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An Extension of the Core Solution Concept

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A solution concept for cooperative games, the extended core, is introduced. This concept is always nonempty yet coincides with the core whenever it is nonempty. Moreover, a non-cooperative framework can generate the extended core. Every transferable utility game is associated with a two-player zero-sum non-cooperative game. The min-max values of the associated zero-sum games characterize when cooperative games have nonempty cores. If the core is empty, the min-max value determines how an exogenous regulator can impose costs on proper coalition formation so that there are no incentives to deviate from extended core imputations, which are necessarily feasible in the original game. In order to choose among the imputations belonging to the extended core, a proportional version of the nucleolus is proposed as a selection device.

1 Introduction

1.1 The Empty Core Problem

The core is by far the cooperative solution concept that is most frequently applied in economics. Numerous economic fields such as public economics,

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political economy and industrial organization to name a few, have successfully applied this solution concept. Nevertheless, there exist important economic situations which translate into environments where the core is empty. In such settings the core is not applicable.

The question of how to handle empty-core situations is of great importance because the number of games in which the core cannot be applied is considerable.[1] Still, the issue is frequently avoided by imposing enough assumptions so that the core of a game is nonempty. Despite including some of the best pieces of research in economic theory,[2] the cooperative game literature does not address the empty-core problem.

In the literature it is possible to find attempts of defining solution concepts that can be applied to empty-core situations. Within this class of concepts the main examples are the bargaining set (Aumann and Maschler 1964) [2] and the kernel (Davis and Maschler 1965) [4]. Although technically sound, the applicability of these alternative solution concepts is debatable. In very few situations, at least when compared against the core, they have been used as a tool to study economic fields outside game theory.

Two main reasons can account for this lack of applicability: First, these solution concepts are not suitable to build upon core applications (which constitute the vast majority of the literature where cooperative games have been utilized) because in games where the core is nonempty, they do not coincide with the core. In cases where the core is nonempty, inclusion of the core (with the bargaining set) or nonempty intersection with the core (with the kernel) are the best results available. Second, these alternative concepts lack the simplicity and intuitive appeal of the core. As a consequence, it is quite burdensome to actually compute these solution concepts, even for very simple examples.[3]

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[1] Bondareva (1963) [3] and Shapley (1967) [14] characterized these types of games.

[2] E.g. The Debreu-Scarf core convergence theorem, the Shapley and Shubik work on market games [16], and the asymptotic results on ϵ-cores, e.g. [20].

[3] According to [10] to calculate the bargaining set of a cooperative game with four players, it is necessary to solve 150 systems of linear inequalities.

1.2 An Alternative

In this project we provide an alternative to deal with empty-core environments: the extended core. The main characteristic of the extended core is that, unlike the core, it is never empty. Nevertheless, in cases where the core is nonempty both concepts coincide. Therefore, the extended core is a good candidate to generalize the numerous applications of the core to settings where the core is empty. Even when this generalization is not possible, we show how the tools that define the extended core can be used to define a notion that measures "how far" a game is from having a nonempty core.

After going through the formal definition of the extended core, we point out different interpretations of the solution concept. The extended core can be justified in three different ways, one of them based on tools taken from non-cooperative game theory and the other two arguing from the cooperative standpoint. This versatility intends to emphasize that the extended core is both simple and intuitive.

1.2.1 Non-cooperative Point of View

The challenge generated by the first concern (see Section 1.1, last paragraph) is to describe the core within a framework that is also applicable to games in which the core is empty. We achieve this goal by using non-cooperative game theory: We associate to every cooperative game (with a nonempty core) a non-cooperative two-player zero-sum game in a similar fashion to Aumann 1989 [1]. We identify Nash equilibrium strategies[4] for the row player with core vectors.[5] One of the main contributions of this work is to understand that exactly the same exercise can be performed with an empty core game. The set of vectors that result from this experiment conform the extended core.

[4] As we are talking about a zero-sum game, the term min-max strategies is more accurate.

[5] One of the most appealing features of the associated game is the ability to link the most important non-cooperative concept, Nash equilibrium, with the most important cooperative concept: the core.

1.2.2 Cooperative Point of View

Besides the theoretical interest of such a solution concept, there exist two natural economic interpretations of the extended core. The first one requires incrementing the aggregate payoff until the game eventually has a nonempty core. Then, the core vectors of this enlarged game are normalized so that they add up to the original aggregate payoff. The result of this procedure is precisely the extended core vectors. This exercise can be thought of as the set of players asking for a loan, choosing a core payoff vector, and then paying the loan according to the proportions established by the chosen payoff vector.

Our second interpretation of the extended core is based on finding a way so that coalitions have fewer incentives to block. This can be interpreted as incrementing coalition formation costs. With a sufficiently high level of costs every game eventually has a nonempty core and moreover, every payoff vector eventually belongs to the core. The extended core consists of those payoff vectors that require the minimum level of costs so that no coalition wants to block them.

1.3 Choosing a Single Outcome

As the extended core is a multiple valued concept, it is important to establish a criterion to select among its payoff vectors. In a similar way as the concept of nucleolus (Schmeidler 1969 [13]) can be used to select a particular payoff vector from the core, the proportional nucleolus (Young et. al. 1982 [21]), always chooses a payoff vector from the extended core. The nucleolus formalizes the idea of a fair distribution of output in the sense of choosing the payoff distribution that minimizes the biggest complaint by any coalition. The proportional nucleolus differs from the original nucleolus in the definition of complaint. This latter version of nucleolus is concerned with coalitions that suffer the biggest proportional loss (as opposed to absolute loss) of their worth.

2 Notation and Basic Definitions

For completeness, we start by defining some basic concepts in cooperative game theory.

2.1 Cooperative Games

Definition 2.1: Let $N = \{1, 2, \ldots, n\}$. A **coalition** is defined as any nonempty subset of N. A **cooperative game** is an ordered pair (N, v) where N is the set of agents[6] and $v : 2^N \longrightarrow \mathbf{R}_+$ is a function such that $v(\emptyset) = 0$. Let $\Gamma(N)$ be the set of all cooperative games that have N as their set of agents. For technical reasons throughout this paper it will be assumed that $v(N) > 0$.

Definition 2.2: An **imputation** is any vector $x \in \mathbf{R}_+^N$ such that $\sum_{i \in N} x_i = v(N)$.[7]

The **core** of a cooperative game (N, v) is the set of imputations which no coalition is able to improve upon. It will be denoted by $C(N, v)$. Rigorously,

$$C(N,v) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S)\ \forall S \subseteq N\}.$$

Not every game has a nonempty core. The following necessary and sufficient conditions for a game to have a nonempty core were given independently by Bondareva [3] and Shapley [14]:

Definition 2.3: A collection of coalitions $\{S_1, S_2, \ldots S_k\}$ is called **balanced** if there exist non-negative numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$\sum_{S_j \ni i} \lambda_j = 1 \qquad \forall i \in N.$$

The numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ are called balancing weights.

