DISCUSSION PAPERS
Institute of Economics
University of Copenhagen

03-39

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Evolutionary Stable Stock Markets*

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October 27, 2003

Abstract

This paper shows that a stock market is evolutionary stable if and only if stocks are evaluated by expected relative dividends. Any other market can be invaded by portfolio rules that will gain market wealth and hence change the valuation. In the model the valuation of assets is given by the wealth average of the portfolio rules in the market. The wealth dynamics is modelled as a random dynamical system. Necessary and sufficient conditions are derived for the evolutionary stability of portfolio rules when (relative) dividend payoffs form a stationary Markov process. These local stability conditions lead to a unique evolutionary stable strategy according to which assets are evaluated by expected relative dividends.

JEL-Classification: G11, D52, D81.
Keywords: evolutionary finance, portfolio theory, incomplete markets.

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*We are grateful to Jarrod Wilcox and William Ziemba for very valuable comments. Financial support by the national center of competence in research "Financial Valuation and Risk Management" is gratefully acknowledged. The national centers in research are managed by the Swiss National Science Foundation on behalf of the federal authorities.

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1 Introduction

The expected discounted dividends model is one of the cornerstones of finance. According to this model the rational and fair value of common stocks is given by the discounted sum of future dividends paid out by the company. Indeed in the very long run (half a century) the trend of stock market prices coincides with the trend of the dividends paid by the companies. Yet over shorter horizons (even for decades) stock market prices can considerably deviate from their dividend fundamentals. This phenomenon, called excess volatility, was first pointed out by Shiller (1981). While models based on complete rationality have difficulties to cope with excess volatility, models based on adaptive behavior typically go to the other extreme and generate too irregular price dynamics. The model considered in this paper suggest a solution in between these two extremes.

We consider a stock market with a heterogenous population of portfolio rules. In the model rationality is important on the level of the market since market selection may ultimately give pressure for selecting the rational portfolio rules. It turns out that only a rational market in which assets are evaluated by expected relative dividends is evolutionary stable. Any other market can be invaded by portfolio rules that will gain market wealth and hence change the valuation. While, as in De Long, Shleifer, Summers, and Waldmann (1990), rational strategies clearly face the risk that there are too many irrational strategies, any set of irrational strategies is however more easily turned over by invasion of even a small fraction of slightly different strategies. That is to say, every now and then the market can be displaced from its rational valuation by a *big* push of irrationality but eventually the market selection pressure will lead the market back to the rational valuation because from any irrational market there exists a sequence of *small* and *nearby* innovations leading back to the rational market. This stability property may be the explanation that on long-term averages stock markets look quite rational while severe departures are possible in the short- and medium-term.

In a sense our results give support to a long hold conjecture by Friedman (1953) and Fama (1965) who argued that the market naturally selects for rational strategies, which, in effect, would lead to market efficiency. However, our paper also makes clear that the mutation force has to be added to the selection argument in order to prove this conjecture. Considering only the market selection process, the economy can get stuck at any situation in which all traders use the same portfolio rule. Moreover, our paper shows which types of mutant strategies can successfully invade which type of market. For example, an irrational market can be turned over by invading strategies that are not themselves the rational strategy. An invasion of the rational strategy may fail in an irrational market.

To make these ideas precise we study an incomplete asset market where a finite number of portfolio rules manage capital by iteratively reinvesting in a fixed set of long-lived assets. In every period assets pay dividends according to the realization of a stationary Markov process in discrete time. In addition to the exogenous wealth increase due to dividends, portfolio rules face endogenously determined capital gains or losses. Portfolio rules are encoded as non-negative vectors of expenditure shares for assets. The set of portfolio rules considered is not restricted to those generated by expected utility maximization. It may as well include investment rules favored by behavioral finance models. Indeed any portfolio rule that is adapted to the information filtration is allowed in our framework.

Portfolio rules compete for market capital that is given by the total value of all assets in every period in time. The endogenous price process provides a *market selection mechanism* along which some strategies gain market capital while others lose. We give a description of the market selection process from a random dynamical systems perspective. In each period in time the evolution of the distribution of market capital, i.e. the wealth shares of the portfolio rules as percentages of total market wealth, is given by a map that depends on the exogenous process determining the asset payoffs. An equilibrium in this model is provided by a distribution of wealth shares across portfolio rules that is invariant under the market selection process. Provided there are no redundant assets every invariant distribution of market shares is generated by a *monomorphic* population, i.e. all traders with strictly positive wealth use the same portfolio rule at such an equilibrium. A criterion for evolutionary stability as well as evolutionary instability is derived for monomorphic populations. Roughly speaking a portfolio rule is evolutionary stable if it has the highest exponential growth rate in any population where itself determines market prices. This implies that an evolutionary stable investment strategy is robust against the entry of new portfolio rules. In a sense an evolutionary stable population plays the "best response against itself."

The stability criterion for the robustness of invariant distributions with respect to the entry of new portfolio rules singles out one portfolio rule, denoted λ^*, that is the unique *evolutionary stable strategy*, i.e. it drives out any mutation. Moreover, any other investment strategy can successfully be invaded by a slightly changed strategy. According to this rule one should divide wealth proportionally to the expected relative dividends of the assets. An explicit formula for this rule is given—applicable in actual markets.

The effect of this rule on asset prices is equalization of assets' expected relative returns—in particular asset pricing is *log-optimal* in the sense of

Luenberger (1997, Chapter 15). It is well known that log-optimal pricing is obtained if all investors have logarithmic von-Neumann–Morgenstern utilities (Kraus and Litzenberger 1975). Hence the portfolio rule λ^* could also be obtained as the outcome of an idealized market with a single representative agent having rational expectations. For a market selection model based on rational expectations see Blume and Easley (2001) and Sandroni (2000). Our paper shows that an idealized market with rational expectations could be justified by evolutionary reasoning.

One implication of our main results is that a rational market is evolutionary stable while an irrational market is evolutionary unstable. In particular we show that any irrational market can already be destabilized by small changes in the existing strategies. A further implication of our evolutionary stability results is that among all proportional investment strategies only λ^* can be a candidate for a rule that starting from any initial distribution of wealth obtain total market wealth in the long-run in competition with any set of other portfolio rules. Indeed, global stability of the λ^* rule has recently been proved for the case of short-lived assets (Evstigneev, Hens, and Schenk-Hoppé 2002). Simulations with simple strategies show that also with long-lived assets λ^* is the unique portfolio rule which among all simple strategies is able to gather total market wealth (Hens, Schenk-Hoppé, and Stalder 2002). An analytical proof of this finding is still warranted.

