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University of Copenhagen

03-40

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Volatility-induced Growth in Financial Markets*

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August 13, 2003

Abstract

We show that the volatility of prices, which is usually regarded as an impediment for financial growth, may serve as a cause of it.

JEL classification: G11

1. Can price volatility, which is present in virtually every financial market and usually thought of as a risky investment's downside, serve as an "engine" of financial growth? Paradoxically, the answer to this question turns out to be positive.

To demonstrate this paradox, we examine the long-run performance of constant proportions investment strategies in financial markets. Such strategies prescribe to rebalance the investor's portfolio, depending on price fluctuations, so as to keep fixed proportions of wealth invested in all the portfolio positions. It is assumed that asset returns form stationary ergodic processes and asset prices grow (or decrease) at a common rate ρ. It is shown that if an investor employs a constant proportions strategy, then the value of his or her portfolio grows almost surely at a rate strictly greater than ρ, provided the investment proportions are strictly positive and the stochastic price process is in a sense non-degenerate. The very mild assumption of non-degeneracy we impose requires some randomness, or volatility, of the price process. If this assumption is violated, then the market is essentially deterministic, and

*Financial support by the National Centre of Competence in Research "Financial Valuation and Risk Management" (NCCR FINRISK) is gratefully acknowledged. The NCCR FINRISK is a research program supported by the Swiss National Science Foundation.

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the result ceases to hold. Thus, in the present context, the price volatility appears to be an endogenous source of acceleration of financial growth. This phenomenon might seem counterintuitive, especially in stationary markets [1], where the asset prices themselves, and not only their returns, are stationary. In this case, $\rho = 0$, i.e. each asset grows at zero rate, while any constant proportions strategy exhibits growth at a strictly positive exponential rate with probability one.

The effect highlighted in this article can be demonstrated in the framework of the conventional, well-studied models of financial markets. Constant proportions strategies are involved in many practical financial computations, cf. [4]. However, to our knowledge, the phenomenon examined here has not been clearly described and systematically investigated in the literature.[1] The aim of the present note is to fill this gap.

2. Consider a financial market with $K \geq 2$ *securities* (*assets*). Let $p_t = (p_t^1, ..., p_t^K)$ be the vector of security prices at time $t \in \{0, 1, 2, ...\}$. Assume that $p_t^k > 0$ for each t and k, and denote by

$$R_t^k = \frac{p_t^k}{p_{t-1}^k} \; (k \in \{1, 2, ..., K\}, \; t \in \{1, 2, ...\}) \tag{1}$$

the (gross) *return on asset* k over the time period $(t-1, t]$. Define $R_t = (R_t^1, ..., R_t^K)$.

At each time period t, an investor chooses a *portfolio* $h_t = (h_t^1, ..., h_t^K)$, where h_t^k is the number of units of asset k in the portfolio h_t. A sequence $H = \{h_0, h_1, ...\}$, specifying a portfolio at each time t, is called a *trading strategy.* Let $\lambda = (\lambda^1, ..., \lambda^K)$ be a vector such that

$$\lambda^k \geq 0 \; (k \in \{1, 2, ..., K\}) \text{ and } \sum_{k=1}^{K} \lambda^k = 1. \tag{2}$$

A trading strategy H is called a *constant proportions strategy* with vector of proportions $\lambda = (\lambda^1, ..., \lambda^K)$ if

$$p_t^k h_t^k = \lambda^k p_t h_{t-1} \; (k \in \{1, 2, ..., K\}, \; t \in \{1, 2, ...\}). \tag{3}$$

If $\lambda^k > 0$ for each k, then H is said to be *completely mixed.* The scalar product $p_t h_{t-1} = \sum_{k=1}^{K} p_t^k h_{t-1}^k$ expresses the value of the portfolio h_{t-1} in terms of the prices p_t^k at time t. According to relation (3), the amounts $p_t^k h_t^k$ invested in

[1] The only reference we can indicate in this connection is the book by Luenberger [3], Chapter 15.2, where some examples related to the topic under consideration are discussed.