[6]We will refer to individuals who take part in a cooperative games as "agents". Individuals who take part in non-cooperative games will be called "players".

[7]Notice that in this setting an imputation is not necessarily individually rational.

Proposition 2.4: (Bondareva [3] / Shapley [14]) The core of a game (N, v) is nonempty if and only if for every balanced collection $\{S_1, S_2, \ldots S_k\}$ with weights $\lambda_1, \lambda_2, \ldots, \lambda_k$, the inequality

$$\sum_{j=1}^{k} \lambda_j v(S_j) \leq v(N)$$

holds.

While the Bondareva-Shapley theorem provides a tractable set of necessary and sufficient conditions for a game to have a nonempty core, it is obviously true that many games — in fact, a nonempty open set of n-player games in $\mathbf{R}^{2^n-1}$ — fail to satisfy the Bondareva-Shapley conditions and therefore necessarily have empty cores.

Definition 2.5: Given a game (N, v), its **cover** $(N, \bar{v})$ (see [16]) is a game defined by:

$$\bar{v}(S) \quad = \quad \max \sum_{j=1}^{k} \lambda_j v(S_j) \qquad \forall S \subseteq N$$

where the maximum is taken over all balanced collections of sub-coalitions of S, $\{S_1, S_2, \ldots S_k\}$ with corresponding weights $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Now we present some examples of games with both nonempty and empty cores. As we define new concepts, we will illustrate them in these examples.

2.1.1 Examples of Games with Nonempty Cores

For our first example, consider a setting with three agents. Agents 1 and 2 own a right glove while agent 3 owns a left glove. The worth of each coalition is either 0 or 1, depending on whether it can form a right-left pair of gloves. Formally, this situation can be modeled as a cooperative game $v : 2^{\{1,2,3\}} \longrightarrow \{0, 1\}$ defined by

$$
\begin{aligned}
v(\emptyset) &= 0 \\
v(i) &= 0 \qquad \forall \quad i \in \{1,2,3\} \\
v(1,2) &= 0 \\
v(1,3) = v(2,3) = v(1,2,3) &= 1
\end{aligned}
$$

For this game the core is nonempty. Indeed, it is easy to show that $C(N, v) = \{(0, 0, 1)\}$. We will refer to this game as the **gloves game**.

2.1.2 Examples of Games with Empty Cores

The second game will be generated by the example given in the motivation [11].

$$
\begin{aligned}
v(\emptyset) &= 0 \\
v(i) &= 7 \times 1 - 6 = 1 \qquad \forall \quad i \in \{1,2,3\} \\
v(i,j) &= 7 \times 2 - 9 = 5 \qquad \forall \quad i,j \in \{1,2,3\} \quad i \neq j \\
v(1,2,3) &= 7 \times 3 - 14 = 7
\end{aligned}
$$

For this game the core is empty. Indeed, any imputation that attains efficiency will not be stable as there will exist a two player coalition that will want to deviate. We will refer to this game as the **phone game**.

For our last example we use the classical game ***n* men and a trunk**. Here, n agents are stuck with a valuable trunk in the middle of the desert. On his own, no agent can carry the trunk out of the desert. Nevertheless, any group constituted by two or more of them is able to do so. The characteristic function generated by these payoffs is as follows:

$$
\begin{aligned}
v(S) &= 0 \text{ if } |S| \leq 1 \\
v(S) &= 1 \text{ if } |S| > 1
\end{aligned}
$$

For this game, the core is empty if $n > 2$. If $n = 2$, then the core will coincide with the set of all imputations.

2.2 The Associated Zero-Sum Non-cooperative Game

In this section we describe a two player non-cooperative zero-sum game $G(N, v)$ that is generated from any cooperative game (N, v). This zero-sum game was first used by Aumann [1] in order to generate an alternative proof of the Bondareva-Shapley theorem. Instead of dealing directly with the linear programming problem posed by the core, he translates the model into an environment where the duality results needed for the proof are embedded in the Min-Max Theorem for zero-sum games. This paper intends to further exploit the analogy between these two settings in order to give a non-cooperative support of the extended core.

In such zero-sum game, the row player chooses an agent and the column player chooses a coalition. The row player gets a strictly positive payoff if and only if the chosen agent belongs to the coalition selected by the column player. This payoff is inversely related to the worth of the coalition chosen by the second player. In any other case, players receive zero payoff.

<u>Definition 2.6:</u> Given any cooperative game (N, v) define the non-cooperative zero-sum game $G(N, v)$ by:

- **Set of Players:** $\{A, B\}$.
- **Strategy Sets:** $\mathbf{S}_A = N$ and $\mathbf{S}_B = \{S \subseteq N \mid v(S) > 0\}$.
- **Payoffs:**

$$u_A(i, S) = \begin{cases} \frac{v(N)}{v(S)} & \text{if} \quad i \in S \\ 0 & \text{if} \quad i \notin S \end{cases}$$

  and $u_B = -u_A$.

Notice that the restriction to coalitions such that $v(S) > 0$ is imposed just to avoid dividing by zero. All of the results go through if, for coalitions of worth equal to zero, $u_A(i,S)$ is replaced by a large enough number M.

Examining the payoff matrices of the associated games that correspond to our examples gives the following results: For the gloves game the payoff matrix of $G(N,v)$ is

A \ B	{ 1,3 }	{ 2,3 }	{ 1,2,3 }
1	(1,-1)	(0,0)	(1,-1)
2	(0,0)	(1,-1)	(1,-1)
3	(1,-1)	(1,-1)	(1,-1)

Similarly, the payoff matrix corresponding to the phone game

A \ B	{ 1 }	{ 2 }	{ 3 }	{ 1,2 }	{ 1,3 }	{ 2,3 }	{ 1,2,3 }
1	(7,-7)	(0,0)	(0,0)	$(\frac{7}{5},-\frac{7}{5})$	$(\frac{7}{5},-\frac{7}{5})$	(0,0)	(1,-1)
2	(0,0)	(7,-7)	(0,0)	$(\frac{7}{5},-\frac{7}{5})$	(0,0)	$(\frac{7}{5},-\frac{7}{5})$	(1,-1)
3	(0,0)	(0,0)	(7,-7)	(0,0)	$(\frac{7}{5},-\frac{7}{5})$	$(\frac{7}{5},-\frac{7}{5})$	(1,-1)

Finally, the payoff matrix for three men and a trunk is

A \ B	{ 1,2 }	{ 1,3 }	{ 2,3 }	{ 1,2,3 }
1	(1,-1)	(1,-1)	(0,0)	(1,-1)
2	(1,-1)	(0,0)	(1,-1)	(1,-1)
3	(0,0)	(1,-1)	(1,-1)	(1,-1)

As the game at hand is zero-sum, any Nash equilibrium (mixed) strategy profile always generates the same expected payoffs. Thus, the associated game can be assigned a value in the same way that was done by von-Neumann and Morgenstern [19].