Our approach is related to the classical finance approach to maximize the expected logarithm of the growth rate of relative wealth for some exogenously given return process. From this perspective we show which portfolio rule maximizes the expected logarithm of the growth rate of wealth in a model with *endogenously* determined returns. Following the work by Kelly (1956) and Breiman (1961), Hakansson (1970), Thorp (1971), Algoet and Cover (1988), and Karatzas and Shreve (1998), among others, have explored this maximum growth perspective. Computing the maximum growth portfolio is a stochastic non-linear programming problem. Even if one restricts attention to i.i.d. returns, when markets are incomplete, with more than two assets, there is no explicit solution to this investment problem in general. To overcome this problem, numerical algorithms to compute the maximum growth portfolio have been provided by Algoet and Cover (1988) and Cover (1984, 1991). Our result is interesting also in this respect because the simple portfolio rule that we obtain shows that considering the equilibrium consequences of this maximization does not make matters more complicated but rather much easier. Indeed, as mentioned above, the portfolio rule λ^* can be characterized as the unique portfolio rule that maximizes the logarithm of the growth rate of relative wealth in a population in which the rule itself determines the returns. Note however, that applying λ^* does not require the solution of any optimization problem. It is the rationality of the market selection and mutation process that makes the simple strategy λ^* a smart strategy.

The next section presents the economic model which has the mathematical structure of a random dynamical system. The model is based on Lucas (1978)'s infinite horizon asset market model with long-lived assets and a single perishable consumption good. In this model we introduce heterogenous portfolio rules that are adapted to the information filtration, and we study the resulting sequence of short run equilibria. In section 3 we define the long run equilibrium concepts and different stability notions. In particular we define invariant distributions of relative wealth and show that those are characterized by monomorphic populations, i.e. an invariant distribution of relative wealth arise if and only if all investors use the same portfolio rule. Then we define evolutionary stability of invariant distributions of relative wealth as those being robust to the innovation of new strategies. Section 4 contains the main result. Section 5 concludes.

2 An Evolutionary Stock Market Model

This section introduces an infinite horizon asset market model with long-lived assets and a single perishable consumption good, as in the seminal paper Lucas (1978).

There are $K \geq 1$ long-lived assets and cash. Time is discrete and denoted by $t = 0, 1, \ldots$. Each asset $k = 1, ..., K$ pays off a dividend per share at the beginning of every period and before trade takes place in this period. $D_t^k \geq 0$ denotes the total dividend paid to all shareholders of asset k at the beginning of period t. Assume that $\sum_k D_t^k > 0$.[1] D_t^k depends on the history of states of the world $\omega^t = (..., \omega_0, ..., \omega_t)$ where $\omega_t \in S$ is the state revealed at the beginning of period t. For technical convenience (and without loss of generality) we assume infinite histories. S is assumed to be finite, and every state is drawn with some strictly positive probability.

Dividend payoffs are in cash. Cash is only used to buy consumption goods—in particular it cannot be used to store value. Assets are issued at time 0. The initial supply of every asset k, s_0^k, is normalized to 1. At any period in time the supply remains constant: $s_t^k = s_0^k$. The supply of cash s_t^0 is given by the total dividends of all assets.

There are finitely many portfolio rules (also referred to as investment strategies) indexed by $i = 1, ..., I$, $I \geq 2$, each is pursued by an investor. The portfolio rule of investor i is a time- and history-dependent proportional

[1] This assumption avoids "dead" periods in which no dividends are paid.

strategy, denoted by $\lambda_t^i(\omega^t) = (\lambda_{t,k}^i(\omega^t))_{k=0,...,K}$ with $0 \leq \lambda_{t,k}^i(\omega^t) \leq 1$ for all k and $\sum_{k=0}^K \lambda_{t,k}^i(\omega^t) = 1$. For each $k \geq 1$, $\lambda_{t,k}^i(\omega^t)$ is the fraction of the wealth investor i assigns to the purchase of the risky asset k in period t, while $\lambda_{t,0}^i(\omega^t)$ is the fraction of wealth held in cash. Investment strategies are distinct across investors.[2]

In the following discussion we assume that everything is well-defined. In particular prices are assumed to be strictly positive. Sufficient conditions are provided after the full derivation of the model.

For a given portfolio rule $\lambda_t^i(\omega^t)$ and wealth w_t^i, the portfolio purchased by investor i at the beginning of period t is

$$\theta_{t,k}^i = \frac{\lambda_{t,k}^i(\omega^t)\, w_t^i}{p_t^k} \qquad k = 0, 1, ..., K. \tag{1}$$

$\theta_{t,0}^i$ is the units of cash and $\theta_{t,k}^i$ is the units of assets held by investor i. Since we have normalized the supply of the long-lived assets to 1, $\theta_{t,k}^i$ is the percentage of all shares issued of asset k that investor i purchases. p_t^k denotes the market clearing price of asset k in period t. We normalize the price for cash $p_t^0 = 1$ in every period t. The price of the consumption good is also the numeraire.

For any portfolio holdings of agents $(\theta_t^i)_{i=1,...,I}$ the market equilibrium conditions for cash and long-lived assets are

$$\sum_{i=1}^I \theta_{t,k}^i = s_t^k, \qquad k = 0, ..., K, \tag{2}$$

where the supply of the risky assets is $s_t^k = 1$, while the supply of cash is

$$s_t^0 = \sum_{k=1}^K D_t^k(\omega^t) > 0 \tag{3}$$

with strict positivity by the assumption that at least one asset pays a dividend.

The budget constraint of investor i in every period $t = 0, 1, ...$

$$\sum_{k=0}^K p_t^k\, \theta_{t,k}^i = w_t^i \tag{4}$$

is fulfilled because the fractions $\lambda_{t,k}^i(\omega^t)$, $k = 0, ..., K$, sum to one, see (1).

[2]The case of investors pursuing the same portfolio rule can be handled as follows: Investors with the same strategy set up a fund with claims equal to their initial share.

Since the consumption good is perishable, the wealth of investor i (in terms of the price of the consumption good) at the beginning of period $t+1$ and after dividends are payed is

$$w_{t+1}^i = \sum_{k=1}^{K} (D_{t+1}^k(\omega^{t+1}) + p_{t+1}^k)\, \theta_{t,k}^i. \tag{5}$$

Wealth can change over time because of dividend payments and capital gains. Since the cash $\theta_{t,0}^i$ held by every investor is consumed, the amount of cash available in any one period stems only from the current's period dividend payments.

The market-clearing price p_t^k for the risky assets ($k \geq 1$) can be derived from (2) by inserting (1). One finds

$$p_t^k = \sum_{i=1}^{I} \lambda_{t,k}^i(\omega^t)\, w_t^i = \lambda_{t,k}(\omega^t)\, w_t \tag{6}$$

where $\lambda_{t,k} = (\lambda_{t,k}^1, ..., \lambda_{t,k}^I)$ and $w_t^T = (w_t^1, ..., w_t^I)$.

