}_j^0 - \sum_s \tilde{y}_j^s \\ \tilde{y}_j^1 \\ \vdots \\ \tilde{y}_j^S \end{pmatrix},$$

such that the convex hull of the normalized gradients co $\{(\pi(\tilde{x}_i^\alpha))_i\}$ is a non-empty simplex of dimension $S-J$ which contains the center $e = (1/(S+1))(1,\ldots,1)$ of Δ. Let production sets $(Y_j^\beta)_j$ be defined by

$$Y_j^\beta \;=\; \{y_j \in \mathbb{R}^{S+1} | \|y_j - (1-\beta)\tilde{y}_j\| \leq \beta\|\tilde{y}_j\|\} - \mathbb{R}_+^{S+1}.$$

Corollary 2 below is a corollary to Theorem 1 in Greenberg (1979).

Corollary 2. *For γ where $\alpha > 0$, if $\bar{\lambda} = e$ and $\bar{x}_i^\alpha = \tilde{x}_i^\alpha$ and $\bar{\theta}_i = \delta$ for all i and $\bar{y}_j = \tilde{y}_j$ and $\bar{\mu}_j = e$ for all j, then $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ is a ρ-majority stable no-arbitrage equilibrium if and only if $\rho \geq (S-J)/(S-J+1)$.*

Proof: First, it is shown that $(\bar{\lambda}, (\bar{x}_i^\alpha, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ satisfies (C'), (F') and (E) for all γ. Second, it is shown that $(\bar{\lambda}, (\bar{x}_i^\alpha, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ satisfies (F") for all γ where $\alpha > 0$, if and only if $\rho \geq (S-J)/(S-J+1)$.

"(C'), (F') and (E)" Clearly, $(\bar{\lambda}, (\bar{x}_i^\alpha, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ satisfies (C'), (F') and (E), because the normalized gradients $(\pi(\bar{x}_i^\alpha))_i$ are orthogonal to the dividend vectors $(\tilde{d}_j)_j$, the tangent space of the boundary of the production set at $\bar{y}_j^\gamma$ is $\langle e\rangle^\perp$ for $\beta > 0$ and $\sum_i \bar{x}_i^\alpha = \sum_i \omega_i^\gamma + \sum_j \bar{y}_j$.

"(F")" Clearly for all $\mu_j \in \Delta$ where $\mu_j \neq \bar{\mu}_j$, there exists at least one consumer k, such that $\mu_j \notin W_k(\bar{x}_k^\alpha, \bar{\mu}_j)$, because $\bar{\mu}_j$ is in the convex hull of the normalized gradients co $\{(\pi_i(\bar{x}_i^\alpha))_i\}$. Therefore

$$\sum_{i \in w((\bar{x}_i^\alpha)_i, \bar{\mu}_j, \mu_j)} \bar{\theta}_{ij} \;\leq\; \frac{S-J}{S-J+1}.$$

Hence, if $\rho \geq (S-J)/(S-J+1)$, then $Q^\rho((\bar{x}_i^\alpha, \bar{\theta}_i)_i, \bar{\mu}_j) = \emptyset$.

There exists μ_j in the convex hull of the normalized gradients co $\{(\pi_i(\bar{x}_i^\alpha))_i\}$ such that $\mu_j \in W_i(\bar{x}_i^\alpha, \bar{\mu}_j)$ for all but one consumer k, because the convex hull of the normalized gradients is a non-empty simplex of dimension $S-J$. Therefore

$$\sum_{i \in w((\bar{x}_i^\alpha), \bar{\mu}_j, \mu_j)} \bar{\theta}_{ij} \;=\; \frac{S-J}{S-J+1}.$$

Hence, if $\rho < (S-J)/(S-J+1)$, then $Q^\rho((\bar{x}_i^\alpha, \bar{\theta}_i)_i, \bar{\mu}_j) \neq \emptyset$.

Q.E.D

For γ where $\alpha = \beta = 0$, if $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ is a ρ-majority stable no-arbitrage equilibrium, then $\bar{x}_i = \tilde{x}_i^\alpha = \tilde{\omega} + \sum_j \tilde{y}_j \delta_{ij}$ and $\theta_i = \delta_i$ for all i and $y_j = \tilde{y}_j$ and $\mu_j = e$ for all j, because the allocation is Pareto optimal and all consumers are identical and have normalized gradients e.

Let S_J be a subset of $\{1, \ldots, S\}$ with J elements such the production plans for date 1 in the states in S_J are linearly independent. Then for γ where $\alpha = \beta = 0$, $((\lambda^s)_{s \in S_J}, (x_i, \theta_i)_i)$ where (C') and (E) are satisfied, is regular in $(\lambda^s)_{s \in S \setminus S_J}$ and $(\omega_i)_i$. Therefore there exists $\bar{\alpha} > 0$ such that for all γ where $\alpha \in]0, \bar{\alpha}[$ and $\beta = 0$, if $(\bar{\lambda}, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ is a ρ-majority stable no-arbitrage equilibrium, then $\bar{x}_i = \tilde{x}_i^\alpha$ and $\bar{\theta}_i = \delta$ for all i, $\bar{y}_j = \tilde{y}_j$ and $\bar{\mu}_j = e$ for all j and $\rho \geq (S - J)/(S - J + 1)$. Hence for all $\tilde{\alpha}$ where $\bar{\alpha} > \tilde{\alpha} > 0$, and $\varepsilon > 0$, there exists $\bar{\beta} > 0$ such that for all γ where $\alpha = \tilde{\alpha}$ and $\bar{\beta} > \beta > 0$, if $(\lambda, (\bar{x}_i, \bar{\theta}_i)_i, (\bar{y}_j, \bar{\mu}_j)_j)$ is a ρ-majority stable equilibrium, then $\rho > (S - J)/(S - J + 1) - \varepsilon$, because the equilibrium correspondence is upper hemi-continuous in β. Thus for all $\rho < (S - J)/(S - J + 1)$, there exists an economy which does not have a ρ-majority stable no-arbitrage equilibrium.