A general theory of decision making

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Abstract

We formulate a general theory of decision making based on a lattice of observable events, and we exhibit a large class of representations called the general model. Some of the representations are equivalent to the so called standard model in which observable events are modelled by an algebra of measurable subsets of a state space, while others are not compatible with such a description. We show that the general model collapses to the standard model, if and only if an additional axiom is satisfied. We argue that this axiom is not very natural and thus assert that the standard model may not be general enough to model all relevant phenomena in economics. Using the general model we are (as opposed to Schmeidler [16]) able to rationalize Ellsberg's paradox without the introduction of non-additive measures.

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1 Introduction

The model developed by Arrow, Debreu and others (the standard model) is accepted by most economists as the foundation of the modelling of the behavior of agents exposed to risk. Knight [11] introduced the distinction between the notions of "measurable uncertainty", which can be expressed by a well-defined probability distribution known to the agents, and "unmeasurable uncertainty" describing the circumstance arising, when agents only have vague or non-existing ideas about the rules of the world. The former concept is by convention denoted risk and the latter uncertainty. Savage [15] formulated axioms that capture the situations where probability distributions can be defined, and the subjects therefore are exposed only to risk and not to uncertainty. Ellsberg [6] constructed some hypothetical experiments designed to check compliance with the Savage axioms and interviewed people about their personal choices and preferences in the experiments. The result was that some people, including Savage himself, gave answers not compatible with adherence to the Savage axioms. Schmeidler and others [16, 13, 9] introduced the notion of non-additive measures, and in the process they extended the reach of mathematical modelling into the realm of uncertainty. Ellsberg's paradox can in particular be explained within this new setting.

In the standard model an event is represented by a measurable subset of a state space, and the event occurs if the true state of nature represented by an element of the state space is contained in the subset representing the event. We note that the one-point set representing the true state of nature may not be a measurable set and therefore not observable to the agents. It is nevertheless assumed that "knowledge" of the true state of nature makes it possible to determine which events have occurred.

This way of modelling events is ubiquitous in much economic thought, although it places severe and unrealistic constraints on the behaviour of agents. For example, it requires agents to agree on which events have occurred, once the state is known. This constraint alone is very far from the situation encountered in every day life. Agents of the real world may very well disagree on the occurrence of historical events even after hundreds of years of study. Confronted with such critique of the standard model, many economists answer that the model in question is not sufficiently specified and with sufficient information given every agent will realize the true state of nature, and they will then all agree on which events have occurred and which have not. Any disagreement is thus due only to lack of knowledge and disappears with the enlightenment of the agents.

1.1 The structure of the paper

We reformulate Arrow and Debreu's model (the standard model) in terms of a lattice of observable events without reference to the state space, and then demonstrate that this formulation is completely equivalent to the traditional description, including the reappearance of the state space as a purely mathematical construction. The theory is based on the notion of an event which we consider more fundamental than the notion of a state.

We give an axiomatic description of the minimal requirements to any sensible theory of decision making, before introducing the so called general model of decision making which is then shown to satisfy eight listed axioms.

We discuss an additional axiom (9) and demonstrate that the general model of decision making is equivalent to the standard model if and only if this additional axiom is satisfied.

We introduce securities, portfolios and expectations in the general model. There is a sharp distinction between the notions of security and portfolio in the general model. Securities or bets are valued solely in terms of a monetary numeraire. The theory does not in its present version support a more advanced notion of utility.

We notice that the additional axiom (9) implies that the expected probability of the majorant event (union) of two mutually exclusive events is the sum of the expected probabilities of each of the events, and we argue that this axiom may be too strong in many situations of interest to economists. It is in particular incompatible with a rational explanation of Elsberg's paradox.

We demonstrate that the behaviour of the agents in Elsberg's paradox can be rationally explained in a very simple representation of the general model. This is done without any additional assumptions or the introduction of non-additive measures as in [16].

We finally demonstrate that the notion of arbitrage free asset evaluation may be introduced in the general model, and that the absence of arbitrage can be characterized in terms which are quite close to the familiar statements in the standard model. If there is riskless borrowing in an arbitrage free economy, then there is an asset equally valued by all agents.

2 The standard model

The observable (or knowable) events in (a representation of) the standard model are given by the measurable subsets of a state space, or equivalently by their indicator functions. These functions are by themselves projections when acting as multiplication operators. Let the state space Ω be equipped with a σ-algebra (or tribe) $\mathcal{F}$ of subsets. An event is thus a set $A \in \mathcal{F}$, or equivalently the indicator function 1_A, or equivalently the projection operator P_A defined by setting

$$(P_A\xi)(\omega) = 1_A(\omega)\xi(\omega) \qquad \omega \in \Omega \tag{2.1}$$

for each $\mathcal{F}$-measurable function ξ on Ω.

If an objective probability measure μ is given, rendering $(\Omega, \mathcal{F}, \mu)$ into a measure space, then P_A becomes a self-adjoint projection on the Hilbert space $L^2(\Omega, \mathcal{F}, \mu)$. We also assume that sets $A, B \in \mathcal{F}$ represent the same event if they only differ on a null set, or equivalently if $P_A = P_B$. Inherent in this formulation is the assumption that the measure space is complete. To avoid excessive generalizations we shall assume that Ω is a locally compact, second countable Hausdorff space and that μ is the completion of the Riesz representation of a Radon measure (the integral of continuous functions with respect to μ). We refer to Bourbaki [3] for a general introduction to integration theory.