Inserting (1) and (6) in (5) yields

$$w_{t+1}^i = \sum_{k=1}^{K} \left(D_{t+1}^k(\omega^{t+1}) + \lambda_{t+1,k}(\omega^{t+1})\, w_{t+1}\right) \frac{\lambda_{t,k}^i(\omega^t)\, w_t^i}{\lambda_{t,k}(\omega^t)\, w_t} \tag{7}$$

This is an implicit equation for the wealth of each investor i, w_{t+1}^i, for a given distribution of wealth w_t across investors. Define

$$A_t^i = \sum_{k=1}^{K} D_{t+1}^k(\omega^{t+1}) \frac{\lambda_{t,k}^i(\omega^t)\, w_t^i}{\lambda_{t,k}(\omega^t)\, w_t}, \quad \text{and} \quad B_t^{i,k} = \frac{\lambda_{t,k}^i(\omega^t)\, w_t^i}{\lambda_{t,k}(\omega^t)\, w_t} \tag{8}$$

The time index refers to the dependence on wealth: A_t^i and $B_t^{i,k}$ both depend on the wealth in period t. (7) can now be written as

$$w_{t+1}^i = A_t^i + \sum_{k=1}^{K} B_t^{i,k}\, \lambda_{t+1,k}(\omega^{t+1})\, w_{t+1} \tag{9}$$

and thus

$$w_{t+1} = A_t + B_t\, \Lambda_{t+1}(\omega^{t+1})\, w_{t+1} \tag{10}$$

where $\Lambda_{t+1}(\omega^{t+1})^T = (\lambda_{t+1,1}(\omega^{t+1})^T, ..., \lambda_{t+1,K}(\omega^{t+1})^T) \in \mathbb{R}^{I \times K}$ is the matrix of portfolio rules, and $B_t \in \mathbb{R}^{I \times K}$ is the matrix of portfolios in period t. $A_t^T = (A_t^1, ..., A_t^I) \in \mathbb{R}^I$ are the dividends payments, and $B_t\, \Lambda_{t+1}(\omega^{t+1})\, w_{t+1}$ are the capital gains.

Solving the linear equation (9) gives an explicit law of motion governing the distribution of wealth across strategies. One has

$$w_{t+1} = \left[\mathrm{Id} - B_t\, \Lambda_{t+1}(\omega^{t+1})\right]^{-1} A_t \tag{11}$$

(assuming existence of the inverse matrix) with Id being the identity matrix in $\mathbb{R}^{I\times I}$. The following result ensures that the evolution of wealth (11) is well-defined.

The next two assumptions are imposed throughout the following.

(A.1) Every investor consumes in every period but less than his entire wealth, i.e. $0 < \lambda^i_{t,0}(\omega^t) < 1$ *for all* i, t *and* ω^t.

(A.2) There is at least one investor with a complete-diversification strategy, i.e. there is a j *such that* $\lambda^j_{t,k}(\omega^t) > 0$ *for all* $k = 1, ..., K$, t *and* ω^t.

Proposition 1 *Suppose* $w_0 > 0$, *(A.1) holds, and (A.2) is satisfied for some investor with* $w^j_0 > 0$. *Then the evolution of wealth (11) is well-defined in all periods in time. Moreover, for every* $i = 1, ..., I$, $w^i_t > 0$ *if and only if* $w^i_0 > 0$.

Proof of Proposition 1. It suffices to prove the following: Suppose $w_t > 0$, (A.1) holds, and (A.2) is satisfied for some investor with $w^j_t > 0$. Then (11) is well-defined, $w_{t+1} > 0$, and, moreover, $w^i_{t+1} > 0$ if and only if $w^i_t > 0$ for every $i = 1, ..., I$.

We show first that the matrix $C := \mathrm{Id} - B_t\, \Lambda_{t+1}(\omega^{t+1})$ is invertible by proving that it has a column dominant diagonal Murata (1977, Corollary p. 22). C has entries

$$C_{jj} = 1 - \sum_{k=1}^{K} \bar{\lambda}^j_k \,\frac{\lambda^j_k w^j}{\lambda_k w} \quad \text{and} \quad C_{ij} = -\sum_{k=1}^{K} \bar{\lambda}^j_k \,\frac{\lambda^i_k w^i}{\lambda_k w} \quad (i \neq j)$$

on the diagonal and off-diagonal, respectively, where $\bar{\lambda}^i_k = \lambda^i_{t+1,k}(\omega^{t+1})$, $\lambda^i_k = \lambda^i_{t,k}(\omega^t)$, and $w = w_t$ for notational ease. All entries are well-defined because prices $\lambda_k w \geq \lambda^j_k w^j > 0$ (for some j) by our assumption.

The condition for a column dominant diagonal is in particular satisfied, if for every $j = 1, ..., I$,

$$|C_{jj}| > \sum_{i \neq j} |C_{ij}|. \tag{12}$$

Off-diagonal entries are non-positive, i.e. $C_{ij} \leq 0$ for $i \neq j$. The diagonal elements are strictly positive, i.e. $C_{jj} > 0$, since $0 \leq \lambda^j_k w^j/(\lambda_k w) \leq 1$ and therefore

$$C_{jj} \geq 1 - \sum_{k=1}^{K} \bar{\lambda}_k^j = 1 - (1 - \bar{\lambda}_0^j) = \bar{\lambda}_0^j > 0$$

according to assumption (A.1).

Thus (12) is equivalent to,

$$1 > \sum_{i=1}^{I} \sum_{k=1}^{K} \bar{\lambda}_k^j \frac{\lambda_k^i w^i}{\lambda_k w}. \tag{13}$$

Since the right-hand side of the last equation is

$$\sum_{k=1}^{K} \bar{\lambda}_k^j \sum_{i=1}^{I} \frac{\lambda_k^i w^i}{\lambda_k w} = \sum_{k=1}^{K} \bar{\lambda}_k^j = 1 - \bar{\lambda}_0^j$$

and $\bar{\lambda}_0^j > 0$ by assumption, (13) holds true. Thus C is invertible.

The matrix C has strictly positive diagonal entries and non-positive off-diagonal entries. Thus, Murata (1977, Theorem 23, p. 24) ensures that $w_{t+1} \geq 0$ if $A_t \geq 0$ (see (8) for the definition of A_t). Clearly, $A_t \geq 0$ if $w_t \geq 0$. This implies $\lambda_{t+1,k}(\omega^{t+1})\, w_{t+1} \geq 0$ for all k.

Let (A.2) hold for investor j and let $w_t^j > 0$. Investor $j's$ portfolio is completely diversified, i.e. $\theta_{t,k}^j > 0$ for all k. Thus, $A_t^j > 0$ because at least one asset pays a strictly positive dividend. Equation (7) implies, together with the above result that prices in period $t+1$ are non-negative, $w_{t+1}^j > 0$.

