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Two agents negotiate, according to the Nash bargaining solution, over the allocation of a single (divisible) commodity (or multiple commodities with fixed ordinal preferences). It has been shown that in this situation agents find dominant to report their least risk averse utility functions. This result depends crucially on the fact that in this kind of "distortion game," agents have been restricted to report risk-averse utility functions. This paper studies the distortion game originated when agents are also allowed to claim non risk-averse utility functions. Contrasting with previous literature, we find multiple Nash equilibria, multiple payoff outcomes and the existence of a first-mover advantage.

1 Introduction

1.1 Motivation

When two persons bargain it is often the case that they have no incentives to reveal their true preferences.[1] Nevertheless, standard bargaining procedures

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[1]Examples of this simple fact can be seen in every day negotiations. A buyer might be willing to pay a high price for a commodity, but actually pretend to be uninterested in order to obtain a better deal.

assume complete knowledge of the agents' utility functions. This issue is addressed in the literature by defining a "distortion game" in which agents strategically report utility functions and the outcome of the negotiation is determined by applying, for example, the Nash bargaining solution concept.

Up to now, the reports of agents have been restricted to very particular families of utility functions. This is acceptable if the distortion game is interpreted as a mechanism used by an arbitrator in order to obtain, despite untruthful reports, an outcome with desirable properties. We think that distortion games can also be used to model situations in which no arbitrator is used. In this setting, it is not reasonable to require from agents to report within a specific class (e.g. concave) of functions.

The main contribution of this paper is to analyze the case in which the reported utility functions are not restricted to be risk-averse. We see no reason for an agent that does not restrain from misrepresenting his utility function to limit himself to risk averse claims. It is even more important to analyze this case taking into account that the conclusions deduced from this work differ substantially from the previous literature. In particular, instead of a unique dominant strategy equilibrium, we find multiple Nash equilibria. In fact, a continuum of Pareto optimal allocations can be supported by Nash equilibrium strategies. Finally, as we are not anymore in a dominant Nash strategy equilibrium, the issue of who plays first becomes relevant.

1.2 Literature Overview

The simplest distortion game[2] that has been studied is that one in which two risk averse agents bargain over the distribution of a single (divisible) commodity. The simplicity of this example includes two aspects: First, when considering just a single commodity, the ordinal preferences of the agents are unique, and so cannot be misrepresented. Thus, the strategic possibilities of the agents are limited to choose the cardinal utility function representing their ordinal preferences. Second, it is well known that as an agent becomes

[2]The distortion game technique was first used by Kurz [11] in reference to a taxation game in Aumann and Kurz [1].

less risk averse, his Nash bargaining outcome actually improves.[3] The fact that bargainers can only claim to have concave (risk averse) utility functions leads to the existence of a dominant strategy equilibrium: agents claim the least risk averse strategy among those admissible by the distortion game.

Following the spirit of the last paragraph, Crawford and Varian [4] (for the single commodity case), and Kannai [7] (for multiple commodities) find dominant strategies for risk averse agents with fixed ordinal preferences. They show that it is optimal for agents to report their least concave (linear in the single commodity case) representation of their ordinal preferences. If only one commodity is distributed, these reports lead to equal division of the commodity.

In this work, despite not existing a dominant strategy, we characterize the set of Nash equilibria of the distortion game. To achieve this we proceed as follows: Proposition 1 gives an explicit method of how to manipulate the utility possibility set, the set of utility vectors over which Nash b argaining will be applied. Then, Proposition 2 recalls a property of Nash Bargaining[4] that puts a lower bound (in terms of the reported utilities) to the worst possible result of the negotiation for either agent. Proposition 3 characterizes the Nash equlibria of the distortion game if the UPS is convex. Finally, Proposition 4 characterizes the allocations that can result from agents playing Nash equilibria. It is particularly appealing that the method of proof is constructive in the sense that we explicitly describe how agents can improve their outcome if they are not playing a Nash equilibrium.