Two events $A, B \in \mathcal{F}$ are represented by commuting projections P_A and P_B. Indeed

$$\begin{aligned}(P_A P_B\xi)(\omega) &= 1_A(\omega)(P_B\xi)(\omega) = 1_A(\omega)1_B(\omega)\xi(\omega)\\ &= (P_B P_A\xi)(\omega) \qquad \forall\omega \in \Omega,\end{aligned}$$

for each $\xi \in L^2(\Omega, \mathcal{F}, \mu)$. We collect together a number of well-known probabilistic concepts related to events in the standard model of decision making and write down equivalent properties in terms of the representing projections.

Proposition 2.1 *Let $A, B \in \mathcal{F}$ be events in (a representation of) the standard model, and let P_A and P_B be the representing self-adjoint projections on the Hilbert space $L^2(\Omega, \mathcal{F}, \mu)$.*

(1) The event B is majorizing the event A, if B occurs with probability one provided A occurs. The property is equivalent to the inequality $P_A \le P_B$.

(2) The events A, B have a minorant event, $P_A \wedge P_B$, which is the maximal event in the set of events majorized by both A and B. It is represented by the orthogonal projection on the intersection of the ranges of P_A and P_B in the Hilbert space $L^2(\Omega, \mathcal{F}, \mu)$.

(3) *The events A, B have a majorant event, $P_A \vee P_B$, which is the minimal event in the set of events majorizing both A and B. It is represented by the orthogonal projection on the closure of the sum of the ranges of P_A and P_B in the Hilbert space $L^2(\Omega, \mathcal{F}, \mu)$.*

(4) *The events A, B are said to be mutually exclusive, if their minorant $A \wedge B = 0$. The property is equivalent to the inequality $P_A \leq 1 - P_B$, where 1 denotes the identity operator on the Hilbert space $L^2(\Omega, \mathcal{F}, \mu)$.*

(5) *The events A, B are said to be complementary, if the probability of exactly one of them occurring is one. The property is equivalent to the equation $P_A = 1 - P_B$.*

The events in the standard model are thus represented by self-adjoint projections on a Hilbert space $L^2(\Omega, \mathcal{F}, \mu)$ given by multiplication operators of the form (2.1). The so called simple functions on Ω are linear combinations of indicator functions for measurable subsets, and they (more precisely, their equivalence classes) are by construction weakly dense in $L^\infty(\Omega, \mathcal{F}, \mu)$. We are now able to reformulate the standard model without reference to the state space.

2.1 The standard model reformulated

Definition 2.2 (the standard model) The observable events are specified by a family $\mathcal{F}$ of commuting (self-adjoint) projections on a separable Hilbert space H satisfying:

(i) The zero projection on H (denoted 0) and the identity projection on H (denoted 1) are both in $\mathcal{F}$.

(ii) $1 - P \in \mathcal{F}$ for arbitrary $P \in \mathcal{F}$.

(iii) $P \wedge Q \in \mathcal{F}$ for arbitrary $P, Q \in \mathcal{F}$.

(iv) $\sum_{i \in I} P_i \in \mathcal{F}$ for any family $(P_i)_{i \in I}$ of mutually orthogonal projections in $\mathcal{F}$.

It is a consequence of Theorem 4.3 to be proved later that a family of projections $\mathcal{F}$ satisfying the assumptions of Definition 2.2 is a Boolean σ-algebra [17, page 10].

Proposition 2.3 *The complex vector space $L_0(\mathcal{F})$ generated by a family of commuting projections $\mathcal{F}$ satisfying the conditions in Definition 2.2 is a commutative $*$-algebra, where each element can be written as a linear combination of mutually orthogonal projection in $\mathcal{F}$.*

Proof: We first notice that $PQ = P \wedge Q$ for projections $P, Q \in \mathcal{F}$. Indeed, since P and Q commute we have $PQ = PQP \leq P$ and $PQ = QPQ \leq Q$, thus $PQ \leq P \wedge Q$. On the other hand $P \wedge Q = P(P \wedge Q)P \leq PQP = PQ$. We thus obtain $PQ \in \mathcal{F}$ from (iii), and since

$$P + Q = P(1 - Q) + (1 - P)Q + 2PQ,$$

we derive that a linear combination of projections in $\mathcal{F}$ can be written as a linear combination of orthogonal projections in $\mathcal{F}$, and that the product of linear combinations of projections in $\mathcal{F}$ again is a linear combination of projections in $\mathcal{F}$. The algebra $L_0(\mathcal{F})$ is invariant under the adjoint operation and becomes an involutive algebra. **QED**

We denote by $L(\mathcal{F})$ the norm closure of $L_0(\mathcal{F})$. Since the sum, the product and the adjoint operations are continuous in the norm topology, we obtain that $L(\mathcal{F})$ is a norm closed commutative $*$-algebra[1] of bounded linear operators on the Hilbert space H.

Proposition 2.4 *The spectral projections of the self-adjoint operators in $L(\mathcal{F})$ are in $\mathcal{F}$.*

Proof: Let X be a self-adjoint element in $L(\mathcal{F})$. There exists a sequence of self-adjoint elements (X_n) in $L_0(\mathcal{F})$ such that $\|X - X_n\| \to 0$ for $n \to \infty$. Since (X_n) is a Cauchy-sequence we can find an increasing sequence $n_1, n_2, \ldots$ such that

$$\|X_{n_{k+1}} - X_{n_k}\| \leq \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \qquad k = 1, 2, \ldots.$$

It follows that

$$X_{n_k} - \frac{1}{k} \leq X_{n_{k+1}} - \frac{1}{k+1} \qquad k = 1, 2, \ldots.$$

The sequence (Y_k), where $Y_k = X_{n_k} - k^{-1} \in L_0(\mathcal{F})$, is thus monotone increasing to X. Since $L(\mathcal{F})$ is commutative any increasing function is monotone on $L(\mathcal{F})$, hence the spectral projections $E_k(t,\infty)$ of Y_k are monotone increasing necessarily to the spectral projection $E(t,\infty)$ of X. It thus follows[2] from Theorem 4.3 that the spectral projection $E(t,\infty)$ is in $\mathcal{F}$ for each $t \in \mathbf{R}$. Any other spectral projection $E(B)$ of X associated with a Borel set B in $\mathbf{R}$ is contained in the Boolian σ-algebra generated by the family $\{E(t,\infty) \mid t \in \mathbf{R}\}$ of commuting spectral projections, and since $\mathcal{F}$ is closed under these operations, cf. Theorem 4.3, we obtain $E(B) \in \mathcal{F}$.