By assumption (A.2) this finding implies $\lambda_{t+1,k}^j(\omega^{t+1}) > 0$ for all k. Since for each investor with $w_t^i > 0$, $B_{t,k}^i > 0$ for some k, (7) further implies that $w_{t+1}^i > 0$ for every investor with $w_t^i > 0$. Obviously, $w_{t+1}^i = 0$ if $w_t^i = 0$. This completes the proof. □

Proposition 1 ensures that the evolution of the wealth distribution on $\mathbb{R}_+^I$ is well-defined: for given w_t, (11) yields the distribution of wealth w_{t+1} in the subsequent period in time. We can state the law of motion in the convenient form[3]

$$w_{t+1} = f_t(\omega^{t+1}, w_t) \tag{14}$$

where

$$f_t(\omega^{t+1}, w_t) = \left[\mathrm{Id} - \left[\frac{\lambda_{t,k}^i(\omega^t) w_t^i}{\lambda_{t,k}(\omega^t) w_t} \right]_i^k \Lambda_{t+1}(\omega^{t+1}) \right]^{-1} \left[\sum_{k=1}^{K} D_{t+1}^k(\omega^{t+1}) \frac{\lambda_{t,k}^i(\omega^t) w_t^i}{\lambda_{t,k}(\omega^t) w_t} \right]_i .$$

[3]It is also convenient to define $w_{t+1} = (0, ..., 0)$, if $w_t = (0, ..., 0)$.

The final step is to derive the law of motion for the investors' market shares. This will complete the derivation of the evolutionary stock market model.

The following assumption is imposed throughout the remainder of this paper.

(B.1) All investors have the same rate of consumption: $\lambda^i_{t,0}(\omega^t) = \lambda_{t,0}(\omega^t)$.

It is clear that, other things being equal, a smaller rate of consumption leads to a higher growth rate of wealth. Without assumption (B.1) the evolution of wealth would be biased in favor of investors with a high saving rate. Since we want to analyze the relative performance of different asset allocation rules no rule should have an disadvantage in terms of the rate at which wealth is withdrawn from it.

Aggregating (7) over investors, one finds

$$\begin{aligned} W_{t+1} &= \sum_{k=1}^{K} D^k_{t+1}(\omega^{t+1}) + \sum_{k=1}^{K} \lambda_{t+1,k}(\omega^{t+1})\, w_{t+1} \\ &= D_{t+1}(\omega^{t+1}) + (1 - \lambda_{t+1,0}(\omega^{t+1}))\, W_{t+1} \end{aligned} \tag{15}$$

where $D_{t+1}(\omega^{t+1}) = \sum_{k=1}^K D^k_{t+1}(\omega^{t+1})$ is the aggregate dividend payment. The last equality holds because $\sum_{k=1}^K \lambda_{t+1,k}\, w_{t+1} = \sum_{i=1}^I \sum_{k=1}^K \lambda^i_{t+1,k}\, w^i_{t+1} = (1-\lambda_{t+1,0}) \sum_{i=1}^I w^i_{t+1}$.

Equation (15) implies

$$W_{t+1} = \frac{D_{t+1}(\omega^{t+1})}{\lambda_{t+1,0}(\omega^{t+1})}. \tag{16}$$

The economy grows (or declines) with rate $D_{t+1}(\omega^{t+1})/(\lambda_{t+1,0}(\omega^{t+1})\, W_t)$. The growth rate is thus the ratio of the rate at which additional wealth is injected by dividends, $D_{t+1}(\omega^{t+1})/W_t$, to the rate at which wealth is withdrawn from the process for consumption, $\lambda_{t+1,0}(\omega^{t+1})$.

The market share of investor i is $r^i_t = w^i_t/W_t$. Using (16) and exploiting the particular structure of the variables (8) that define the law of motion (14), we obtain

$$r_{t+1} = \frac{\lambda_{t+1,0}(\omega^{t+1})}{D_{t+1}(\omega^{t+1})}\; f_t(\omega^{t+1}, r_t) \tag{17}$$

or, equivalently,

$$r_{t+1} = \lambda_{t+1,0}(\omega^{t+1}) \left(\mathrm{Id} - \left[\frac{\lambda^i_{t,k}(\omega^t) r^i_t}{\lambda_{t,k}(\omega^t) r_t} \right]_i^k \Lambda_{t+1}(\omega^{t+1}) \right)^{-1} \left[\sum_{k=1}^{K} d^k_{t+1}(\omega^{t+1}) \frac{\lambda^i_{t,k}(\omega^t) r^i_t}{\lambda_{t,k}(\omega^t) r_t} \right]_i$$

where

$$d_{t+1}^k(\omega^{t+1}) = \frac{D_{t+1}^k(\omega^{t+1})}{D_{t+1}(\omega^{t+1})}$$

is the relative dividend payment of asset k. Equation (17) is referred to as the *market selection process.*

The wealth of an investor i in any period in time can be derived from their market share and the aggregate wealth, defined by (16), as

$$w_{t+1}^i = \frac{D_{t+1}(\omega^{t+1})}{\lambda_{t+1,0}(\omega^{t+1})}\, r_{t+1}^i. \tag{18}$$

3 Evolutionary Stability

We next introduce the stability concepts needed to analyze the long run behavior of the wealth shares under the market selection process. The analysis is restricted to the stationary case. The following assumptions are imposed to ensure that the calender date does not enter in strategies and dividends, i.e. the model becomes stationary; only the observed history matters.

(B.2) Strategies are stationary, i.e. $\lambda_{t,k}^i(\omega^t) = \lambda_k^i(\omega^t)$ *for all* $i = 1, ..., I$ *and* $k = 0, 1, ..., K$.

(B.3) Relative dividend payments are stationary and depend only on the current state of natue, i.e. $d_t^k(\omega^t) = d^k(\omega_t)$ *for all* $k = 1, ..., K$.

Assumption (B.3) is fulfilled, for instance, if $D_{t+1}^k(\omega^{t+1}) = d^k(\omega_{t+1})\, W_t$ with $W_t = \sum_i w_t^i$, i.e. the dividend payment of every asset has an idiosyncratic component $d^k(\omega_{t+1})$ (depending only on the state of nature in the respective period) and an aggregate component W_t. Dividends grow or decline with the same rate as aggregate wealth.

Under these assumptions, the market selection process (17) generates a *random dynamical system* (Arnold 1998) on the simplex $\Delta^I = \{r \in \mathbb{R}^I \mid r^i \geq 0, \sum_i r^i = 1\}$. For any initial distribution of wealth $w_0 \in \mathbb{R}_+^I$, (17) defines the path of market shares on the event tree with branches ω^t. The initial distribution of market shares is $(r_0^i)_i = (w_0^i/W_0)_i$. Formally, this can be stated as follows.

