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Exponential health utility: A characterization and comments on a paper by Happich and Mühlbacher

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Abstract

In a recent paper Happich and Mühlbacher [Eur J Health Econom (2003) 4:292-294] proposed an axiom of constant absolute trade-off in life years, and studied the family of QALY models satisfying this axiom under expected utility and mutual utility independence between life years and health state. In this paper, we provide a complete characterization of the above-mentioned family of QALY models. This family should not be mistaken for the family of multiplicative exponential QALY models; in particular, it violates the zero-condition.

1 Introduction

In a recent paper, Happich and Mühlbacher [3] (hereafter HM) proposed the axiom *constant absolute trade-off*, which means that the number of life years that one is willing to give up in exchange for an improvement of the health state from one given level to another is independent of the initial number of remaining life years.

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In this paper, we give a complete characterization of QALY models consistent with constant absolute trade-off under expected utility and mutual utility independence. The characterization is very simple and gives a clear picture of the implications of the axiom of constant absolute trade-off.

HM did not provide a characterization (in the sense of deriving the family of functions satisfying the above-mentioned conditions), but in their Theorem 1 they showed that any QALY model satisfying the above mentioned axioms exhibits *constant absolute risk posture* over life years. Our characterization confirms this observation, but we notice that the conditional utility function for life years may also be linear, a possibility not taking into account by HM. More importantly, and contrary to what seems to be the message by HM, we find that the family of QALY models characterized by the above mentioned axioms is not the same as, indeed it is very different from, the family of multiplicative exponential QALY models which was characterized by constant absolute risk posture in a paper by Cher et al. [2].[1] In fact, any member of the family of multiplicative exponential QALY models fails to satisfy the axiom of constant absolute trade-off.

2 The model

A *health profile* is a pair (y, q) where y is a non-negative number of life years and q is a health state. *Expected utility* holds if there is a function U from pairs (y, q) to the real numbers such that preferences for gambles over a finite set of possible health profiles are governed by the expectation of U (see e.g. [4]).

Mutual utility independence holds if conditional preferences for lotteries over y do not depend on the particular level of q and vice versa. As noted by HM, under expected utility this conditions holds if and only if there are constants a and b and functions $U_Y(y)$ and $U_Q(q)$ such that

$$U(y, q) = aU_Y(y) + bU_Q(q) + (1 - a - b)U_Y(y)U_Q(q). \tag{1}$$

(cf. [4, Theorem 5.2]). However, we shall consider a formulation which is much more convenient for our purpose ([4] p. 238). Under expected utility and mutual utility independence there are constants a and b and functions $U_Y(y)$ and $U_Q(q)$ such that

[1] For a review of other axioms systems under expected utility and rank-dependent utility, see Miyamoto (1999).

$$U(y,q) = \begin{cases} U_Y(y)U_Q(q), & \text{or} \\ U_Y(y) + U_Q(q). \end{cases} \tag{2}$$

The *constant absolute trade-off* axiom (as defined by HM) holds when $U(y,q) = U(y',q')$ implies $U(y+p,q) = U(y'+p,q')$ for all $p > 0$.

We impose two minor conditions which simplifies exposition, but they are not essential and the characterization below could be adapted to cover situations where these conditions do not hold: We assume that there is at least one health state for which U is strictly increasing in life years y; and we assume that there is health state q_* which is the worst possible and a health state q^* which is the best possible, in the sense that $U_Q(q_*) \leq U_Q(q) \leq U_Q(q^*)$ for all q.

Finally, we must impose a richness condition which, on the other hand, plays some role: We assume that there is a continuum of health states such that for any $U(q_*) < \alpha < U(q^*)$ there is q such that $U_Q(q) = \alpha$.[2]

3 A characterization

If constant absolute trade-off holds, define $\Delta(q)$ to be the number of life years that one is willing to give up in order to replace q_* with q, i.e. $U(t,q_*) = U(t-\Delta(q),q)$, for all q and all $t - \Delta(q) \geq 0$.

The following characterization is now obtained making use of variations of well-known functional equation arguments (see Aczél [1]).