Theorem 2.5 *Let the observable events be given by a family $\mathcal{F}$ of commuting self-adjoint projections on a separable Hilbert space as specified in Definition 2.2. There exists a probability space $(\Omega, \mathcal{S}, \mu)$, where Ω is a locally compact, second countable Hausdorff space and μ is the completion of the Riesz representation of a Radon measure, and an isomorphism $\Phi : L(\mathcal{F}) \to L^\infty(\Omega, \mathcal{S}, \mu)$ such that $P \in \mathcal{F}$ if and only if $\Phi(P)$ is (the equivalence class of) the indicator function for a set in $\mathcal{S}$.*

Proof: We have shown that the subspace $L(\mathcal{F})$ is a commutative C^*-algebra with the special property that it contains all spectral projections of each of its self-adjoint elements. In fact, we proved that each such projection is already included in $\mathcal{F}$. In particular, each projection in $L(\mathcal{F})$ is in $\mathcal{F}$, and since $\mathcal{F}$ is stable under arbitrary sums of orthogonal projections, cf.

[1] A norm closed $*$-algebra of linear operators on a Hilbert space is called a (concrete) C^*-algebra. Thus $L(\mathcal{F})$ is an abelian (commutative) C^*-algebra.

[2] None of the theorems in Section 4 depends on the material in this section.

Definition 2.2 (*iv*), we derive that $L(\mathcal{F})$ has the same property. But a C^*-algebra of linear operators on a Hilbert space which contains all the spectral projections of its self-adjoint elements and is stable under arbitrary sums of mutually orthogonal projections is necessarily strongly (and weakly) closed, cf. [12, 2.8.4 (*iv*) Theorem] and is therefore a von Neumann algebra[3]. Since it is also abelian (commutative) and H is separable there exists, cf. [12, 3.4.4 Theorem], a probability space $(\Omega, \mathcal{S}, \mu)$ where Ω is a locally compact, second countable Hausdorff space and μ is the completion of the Riesz representation of a Radon measure, and an isomorphism $\Phi : L(\mathcal{F}) \to L^\infty(\Omega, \mathcal{S}, \mu)$. Since trivially the set of projections in $L(\mathcal{F})$ is $\mathcal{F}$, cf. Proposition 2.4, the last part of the statement follows. **QED**

In this formulation of the standard model we no longer rely on the notion of a state space, which may be inaccessible to observation. The probability space $(\Omega, \mathcal{S}, \mu)$ is a purely mathematical construction generated by the lattice of observable events.

3 The lattice of events

In this section we list the most basic and intuitive criteria associated with the notion of an event. They constitute the minimal requirements to any sensible theory of decision making. The (observable) events are represented by a lattice $(\mathcal{F}, \leq)$, which is a partially ordered set such that

(1) There are elements 0 and 1 in $\mathcal{F}$ such that $0 \leq a \leq 1$ for all $a \in \mathcal{F}$.

(2) To arbitrary events $a, b \in \mathcal{F}$ there is a minorant event $a \wedge b \in \mathcal{F}$. It has the property that $c \leq a \wedge b$ for any event $c \in \mathcal{F}$ with $c \leq a$ and $c \leq b$.

(3) To arbitrary events $a, b \in \mathcal{F}$ there is a majorant event $a \vee b \in \mathcal{F}$. It has the property that $a \vee b \leq c$ for any event $c \in \mathcal{F}$ with $a \leq c$ and $b \leq c$.

If $a \leq b$ for events $a, b \in \mathcal{F}$, then we consider b to be a larger or more comprehensive event than a. The minorant event $a \wedge b$ is for arbitrary events $a, b \in \mathcal{F}$ interpreted as the combination of a and b, while the majorant event (union) $a \vee b$ represents the event of either a or b.

The observable events are divided into two classes: The obtaining (occurring) events and the non-obtaining events. We assert by convention that 0 represents the vacuous (empty) event, while 1 represents the universal (sure) event. We assume that if $a \leq b$ for events $a, b \in \mathcal{F}$ and a is obtaining, then b is also obtaining. This is in line with the interpretation of b as being a more comprehensive event than a.

We furthermore assume the existence of an orthocomplementation of $\mathcal{F}$, that is a bijective mapping $a \to a^\perp$ of $\mathcal{F}$ onto itself such that

(4) $a \leq b \Rightarrow b^\perp \leq a^\perp \quad$ for all $a, b \in \mathcal{F}$.

[3] A von Neumann algebra is a weakly closed $*$-algebra of linear operators on a Hilbert space.

(5) $a^{\perp\perp} = a$ for all $a \in \mathcal{F}$.

(6) $a \wedge a^{\perp} = 0$ for all $a \in \mathcal{F}$.

(7) $a \vee a^{\perp} = 1$ for all $a \in \mathcal{F}$.

Definition 3.1 *We say that events a and b in $\mathcal{F}$ are mutually exclusive if $a \wedge b = 0$. They are said to be orthogonal if $a \leq b^{\perp}$.*

Proposition 3.2 *We list the following immediate consequences of the preceding axioms:*

(i) $1^{\perp} = 0$ *and* $0^{\perp} = 1$.