Denote by $\Omega = S^{\mathbb{Z}}$ the set of all sequences of states of nature $\omega = (\omega_t)_{t\in\mathbb{Z}}$. Denote the right-hand side of (17) by $h(\omega^{t+1}, r_t) : \Delta^I \to \Delta^I$. This map is independent of the calender date by assumptions (B.2-B.3). Define $\varphi(t, \omega, r) =$

$h(\omega^{t+1}, \cdot) \circ \ldots \circ h(\omega^1, r)$ for all $t \geq 1$, and $\varphi(0, \omega, r) = r$. $\varphi(t, \omega, r)$ is the vector of investor's wealth shares at time t for the initial distribution of market shares r and the sequence of realizations of states ω.

Given a random dynamical system for a set of stationary trading strategies (λ^i), one is particularly interested in those wealth shares that evolve in a stationary fashion over time. Here we restrict ourselves to deterministic distributions of market shares, i.e those that are fixed under the market selection process (17).[4] To specify this notion, we recall the definition of a deterministic fixed point in the framework of random dynamical systems. Let a set of strategies (λ^i) be given, and denote by φ the associated random dynamical system.

Definition 1 *$\bar{r} \in \Delta^I$ is called a (deterministic)* fixed point *of φ if, for all $\omega \in \Omega$ and all t,*

$$\bar{r} = \varphi(t, \omega, \bar{r}). \tag{19}$$

The distribution of market shares $\bar{r}$ is said to be invariant *under the market selection process (17).*

By the definition of $\varphi(t, \omega, r)$ the condition (19) is equivalent to $\bar{r} = \varphi(1, \omega, \bar{r})$ for all ω, i.e. a deterministic state is fixed under the one-step map if and only if it is fixed under all t-step maps.

It is straightforward to see that the state in which one investor possesses the entire market does not change over time. In any set of trading strategies each unit vector (i.e. each vertex of Δ^I) is a fixed point. This follows from Proposition 1 which shows that $r^i = 0$ implies $\varphi^i(t, \omega, r) = 0$.

The following result even holds without conditions (B.2-B.3).

Proposition 2 *Suppose the dividend and capital gains matrix has full rank at a deterministic fixed point. Then all investors use the same portfolio rule.*

Proof of Proposition 2. Equations (7) and (16) give

$$r^i_{t+1} = \sum_{k=1}^{K} (\lambda_0 \, d^k_{t+1}(\omega_{t+1}) + q^k_{t+1}(\omega^{t+1})) \, \frac{\lambda^i_{t,k}(\omega^t) r^i_t}{q^k_t(\omega^t)} \tag{20}$$

with

$$q^k_t(\omega^t) = \sum_{i=1}^{I} \lambda^i_{t,k}(\omega^t) \, r^i_t. \tag{21}$$

[4]See e.g. Schenk-Hoppé (2001) for an application and discussion of stochastic fixed points.

Suppose $r^i_{t+1} = r^i_t = r^i > 0$ for all i. Then equation (20) can be written as

$$\left(\sum_{k=1}^{K} \left[\lambda_0 \, d^k_{t+1}(\omega_{t+1}) + q^k_{t+1}(\omega^{t+1}) \right] \frac{\lambda^i_{t,k}(\omega^t)}{q^k_t(\omega^t)} - 1 \right) r^i = 0. \tag{22}$$

If the dividend and capital gain matrix

$$\lambda_0 \, d^k_{t+1}(\omega_{t+1}) + q^k_{t+1}(\omega^{t+1})$$

has full rank (as a function of k and ω_{t+1} for each given history ω^t), then (22) implies $\lambda^i_{t,k}(\omega^t) = q^k_t(\omega^t)$. In light of (21), this means that for all $i = 1, ..., I$

$$\lambda^i_{t,k} = \sum_{j=1}^{I} \lambda^j_{t,k} \, r^j.$$

Hence λ^i and λ^j are identical for all i, j. □

We are particularly interested in *stable* fixed points of the market selection process. Loosely speaking, stability means that small perturbations of the initial market share distribution do not have a long-run effect. If a fixed point of market shares is stable, every path of market shares starting in a neighborhood of this fixed point becomes asymptotically identical to it. Since fixed points are associated to unique trading strategies by Proposition 2 (the total wealth being concentrated on this trading strategy), the natural definition of a trading strategy's stability is that of the fixed point's stability. Two different notions of stability will be needed. They are defined as follows.

It is assumed that for any given incumbent strategy λ^i, the mutant strategy λ^j is distinct in the sense that $\lambda^j(\omega^0) \neq \lambda^i(\omega^0)$ with strictly positive probability. Moreover, the first entry in the tuple of relative wealth shares $r = (r^i, r^j)$ refers to the incumbent's strategy, while the second refers to the entrant's wealth share.

Definition 2 *A trading strategy λ^i is called* evolutionary stable *if, for all λ^j, there is a random variable $\varepsilon > 0$ such that $\lim_{t\to\infty} \varphi^i(t, \omega, r) = 1$ for all $r^i \geq 1 - \varepsilon(\omega)$ ($r^j = 1 - r^i \leq \varepsilon(\omega)$) almost surely.*

For each evolutionary stable distribution of market shares there exits an entry barrier (a random variable here) below which the new portfolio rule does not drive out the incumbent player. Any perturbation of the distribution of market shares, if sufficiently small, does not change the long-run behavior. The market selection process asymptotically leaves the mutant with no market share.

Finally, a corresponding stability criterion for local mutations is introduced.

Definition 3 *A trading strategy λ^i is called* locally evolutionary stable *if for all λ^j there exists a random variable $\delta(\omega) > 0$ such that λ^i is evolutionary stable for all portfolio rules λ^j with $\|\lambda^i(\omega) - \lambda^j(\omega)\| < \delta(\omega)$ for all ω.*

A locally evolutionary stable distribution of market shares is evolutionary stable with respect to local mutations. That is, the mutant's strategies are restricted to small deviations from the status quo strategy.

4 The Main Result

We turn now to a detailed analysis of the evolutionary stability of stationary portfolio rules. The local (in)stability conditions obtained here lead to a unique evolutionary stable investment rule, provided the relative dividend payoffs are governed by a stationary Markov process.