Definition 2.7: Let the pair of strategies (σ^*, τ^*) be any Nash equilibrium of the zero-sum game $G(N,v)$. Define $\omega(N,v) = u_A(\sigma^*, \tau^*)$. The Min-Max Theorem guarantees that the value $\omega(N,v)$ is well defined.

After some arithmetic, $\omega(N,v)$ can be calculated for each of our examples. In the gloves game it is evident that a Nash equilibrium occurs when player A chooses agent 3 and player 2 selects the grand coalition. Thus, $\omega(N,v) = 1$.

In the phone game, consider the Nash equilibrium in which player A gives weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ to her pure strategies and player B gives the same weights to the coalitions $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$. Then, the expected utility for the first player is equal to $\omega(N,v) = \frac{7}{5} \times \frac{2}{3} = \frac{14}{15}$.

For n men and a trunk, the unique Nash equilibrium occurs when the first player gives equal weights $\frac{1}{n}$ to all agents and the second player gives equal weights $\frac{2}{n(n-1)}$ to all coalitions of cardinality two. In such instance, the expected utility for the first player is equal to $\omega(N,v) = (n-1) \times \frac{2}{n(n-1)} = \frac{2}{n}$.

Notice that the fact that $0 < \omega(N,v) \leq 1$ for every example is not a coincidence. First, as the first player's expected payoff is always strictly positive, $\omega(N,v)$ must also be. Second, if things come to worst, player B can always limit her opponent's payoff to 1 by choosing the grand coalition. This reasoning proves our first result.

Proposition 2.8: For any cooperative game (N,v), $\omega(N,v)$ belongs to the interval $(0,1]$.

3 Relationship Between (N,v) and $G(N,v)$

3.1 Nash Equilibria and Core Imputations

We begin by characterizing what kind of associated games are generated by cooperative games with a nonempty core. The first result in this direction is used in Aumann's alternative proof of the Bondareva-Shapley Theorem [1].

Proposition 3.1: (Aumann) Let (N,v) be any cooperative game. If the value $\omega(N,v)$ of the associated game is equal [8] to one, then the core of the original game is not empty.

An example to illustrate this proposition is the gloves game. For this game, $\omega(N,v) = 1$ and the core is nonempty. The same works for the game of two men and a trunk. The fact that our examples with an empty core have a value strictly less than one suggests that the converse of Proposition 3.1 holds. In fact, this is true, as stated formally in the following proposition:

[8] In the original [1] this result is stated with $\omega(N,v) \geq 1$ instead of $\omega(N,v) = 1$

Proposition 3.2: If the core of a cooperative game (N, v) is nonempty, then the value $\omega(N, v)$ of the associated game is equal to one.

The method used to prove Propositions 3.1 and 3.2 is constructive. To prove Proposition 3.1, a core imputation is assigned to any Nash strategy corresponding to player A. Conversely, the proof for Proposition 3.2 shows how to generate a Nash strategy for player 1 starting from any core imputation. Thus, we can say even more about the relationship between (N, v) and $G(N, v)$. When $\omega(N, v) = 1$, there is a natural bijection between core imputations and Nash strategy profiles for player A.

In symbols, given an imputation x define a strategy profile σ for player A by

$$\sigma_i = \frac{x_i}{\sum_{j \in N} x_j} \quad \forall i \in N$$

Then,

$$x \in C(N, v) \text{ iff } \exists \tau \text{ s.t. } (\sigma, \tau) \text{ is a Nash eqm. of } G(N, v)$$

We proceed to examine this relationship in the particular context of the nonempty core examples previously described. In the zero-sum game associated with the gloves game, it is clear that it is optimal for player A to choose agent 3 in a Nash equilibrium. In the same way, the unique core imputation of the cooperative game is $(0, 0, 1)$. In the game two men and a trunk, any imputation of the form $(a, 1 - a)$ belongs to the core for $a \in [0, 1]$. In turn, it is clear that no matter what mixed strategy player A uses, it will be part of a Nash equilibrium.

3.2 Cooperative Interpretation of $\omega(N, v)$

The fact that a cooperative game (N, v) has an empty core is not an impediment to associate it with it a zero-sum game $G(N, v)$ and a quantity $\omega(N, v)$. Various natural questions arise. Are the Nash strategy profiles in $G(N, v)$ related to some relevant set of imputations of the cooperative game?

If so, does this set preserve, in some sense, the stability properties of the core? How can $\omega(N, v)$ be interpreted in the context of the cooperative game (N, v)? In what follows we attempt to answer each of the previous questions.

Definition 3.3: Given a game (N, v) and a real number $k \geq 1$, define the **k-expanded game** $(N, \hat{v}_k)$ in the following way:

$$\begin{aligned} \hat{v}_k(S) &= v(S) \quad \text{if } S \neq N \\ \hat{v}_k(N) &= kv(N) \end{aligned}$$

Define also

$$\hat{k}(N, v) = \min\{k \geq 1 \mid C(N, \hat{v}_k) \neq \emptyset\}.$$

Intuitively, consider any cooperative game (N, v). If the worth of the grand coalition is multiplied by a number greater than one, eventually the core will become nonempty. Then $\hat{k}(N, v)$ is defined as the smallest number k that does the trick. Proposition 2.4 and Definition 2.5 imply that $\hat{v}_{\hat{k}}(N)$ is exactly equal to $\bar{v}(N)$, the worth the grand coalition takes in the cover game $(N, \bar{v})$. This characterization gives an easy way to check that $\hat{k}(N, v)$ is well defined.[9] Furthermore, $\hat{v}_{\hat{k}}(N)$ also coincides with the concept that Zhao [22] denotes as minimum no-blocking payoff (MNBP). In that particular paper he shows that the core has a nonempty relative interior if and only if $\hat{k}(N, v) = 1$.

Notice that $\hat{k}(N, v)$ is defined using concepts that do not refer at all to the associated zero-sum game. Nevertheless, it will be closely related to $\omega(N, v)$. Indeed the following proposition formalizes this fact.

Proposition 3.4: For any cooperative game (N, v), $\omega(N, v) = \frac{1}{\hat{k}(N,v)}$.

The key to the proof of Proposition 3.4 is to relate the cooperative game to the non-cooperative environment. This is achieved by characterizing $\omega(N, \hat{v}_k)$, the expected utility of player A in the associated zero-sum game corresponding to the k-expanded game.