(ii) The events a and $a^{\perp}$ are complementary, meaning that exactly one of them occurs.

(iii) The orthocomplementation mapping $a \to a^{\perp}$ is a bijection from the set of occurring events onto the set of non-occurring events.

(iv) Orthogonal events are mutually exclusive.

Proof: We deduce (i) from (1) and (4), and (ii) from (6) and (7). In combination with (5) we then obtain (iii). Suppose $a \leq b^{\perp}$ and c is an event such that $c \leq a$ and $c \leq b$. We then have $c \leq a \leq b^{\perp}$ and thus $c \leq b \wedge b^{\perp} = 0$ by (6). Therefore $a \wedge b = 0$ which proves (iv). **QED**

We furthermore assume the following extension of axioms (2) and (3):

(8) To any family $(a_i)_{i \in I}$ of events in $\mathcal{F}$ there is a minorant event $\wedge_{i \in I} a_i$ in $\mathcal{F}$ and a majorant event $\vee_{i \in I} a_i$ in $\mathcal{F}$.

We also consider the following axiom, which we do not in general require to be valid:

(9) $a \wedge b = 0 \Rightarrow a \leq b^{\perp}$ for all $a, b \in \mathcal{F}$.

Axiom (9) states that mutually exclusive events are orthogonal.

4 The general model

We will not try to determine all representations of the "lattice of events" satisfying the eight axioms in the preceding section, but rather exhibit a large class of representations.

Definition 4.1 (the general model) A representation of the general model is specified by a family $\mathcal{F}$ of projections (the observable events) on a separable Hilbert space H satisfying:

(i) The zero projection on H (denoted 0) and the identity projection on H (denoted 1) are both contained in $\mathcal{F}$.

(ii) $1 - P \in \mathcal{F}$ for arbitrary $P \in \mathcal{F}$.

(iii) $P \wedge Q \in \mathcal{F}$ for arbitrary $P, Q \in \mathcal{F}$.

(iv) $\sum_{i \in I} P_i \in \mathcal{F}$ for any family $(P_i)_{i \in I}$ of mutually orthogonal projections in $\mathcal{F}$.

The minorant event to $P, Q \in \mathcal{F}$ is the minorant projection $P \wedge Q$ on the intersection of the ranges of P and Q. The majorant event is the majorant projection $P \vee Q$ on the closure of the sum of the ranges. The orthocomplementation is given by $P^\perp = 1 - P$.

We first collect some useful facts.

Lemma 4.2 *If P and Q are self-adjoint projections on a Hilbert space, then*

(i) $P \wedge Q = 1 - (1 - P) \vee (1 - Q)$

(ii) $P \leq Q \quad \Rightarrow \quad Q - P = Q \wedge (1 - P)$.

Proof: By construction the majorant $(1 - P) \vee (1 - Q)$ majorizes $1 - P$ and $1 - Q$ hence $1 - (1 - P) \vee (1 - Q)$ is majorized by P and Q, thus $P \wedge Q \geq 1 - (1 - P) \vee (1 - Q)$. On the other hand $1 - (1 - P) \vee (1 - Q) \geq 1 - (1 - P \wedge Q) \vee (1 - P \wedge Q) = P \wedge Q$. This proves (i).

Suppose $P \leq Q$. The difference $Q - P$ is a projection and since $Q \geq Q - P$ and $1 - P \geq Q - P$, we obtain $Q \wedge (1 - P) \geq Q - P$. Let R be a projection such that $R \leq Q$ and $R \leq 1 - P$. We obtain

$$R + P = QRQ + QPQ = Q(R + P)Q \leq Q,$$

hence $R \leq Q - P$ and thus $Q \wedge (1 - P) \leq Q - P$. This proves (ii). **QED**

Theorem 4.3 *A family $\mathcal{F}$ of projections on a separable Hilbert space H satisfying the conditions in Definition 4.1, with lattice operations and orthocomplementation as specified, satisfy the axioms (1) to (8).*

Proof: Axioms (1) and (2) follow from conditions (i) and (iii) respectively, and axiom (3) then follows by invoking Lemma 4.2 (i). The orthocomplementation map defined by $P^\perp = 1 - P$ is by condition (ii) a bijection of $\mathcal{F}$, and the axioms (4), (5), (6) and (7) are well-known properties of projections.

We shall finally consider axiom (8). Let $(P_i)_{i \in I}$ be a family of projections in $\mathcal{F}$. We consider the subsets $J \subseteq I$ such that the majorant $P_J = \vee_{j \in J} P_j$ is in $\mathcal{F}$. The set $\mathcal{D}$ of such subsets of I is inductively ordered. Indeed, if $J(\alpha)$ is a monotone increasing net in $\mathcal{D}$ with union J, then the majorant projection P_J is the limit of the monotone increasing net $(P_{J(\alpha)})$ of projections in $\mathcal{F}$. But since a monotone increasing net in $\mathcal{F}$ by transfinite induction can be replaced with a net in $\mathcal{F}$ of orthogonal projections with the same majorant, cf. Lemma 4.2

(ii), we obtain from condition (iv) that the majorant P_J is in $\mathcal{F}$ and thus $J \in \mathcal{D}$. Therefore, by Zorn's lemma, $\mathcal{D}$ has a maximal element J_0. If the majorant

$$P = \bigvee_{j \in J_0} P_j < \bigvee_{i \in I} P_i$$

then there is a $i \in I$ such that $P < P \vee P_i$. But since $P \in \mathcal{F}$ and thus $P \vee P_i \in \mathcal{F}$ this contradicts the maximality of J_0. We conclude that $\vee_{i \in I} P_i = P \in \mathcal{F}$. The statement for minorants then follows by Lemma (4.2) (i). **QED**

Theorem 4.4 *A representation $\mathcal{F}$ of the general model is commutative, if and only if axiom (9) holds.*

Proof: If $\mathcal{F}$ is commutative, then $P \wedge Q = PQ$ for any projections P and Q in $L(\mathcal{F})$. Therefore, if $P \wedge Q = 0$ the projections are orthogonal, and thus $P \leq Q^{\perp}$.