It is important to mention other distortion games analyzed in the literature. In particular, papers by Sobel [15], [16],[5] and Kibris [9] study the multiple commodity case when the ordinal preferences are not fixed. In this instance, the set of Nash equilibria has not been characterized. It is necessary to restrict agents to report linear utility functions in order to characterize Nash equilibrium outcomes. In particular, Sobel [15] gives a "non-pathological example" of non-linear Nash equilibria.

[3]The first references that prove this are Kihlstrom, Roth and Schmeidler [10](for two agents), and Nielsen [14] (for multiple agents).

[4]This property is known in the literature as symmetric monotonicity [15] or midpoint dominance [17].

[5]In this particular work the bargaining concept used is the utilitarian solution.

2 Notation and the Environment

Consider an environment with two agents and n commodities. Without loss of generality assume there is one (divisible) unit of each commodity so that the aggregate endowment is described by the vector $\vec{1} = (1, \ldots, 1) \in \mathbf{R}^n$. Let the complete preorders $\succeq_1$ and $\succeq_2$ denote the agents' continuous, convex and strictly monotone preferences over the set $X = [0,1]^n$. Let u_1 and u_2 denote the von Neumann-Morgenstern utility functions of the agents. Normalize the original utility functions so that $u_i(\vec{0}) = 0$ and $u_i(\vec{1}) = 1$ for $i = 1, 2$. As the Nash bargaining solution satisfies invariance under positive affine transformations, there is no loss of generality in using this normalization. Define:

$$\mathbf{U} = \{u : X \to [0,1] \quad | \quad u(\vec{0}) = 0, u(\vec{1}) = 1, u \text{ continuous, strictly increasing}\}.$$

$$\mathbf{U}_i = \{u \in \mathbf{U} \quad | \quad u \text{ represents } \succeq_i\} \text{ for } i = 1, 2.$$

Assume that $u_i \in \mathbf{U}_i$ for $i = 1, 2$. Notice that concavity of u_i is not required.

Given two utility functions $u_1, u_2 \in \mathbf{U}$ the utility possibility set (UPS) is given by

$$UPS(u_1, u_2) = \{(v, w) \in [0,1]^2 \ | \ \exists x \in X \text{ s.t. } v \leq \tilde{u}_1(x) \text{ and } w \leq \tilde{u}_2(\vec{1} - x)\}.$$

The set $UPS(u_1, u_2)$ is not necessarily convex. The Pareto frontier of this set is described by the function $h : [0,1] \to [0,1]$ where for any given level of utility $v \in [0,1]$

$$h(v) = \max\{\tilde{u}_2(\vec{1} - x) \ | \ x \in X \text{ and } \tilde{u}_1(x) = v\}.$$

The following lemma describes the properties of the function h. For completeness, the proof is given in the appendix.

Lemma 1 *Given two utility functions $u_1, u_2 \in \mathbf{U}$, the function h that describes the Pareto frontier of the set $UPS(u_1, u_2)$ satisfies $h(0) = 1$, $h(1) = 0$, and is strictly decreasing (thus invertible). Furthermore, if u_1 and u_2 are (strictly) concave, then h is (strictly) concave.*

An immediate implication of the lemma is that the UPS generated by two concave utility functions is a convex set.[6] (See Billera and Bixby [2] or Chipman and Moore [3].)

Our objective is to study a bargaining problem generated by the strategic choice of a utility function representing fixed ordinal preferences. For example, suppose that agent 1 is interested in changing the UPS Pareto frontier by reporting utility function $\tilde{u}_1 \in \mathbf{U}$ instead of the real u_1.[7] In particular, assume that his intention is to transform the Pareto frontier into one that is characterized by the equation $w = \tilde{h}(v)$. We claim that if agent 1 knows the function h describing the original Pareto frontier, then he is able to report a utility function that achieves the desired transformation. Furthermore, the explicit formula for $\tilde{u}_1$ is given. As a byproduct, the proposition also shows that any continuous, strictly decreasing function $\tilde{h}$ once normalized, is a possible description of some UPS Pareto frontier. This is an important difference with the case in which agents are restricted to report concave utility functions.