[9] $\hat{k}(N, v)$ is bounded above by $\sum_{S \subseteq N} v(S)$.

Lemma 3.5: For any number $k \geq 1$, the associated value of the k-expanded game $\omega(N, \hat{v}_k)$ is equal to the minimum between one and the product of k times the associated value of the original game $\omega(N, v)$.

The intuition used in the proof of this lemma is as follows. Suppose that (σ^*, τ^*) is a Nash equilibrium of $G(N, v)$. The interesting case occurs when $u_A(\sigma^*, \tau^*) < 1$. The only difference between the zero-sum games $G(N, v)$ and $G(N, \hat{v}_k)$ is that when player B does not use the grand coalition the payoffs are multiplied by k. If $ku_A(\sigma^*, \tau^*) = k\omega(N, v)$ is still less than one, then (σ^*, τ^*) is also a Nash equilibrium of $G(N, \hat{v}_k)$ and $\omega(N, \hat{v}_k) = k\omega(N, v)$. Otherwise, player B chooses the grand coalition so that $\omega(N, \hat{v}_k) = 1$.

4 The Extended Core

The previous section shows how, if the core is nonempty, imputations in the core are generated from Nash equilibrium profiles for player A. But, even when the cooperative game has an empty core, it can still be associated with a zero-sum game. The described non-cooperative framework now becomes useful because it is also applicable to cooperative games that have an empty core. The analogous to the core in this setting is the set of imputations generated by Nash equilibrium profiles for player A. We call it the extended core.

Definition 4.1: For any game (N, v), define its **extended core** $EC(N, v)$ as the set

$$\{x \in \mathbf{R}^n_+ \ | \ \sum_{i \in N} x_i = v(N) \text{ and } \exists \tau^* \text{ s.t. } (\frac{x}{v(N)}, \tau^*) \text{ is a Nash eqm. of } G(N, v)\}$$

Of course, in the particular case in which the core is nonempty, Propositions 3.1 and 3.2 imply that the extended core and the core coincide. In this sense, the extended core is a generalization of the core with a very important feature:

Proposition 4.2: The extended core is nonempty for any game (N, v).

Up to now the extended core is a mathematical concept with desirable properties but no economic intuition. The first step towards interpreting the extended core in a cooperative setting is to characterize it without referring to the associated zero-sum game. The following proposition shows how the extended core of a cooperative game (N, v) can be found directly:

Proposition 4.3: To calculate directly the extended core of a cooperative game (N, v), first find the smallest k-expanded game with a nonempty core and then normalize the imputations in its core multiplying them by $\omega(N, v)$. More concisely, for any game (N, v), $EC(N, v) = \omega(N, v)C(N, \hat{v}_{\hat{k}(N,v)})$.

An objection could be raised about the fact that this concept is defined by changing the original worths of coalitions somewhat arbitrarily. This argument can be contested by the fact that both $\omega(N, v)$ and $\hat{k}(N, v)$ can be expressed, using Proposition 2.4, in terms of the original worths of the game (N, v) by the formula

$$\frac{1}{\omega(N, v)} = \hat{k}(N, v) = \max_{\mathcal{B}} \sum_{S \in \mathcal{B}} \lambda_S v(S)$$

where the maximum ranges over all balanced families of coalitions $\mathcal{B}$ with corresponding weights $\{\lambda_S\}_{S \in \mathcal{B}}$.

Notice that expanding the worth $v(N)$ of the grand coalition by a factor of $\hat{k}(N, v)$ is strategically equivalent to shrinking the worths of all proper coalitions $S \subsetneqq N$ by a factor of $\frac{1}{\hat{k}(N,v)} = \omega(N, v)$.[10] This observation leads to yet another cooperative interpretation of the extended core.

Proposition 4.4: The game that results from multiplying the worth of all proper coalitions by $\omega(N, v)$ has a nonempty core and it coincides with the extended core of the original game.

Suppose that an economic situation (e.g. the phone game) generates a game in which for every imputation there exists a coalition with incentives to deviate, leading to an inefficient outcome. If multiplying by $\omega(N, v)$ is interpreted as imposing costs (or taxes) on proper coalition formation, the previous proposition describes a method for an exogenous regulatory institution

[10] If we look at the games as $2^{|N|}$-dimensional vectors, one is a scalar multiple of the other

to attain a stable and efficient situation. In other words, if proper coalition formation is taxed a fraction $1-\omega(N,v)$ of its worth, it is impossible to block imputations in the extended core. Furthermore $1-\omega(N,v)$ is the smallest (least coercive) tax that is required to achieve this objective.

This kind of proportional taxation has been previously mentioned in the literature. Tijs and Driessen [18] propose to tax proportionally the surplus $v(S) - \sum_{i\in S} v(\{i\})$ generated by creating coalition S. The extended core coincides with this concept when the worth of all singleton coalitions is zero (e.g. if the original game is zero-one normalized.) Faigle and Kern [6] also propose to take away a fraction of the worth of proper coalitions and apply the concept to operation research topics such as traveling salesman games.

4.1 Applying the Extended Core to Empty Core Games

We proceed to calculate the extended core of the examples that have been studied. Consider n men and a trunk for $n \geq 3$. Recall from previous calculations that

$$\begin{aligned} \omega(N,v) &= \frac{2}{n} \\ C(N,\hat{v}_{\frac{1}{\omega(N,v)}}) &= \{(\frac{1}{2},\ldots,\frac{1}{2})\} \end{aligned}$$

Hence, we conclude that $EC(N,v) = \{\frac{2}{n}(\frac{1}{2},\ldots,\frac{1}{2})\} = \{(\frac{1}{n},\ldots,\frac{1}{n})\}$. This result seems to be plausible considering the symmetry of the game.

Now consider the phone game. In this particular case

$$\begin{aligned} \omega(N,v) &= \frac{14}{15} \\ C(N,\hat{v}_{\frac{1}{\omega(N,v)}}) &= \{(\frac{5}{2},\ldots,\frac{5}{2})\} \end{aligned}$$

The previous data imply that $EC(N,v) = \{(\frac{7}{3},\frac{7}{3},\frac{7}{3})\}$.

Contrary to what the previous examples might suggest, the extended core is not always a singleton. Any game with a multiple-valued core is an example of this. Moreover, even when the extended core is single valued, it will generally not coincide with the Shapley value. An example that illustrates these points is the following:

$$\begin{array}{ccccc} v(1)=5 & v(2) & = & 2 & v(3)=2 \\ v(12)=5 & v(13) & = & 5 & v(23)=5 \\ & v(123) & = & 10 & \end{array}$$

Here the extended core is multiple valued (it actually coincides with the core) and is given by $C(N,v) = \{(5, x, 5-x) \in \mathbf{R}^3 \mid x \in [2,3]\}$. The Shapley Value of this game is $(4\frac{1}{3}, 2\frac{5}{6}, 2\frac{5}{6})$ so it is not included in the extended core. Even if we increase $v(2) = v(3) = 2\frac{1}{2}$ so that the extended core is single valued, the Shapley Value will not coincide with the extended core.