Let P and Q be arbitrary events in $\mathcal{F}$. Since both $P \leq P \vee Q$ and $Q \leq P \vee Q$ we obtain

$$(4.2) \qquad (P \vee Q) - P = (P \vee Q) \wedge (1 - P) \quad \text{and} \quad (P \vee Q) - Q = (P \vee Q) \wedge (1 - Q)$$

by Lemma 4.2 (ii). The projections $(P \vee Q) - P$ and $(P \vee Q) - Q$ are therefore events in $\mathcal{F}$ by axioms (3) and (2). Let R be a projection majorized by both $(P \vee Q) - P$ and $(P \vee Q) - Q$. We have in particular

$$R \leq P \vee Q = 1 - (1 - P) \wedge (1 - Q),$$

cf. Lemma 4.2 (i). But we also have $R \leq 1 - P$ and $R \leq 1 - Q$, hence $R \leq (1 - P) \wedge (1 - Q)$. Adding the two inequalities we obtain $2R \leq 1$, thus $R = 0$ since R is a projection. Therefore

$$((P \vee Q) - P) \wedge ((P \vee Q) - Q) = 0.$$

The events $(P \vee Q) - P$ and $(P \vee Q) - Q$ are thus mutually exclusive. If we assume axiom (9) they are therefore also orthogonal, hence

$$((P \vee Q) - P)((P \vee Q) - Q) = (P \vee Q) - Q - P + PQ = 0$$

or $PQ = P + Q - (P \vee Q)$. Since the right hand side is symmetric in P and Q we obtain $PQ = QP$. Therefore $\mathcal{F}$ is commutative. **QED**

Corollary 4.5 *A representation of the general model is a representation of the standard model, if and only if axiom (9) is satisfied.*

4.1 Securities and portfolios

Let $(\mathcal{F}, H)$ be a representation of the general model. An event P is also called a pure security. It pays one unit if it obtains and zero if it does not. In the standard model a pure security is called an Arrow security. The pure securities P and $1-P$ are complementary events. An element $A \in B(H)$ of the form

$$A = \sum_{i=1}^{k} \lambda_i P_i, \tag{4.3}$$

where $P_1, \dots, P_k$ are mutually orthogonal projections in $\mathcal{F}$ with sum $P_1 + \cdots + P_k = 1$ and $\lambda_1, \dots, \lambda_k$ are real numbers, is called a simple security. The projections $P_1, \dots, P_k$ are the events associated with the security and the numbers $\lambda_1, \dots, \lambda_k$ are the corresponding dividends. Since exactly one of the events $P_1, \dots, P_k$ obtains, we know that A pays one of the numbers $\lambda_1, \dots, \lambda_k$ as dividend.

Theorem 4.6 *Let $(\mathcal{F}, H)$ be a representation of the general model, and let P be a self-adjoint projection on H. If there exists an upwards filtering net $(A_i)_{i\in I}$ of positive semi-definite simple securities such that $A_i \nearrow_i P$, then $P \in \mathcal{F}$.*

Proof: Suppose A is a positive-definite simple security majorized by P, that is

$$A = \sum_{j=1}^{k} \lambda_j Q_j \le P,$$

where $\lambda_1, \dots, \lambda_k$ are non-negative numbers and $Q_1, \dots, Q_k$ are orthogonal projections in $\mathcal{F}$ with sum one. Then $\lambda_j Q_j \le A \le P$ for $j = 1, \dots, k$ and thus, for $\lambda_j > 0$, we obtain $\lambda_j \le 1$ and $Q_j \le P$. Consequently,

$$A = \sum_{j=1}^{k} \lambda_j Q_j \le \sum_{\lambda_j > 0} Q_j = Q \le P.$$

But since Q is a projection in $\mathcal{F}$, we derive that P is the majorant projection of a family of projections in $\mathcal{F}$, hence $P \in \mathcal{F}$ by Theorem 4.3.

Definition 4.7 *Let $(\mathcal{F}, H)$ be a representation of the general model. An operator $A \in B(H)$ is called a security, if all its spectral projections are events in $\mathcal{F}$. The set of securities is denoted by $A(\mathcal{F})$.*

Definition 4.8 *Let $(\mathcal{F}, H)$ be a representation of the general model. The space of portfolios $L(\mathcal{F})$ is the norm closed linear subspace of $B(H)$ generated by $A(\mathcal{F})$.*

The spectral projections of a portfolio are not generally included in $\mathcal{F}$ and may therefore not be observable events. There is hence a sharp distinction between securities and portfolios in the general model, while the distinction is more blurred and non-essential in the standard model. We find this situation quite natural as we would not expect to be able to replicate a general portfolio as a single security with a well-defined associated family of events and dividends. An example of a portfolio is an operator of the form

$$A = \int A_t \, d\mu(t), \tag{4.4}$$

where $t \to A_t$ is a norm continuous family of securities and μ is a bounded positive measure.

4.2 Expectations

We consider a representation $(\mathcal{F}, H)$ of the general model. Subjective beliefs are specified by norm continuous linear mappings $\varphi : L(\mathcal{F}) \to \mathbf{R}$ satisfying

(1) $\varphi(1) = 1$

(2) $\varphi(A) \geq 0$ for any positive semi-definite $A \in L(\mathcal{F})$

(3) $\varphi(A_i) \nearrow_i \varphi(A)$ for each upwards filtering net $(A_i)_{i \in I}$ in $L(\mathcal{F})$ converging to a portfolio $A \in L(\mathcal{F})$,

and we denote by $S(\mathcal{F})$ the set of such mappings. In the case of the standard model these mappings are given by probability measures on the state space.