Theorem 1 *Under expected utility and mutual utility independence, then constant absolute trade-off holds if and only if there is U_Q and $\lambda > 1$ such that either i) $U(y,q) = \lambda^{(y+U_Q(q))}$ or ii) $U(y,q) = y + U_Q(q)$.*

Proof of Theorem:

Step 1: By constant absolute trade-off there is a function $\Delta(q)$ such that $U(y,q) = U(y+\Delta(q),q_*)$ for all y and all q. Let $f(y) \equiv U(y,q_*)$ for all y. Then we have

$$U(y,q) = f(y+\Delta(q)). \tag{3}$$

As noted previously, by expected utility and mutual utility independence we have a representation (2), and we shall consider each case in turn.

[2]Indeed, this conditions seems also to be an implicit assumption in the second part of HM's proof.

Step 2: First, assume that

$$U(y,q) = U_Y(y)U_Q(q). \tag{4}$$

Note that from the assumption that there is at least one health state for which U is strictly increasing in y we can assume $U_Y, U_Q > 0$. Combining (3) and (4) we have

$$f(y+\Delta(q)) = U_Y(y)U_Q(q), \tag{5}$$

for some function f. Note that from (5) it follows that $\Delta(q) = \Delta(q')$ if and only if $U_Q(q) = U_Q(q')$. Hence there is $\widetilde{U}_Q : \mathbb{R}_+ \to \mathbb{R}_{++}$ such that

$$U(y,q) = f(y+\Delta(q)) = U_Y(y)\widetilde{U}_Q(\Delta(q)). \tag{6}$$

Taking logarithms we have

$$\log f(y+\Delta(q)) = \log U_Y(y) + \log \widetilde{U}_Q(\Delta(q)). \tag{7}$$

Now we can make use of the following lemma.

Lemma 1 *Let $f, g, h : \mathbb{R}_+ \to \mathbb{R}$ be strictly increasing functions satisfying the Pexider equation*

$$h(x+x') = f(x) + g(x'), \tag{8}$$

for all $x \geq 0$ and $0 \leq x' \leq \varepsilon$. Then there is some $a > 0$ such that $f(x) = ax + b$, $g(x) = ax + c, h(x) = ax + b + c$.

Proof of Lemma 1: Follows from Corollary 1.8 and Theorem 5.4 in [1]. ■

By Lemma 1, we have $\log U_Y(x) = ax + b$, $\log \widetilde{U}_Q(x) = ax + c$, and $\log f(x) = ax + b + c$. Hence $U_Y(y) = e^{ay+b} = \overline{b}e^{ay}$. $\widetilde{U}_Q(\Delta(q)) = e^{a\Delta(q)+c} = \overline{c}e^{a\Delta(q)}$, and $f(x) = \overline{b}\overline{c}e^{ax}$.

From (6) we have

$$U(y,q) = \overline{b}e^{ay}\overline{c}e^{a\Delta(q)} = \overline{b}\overline{c}e^{a(y+\Delta(q))},$$

and because U is unique up to a positive affine transformation we have a representation of the form i) with $U_Q(q) \equiv \Delta(q)$.

Step 3: Second, assume that

$$U(y,q) = U_Y(y) + U_Q(q) \tag{9}$$

By (3) we have

$$f(y + \Delta(q)) = U_Y(y) + U_Q(q),$$

for some function f. Now, consider $q = q_*$, and without loss of generality assume that $U_Q(q_*) = 0$. Then, because $\Delta(q_*) = 0$, we have

$$f(y) = U_Y(y),$$

for all y, i.e. $f = U_Y$, and for an arbitrary q we thus have

$$U(y, q) = U_Y(y + \Delta(q)) = U_Y(y) + U_Q(q). \tag{10}$$

We have $U(0, q) = U(\Delta(q), q_*)$, and because $U(y, q) = U_Y(y) + U_Q(q)$, we have $U_Q(q) = U_Y(\Delta(q))$. Thus we have

$$U_Y(y + \Delta(q)) = U_Y(y) + U_Y(\Delta(q)). \tag{11}$$

It remains to show that U_Y is linear. For this we formulate the following lemma.