To analyze evolutionary stability of a trading strategy one has to consider the random dynamical system (17) with an incumbent (with market share r_t^1) and a mutant (with market share $r_t^2 = 1 - r_t^1$). The resulting one-dimensional system governing the market selection process for two investors with stationary portfolio rules is given by

$$r_{t+1}^1 = \frac{\lambda_0}{\delta_{t+1}} \left(\left[1 - \sum_{k=1}^K \lambda_{t+1,k}^2 \theta_{t,k}^2 \right] \sum_{k=1}^K d_{t+1}^k \theta_{t,k}^1 + \left[\sum_{k=1}^K \lambda_{t+1,k}^2 \theta_{t,k}^1 \right] \sum_{k=1}^K d_{t+1}^k \theta_{t,k}^2 \right) \quad (23)$$

where $\lambda_{t,k}^i = \lambda_k^i(\omega^t)$, $d_{t+1}^k = d^k(\omega_{t+1})$ and

$$\theta_{t,k}^1 = \frac{\lambda_{t,k}^1 \, r_t^1}{\lambda_{t,k}^1 \, r_t^1 + \lambda_{t,k}^2 \, (1 - r_t^1)} \quad , \quad \theta_{t,k}^2 = \frac{\lambda_{t,k}^2 \, (1 - r_t^1)}{\lambda_{t,k}^1 \, r_t^1 + \lambda_{t,k}^2 \, (1 - r_t^1)}$$

$$\delta_{t+1} = \left[1 - \sum_{k=1}^K \lambda_{t+1,k}^1 \theta_{t,k}^1 \right] \left[1 - \sum_{k=1}^K \lambda_{t+1,k}^2 \theta_{t,k}^2 \right] - \left[\sum_{k=1}^K \lambda_{t+1,k}^2 \theta_{t,k}^1 \right] \left[\sum_{k=1}^K \lambda_{t+1,k}^1 \theta_{t,k}^2 \right].$$

The derivative of the right-hand side of (23), denoted by $h(\omega^{t+1}, r_t^1)$, with respect to r_t^1 evaluated at $r_t^1 = 1$ is [5]

$$\left.\frac{\partial h(\omega^{t+1}, r_t^1)}{\partial r_t^1}\right|_{r_t^1 = 1} = \sum_{k=1}^K \left(\lambda_k^1(\omega^{t+1}) + \lambda_0 \, d^k(\omega_{t+1}) \right) \frac{\lambda_k^2(\omega^t)}{\lambda_k^1(\omega^t)}. \quad (24)$$

¿From (24) one can read off the exponential growth rate of investment strategy 2's wealth share in a small neighborhood of $r^1 = 1$, i.e. the state in which investment strategy 1 owns total market wealth.

[5] The necessary calculations are lengthy but elementary and therefore omitted.

Throughout the following we restrict the analysis to the stationary Markov case. It is imposed that

(C) The state of nature follows a Markov process with strictly positive transition probabilities, i.e. $\pi_{s\tilde{s}} > 0$ *for all* $s, \tilde{s} \in S$.

The growth rate of investment strategy 2's wealth share in a small neighborhood of $r^1 = 1$ is

$$g_{\lambda^1}(\lambda^2) = \int_{S^N} \sum_{s \in S} \pi_{\omega_0 s} \ln \left[\sum_{k=1}^{K} \left(\lambda_k^1(\omega^0, s) + \lambda_0 \, d^k(s) \right) \frac{\lambda_k^2(\omega^0)}{\lambda_k^1(\omega^0)} \right] \mathbb{P}(d\omega^0). \quad (25)$$

It can be interpreted as the growth rate of strategy λ^2 at λ^1-prices because for $r^1 = 1$ the asset prices correspond to the budget shares of investor 1 and thus to the vector λ^1. Clearly asset prices change over time due to changes in the status quo strategy λ^1. For instance at time t, prices are $\lambda^1(\omega^t)$.

This growth rate determines the local stability of the fixed point $r^1 = 1$. If the growth rate is negative, $g_{\lambda^1}(\lambda^2) < 0$, investor 2 looses market share and investor 1's market share tends to one. In this case the portfolio rule λ^1 is stable against λ^2. If the growth rate is positive, $g_{\lambda^1}(\lambda^2) > 0$, investor 2 gains market share while that of investor 1 falls. In this case the portfolio rule λ^1 is not stable against λ^2.

Our main result shows that this (in)stability condition can be employed to single out a unique evolutionary stable strategy. Moreover, an explicit representation can be given for this strategy.

Theorem 1 *Define the stationary portfolio rule* λ^* *by* $\lambda_0^* = \lambda_0$ *and, for all* $s \in S$ *and all* $k = 1, ..., K$, *by*

$$\lambda^* = (1 - \lambda_0)\, \lambda_0 \left[\mathrm{Id} - (1 - \lambda_0)\, \pi \right]^{-1} \pi \, d \quad (26)$$

using the matrix notation $\lambda^* = (\lambda_k^*(s))_s^k$.

Instability results

(i) *Every strategy distinct from* λ^* *is not stable against some arbitrarily close strategy, i.e. for all* $\lambda \neq \lambda^*$ *and for all* $\varepsilon > 0$ *there exists a stationary strategy* μ *with* $|\mu(\omega^0) - \lambda(\omega^0)| < \varepsilon$ *for all* ω^0 *such that* $g_\lambda(\mu) \geq 0$.

(ii) *Suppose there is a set of positive measure* $\tilde{\Omega}$ *such that for all* $\omega^0 \in \tilde{\Omega}$, $\lambda(\omega^0) \neq \lambda^*(\omega_0)$ *and* $([(1 - \lambda_0)\lambda_k(\omega^0, s) + \lambda_0 \, d^k(s)]/\lambda_k(\omega^0))_s^k$ *has full rank. Then the strategy* λ *is unstable against some arbitrarily close strategy* μ, *i.e.* $g_\lambda(\mu) > 0$.

Stability results

(iii) λ^* *is never unstable, i.e.* $g_{\lambda^*}(\lambda) \leq 0$ *for every strategy* λ.

(iv) λ^* *is stable, i.e.* $g_{\lambda^*}(\lambda) < 0$ *for every strategy* $\lambda \neq \lambda^*$ *provided* $([(1 - \lambda_0)\,\mathbb{E}(\lambda^*_k \mid s) + \lambda_0\,\mathbb{E}(d^k \mid s)]/\lambda^*_k(s))^k_s$ *has full rank.*[6]

Theorem 1 defines a Markov strategy λ^*, i.e. a portfolio rule such that only the current state of nature is taken into account when making an investment decision today. The strategy λ^* is well-defined because existence of the inverse in (26) follows immediately from the property that $\mathrm{Id} - (1 - \lambda_0)\,\pi$ has a row dominant diagonal (recalling that $0 < \lambda_0 < 1$). The budget shares of the λ^* investment rule have the following properties. First, note that the definition (26) implies that the strategy λ^* satisfies

$$\mathbb{E}(\lambda^*_k \mid s) + \lambda_0\,\mathbb{E}(d^k \mid s) = \lambda^*_k(s)/(1 - \lambda_0)$$

for all k and all s.[7] That is, the λ^* portfolio rule "balances" capital and dividend gains. Due to the Markov structure of dividends, an adjustment is necessary whenever the conditional expected future payoff $\mathbb{E}(d^k \mid s)$ changes. Second, one observes that (26) takes on the form of the limit of a geometric series. Indeed, there is the following alternative expression for λ^* which we state as a remark.