4.2 How Far is a Game from Having a Nonempty Core?

The parameter $\omega(N,v)$ is useful to determine if the game (N,v) is near or far from having a nonempty core. Intuitively, a game with a higher $\omega(N,v)$ than another needs less costs to generate unblocked imputations, and so can be considered closer to have a nonempty core. Moreover, the difference between the associated values can be used to tell if two games are near or far from each other. In what follows we formalize these ideas, imposing a metric-like structure over the space of games.

Notice that assigning numbers $\omega(N,v) \in (0,1]$ to any game (N,v) induces an equivalence relation on $\Gamma(N)$, the set of all cooperative games that have N as their set of agents. The equivalence class corresponding to $\omega(N,v) = 1$ is precisely the set of games with nonempty cores.

Definition 4.5: Define the equivalence relation $R \subseteq \Gamma(N) \times \Gamma(N)$ as follows: For any two games $(N,v_1),(N,v_2) \in \Gamma(N)$ it is true that $((N,v_1),(N,v_2)) \in$

R if and only if $\omega(N, v_1) = \omega(N, v_2)$. The equivalence classes in which $\Gamma(N)$ is partitioned are of the form

$$\Gamma_\alpha(N) := \{(N, v) \in \Gamma(N) \mid \alpha = 1 - \omega(N, v)\}$$

The definition of $\Gamma_\alpha(N)$ uses $\alpha = 1 - \omega(N, v)$ (as opposed to $\alpha = \omega(N, v)$) so that the parameter α can be used to decide how far is a game from having a nonempty core.

Definition 4.6: Define the function $d : \Gamma(N)/R \times \Gamma(N)/R \longrightarrow [0, 1]$ as follows: For any $\alpha_1, \alpha_2 \in [0, 1]$, let $d(\Gamma_{\alpha_1}(N), \Gamma_{\alpha_2}(N)) = |\alpha_1 - \alpha_2|$.

Proposition 4.7: The function $d : \Gamma(N)/R \times \Gamma(N)/R \longrightarrow [0, 1]$ satisfies all the properties of a distance function.

The proof of this proposition is straightforward. We have thus proposed a well defined concept of distance between equivalence classes of games.

4.3 How Far is an Imputation from Being in the Core?

Suppose that an exogenous regulator wants to impose a particular imputation x on the agents of the cooperative game. A question of economic interest is: What is the minimum level of taxes needed so that no coalition has incentives to deviate from x? The non-cooperative framework can also be used to answer this question. Using tools that are very similar to those used in Subsection 4.2, now we impose a metric-like structure over the space of imputations of a given game.

Definition 4.8: (Analogous to Definition 2.7) Let (N, v) be a game with $G(N, v)$ and u_A as previously defined. Let x be an imputation of (N, v). Define

$$\omega(x) = \min_S u_A(\frac{x}{v(N)}, S).$$

Definition 4.9: (Analogous to the last part of Definition 3.3) Let (N, v) be a game. Given any imputation x, define

$$\hat{k}(x) = \min\{k \geq 1 \mid x \in \frac{1}{k}C(N, \hat{v}_k)\}$$

Proposition 4.10: (Analogous to Proposition 3.4) For any imputation x, $\omega(x) = \frac{1}{\hat{k}(x)}$.

Definition 4.11: (Analogous to Definition 4.5) Let $\gamma(N, v)$ denote the set of imputations of the game (N, v). Define the equivalence relation $R' \subseteq \gamma(N, v) \times \gamma(N, v)$ as follows: For any two imputations $x_1, x_2 \in \gamma(N, v)$, it holds that $(x_1, x_2) \in R'$ if and only if $\omega(x_1) = \omega(x_2)$. The equivalence classes in which $\gamma(N, v)$ is partitioned are of the form

$$\gamma_\alpha(N, v) := \{x \in \gamma(N, v) \mid \alpha = 1 - \omega(x)\}$$

Definition 4.12: (Analogous to Definition 4.6) Define the function $d : \gamma(N, v)/R' \times \gamma(N, v)/R' \longrightarrow [0, 1]$ as follows: For any $\alpha_1, \alpha_2 \in [0, 1]$, let $d(\gamma_{\alpha_1}(N, v), \gamma_{\alpha_2}(N, v)) = |\alpha_1 - \alpha_2|$.

Proposition 4.13: (Analogous to Proposition 4.7) The function $d : \gamma(N, v)/R' \times \gamma(N, v)/R' \longrightarrow [0, 1]$ satisfies all the properties of a distance function.

5 The Proportional Nucleolus

The purpose of this section is to provide criteria for choosing a particular imputation from all those that belong to the extended core. This line of research has already been explored when the question is how to pick one among all core imputations, giving rise to the nucleolus. We can solve the extended core selection problem by using a modified version of this solution concept.

The original nucleolus chooses the particular payoff distribution that, in some sense, minimizes protests of all coalitions. Given an imputation, different coalitions will protest (or be satisfied) with their corresponding shares. One way to model the different level of this complaints is with the concept of excess, i.e. how much more can a coalition generate by itself compared to the share assigned to it.

<u>Definition 5.1:</u> Let S be a coalition and x an imputation in a given cooperative game (N, v). Define the **<u>excess</u>** of coalition S with respect to imputation x by

$$e(S, x) = v(S) - \sum_{i \in S} x_i$$

Select (from a given set of payoff vectors) all imputations that minimize the magnitude of the protest of the unhappiest coalition. Among those, select all imputations that minimize the magnitude of the protest of the second most unhappy coalition. The set of imputations that is left after iterating this process $2^{|N|} - 1$ times is the nucleolus. Intuitively, minimizing the maximum protest implies, in some sense, minimizing all of the protests. No coalition can be too unsatisfied, so the payoff vectors in the nucleolus are "fair" for all.