Proposition 1 *Assume that $\tilde{h}$ is a strictly decreasing (hence invertible) concave function such that $\tilde{h}(0) = 1$ and $\tilde{h}(1) = 0$. Then agent 1 can transform the original UPS Pareto frontier into $\tilde{h}$ by claiming that his (unchanged) ordinal preferences are represented by the utility function*

[6] Concavity of the utility functions is essential for Lemma 1 to hold in an environment with more than two players. Kannai and Mantel [8] provide a three player counterexample in which agents with continuous, convex and strictly monotone preferences never generate a convex UPS, no matter the utility functions used to represent the ordinal preferences.

[7] From now on, u_i will be used to denote the true utility functions of the agents.

$$\tilde{u}_1(x) = \tilde{h}^{-1}(h(u_1(x))).$$

Proof: The value that the new function takes at $v \in [0,1]$ is

$$\max_{\tilde{u}_1(x)=v} u_2(\vec{1}-x) = \max_{\tilde{h}^{-1}(h(u_1(x)))=v} u_2(\vec{1}-x) = \max_{h(u_1(x))=\tilde{h}(v)} u_2(\vec{1}-x).$$

Using the fact that the maximizer $x \in X$ is Pareto optimal we conclude that

$$\max_{h(u_1(x))=\tilde{h}(v)} u_2(\vec{1}-x) = \max_{u_2(\vec{1}-x)=\tilde{h}(v)} u_2(\vec{1}-x) = \tilde{h}(v).$$

The transformation applied to u_1 so that it transforms into $\tilde{u}_1$ is $\tilde{h}^{-1} \circ h$. This is a monotone transformation because it is the composition of two strictly decreasing functions[8], so the preferences of agent 1 remain unchanged. ■

3 The Distortion Game

The question now is what happens if both agents act strategically and the outcome is determined by Nash bargaining. We use the utility vector $(0,0)$ as the threat point in case of disagreement. If the UPS is not convex, then the outcome of the distortion game is decided by the toss of a coin: each agent has equal probability of being allocated the aggregate output.

Next, we formally describe the distortion game.

Definition 1 *Let S be a nonempty convex subset of $\mathbf{R}^2$ and let $d = (d_1, d_2) \in S$. Define the* **Nash bargaining utility vector** *[13] as*

$$NB(S,d) = argmax_{v \in S,\, v \geq d}(v_1 - d_1)(v_2 - d_2).$$

Convexity of S implies that $NB(S,d)$ is always a singleton.

[8] As $\tilde{h}$ is strictly decreasing, its inverse also has this property.

Definition 2 *The* **distortion game** *is a non-cooperative game in which agents report utility functions* $\tilde{u}_1, \tilde{u}_2 \in \mathbf{U}_1 \times \mathbf{U}_2$*. If* $UPS(\tilde{u}_1, \tilde{u}_2)$ *is convex, there must exist a vector* $x \in X$ *such that*

$$NB(UPS(\tilde{u}_1, \tilde{u}_2), (0,0)) = (\tilde{u}_1(x), \tilde{u}_2(\vec{1} - x)).$$

The payoffs of the game are given by the vector $\psi \in \mathbf{R}^2$ *defined as:*

$$\psi(\tilde{u}_1, \tilde{u}_2) = \begin{cases} (u_1(x), u_2(\vec{1} - x)) & \textit{if } UPS(\tilde{u}_1, \tilde{u}_2) \textit{ is convex} \\ \\ (\frac{1}{2}, \frac{1}{2}) & \textit{otherwise} \end{cases}$$

Notice that because ordinal preferences are known, there is no need to specify a tie-breaking mechanism as in Sobel [15]: although there may be more than one outcome that generates utility vector $\psi(\tilde{u}_1, \tilde{u}_2)$, all of them generate the same utility when evaluated in the true utility functions. Thus, the distortion game is well defined.