assets $k = 1, 2, ..., K$ are proportional to $\lambda^1, ..., \lambda^K$. It is immediate from (2) and (3) that a constant proportions strategy is *self-financing*:

$$p_t h_t = p_t h_{t-1},\ t \in \{1, 2, ...\}. \tag{4}$$

We fix λ and H satisfying (2) and (3) and denote by $V_t = p_t h_t$ the value of the portfolio h_t at time $t \in \{0, 1, 2, ...\}$ expressed in terms of the current prices p_t^k. It will be assumed that $V_0 > 0$, which implies $V_t > 0$ for each t (see formula (6) below). We will suppose that the price vectors p_t, and hence the return vectors R_t, are random, i.e., they evolve in time as stochastic processes. Then the trading strategy $h_t,\ t \in \{0, 1, 2, ...\}$, generated by the investment rule (3) and the value $V_t = p_t h_t,\ t \in \{0, 1, 2, ...\}$, of the portfolio h_t are stochastic processes as well. We are interested in the asymptotic behavior of V_t as $t \to \infty$.

3. We will assume:
(R) The vector stochastic process R_t is stationary and ergodic. The expected values $E|\ln R_t^k|$ are finite.

It follows from (R) that $p_t^k = p_0^k R_1^k ... R_t^k$, where the random sequence R_t^k is stationary. This assumption on the structure of the price process is a fundamental hypothesis widely accepted in mathematical finance (in particular, it lies in the basis of the famous Black-Scholes formula, see e.g. [3]). By virtue of Birkhoff's ergodic theorem, we have

$$\lim_{t\to\infty} \frac{1}{t} \ln p_t^k = \lim_{t\to\infty} \frac{1}{t} \sum_{j=1}^{t} \ln R_j^k = E \ln R_t^k \tag{5}$$

almost surely (a.s.) for each $k \in \{1, 2, ..., K\}$. This means that each asset k has a well-defined and finite *(exponential) growth rate*, which turns out to be equal to the expectation $E \ln R_t^k$. This expectation can be positive, zero or negative. It does not depend on t in view of the stationarity of R_t.

In addition to (R), we will assume that all the assets under consideration have *the same* growth rate:
(R1) There exists a number ρ such that, for each $k \in \{1, ..., K\}$, we have $E \ln R_t^k = \rho$.

This assumption allows to concentrate, for example, on those assets in the market that grow at the maximum rate. It is natural to suppose that all the others, growing slower, will eventually be driven out of the market. As long as we deal with an infinite time horizon, we can exclude such assets from consideration.

4. The main results are presented in the following theorem.

Theorem 1. *(i) The growth rate* $\lim_{t\to\infty}(1/t)\ln V_t$ *of a constant proportions strategy with vector of proportions* $\lambda \geq 0$ *is equal to* $E\ln(R_t\lambda)$ *(a.s.).*

(ii) Suppose all the coordinates λ^k *of the vector* λ *are strictly positive, i.e. the strategy under consideration is completely mixed. Let the following condition hold:*

(V) With strictly positive probability,

$$p_t^k/p_t^m \neq p_{t-1}^k/p_{t-1}^m \text{ for some } 1 \leq k, m \leq K \text{ and } t \geq 1.$$

Then the growth rate $E\ln(R_t\lambda)$ *of the constant proportions strategy is strictly greater than* ρ*.*

(iii) If condition (V) does not hold, then $E\ln(R_t\lambda) = \rho$*.*

Since the process R_t is stationary, the expectation $E\ln(R_t\lambda)$ involved in the statement of the theorem does not depend on t. Condition (V) means that, for at least one moment of time t, the ratio p_t^k/p_t^m of the prices of at least two assets k and m is not the same as the analogous ratio at the previous moment of time $t-1$. This is a very mild assumption of volatility of the price process. If this assumption does not hold, then the price ratios of all the assets are constant over time (a.s.) and, for each t, the return $R_t^k = p_t^k/p_{t-1}^k$ on each asset k is equal to one and the same number, α_t (a.s.).

In the context of Theorem 1, the volatility of the price process appears to be the only cause for any completely mixed constant proportions strategy to grow at a rate strictly greater than ρ – the growth rate of each particular asset. This result looks at first glance unexpected, since the volatility of asset prices is usually regarded as an impediment for financial growth, while here it serves as an "engine" of it. In a stationary market, where the process p_t (and not only R_t) is ergodic and stationary and where $E|\ln p_t^k| < \infty$, the growth rate of each asset is zero,

$$E\ln R_t^k = E\ln p_t^k - E\ln p_{t-1}^k = 0,$$

while any completely mixed constant proportions strategy grows at a strictly positive exponential rate.