<u>Definition 5.2:</u> (Schmeidler [13]) Given a game (N, v) and a fixed imputation x, let $\theta(x)$ be the $2^{|N|} - 1$ dimensional vector that arranges all possible excesses generated by x in decreasing order. If X is a nonempty set of imputations, define the **<u>nucleolus</u>** of X $\mathcal{N}(X)$ as the set

$$\mathcal{N}(X) = \{x \in X \mid \theta(x) \leq_L \theta(y) \quad \forall y \in X\}$$

where $\leq_L$ denotes the lexicographic order. Finally, define the nucleolus of the game (N, v) as the set $\mathcal{N}(N, v)$ equal to the nucleolus of the set of imputations of the game.[11]

Moreover, the nucleolus satisfies the following:

<u>Proposition 5.3:</u> (Schmeidler [13]) The nucleolus $\mathcal{N}(N, v)$ of a game satisfies:

[11] Notice that according to this definition the nucleolus of a game can contain imputations that are not individually rational. This concept is also known as prenucleolus.

- $\mathcal{N}(N, v)$ is nonempty.
- $\mathcal{N}(N, v)$ is single-valued.
- If the core $C(N, v)$ is nonempty, $\mathcal{N}(N, v)$ belongs to the core.

The previous properties justify that, if an imputation needs to be selected from the core, the nucleolus is a good choice. Now, we go back to the question of how to choose from the set of imputations of the extended core. We claim that changing the definition of excess in a suitable manner solves the problem.

To give intuition about how the excess definition needs to be changed we will go back to the non-cooperative framework previously discussed. The next proposition gives an interpretation of the behavior of the column player in the associated game $G(N, v)$. Indeed, it is optimal for agent B to choose coalitions that suffer the biggest proportional loss given the payoff vector generated by strategy profile chosen by player A.

Proposition 5.4: Given a cooperative game (N, v), let (σ^*, τ^*) be a Nash equilibrium of the associated game $G(N, v)$. If $\tau^*(S) > 0$ then the vector x defined by $x_i = \sigma_i^* v(N) \ \forall i \in N$ satisfies

$$\frac{v(S) - \sum_{i \in S} x_i}{v(S)} \geq \frac{v(T) - \sum_{i \in T} x_i}{v(T)} \quad \forall T \text{ s.t. } v(T) > 0.$$

In this context, it is more important what percentage of the original worth is lost than the actual magnitude of the loss. Suppose, for the sake of example, a game (N, v) such that two coalitions S and T satisfy $v(S) = 4$ and $v(T) = 100$. If a payoff vector takes away two units from both coalitions, according to the conventional notion of excess they will complain equally. In contrast, according to the proportional excess, S protest is far more important because S is loosing fifty percent of its worth (compared with a two percent loss of T.) To rigorously define a notion of proportional excess, we will assume that for every coalition S we have that $v(S) > 0$ to avoid division by zero.

Definition 5.5: Let S be a coalition of positive worth and x an imputation in a given cooperative game (N, v). Define the **proportional excess** of coalition S with respect to imputation x by

$$\hat{e}(S, x) = \frac{v(S) - \sum_{i \in S} x_i}{v(S)}.$$

Definition 5.6: Given a game (N, v) and a fixed imputation x, let $\hat{\theta}(x)$ be the $2^{|N|}-1$ dimensional vector that arranges all possible proportional excesses generated by x in decreasing order. If X is a nonempty set of imputations, define the **proportional nucleolus** of X $\hat{\mathcal{N}}(X)$ as the set

$$\hat{\mathcal{N}}(X) = \{x \in X \mid \hat{\theta}(x) \leq_L \hat{\theta}(y) \ \forall y \in X\}$$

where $\leq_L$ denotes the lexicographic order. Finally, define the proportional nucleolus of the game (N, v) as the set $\hat{\mathcal{N}}(N, v)$ equal to the proportional nucleolus of the set of imputations of the game.

The proportional nucleolus is originally attributed to Young, Okada and Hashimoto [21], and has been used in solving cost allocation problems (see [9].) If the game is not assumed to be strictly positive, single-valuedness is impossible to obtain. For non-negative games, a similar but multiple-valued concept called the nucleon is described in [7]. This paper deals with the computational issues of the iterative process used to trim down the original set of imputations. With the strictly positive assumption on coalitions' worths we have:

Proposition 5.7: The proportional nucleolus $\hat{\mathcal{N}}(N, v)$ of a strictly positive game satisfies:

- $\hat{\mathcal{N}}(N, v)$ is nonempty.
- $\hat{\mathcal{N}}(N, v)$ is single-valued.
- $\hat{\mathcal{N}}(N, v)$ always belongs to the extended core.
- If the core $C(N, v)$ is nonempty, $\hat{\mathcal{N}}(N, v)$ belongs to the core.

Thus, the proportional nucleolus is a sensible solution to the extended core selection problem. This ability to select an imputation is another advantage of the extended core as a solution concept.

6 Relation to the ε-core

The ε-core was first suggested by Shapley and Shubik in [15] and numerous variants have been defined in subsequent research. Most of the literature[12] has focused on obtaining asymptotic results when the number of agents is big. In this section we compare the proportional tax interpretation of the extended core with the lump-sum taxation that occurs in the ε-core. First, we proceed to review the definitions of the two main variations of ε-core starting with the more restrictive. My reference for these definitions is [8].

Definition 6.1: Given a cooperative game (N, v), the **strong ε-core** of (N, v) is given by

$$\{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) - \varepsilon \quad \forall S \subseteq N\}.$$

and the **weak ε-core** of (N, v) is given by

$$\{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) - |S|\varepsilon \quad \forall S \subseteq N\}.$$

The (weak) inclusion chain formed by these concepts is given as follows:

$$\text{Core } \subseteq \text{Weak } \frac{\varepsilon}{n}\text{-core} \subseteq \text{Strong } \varepsilon\text{-core} \subseteq \text{Weak } \varepsilon\text{-core}$$

To show that (any of the variants of) the ε-core is essentially different from the extended core, we examine the following game and check that it is impossible to approximate the extended core by set containment between two arbitrarily close (weak or strong) ε-cores.

[12] e.g. see [20].

Example: Consider the cooperative game defined as follows:

$$\begin{array}{rcrcl} v(1)=1 & & v(2) & = & 2 & & v(3)=3 \\ v(12)=7 & & v(13) & = & 8 & & v(23)=9 \\ & & v(123) & = & 10 & & \end{array}$$

For this game, computing the value yields $\omega(N,v)=\frac{5}{6}$. The extended core of this game is then the singleton $\{(\frac{15}{6},\frac{20}{6},\frac{25}{6})\}$. On the other hand, consider the strong ε-core. Of course, if ε is big enough, at some point the extended core (and actually any imputation) will be included in the strong ε-core. Nevertheless, we claim that no nonempty (weak or strong) ε-core is contained in the extended core of this game.