Remark 1 *The definition of the unique evolutionary stable strategy* λ^* *in (26) is equivalent to*

$$\lambda^* = \lambda_0 \sum_{m=1}^{\infty} (1 - \lambda_0)^m\,\pi^m\,d. \tag{27}$$

It is obvious from the last representation that according to the strategy λ^* one has to divide wealth across assets according to the present expected value of their (relative) future dividend payoffs. The discounting rate is the inverse of the saving rate $1 - \lambda_0$. If the λ^* portfolio rule manages all market wealth then all asset prices are given by this vector of fundamental values. In this respect the λ^* strategy corresponds to a rational market.

The main result of this paper, Theorem 1, shows that this investment strategy has the following properties. First, a rational market is evolutionary stable: it cannot be invaded by a portfolio rule that is distinct from λ^*. Second, any irrational market, i.e. one in which λ^* does not manage

[6] The conditional expected value is defined as $\mathbb{E}(d^k \mid s) = \sum_{\tilde{s}} \pi_{s\tilde{s}}\, d^k(\tilde{s})$.

[7] The term $(1 - \lambda_0)$ on the right-hand side appears because investment shares add up to one minus the consumption share, i.e. $\sum_{k=1}^{K} \lambda^*_k(s) = 1 - \lambda_0$.

all wealth, is not robust against small deviations from the status quo. The stronger version of both findings requires that every investment strategy distinct from λ^* generates a different payoff stream which is ensured by the full rank condition.

The intuition for the main result is immediate from the exponential growth rate (25). From this equation one can read off that λ^* is the portfolio rule with the highest exponential growth rate in any population where itself determines market prices. In this sense the λ^* portfolio rule plays the "best response against itself." The expression (25) for the exponential growth rate also show that the λ^* portfolio rule is an equilibrium in a standard two-period economy with a log-utility investor. Therefore the λ^* portfolio rule yields the maximum growth portfolio and λ^*-prices prevail in such an economy.

A short remark on the case of i.i.d. dividend payments is in order. It is apparent from representation (27) in Remark 1 that for an i.i.d. dividend process the unique evolutionary stable strategy λ^* is given by $\lambda^*_k = (1-\lambda_0)\mathbb{E}d^k$, $k = 1, ..., K$. Again each asset is priced at its fundamental value. Moreover, in the i.i.d. case the budget shares are fixed since the current state of nature does not provided any valuable information on the future dividend payoffs.

Proof of Theorem 1. We first show (ii), pointing out how to derive assertion (i) in the proof. Suppose $([(1-\lambda_0)\lambda_k(\omega^0, s) + \lambda_0\, d^k(s)]/\lambda_k(\omega^0))^k_s$ has full rank for almost all ω^0. For notational simplicity, we first normalize strategies with $1-\lambda_0$ to make $\sum_{k=1}^K \lambda_k = 1$, and we also denote $\lambda = (\lambda_1, .., \lambda_K) \in \Delta^K$.

Let $\lambda \neq \lambda^*$. To prove assertion (ii), it suffices to show that $g_\lambda(\mu) > 0$ for all μ in some neighborhood of λ.

Using (25), the growth rate of strategy μ at λ-prices, cf. (25), can be written as

$$g_\lambda(\mu) = \int_{S^N} \tilde{g}_\lambda(\mu(\omega^0), \omega_0)\, \mathbb{P}(d\omega^0) \tag{28}$$

with

$$\tilde{g}_\lambda(\mu(\omega^0), \omega_0) = \sum_{s\in S} \pi_{\omega_0 s} \ln\left[\sum_{k=1}^{K} \left[(1-\lambda_0)\lambda_k(\omega^0, s) + \lambda_0\, d^k(s)\right] \frac{\mu_k(\omega^0)}{\lambda_k(\omega^0)}\right] \quad .$$

For given ω_0, $\mu(\omega^0) \to \tilde{g}_\lambda(\mu(\omega^0), \omega_0)$ is a concave function. The full rank assumption ensures that it is actually strictly concave with positive probability.

One further has

$$\frac{\partial \tilde{g}_\lambda(\mu(\omega^0), \omega_0)}{\partial \mu_n(\omega^0)} = \sum_{s\in S} \pi_{\omega_0 s} \frac{\left[(1-\lambda_0)\, \lambda_n(\omega^0, s) + \lambda_0\, d^n(s)\right]/\lambda_n(\omega^0)}{\sum_{k=1}^K \left[(1-\lambda_0)\, \lambda_k(\omega^0, s) + \lambda_0\, d^k(s)\right]\, \mu_k(\omega^0)/\lambda_k(\omega^0)}$$

Thus

$$\sum_{n=1}^{K}\left(\frac{\partial\tilde{g}_\lambda(\mu(\omega^0),\omega_0)}{\partial\mu_n(\omega^0)}\Big|_{\mu(\omega^0)=\lambda(\omega^0)}\right)\, d\mu_n(\omega^0)$$
$$= \sum_{n=1}^{K}\frac{\sum_{s\in S}\pi_{\omega_0 s}\,\left[(1-\lambda_0)\,\lambda_n(\omega^0,s)+\lambda_0\,d^n(s)\right]}{\lambda_n(\omega^0)}\, d\mu_n(\omega^0) \quad (29)$$

for every $d\mu_1(\omega^0),...,d\mu_K(\omega^0)$ with $\sum_{n=1}^{K} d\mu_n(\omega^0)=0$.

We final step in proving (ii) is to show that for each $\omega^0\in\tilde{\Omega}$, (29) is strictly positive for some $(d\mu_1(\omega^0),...,d\mu_K(\omega^0))$ with $\sum_{n=1}^{K} d\mu_n(\omega^0)=0$. This property implies that there is some $\mu(\omega^0)$, arbitrarily close to $\lambda(\omega^0)$ with $\tilde{g}_\lambda(\mu(\omega^0),\omega_0)>0$ on $\tilde{\Omega}$. For $\omega^0\notin\tilde{\Omega}$ we let $\mu(\omega^0)=\lambda(\omega^0)$. The strategy μ is arbitrarily close to λ. Measurability of μ follows from the fact that, due to finiteness of S, the sigma algebra of the probability space under consideration is the power set (and thus every function is measurable). By construction the strategy μ satisfies $g_\lambda(\mu)>0$, which verifies assertion (ii).

It is clear that (29) is strictly positive for some $(d\mu_1(\omega^0),...,d\mu_K(\omega^0))$ with $\sum_{n=1}^{K} d\mu_n(\omega^0)=0$ if and only if

$$\frac{\sum_{s\in S}\pi_{\omega_0 s}\,\left[(1-\lambda_0)\,\lambda_n(\omega^0,s)+\lambda_0\,d^n(s)\right]}{\lambda_n(\omega^0)} \quad (30)$$

is not constant in n (for given ω^0).

We will show that (30) is constant, i.e.

$$\sum_{s\in S}\pi_{\omega_0 s}\,\left[(1-\lambda_0)\,\lambda_n(\omega^0,s)+\lambda_0\,d^n(s)\right]=c\lambda_n(\omega^0)$$

for all n, if and only if $\lambda=\lambda^*$.