An appealing property of the Nash bargaining solution is that the outcome guarantees a certain amount ($\frac{1}{2}$) of the 0-1 normalized (reported) utilities to the parties involved. This characteristic helps the agents to be in some sense protected against misrepresentation of utilities by their opponents. This property has been known in the literature as symmetric monotonicity (Sobel [15]) or midpoint dominance axiom (Thomson [17]). Actually, it was Sobel who first discovered its relevance for distortion games. The next proposition indicates that Nash bargaining satisfies this property. Although Sobel proved it first, in the appendix we follow Moulin's proof because it does not use risk sensitivity considerations.

Proposition 2 *(Sobel [15], Moulin [12]) For any pair of utility functions* $u_1, u_2 \in \mathbf{U}$*, then* $NB(UPS(u_1, u_2), (0,0)) \geq \frac{1}{2}$*.*

The previous result gives a lower bound for an agent's payoff in "reported utility units". One might suspect that when agents only manage to obtain this lower bound, it is because they are at a Nash equilibrium of the distortion game. Indeed, the following proposition proves that guess is correct.

4 Characterization of Nash Equilibria

4.1 Nash Equilibrium Strategies

We will now characterize the set of Nash equilibria in the distortion game. The first case we consider is Nash equilibrium strategies (u_1^*, u_2^*) such that $UPS(u_1^*, u_2^*)$ is not a convex set. In this case, the conditions needed to reach this potentially inefficient equilibrium are:

1. $u_1^*(x) \geq \frac{1}{2}$ implies that $u_2(\vec{1} - x) \leq \frac{1}{2}$.
2. $u_2^*(x) \geq \frac{1}{2}$ implies that $u_1(\vec{1} - x) \leq \frac{1}{2}$.

In this case, neither agent has any incentive to "convexify" the UPS because the opponent's strategy implies, by Proposition 2, that this kind of deviation will not be an improvement. In a sense, agents were too greedy with their claims and this precluded an agreement. On the other hand, notice that if agents are truthful and play this kind of Nash equilibrium, the outcome will be ex-ante efficient.

Proposition 3 *Assume $UPS(u_1^*, u_2^*)$ is convex. Then, the pair (u_1^*, u_2^*) is a Nash equilibrium of the distortion game if and only if*

1. $NB(UPS(u_1^*, u_2^*), (0, 0)) = (\frac{1}{2}, \frac{1}{2})$.
2. $\psi(u_1^*, u_2^*) \geq \frac{1}{2}$.

Proof: First, suppose that (u_1^*, u_2^*) is a Nash equilibrium of the distortion game. Clearly, $\psi(u_1^*, u_2^*) \geq \frac{1}{2}$ so that the agents do not want to deviate to the coin outcome. Now, suppose that $NB(UPS(u_1^*, u_2^*), (0, 0)) \neq (\frac{1}{2}, \frac{1}{2})$. Without loss of generality, assume that $NB_2((u_1^*, u_2^*), (0, 0)) > \frac{1}{2}$. We will show that agent 1 can improve his payoff by choosing $\tilde{u}_1$ in such a way that the Pareto frontier of the set $UPS((\tilde{u}_1, u_2^*), (0, 0))$ is linear. Proposition 1 implies that this can be done by setting $\tilde{u}_1(x) = 1 - h(u_1^*(x))$ where h is defined as the function that describes the original Pareto frontier (see Section 2). To show the previous claim, consider the Pareto optimal allocations $(x^*, \vec{1} - x^*)$ and $(\tilde{x}, \vec{1} - \tilde{x})$ such that $\psi(u_1^*, u_2^*) = (u_1^*(x^*), u_2^*(\vec{1} - x^*))$ and

$\psi(\tilde{u}_1, u_2^*) = (\tilde{u}_1(\tilde{x}), u_2^*(\vec{1} - \tilde{x}))$. We know that $u_2^*(\vec{1} - x^*) > u_2^*(\vec{1} - \tilde{x}) = \frac{1}{2}$. Then, if $u_1^*(x^*) \geq u_1^*(\tilde{x})$, allocation $(x, \vec{1} - x)$ would Pareto dominate allocation $(\tilde{x}, \vec{1} - \tilde{x})$, a contradiction. We conclude that agent 1 increases his payoff by choosing utility function $\tilde{u}_1$.