Indeed, the smallest ε for which there is a nonempty strong ε-core is equal to $\frac{4}{3}$. Similarly, the smallest ε for which there is a nonempty weak ε-core is equal to $\frac{2}{3}$. In either case, the strong or weak ε-core is equal to $\{(\frac{7}{3},\frac{10}{3},\frac{13}{3})\}$. This means that any nonempty weak or strong ε-core will include $(\frac{7}{3},\frac{10}{3},\frac{13}{3})$, making it impossible that a nonempty ε-core is contained in the extended core. Furthermore, if $\varepsilon<\frac{3}{2}$ (respectively if $\varepsilon<\frac{3}{4}$) the strong ε-core (respectively weak ε-core) is disjoint from the extended core.

In subsection 4.1 a taxing procedure to restore stability of efficient payoff vectors was proposed. An important reason to prefer the extended core over the ε-cores is that the latter may generate negative worths of coalitions after taxation. In this example, to take away $\varepsilon=\frac{4}{3}$ from $v(1)=1$ would imply to give this singleton coalition a worth of $-\frac{1}{3}$. This problem will not occur with the extended core because in that case the quantity taken away from a coalition is always a fraction of its worth.

7 Open Questions

If the extended core is generalized to the case of games with nontransferable utility, the concept may be applied to economic situations in which existence of competitive equilibrium is not a given. Non-convexities, externalities, and public goods come to mind. It would be very interesting to compare the method used by the extended core to restore efficiency with the solutions already proposed by the literature. In particular, is there any relationship between Pigouvian taxation and the kind of taxes we have studied?

A vast literature[13] has studied the branch of the Nash Program that refers to the non-cooperative implementation of the core. As a natural extension of the core, there is a good chance that such implementations can be modified for the extended core.

Two important objections have been made against the concept of the core. The first one is that, although it has been axiomatized, usually the set of axioms only refers to the subset of games that have a nonempty core (see [12].) The second objection is that the core assumes that the grand coalition always forms. We foresee the possibility of making one good thing out of two bad ones. From my point of view, the problem lies in requiring a Pareto optimality axiom (this is equivalent to the second objection) in games with an empty core. A concept that besides payoff vectors explains coalition formation can deal with both objections simultaneously. The present work might play a role in this project by relating optimal strategies for player B (in the associated game) with those coalitions that are most likely to form.

[13] For a survey on the subject, see [17].

Appendix: Proofs

Proposition 2.8: $\omega(N, v) \leq 1$ for any (N, v).

Proof: If (σ^*, τ^*) is a Nash equilibrium of $G(N, v)$ then

$$\omega(N, v) = u_A(\sigma^*, \tau^*) \leq u_A(\sigma^*, N) = 1.$$

Q.E.D.

Proposition 3.1: $\omega(N, v) = 1$ implies that $C(N, v) \neq \emptyset$.

Proof: Let (σ^*, τ^*) be any Nash equilibrium of $G(N, v)$. We show that the imputation defined by $x^* = v(N)\sigma^*$ belongs to $C(v, N)$.[14] Notice that for any $S \subseteq N$ such that $v(S) > 0$, it is true that[15]

$$1 = u_A(\sigma^*, \tau^*) \leq u_A(\sigma^*, S) = \sum_i \sigma^*(i)\chi_S(i)\frac{v(N)}{v(S)}.$$

Then

$$x^*(S) = \sum_{i \in N} \sigma^*(i)\chi_S(i)v(N) \geq v(S),$$

so that $x^* \in C(N, v)$. **Q.E.D.**

Proposition 3.2: $C(N, v) \neq \emptyset$ implies that $\omega(N, v) = 1$.

Proof: Let $x^* \in C(N, v)$ and (σ^*, τ^*) be a Nash equilibrium of $G(N, v)$. We prove that $(\frac{x^*}{v(N)}, \tau^*)$ is also a Nash equilibrium of $G(N, v)$ and that the expected utility for player A for this Nash strategy profile is equal to one.

As $x^* \in C(N, v)$, then for any $S \subseteq N$,

[14] x^* is defined by distributing the worth of the grand coalition according to the weights given by player A to the different strategies.

[15] In what follows χ_S denotes the indicator function of the set S.

$$
\begin{aligned}
x^*(S) &\geq v(S) \\
\sum_{i\in N} \chi_S(i) x_i^* &\geq v(S) \\
\sum_{i\in N} \chi_S(i) \frac{x_i^*}{v(N)} \cdot \frac{v(N)}{v(S)} &\geq 1 \\
\text{and } u_A(\frac{x^*}{v(N)}, S) &\geq 1,
\end{aligned}
$$

but this implies that

$$\omega(N, v) = u_A(\sigma^*, \tau^*) \geq u_A(\frac{x^*}{v(N)}, \tau^*) \geq 1.$$

Now, Proposition 2.8 implies $\omega(N, v) \leq 1$, so that the previous inequalities are binding. Finally, as $u_A(\frac{x^*}{v(N)}, \tau^*) = 1$, $(\frac{x^*}{v(N)}, \tau^*)$ is a Nash equilibrium of $G(N, v)$. **Q.E.D.**

Lemma 3.5: (Used to prove Proposition 3.4) For any $k \geq 1$, $\omega(N, \hat{v}_k) = \min\{1, k\omega(N, v)\}$.

Proof: Denote the expected payoff functions of the game $G(N, \hat{v}_k)$ by $\hat{u}_A$ and $\hat{u}_B$. Using the Min-Max Theorem, the following holds:

$$
\begin{aligned}
\min\{1, k\omega(N, v)\} &= \min\{1, k \min_{\tau} \max_{\sigma} u_A(\sigma, \tau)\} \\
&= \min\{1, k \min_{\tau(N)=0} \max_{\sigma} u_A(\sigma, \tau)\} \\
&= \min\{1, \min_{\tau(N)=0} \max_{\sigma} k u_A(\sigma, \tau)\} \\
&= \min\{1, \min_{\tau(N)=0} \max_{\sigma} \hat{u}_A(\sigma, \tau)\} \\
&= \min\{\min_{\tau(N)>0} \max_{\sigma} \hat{u}_A(\sigma, \tau), \min_{\tau(N)=0} \max_{\sigma} \hat{u}_A(\sigma, \tau)\} \\
&= \min_{\tau} \max_{\sigma} \hat{u}_A(\sigma, \tau) \\
&= \omega(N, \hat{v}_k)
\end{aligned}
$$

Q.E.D.

Proposition 3.4: For any game (N, v), $\omega(N, v) = \frac{1}{\hat{k}(N,v)}$.

Proof: Using the previous propositions and the Lemma 3.5,

$$\begin{aligned}
\hat{k}(N, v) &= \min\{k \geq 1 \mid C(N, \hat{v}_k) \neq \emptyset\} \\
&= \min\{k \geq 1 \mid \omega(N, \hat{v}_k) = 1\} \\
&= \min\{k \geq 1 \mid \min\{1, k\omega(N, v)\} = 1\} \\
&= \min\{k \geq 1 \mid k\omega(N, v) \geq 1\} \\
&= \frac{1}{\omega(N, v)}
\end{aligned}$$

which clearly implies the result. **Q.E.D.**

Proposition 4.2: The extended core is nonempty for any game (N, v).