Taking the sum over n on both sides of the last equality shows that $c=1$. The condition that (30) is constant therefore becomes

$$\sum_{s\in S}\pi_{\omega_0 s}\,\left[(1-\lambda_0)\,\lambda_n(\omega^0,s)+\lambda_0\,d^n(s)\right]=\lambda_n(\omega^0) \quad (31)$$

By definition of λ^*, (31) holds if $\lambda=\lambda^*$. To show that (31) implies $\lambda=\lambda^*$, we need to consider three distinct cases: (a) $\lambda(\omega^0)$ does not depend on ω^0; (b) $\lambda(\omega^0)$ depends only on a finite history, i.e. $\lambda(\omega^0)=\lambda(\omega_{-T},...,\omega_0)$ for some $T\geq 0$; and (c) $\lambda(\omega^0)$ depends on an infinite history.

Case (a): In this case (31) takes the form

$$(1-\lambda_0)\,\lambda_n+\lambda_0\,\mathbb{E}(d^n\mid\omega_0)=\lambda_n \quad (32)$$

which is equivalent to $\mathbb{E}(d^n \mid \omega_0) = \lambda_n$. If, as we have assumed in the Theorem, the dividend process is a non-degenerate Markov process, μ has to depend on ω_0. This is a contradiction.

Case (b): Obviously, if $\lambda(\omega^0)$ is only a function of ω_0, then $\lambda = \lambda^*$. For a strategy $\lambda(\omega^0)$ that depends on a history of length $T \geq 1$, (31) becomes

$$\sum_{s \in S} \pi_{\omega_0 s} \left[(1-\lambda_0)\, \lambda_n(\omega_{-T+1}, ..., \omega_0, s) + \lambda_0\, d^n(s)\right] = \lambda_n(\omega_{-T}, ..., \omega_0) \tag{33}$$

If λ_n would vary with ω_{-T}, (33) could not hold for all ω^0. Thus (33) implies that $\lambda(\omega^0) = \lambda(\omega_{-T+1}, ..., \omega_0)$. Repeated application shows that $\lambda(\omega^0) = \lambda(\omega_0)$. However, this implies $\lambda = \lambda^*$, as discussed above.

Case (c): In this case (31) reads

$$\sum_{\omega_1 \in S} \pi_{\omega_0 \omega_1} \left[(1-\lambda_0)\, \lambda_n(\omega^1) + \lambda_0\, d^n(\omega_1)\right] = \lambda_n(\omega^0). \tag{34}$$

An analogous equation holds with $\lambda_n(\omega^1)$ on the right-hand side,

$$\sum_{\omega_2 \in S} \pi_{\omega_1 \omega_2} \left[(1-\lambda_0)\, \lambda_n(\omega^2) + \lambda_0\, d^n(\omega_2)\right] = \lambda_n(\omega^1). \tag{35}$$

Inserting (35) in (34) yields

$$\lambda_n(\omega^0) = (1-\lambda_0)^2 \pi^2_{\omega_0\omega_2} \lambda_n(\omega^2) + \lambda_0 \left[(1-\lambda_0) \sum_{\omega_2} \pi^2_{\omega_0\omega_2} d^n(\omega_2) + \sum_{\omega_1} \pi^1_{\omega_0\omega_1} d^n(\omega_1)\right]$$

where $\pi^m_{\omega_0\omega_m} = \sum_{\omega_1,...,\omega_m} \pi_{\omega_0\omega_1} ... \pi_{\omega_{m-1}\omega_m}$.

Repeating this procedure and observing that

$$(1-\lambda_0)^m \sum_{\omega_m} \pi^m_{\omega_0\omega_m}\, \lambda_n(\omega^m) \to 0 \text{ as } m \to \infty$$

we find

$$\lambda_n(\omega^0) = \frac{\lambda_0}{1-\lambda_0} \sum_{m=1}^{\infty} (1-\lambda_0)^m \sum_{\omega_m} \pi^m_{\omega_0\omega_m}\, d^n(\omega_m) \tag{36}$$

Thus $\lambda_n(\omega^0)$ is a function of ω_0 only, implying that $\lambda = \lambda^*$, as discussed in case (b). The equivalence of (36) and the definition of λ^* in the Theorem 1 has been established in Remark 1.

Assertion (iii) and (iv) follow immediate from the above results. We consider the case (iv) in which the full rank condition is satisfied. As observed above, the growth rate $\tilde{g}_{\lambda^*}(\lambda(\omega^0), \omega_0)$ is concave in $\lambda(\omega^0)$ and strictly concave under the full rank assumption in (iv). Assertion (iv) is immediate if we can show that its maximum over Δ^K is equal to zero and that this maximum is attained at $\lambda(\omega^0) = \lambda^*(\omega_0)$.

At the maximum (29) is equal to zero. This is equivalent to (30) being constant, i.e. (31) is fulfilled. However, we have already proved above that this is true if and only if $\lambda = \lambda^*$. Thus $g_{\lambda^*}(\lambda)$ takes on its maximum at $\lambda = \lambda^*$. Obviously the maximum is zero. This is (iv).

(iii) is obvious from the proof of (iv). □

5 Conclusion and Outlook

We have studied the evolution of wealth shares of portfolio rules in incomplete markets with long-lived assets. Prices are determined endogenously. The performance of a portfolio rule in the process of repeated reinvestment of wealth is determined by the wealth share eventually conquered in competition with other portfolio rules. Using random dynamical systems theory, we derived necessary and sufficient conditions for the evolutionary stability of portfolio rules. In the case of Markov payoffs these local stability conditions lead to a simple portfolio rule that is the unique evolutionary stable strategy. This rule possesses an explicit representation as it invests proportionally to the expected relative dividends. This stability property may help to explain why on long-term averages stock markets look quite rational while severe departures are possible in the short- and medium-term.

As in many other papers on economic theory, our results are based on a couple of assumptions and modeling choices that shall be extended in future research. For example, we have restricted strategies to be adapted to the information filtration given by the exogenous revelation of the states of the world. Hence, we did not allow for price dependent strategies as for example simple momentum strategies like "buy (sell) if prices have gone up (down)." Moreover, we made a clear distinction between the market selection process and the mutations. The latter act at the selection process only once the former has settled at a point of rest. It would be desirable to consider a selection process with ongoing mutations. Finally, in our model the wealth shares of the strategies increase due to "internal growth," i.e. they increase by the returns they have generated. This process shall be augmented by a process of "external growth" in which strategies increase their wealth share by attracting wealth from less successful strategies. Data from Hedge Funds, for example, show that internal growth leads external growth so that one effect of this extension my be speeding up the market selection process. However, this and the other possible extensions mentioned have to be checked carefully in future research.

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