For the converse, suppose (u_1^*, u_2^*) is not a Nash equilibrium of the distortion game but conditions 1 and 2 hold. Then, without loss of generality, assume there exists $\tilde{u}_1 \in \mathbf{U}_1$ a better option for agent 1. Because $\psi(u_1^*, u_2^*) \geq \frac{1}{2}$, agent 1 will have incentives to deviate only if $UPS(u_1', u_2^*)$ is also convex. Let $(x^*, \vec{1} - x^*)$ and $(\tilde{x}, \vec{1} - \tilde{x})$ be allocations that generate utility vectors $NB(UPS(u_1^*, u_2^*), (0,0))$ and $NB(UPS(\tilde{u}_1, u_2^*), (0,0))$ respectively. Agent 1 is better off, so $u_1(x^*) < u_1(\tilde{x})$. As $u_1^* \in \mathbf{U}_1$, we have $u_1^*(\tilde{x}) > u_1^*(x^*) = NB_1(UPS(u_1^*, u_2^*), (0,0)) = \frac{1}{2}$. Now, by Proposition 2 $u_2^*(\vec{1} - \tilde{x}) = NB_2(UPS(\tilde{u}_1, u_2^*), (0,0)) \geq \frac{1}{2}$. But then, the product $u_1^*(\tilde{x})u_2^*(\vec{1} - \tilde{x})$ would be strictly greater than $\frac{1}{4}$, contradicting the fact that the vector generated by Nash bargaining was $(\frac{1}{2}, \frac{1}{2})$. ■

Finally, we conclude that if agents are in a Nash equilibrium of the distortion game and the UPS constructed with the reported utilities is convex, then the Pareto frontier of the UPS must be linear. Let us emphasize that unlike the case where onle concave utility functions are admissible, a linear Pareto frontier does not imply linear reported utilities. A possible case of a Nash equilibrium is constituted by truth-telling agents, one risk averse and the other one risk loving, compounding into a linear Pareto-frontier.

Corollary 1 *Assume $UPS(u_1^*, u_2^*)$ is convex. If the pair (u_1^*, u_2^*) is a Nash equilibrium of the distortion game then the Pareto frontier of $UPS(u_1^*, u_2^*)$ must be linear.*

Proof: Suppose that $h(v) \neq 1 - v$. Then $UPS(u_1^*, u_2^*)$ must contain a utility pair (v, w) such that $v + w > 1$. Without lost of generality, assume $w > v$. Convexity implies that $\big(w/(1 + w - v), w/(1 + w - v)\big)$ is an element of the UPS as it is a linear combination of (v, w) and $(1, 0)$. Furthermore, $v + w > 1$ implies that $w/(1 + w - v) > \frac{1}{2}$. But we know that the utility vector generated by Nash bargaining must be $(\frac{1}{2}, \frac{1}{2})$. Thus the vector $\big(w/(1 + w - v), w/(1 + w - v)\big) \leq \frac{1}{2}$, a contradiction. ■

4.2 Nash Equilibrium Outcomes

Given that ordinal preferences are fixed, the allocation generating the Nash bargaining utility vector will always be Pareto optimal. Nevertheless, if the game is decided by throwing a coin, the outcome is (possibly) inefficient. In what follows, we characterize which Pareto optimal allocations can arise as the result of Nash equilibrium of the distortion game. We conclude that in this setting agents may arrive to multiple of efficient allocations. The only condition for a Pareto optimal allocation to be supported as a Nash equilibrium outcome is that both agents actually prefer their outcome to the (expected) utility generated by throwing a coin.