If an agent's expectations are given by $\varphi \in S(\mathcal{F})$ and $P \in \mathcal{F}$ then $\varphi(P)$ is the expected probability of P occurring. Moreover, for events $P, Q \in \mathcal{F}$ we have the implication

$$P \leq Q \quad \Rightarrow \quad \varphi(P) \leq \varphi(Q).$$

It follows, since $\varphi(Q) - \varphi(P) = \varphi(Q - P) \geq 0$ where we used the linearity of φ and (2). Agents in the economy are thus forced to consider more comprehensive events to be also more likely. The following result is standard[4]

Lemma 4.9 *If A is an element in $L(\mathcal{F})$ and $\varphi(A) = 0$ for all $\varphi \in S(\mathcal{F})$, then $A = 0$.*

If $P, Q \in \mathcal{F}$ are two events such that $\varphi(P) = \varphi(Q)$ for every $\varphi \in S(\mathcal{F})$, then $P = Q$ according to the above lemma. That is, if all possible agents agree on the likelihood of two events, then the events coincide.

If the condition in Axiom (9), that is the implication

$$P \wedge Q = 0 \quad \Rightarrow \quad P \leq 1 - Q$$

holds for some family of events in $\mathcal{F}$, then

$$P \wedge Q = 0 \quad \Rightarrow \quad \varphi(P \vee Q) = \varphi(P) + \varphi(Q)$$

for such events just like in the standard model. This follows because the majorant event $P \vee Q$ to two mutually orthogonal events P and Q is the algebraic sum $P + Q$. In general, as we shall demonstrate in the next section, there exist mutually exclusive events $P, Q \in \mathcal{F}$ such that

$$\varphi(P \vee Q) \neq \varphi(P) + \varphi(Q)$$

for some $\varphi \in S(\mathcal{F})$.

[4]Indeed, the functional φ_ξ defined by setting $\varphi_\xi(B) = (B\xi \mid \xi)$ for $B \in L(\mathcal{F})$ is for each unit vector $\xi \in H$ in $S(\mathcal{F})$, so we have $(A\xi \mid \xi) = 0$ by assumption. By polarization we obtain $(A\xi \mid \eta) = 0$ for all $\xi, \eta \in H$, hence $A = 0$ as desired.

We calculate the expected dividend for a $\varphi \in S(\mathcal{F})$ of the simple security in (4.3) to be

$$\sum_{i=1}^{k} \lambda_i \varphi(P_i) = \varphi(A).$$

If A is a security with spectral measure E it follows that $B \to \varphi(E(B))$ is a bounded positive Borel measure with compact support and the expected dividend

$$\int \lambda \, d\varphi(E(\lambda)) = \varphi(A) \tag{4.5}$$

for each $\varphi \in S(\mathcal{F})$.

The von Neumann algebra $L^\infty(\Omega, \mathcal{F}, \mu)$ in the standard model is finite dimensional when the state space Ω is essentially finite. Corresponding to this case we may in the general model consider representations $(\mathcal{F}, H)$ where H is a Hilbert space of finite dimension.

Proposition 4.10 *Let $(\mathcal{F}, H)$ be a representation of the general model and assume that the Hilbert space H is of finite dimension. Then there exists to each $\varphi \in S(\mathcal{F})$ a positive semi-definite operator B in $L(\mathcal{F})$ with trace $\operatorname{Tr} B = 1$ such that*

$$\varphi(A) = \operatorname{Tr}(AB)$$

for each $A \in L(\mathcal{F})$.

Proof: Applying Hahn-Banach's theorem we may extend φ, defined on the closed linear subspace $L(\mathcal{F})$, to a positive linear functional $\tilde{\varphi}$ on $B(H)$ with $\tilde{\varphi}(1) = 1$. Hence there exists [10, 4.6.18] a positive semi-definite operator $C \in B(H)$ with trace $\operatorname{Tr} C = 1$ such that

$$\tilde{\varphi}(A) = \operatorname{Tr}(AC) \qquad \forall A \in B(H).$$

The linear space $B(H)$ is a (finite dimensional) Hilbert space with the inner product defined by $(A \mid B) = \operatorname{Tr}(A^* B)$. The orthogonal projection (with respect to this inner product) onto $L(\mathcal{F})$ is thus a conditional expectation $\Phi : B(H) \to L(\mathcal{F})$. Setting $B = \Phi(C)$ we obtain

$$\varphi(A) = \tilde{\varphi}(A) = \operatorname{Tr}(AC) = \operatorname{Tr}(A\,\Phi(C)) = \operatorname{Tr}(AB)$$

for each $A \in L(\mathcal{F})$. Furthermore, $B \in L(\mathcal{F})$ and $\operatorname{Tr} B = \varphi(1) = 1$. **QED**

4.3 The interpretation of a portfolio

Let $(\mathcal{F}, H)$ be a representation of the general model, and suppose that an agent with beliefs given by $\varphi \in S(\mathcal{F})$ consider two securities A_1 and A_2 in $A(\mathcal{F})$.

The expected dividend of the portfolio consisting of the two securities is given by the sum $\varphi(A_1) + \varphi(A_2)$, and since φ is linear this figure is equal to $\varphi(A_1 + A_2)$. The operator $A_1 + A_2 \in L(\mathcal{F})$ may therefore be used to calculate the agent's expected dividend coming from the portfolio of the two securities. In fact, it follows from Lemma 4.9 that the algebraic sum $A_1 + A_2$ is the only operator in $L(\mathcal{F})$ reproducing the expected dividend of the portfolio for all possible agents.