Common intuition suggests that if the market is stationary, then the portfolio value V_t for a constant proportions strategy must converge in one sense or another to a stationary process. The common argument in support of this conclusion appeals to the self-financing property (4). This property seems to exclude possibilities of unbounded growth. The truth, however, lies in the opposite direction: unbounded exponential growth is not only compatible with self-financing, but is characteristic for any completely mixed constant proportions strategy.

Our result bears some similarity with the concept of asymptotic arbitrage, see e.g. [2]. Three features however stand out: growth is exponentially fast, unbounded wealth is achieved with probability one, and the effect of growth is demonstrated for specific (constant proportions) strategies. None of these properties can directly be deduced from asymptotic arbitrage.

5. *Proof of Theorem 1.* (i) We have

$$V_t = p_t h_t = \sum_{m=1}^{K} p_t^m h_{t-1}^m = \sum_{m=1}^{K} \frac{p_t^m}{p_{t-1}^m} p_{t-1}^m h_{t-1}^m =$$

$$\sum_{m=1}^{K} \frac{p_t^m}{p_{t-1}^m} \lambda^m p_{t-1} h_{t-1} = V_{t-1} \sum_{m=1}^{K} R_t^m \lambda^m = (R_t\lambda) V_{t-1}.$$

Thus

$$V_t = V_0 (R_1\lambda)(R_2\lambda)...(R_t\lambda), \tag{6}$$

and so

$$\lim_{t\to\infty} \frac{1}{t} \ln V_t = \lim_{t\to\infty} \frac{1}{t} \sum_{j=1}^{t} \ln(R_j\lambda) = E \ln(R_t\lambda) \text{ (a.s.)}, \tag{7}$$

which proves the first assertion of the theorem.

(ii) Observe that condition (V) is equivalent to the following one:

(V1) For some $t \geq 1$ (and hence for each $t \geq 1$), the probability

$$P\{R_t^k \neq R_t^m \text{ for some } 1 \leq k, m \leq K\}$$

is strictly positive.

Indeed, we have $p_t^k/p_t^m \neq p_{t-1}^k/p_{t-1}^m$ if and only if $p_t^k/p_{t-1}^k \neq p_t^m/p_{t-1}^m$, which can be written as $R_t^k \neq R_t^m$. Denote by δ_t the random variable that is equal to 1 if the event $\{R_t^k \neq R_t^m$ for some $1 \leq k, m \leq K\}$ occurs and 0 otherwise. Condition (V) means that $P\{\max_{t\geq 1} \delta_t = 1\} > 0$, while (V.1) states that, for some t (and hence for each t)), $P\{\delta_t = 1\} > 0$. The latter property is equivalent to the former because

$$\{\max_{t\geq 1} \delta_t = 1\} = \cup_{t=1}^{\infty} \{\delta_t = 1\}.$$

By using Jensen's inequality and (V1), we find that

$$\ln \sum_{k=1}^{K} R_t^k \lambda^k > \sum_{k=1}^{K} \lambda^k (\ln R_t^k)$$

with strictly positive probability, while the non-strict inequality holds always. Consequently,

$$E\ln(R_t\lambda) > \sum_{k=1}^{K} \lambda^k E(\ln R_t^k) = \rho,$$

which proves (ii).

(iii) Suppose assumption (V) does not hold. Then, as has been noted above, the return $R_t^k = p_t^k/p_{t-1}^k$ on each asset k is equal to the same number a_t (a.s.). In this case, $E\ln(R_t\lambda) = E\ln\alpha_t = E\ln R_t^k = \rho$.

The proof is complete.

References

[1] I.V. Evstigneev and K.R. Schenk-Hoppé, From rags to riches: On constant proportions investment strategies, International Journal of Theoretical and Applied Finance, **5** (2002), 563-573.

[2] I. Klein and W. Schachermayer, A quantitative and a dual version of the Halmos-Savage theorem with applications to mathematical finance, Annals of Probability **24** (1996), 867-881.

[3] D. Luenberger, Investment Science, Oxford University Press, New York, 1998.

[4] A.F. Perold and W.F. Sharpe, Dynamic strategies for asset allocation, Financial Analysts Journal **51**(1) (1995), 149-160.