Lemma 2 *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing positive function satisfying the Cauchy equation*

$$f(x + x') = f(x) + f(x'), \tag{12}$$

for all $x \geq 0$ and $0 \leq x' \leq \varepsilon$. Then $f(x) = ax$, for some $a > 0$.

Proof of Lemma 2: It is well-known (Corollary 1.8 in [1]) that the conditions imply $f(x) = ax$ for $x \in [0, \varepsilon]$ for some $a > 0$. Now, consider $x > \varepsilon$. Then there is an integer $n \geq 2$ such that $x = x_1 + ... + x_n$ and $0 < x_i \leq \varepsilon$ for all $i = 1, ..., n$. From (12) we obtain by induction that

$$f(x_1 + ... + x_n) = f(x_1) + ... + f(x_n).$$

Hence

$$f(x) = f(x_1 + ... + x_n) = f(x_1) + ... + f(x_n) = ax_1 + ... + ax_n = ax. \blacksquare$$

Since (11) holds for all $y \geq 0$ and since there is ε (for example $\varepsilon \equiv \Delta(q^*)$) such that for any $0 < \alpha < \varepsilon$ there is q for which $\Delta(q) = \alpha$, it follows from Lemma 2 that $U_Y(y) = ay$ for some $a > 0$. Thus

$$U(y, q) = ay + a\Delta(q),$$

and since U is unique up to a positive affine transformation we have a representation of the form ii) with $U_Q(q) \equiv \Delta(q)$.

This completes the proof of Theorem 1. $\blacksquare$

4 Discussion

We have obtained a complete characterization of the family of QALY models satisfying constant absolute trade-off under the same set of conditions as those outlined by HM, and where some regularity conditions have been made precise.

It is important not to confuse constant absolute trade-off with constant absolute risk posture. Even assuming expected utility and mutual utility independence these two axioms are not equivalent. The family of exponential multiplicative QALY models

$$U(y,q) = \begin{cases} (e^{\lambda y} - 1)U_Q(q), & \lambda > 0 \\ yU_Q(q), & \lambda = 0 \\ (1 - e^{\lambda y})U_Q(q), & \lambda < 0, \end{cases}$$

has been characterized by constant absolute risk posture [2][5] (see also [7]), but note that here the zero-condition[3] is satisfied and the constant absolute trade-off axiom is violated. The zero-condition is not satisfied for the family of models characterized in Theorem 1 but constant absolute risk posture holds.

Constant absolute trade-off is not sensible when remaining life time becomes relatively short, because at some point it means that one is willing to die immediately in exchange for an improvement in health state. This is absurd when initial health is better than death.

Note also that for the family of QALY models characterized in Theorem 1, marginal utility in life years is (weakly) increasing, i.e. discounting of life years is negative.

To conclude, constant absolute trade-off does not seem to have normative appeal; in addition, the family of QALY models characterized by this axiom is a bit strange in a health context, and one might therefore suspect that it has only limited empirical relevance as well.

References

[1] Aczél J (1987) A Short Course on Functional Equations. D. Reidel Publishing Company.

[3] The zero-conditions is satisfied if all health states are equally preferred when $y = 0$, see Miyamoto et al. [6].

[2] Cher DJ, Miyamoto JM, Lenert LA (1997) Risk adjustment of Markov process models: importance in individual decision making. Med Dec Making 17:340-350.

[3] Happich M, Muehlbacher A (2003) An exponential representation of health state utility. Eur J Health Econom 4:292-294.

[4] Keeney R, Raiffa H (1976) Decisions With Multiple Objectives. John Wiley & Sons.

[5] Miyamoto JM (1999) Quality-adjusted life years (QALY) under expected utility and rank-dependent utility assumptions. J Math Psychol 43:201-237.

[6] Miyamoto JM, Wakker P, Bleichrodt H, Peters H (1998) The zero-condition: A simplifying assumption in QALY measurement and multiattribute utility. Management Sci 44:839-849.

[7] Pliskin JS, Shepard DS, Weinstein MC (1980) Utility functions for life years and health status. Operations Res 28: 206-224.