Scholium 4.11 *A portfolio of two securities A_1 and A_2 in $A(\mathcal{F})$ is represented by the algebraic sum $A_1 + A_2 \in L(\mathcal{F})$.*

We should notice that the representing operator $A_1 + A_2$ may not be a security since it is entirely possible that some of its spectral projections are not in $\mathcal{F}$. In fact every pair of events, associated with A_1 and A_2 respectively, may be mutually exclusive. In the standard model we may add dividends in each state, but such a description is not generally possible in the general model.

5 Ellsberg's paradox

Let $(\mathcal{F}, H)$ be the representation of the general model such that H is of dimension three and $\mathcal{F}$ is the set of all self-adjoint projections on H. Note that in this case $L(\mathcal{F}) = B(H)$.

We consider three events A, B and C representing the drawing of a red, a blue, and a black ball from a box. They are given by the projections:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

It is an easy calculation to show that the events are mutually exclusive, that is $A \wedge B = 0$, $B \wedge C = 0$, and $A \wedge C = 0$. The event of drawing either a red ball or a black ball, or a blue ball or a black ball are given by the projections

$$A \vee C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \qquad \text{and} \qquad B \vee C = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

We consider expectations $\varphi \in S(\mathcal{F})$ by setting $\varphi(A) = (A\xi \mid \xi)$ for $A \in L(\mathcal{F})$ where the unit vector $\xi = 2^{-1}(\sqrt{2}, -1, 1)$ and calculate

$$\begin{aligned} \varphi(A) &= \frac{1}{2} \qquad & \varphi(A \vee C) &= \frac{1}{2} \\ \varphi(B) &= \frac{1}{4} \qquad & \varphi(B \vee C) &= \frac{5}{8} + \frac{\sqrt{2}}{4} \simeq 0.9786. \end{aligned}$$

If the pay-offs of the events A, B and C are equal, then a rational agent with monotone preferences will choose the events with the highest probabilities and therefore prefer A for B, but $B \vee C$ for $A \vee C$. It is thus rational to prefer a red ball for a blue, but a blue ball or a black ball for a red ball or a black ball, although such preferences contradict Savage's Postulate 2 or the "Sure-Thing Principle". This is obtained without using non-additive measures.

Since $\varphi(C) = 1/6$ we have $\varphi(A \vee C) < \varphi(A) + \varphi(C)$ while $\varphi(B \vee C) > \varphi(B) + \varphi(C)$. We realize that there is no simple relationship between the probabilities of two mutually exclusive events and the probability of their majorant event. This is very natural in every day life. The expected probability of building either of two competing bridge designs at a certain location is not the sum of the expected probabilities of building each of them.

6 Arbitrage free asset valuation

6.1 A one-period model

Let $(\mathcal{F}, H)$ be a representation of the general model. We shall assume that the Hilbert space H is of finite dimension which corresponds to a finite state space in the standard model. This is far from the most general situation that can be handled, but it has the advantage that we are able to explain the theory without using general tools from functional analysis.

We consider an economy consisting of n securities $A = (A_1, \dots, A_n)$ in $A(\mathcal{F})$ with beginning-of-period price vector $q = (q_1, \dots, q_n)$. A portfolio $\theta \cdot A$ is given by a vector of weights $\theta = (\theta_1, \dots, \theta_n)$ and the beginning-of-period price of the portfolio is $\theta \cdot q$. The expectations of an agent are specified by a $\varphi \in S(\mathcal{F})$, and the agent's expected return of the portfolio is thus

$$\varphi(\theta \cdot A) = \sum_{i=1}^{n} \theta_i \varphi(A_i).$$

Let Tr denote the trace on $B(H)$.

Definition 6.1 An arbitrage is a portfolio vector $\theta \in \mathbf{R}^n$ such that $\theta \cdot A \geq 0$ and either the price $\theta \cdot q < 0$, or the price $\theta \cdot q = 0$ and $\theta \cdot A$ is non-vanishing.

Theorem 6.2 *There is no arbitrage in the economy $(\mathcal{F}, H, A, q)$, if and only if there exists a positive definite element $B \in L(\mathcal{F})$ such that the asset prices $q_i = \mathrm{Tr}\,(A_i B)$ for $i = 1, \dots, n$.*

Proof: Suppose there is a positive definite element $B \in L(\mathcal{F})$ such that $q_i = \mathrm{Tr}\,(A_i B)$ for $i = 1, \dots, n$ and let $\theta \in \mathbf{R}^n$ be any portfolio vector. Then

$$\begin{aligned}\theta \cdot q &= \theta_1 q_1 + \cdots + \theta_n q_n = \theta_1 \mathrm{Tr}\,(A_1 B) + \cdots + \theta_n \mathrm{Tr}\,(A_n B)\\ &= \mathrm{Tr}\,((\theta \cdot A) B) = \mathrm{Tr}\,(B^{1/2}(\theta \cdot A) B^{1/2}).\end{aligned}$$