Proposition 4 *A Pareto optimal allocation* $(x^*, \vec{1} - x^*)$ *can be supported as a Nash equilibrium outcome of the distortion game if and only if* $u_1(x^*) \geq \frac{1}{2}$ *and* $u_2(\vec{1} - x^*) \geq \frac{1}{2}$.

Proof: Let $u_2^* \in \mathbf{U}_2$ be such that $u_1^*(1 - x^*) = \frac{1}{2}$. Let $h : [0,1] \rightarrow [0,1]$ describe the Pareto frontier of $UPS(u_1, u_2^*)$. Define $u_1^*(x) = 1 - h(u_1(x))$. Proposition 1 implies $u_1^* \in \mathbf{U}_1$. The new Pareto frontier is linear so we have that $NB(UPS(u_1^*, u_2^*), (0,0)) = (\frac{1}{2}, \frac{1}{2})$. Finally, because by assumption $\psi(u_1^*, u_2^*) \geq \frac{1}{2}$, Proposition 3 implies that (u_1^*, u_2^*) is a Nash equilibrium supporting the allocation $(x^*, \vec{1} - x^*)$. ■

5 Implications of the Results

Assume that one of the two bargainers is strictly more risk averse than his opponent. When agents are only allowed risk-averse claims, the unique outcome of the distortion game will be biassed in favor of the more risk-averse agent: the closer an agent is to being risk-neutral, the less room he has to manipulate his utility function. We now present an example to illustrate our point.

Two agents are deciding the allocation of two commodities. The true utility functions over X are $u_1(x) = (x_1 x_2)^{\frac{1}{4}}$ and $u_2(x) = (x_1 x_2)^{\frac{1}{2}}$. The true Pareto frontier is then described by the function $h(v) = 1 - v^2$. The utility vector generated by Nash bargaining and the true utility functions is $(\sqrt{\frac{1}{3}}, \frac{2}{3})$. The corresponding outcome is given by the allocation $(\frac{1}{3}\omega, \frac{2}{3}\omega)$ where ω represents the aggregate output.

If agents can only report risk-averse utility functions, in the dominant strategy equilibrium agent 2 remains being truthful (notice that this utility fuction is already linear along the diagonal) while agent 1 chooses $\tilde{u_1}(x) = \tilde{h}^{-1}(h(u_1(x))) = (x_1x_2)^{\frac{1}{2}}$ to represent his preferences. Symmetry then implies that the new utility vector will be $(\frac{1}{2}, \frac{1}{2})$ and equal division of both goods will occur. This outcome is biassed in favor of the agent that was able to lie about his utility function.

Now let us analyze what happens if agents are allowed to report utility functions that are not concave. In this case, the set of Pareto optimal allocations that arise from Nash equilibrium is

$$\{(\alpha\omega, (1-\alpha)\omega) \in X \times X \ \mid \ \alpha \in [\frac{1}{4}, \frac{1}{2}]\}.$$

We observe that the outcome selected by the dominant strategies implied by the concavity restriction lies, in this particular case, on one end of the interval. The more risk-averse agent is clearly aided by the procedure used to determine the final allocation.

Although in our environment we must still select an outcome among many, at least the chosen outcome will not invariably favor more risk-averse agents. In our setting the outcome of the negotiation will depend much more on the bargaining abilities of the parties unlike the previous model where agents had a predetermined result, independent of their level of risk aversion.

Another important implication from the results is that they clearly point towards a first-mover advantage. If one of the agents is able to credibly commit to a strategy, that party is in position to decide which Pareto optimal allocation is used. The midpoint dominance axiom (Propostition 2) transforms a report $\tilde{u}_1$ into a non-agreement threat unless the bargaining outcome assigns at least a certain amount of utility (namely the utility obtained from allocations $x \in X$ such that $\tilde{u}_1(x) = \frac{1}{2}$) to agent i. This kind of advantage is not present when agents are restricted to claim concave utility functions because the order of play is not important when dominant strategies exist.