Proof: Non-emptiness follows directly from the Min-Max Theorem. **Q.E.D.**

Proposition 4.3: For any game (N, v), $EC(N, v) = \omega(N, v)C(N, \hat{v}_{\hat{k}})$.

Proof: Let (σ^*, τ^*) be a Nash equilibrium of $G(N, v)$. Assume that $\tau^*(N) = 0$; otherwise $C(N, v) \neq \emptyset$ and the result follows. Denote the expected payoff function of player A in $G(N, \hat{v}_{\frac{1}{w}})$ and (N, v) by $\hat{u}_A$ and u_A respectively. Let x be an imputation of (N, v). Then

$$\begin{aligned}
x \in EC(N,v) &\iff u_A(\frac{x}{v(N)}, \tau^*) = \omega(N,v) \\
&\iff \frac{1}{\omega(N,v)} u_A(\frac{x}{v(N)}, \tau^*) = 1 \\
&\iff \hat{u}_A(\frac{x}{v(N)}, \tau^*) = 1 \\
&\iff \hat{v}_{\hat{k}}(N) \cdot \frac{x}{v(N)} \in C(N, \hat{v}_{\hat{k}}) \\
&\iff \hat{k} v(N) \cdot \frac{x}{v(N)} \in C(N, \hat{v}_{\hat{k}}) \\
&\iff x \in \omega(N,v) C(N, \hat{v}_{\hat{k}}).
\end{aligned}$$

Q.E.D.

Proposition 4.4: The game that results from multiplying the worth of all proper coalitions by $\omega(N,v)$ has a nonempty core and it coincides with the extended core of the original game.

Proof: Start with the $\hat{k}$-expanded game (which has a nonempty core) and multiply the worth of every coalition by $\omega(N,v)$. The new game, which coincides with the game that results after taxing proper coalitions, must still have a nonempty core. Proposition 4.3 ensures that the core of the taxed game is exactly the extended core. **Q.E.D.**

Proposition 4.7: The function $D : \Gamma(N)/R \times \Gamma(N)/R \longrightarrow [0,1]$ satisfies all the properties of a distance.

Proof: The distance properties of D are directly inherited from the distance properties of the Euclidean distance on $[0,1]$. **Q.E.D.**

Proposition 4.10: (Analogous to Proposition 3.4) For any imputation $x \in \gamma(N,v)$, $\omega(x) = \frac{1}{\hat{k}(x)}$.

Proof: Let $x \in \gamma(N,v)$. The definition of $\hat{k}(x)$ implies that:

$$\begin{aligned}
\sum_{i \in S} x_i &\geq \frac{1}{\hat{k}(x)} v(S) \qquad \forall S \subsetneqq N \\
\frac{1}{v(S)} \sum_{i \in S} x_i &\geq \frac{1}{\hat{k}(x)} \qquad \forall S \subsetneqq N \\
u_A(\frac{x}{v(N)}, S) &\geq \frac{1}{\hat{k}(x)} \qquad \forall S \subsetneqq N \\
\min_S u_A(\frac{x}{v(N)}, S) &\geq \frac{1}{\hat{k}(x)}
\end{aligned}$$

and the last inequality has to be binding, otherwise $\hat{k}(x)$ would not be minimal. **Q.E.D.**

Proposition 4.13: (Analogous to Proposition 4.7) The function $d : \gamma(N)/R \times \gamma(N)/R \longrightarrow [0,1]$ satisfies all the properties of a distance.

Proof: The distance properties of d are directly inherited from the distance properties of the Euclidean distance on $[0,1]$. **Q.E.D.**

Proposition 5.4: Given a cooperative game (N, v), let (σ^*, τ^*) be a Nash equilibrium of the associated game $G(N, v)$. If $\tau^*(S) > 0$ then the vector x defined by $x_i = \sigma_i^* v(N) \ \forall i \in N$ satisfies

$$\frac{v(S) - \sum_{i \in S} x_i}{v(S)} \geq \frac{v(T) - \sum_{i \in T} x_i}{v(T)} \quad \forall T \text{ s.t. } v(T) > 0.$$

Proof: If $\tau^*(S) > 0$ then for any coalition T that has positive worth

$$\begin{aligned}
u_A(\sigma^*, S) &\leq u_A(\sigma^*, T) \\
\sum_{i \in S} \sigma_i^* \frac{v(N)}{v(S)} &\leq \sum_{i \in T} \sigma_i^* \frac{v(N)}{v(T)} \\
\frac{\sum_{i \in S} x_i}{v(S)} &\leq \frac{\sum_{i \in T} x_i}{v(T)} \\
\frac{v(S) - \sum_{i \in S} x_i}{v(S)} &\geq \frac{v(T) - \sum_{i \in T} x_i}{v(T)}
\end{aligned}$$

as we wanted. **Q.E.D.**

Proposition 5.4: The nucleolus $\mathcal{N}(N, v)$ of a game satisfies:

- $\mathcal{N}(N, v)$ is nonempty.
- $\mathcal{N}(N, v)$ is single-valued.
- If the core $C(N, v)$ is nonempty, $\mathcal{N}(N, v)$ belongs to the core.

Proof: See Schmeidler [13].

Proposition 5.7: The proportional nucleolus $\hat{\mathcal{N}}(N, v)$ of a game satisfies:

- $\hat{\mathcal{N}}(N, v)$ is nonempty.
- $\hat{\mathcal{N}}(N, v)$ is single-valued.
- $\hat{\mathcal{N}}(N, v)$ always belongs to the extended core.

Proof: The fact that the game is positive implies that the proportional excess function $\hat{e}(S, x)$ is continuous on both arguments. This allows the arguments used by Schmeidler to be applicable in this setting.

For the last claim, notice that the proportional nucleolus $\hat{\mathcal{N}}(N, v)$ is a solution to the following equivalent problems:

$$\begin{gathered}
\min_x \max_S \hat{e}(S, x) \\
\max_x \min_S -\hat{e}(S, x) \\
\max_x \min_S \frac{\sum_{i \in S} x_i}{v(S)} \\
\max_\sigma \min_S \sum_{i \in S} \sigma_i \frac{v(N)}{v(S)} \\
\max_\sigma \min_S u_A(\sigma, S) \\
\max_\sigma \min_\tau u_A(\sigma, \tau)
\end{gathered}$$

so that player A is optimizing when his strategy profile is based on the proportions in which the nucleolus distributes $v(N)$. This is precisely the definition of belonging to the extended core. **Q.E.D.**

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