The trace is a positive linear functional, thus $\theta \cdot q \geq 0$ for $\theta \cdot A \geq 0$. If in addition $\theta \cdot A$ is non-vanishing, then the operator $B^{1/2}(\theta \cdot A)B^{1/2}$ is positive semi-definite and non-vanishing, thus it has positive trace. Therefore the portfolio price $\theta \cdot q > 0$ and there are no arbitrage possibilities. We consider the real vector space $\mathbf{R} \times L(\mathcal{F})_{sa}$ with the positive cone $K = [0, \infty) \times L(\mathcal{F})_+$ and the subspace

$$M = \{(-\theta \cdot q,\, \theta \cdot A) \mid \theta \in \mathbf{R}^n\}.$$

Suppose there are no arbitrage possibilities. Then $M \cap K = \{(0,0)\}$ and since K is convex, there exists a hyperplane U in $\mathbf{R} \times L(\mathcal{F})_{sa}$ such that

$$M \subseteq U \quad \text{and} \quad U \cap K = \{(0,0)\}.$$

Consequently, there exists a continuous linear functional F on $\mathbf{R} \times L(\mathcal{F})_{sa}$ with kernel U, and we may without loss of generality assume $F(\lambda, x) > 0$ for each $(\lambda, x) \in K \backslash \{(0,0)\}$ and $F(0,1) = 1$. In particular, the number $\alpha = F(1,0) > 0$. Since F is linear we obtain

$$F(\lambda, x) = F(\lambda, 0) + F(0, x) = \lambda\alpha + F(0, x) \qquad \lambda \in \mathbf{R},\, x \in L(\mathcal{F})_{sa}.$$

The linear functional $(0, x) \to F(0, x)$ may be extended to a positive linear functional φ in $S(\mathcal{F})$ and this implies, cf. Proposition 4.10, the existence of a positive definite element $C \in L(\mathcal{F})$ such that $\varphi(0, x) = \mathrm{Tr}\,(xC)$ for any $x \in L(\mathcal{F})$. Since $M \subseteq U$ we thus have

$$F(-\theta \cdot q, \theta \cdot A) = -(\theta \cdot q)\alpha + \mathrm{Tr}\,((\theta \cdot A)C) = 0$$

for any portfolio vector $\theta \in \mathbf{R}^n$. Setting $B = \alpha^{-1}C$ we obtain $B > 0$ and

$$\theta \cdot q = \mathrm{Tr}\,((\theta \cdot A)B) \qquad \forall \theta \in \mathbf{R}^n.$$

In particular $q_i = \mathrm{Tr}\,(A_i B)$ for $i = 1, \cdots, n$. **QED**

Suppose there is riskless borrowing in the arbitrage free economy $(\mathcal{F}, H, A, q)$, that is a portfolio $\theta \cdot A$ given by a portfolio vector $\theta \in \mathbf{R}^n$ such that the expected end-of-period dividend $\varphi(\theta \cdot A) = 1$ for any $\varphi \in S(\mathcal{F})$. Then $\theta \cdot A$ is the identity matrix by Lemma 4.9, and the beginning-of-period price of the portfolio

$$\theta \cdot q = \mathrm{Tr}\,((\theta \cdot A)B) = \mathrm{Tr}\, B$$

for a positive definite $B \in L(\mathcal{F})$. The trace of B is therefore the discount on riskless borrowing. Let us in this situation define the density matrix

$$Q = (\mathrm{Tr}\, B)^{-1} B$$

and set $\rho = \mathrm{Tr}\,(B)$. We define expectations $E^Q \in S(\mathcal{F})$ by setting

$$E^Q(X) = \mathrm{Tr}\,(QX) \qquad X \in L(\mathcal{F}).$$

Note that $E^Q(1) = \mathrm{Tr}\, Q = 1$. The mapping E^Q is equivalent to the trace since B is positive definite, and the security prices

$$q_i = \mathrm{Tr}\,(A_i B) = \rho \mathrm{Tr}\,(QA_i) = \rho E^Q(A_i) \qquad i = 1, \ldots, n$$

are the discounted expected dividends with respect to the equivalent functional $E^Q \in S(\mathcal{F})$.

References

[1] F.J. Anscombe and R.J. Aumann. A definition of subjective probability. *The annals of mathematical statistics.*, 34:199–205, 1963.

[2] K.J. Arrow. Le rôle des valeurs boursières pour la répartition la meilleure des risques. *Econométrie, Paris, Centre National de la Recherche Scientifique*, pages 41–48, 1953.

[3] N. Bourbaki. *Éléments de Mathématique, Livre VI Intégration.* Hermann, Paris, 1973.

[4] G. Debreu. *Theory of Value.* Yale University Press, New Haven and London, 1959.

[5] D. Duffie. *Dynamic Asset Pricing Theory.* Princeton University Press, Princeton, New Jersey, 1996.

[6] D. Ellsberg. Risk, ambiguity, and the Savage axioms. *Quarterly Journal of Economics*, 75:643–669, 1961.

[7] L.G. Epstein and T. Wang. Intertemporal asset pricing under Knightian uncertainty. *Econometrica*, 62:283–322, 1994.

[8] W. Fellner. Distortion of subjective probabilities as a reaction to uncertainty. *Quarterly Journal of Economics*, 75:670–689, 1961.

[9] I. Gilboa. Expected utility with purely subjective non-additive probabilities. *Journal of Mathematical Economics.*, 16:65–88, 1987.

[10] R.V. Kadison and J.R. Ringrose. *Fundamentals of the Theory of Operator Algebras, I-II.* Academic Press, 1982 & 1986.

[11] F.H. Knight. *Risk, Uncertainty and Profit.* Houghton Mifflin, Boston, 1921.

[12] G.K. Pedersen. *C^*-Algebras and their Automorphism Groups.* Academic Press, London, 1979.

[13] J. Quiggin. A theory of anticipated utility. *Journal of economic behaviour and organization*, 3:323–343, 1982.

[14] H. Raiffa. Risk, ambiguity, and the Savage axioms: Comment. *Quarterly Journal of Economics*, 75:690–694, 1961.

[15] L. Savage. *The Foundations of Statistics.* John Wiley, New York, 1954.

[16] D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57:571–587, 1989.

[17] V.S. Varadarajan. *Geometry of Quantum Theory*, volume I. D. Van Nostrand Company, Inc., Princeton, New Jersey-Toronto-London-Melbourne, 1968.