Appendix: proofs

Lemma 1 *Given two utility functions $u_1, u_2 \in \mathbf{U}$, the function h that describes the Pareto frontier of the set $UPS(u_1, u_2)$ satisfies $h(0) = 1$, $h(1) = 0$, and is strictly decreasing (thus invertible). Furthermore, if u_1 and u_2 are (strictly) concave, then h is (strictly) concave.*

Proof: By strict monotonicity, $u_1(x) = 0$ implies $x = \vec{0}$. Thus, $h(0) = u_1(\vec{1} - \vec{0}) = 1$. Similarly, $h(1) = 0$.

Now we prove that h is strictly decreasing. Take any $\bar{v}, \underline{v} \in [0, 1]$ such that $\bar{v} > \underline{v}$. The definition of h implies that there is a vector $\bar{x} \in X$ such that $(\bar{v}, h(\bar{v})) = (u_1(\bar{x}), u_2(\vec{1} - \bar{x}))$. By strict monotonicity and continuity of u_1, there exists an $\alpha \in [0, 1)$ such that $u_1(\alpha\bar{x}) = \underline{v}$. Then

$$h(\bar{v}) \quad = \quad u_2(\vec{1} - \bar{x}) < u_2(\vec{1} - \alpha\bar{x}) \leq \max_{u_1(x) = \underline{v}} u_2(\vec{1} - x) = h(\underline{v}),$$

so that h is strictly decreasing.

For concavity, let $\hat{v}, \bar{v}, \alpha \in [0, 1]$. Let $\hat{x}$ be a vector satisfying $u_1(\hat{x}) = \hat{v}$ and $h(\hat{v}) = u_2(\vec{1} - \hat{x})$. In a similar manner, let $\bar{x}$ be a vector satisfying $u_1(\bar{x}) = \bar{v}$ and $h(\bar{v}) = u_2(\vec{1} - \bar{x})$. Then

$$\begin{aligned}
h(\alpha\hat{v} + (1 - \alpha)\bar{v}) &= h(\alpha u_1(\hat{x}) + (1 - \alpha)u_1(\bar{x})) \\
&\geq h(u_1(\alpha\hat{x} + (1 - \alpha)\bar{x})) \\
&= \max_{u_1(x) = u_1(\alpha\hat{x} + (1 - \alpha)\bar{x})} u_2(\vec{1} - x) \\
&\geq u_2(\vec{1} - (\alpha\hat{x} + (1 - \alpha)\bar{x})) \\
&= u_2(\alpha(\vec{1} - \hat{x}) + (1 - \alpha)(\vec{1} - \bar{x})) \\
&\geq \alpha u_2(\vec{1} - \hat{x}) + (1 - \alpha)u_2(\vec{1} - \bar{x}) \\
&= \alpha h(\hat{v}) + (1 - \alpha)h(\bar{v}),
\end{aligned}$$

as desired. If either u_1 or u_2 is strictly concave, the inequality is strict. ■

Proposition 2 *(Sobel [15], Moulin [12]) For any pair of utility functions $u_1, u_2 \in \mathbf{U}$, then $NB(UPS(u_1, u_2), (0, 0)) \geq \frac{1}{2}$.*

Proof: Let $(v^*, w^*) = NB(UPS(u_1, u_2), (0, 0))$. Consider the sets

$$A = UPS(u_1, u_2) \text{ and } B = \{(v, w) \in \mathbf{R}^2_+ : vw \geq v^* w^*\}.$$

The sets A and B are convex and have disjoint interiors. Hence, there must exist a hyperplane H separating A from B. As (v^*, w^*) lies on the frontier of both sets, $(v^*, w^*) \in H$. Moreover, H must be tangent to B at (v^*, w^*), which implies that its slope is $-\frac{w^*}{v^*}$. Thus, H must be described by the equation

$$w = -\frac{w^*}{v^*}v + 2w^*.$$

This implies that $(2v^*, 0), (0, 2w^*) \in H$. However, for $(2v^*, 0)$ or $(0, 2w^*)$ not to lie in the interior of A, it must be true that $(v^*, w^*) \geq \frac{1}{2}$, as desired. ■

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