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Adaptive Learning in Stochastic Nonlinear Models when Shocks Follow a Markov Chain

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The participants in a September 2002 Workshop on *Topics in Mathematical Economics* in honor of Birgit Grodal decided to have a series of papers appear on Birgit Grodal's 60'th birthday, June 24, 2003.

The Institute of Economics suggested that the papers became Discussion Papers from the Institute.

The editor of *Economic Theory* offered to consider the papers for a special Festschrift issue of the journal with Karl Vind as Guest Editor.

This paper is one of the many papers sent to the Discussion Paper series.

Most of these papers will later also be published in a special issue of *Economic Theory*.

Tillykke Birgit

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Adaptive Learning in Stochastic Nonlinear Models When Shocks Follow a Markov Chain*

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Abstract

Local convergence results for adaptive learning of stochastic steady states in nonlinear models are extended to the case where the exogenous observable variables follow a finite Markov chain. The stability conditions for the corresponding nonstochastic model and its steady states yield convergence for the stochastic model when shocks are sufficiently small. The results are applied to asset pricing and to an overlapping generations model. Large shocks can destabilize learning even if the steady state is stable with small shocks.

Key words: bounded rationality, recursive algorithms, steady state, linearization, asset pricing, overlapping generations

JEL Classification: C62, C61, D83

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1 Introduction

We consider stability under adaptive learning of stochastic steady state equilibria for nonlinear expectations models of the form

$$y_t = H(E_t^* G(y_{t+1}, w_{t+1}), w_t), \tag{1}$$

where y_t is a scalar endogenous variable and w_t is an exogenous random shock. Expectations $E_t^*(.)$ may not always be rational and under rational expectations we denote them by $E_t(.)$. Explicit stability results for this model have been obtained by (Evans and Honkapohja 1995) under the restrictive assumption that w_t is an *iid* process. In this paper we extend the stability results to the case where w_t is a time-dependent process taking the form of a finite Markov chain. The extension is useful since many applied models make the Markov chain assumption for the shock process, see e.g. (Hamilton 1989) and (Mehra and Prescott 1985).

The finiteness of the Markov chain is a limitation in our results, but it allows the formulation of adaptive learning in terms of a finite number of parameters. More general assumptions for shocks would lead to the description of stochastic states in terms of infinite dimensional parameters. For example, if w_t in (1) is a general Markov process the stochastic steady state is likely to be a function $y(w_t)$ that cannot in general be expressed in parametric form. Agents would have to be estimate $y(w_t)$ using advanced nonparametric methods. Such an approach has been studied by (Chen and White 1998), but they do not develop explicit stability conditions in terms of the properties of the underlying economic model. Moreover, models of adaptive learning are based on the hypothesis of boundedly rational agents and the assumption that such agents are sufficiently sophisticated to use non-parametric techniques seems to go against the spirit of the hypothesis.[1]

The class of models (1) with autocorrelated shocks is also important for a different reason. The applied literature very often studies linearizations or log-linearizations of nonlinear Euler equations. The linearization is usually done around a non-stochastic steady state under the assumption that the support of the exogenous shock process is small. We provide existence results for models with small Markov chain shocks and then relate the general stability conditions to the linearized case and thus provide a bridge between

[1] Instead the agents might use a simpler but mis-specified parametric model. For brevity, we do not consider this possibility further.

the nonlinear and the linearized settings.[2]

The results are applied to a standard model of asset pricing and a stochastic overlapping generations model. The stochastic steady state in the asset pricing model is locally stable under learning irrespective of the size of the shocks. In contrast, in the overlapping generations model it is possible that the steady state is stable under learning with small shocks but unstable with sufficiently large shocks.

2 The Model

The precise assumptions for model (1) are as follows. $H(.)$ and $G(.)$ are twice differentiable with bounded first and second derivatives in some open rectangles. $G(y_t, w_t)$ is assumed to be observable. w_t is an observable exogenous variable that follows a finite Markov chain with states $\{\hat{w}_1, ..., \hat{w}_K\}$ and transition probabilities $\pi_{ij} = \Pr\{w_{t+1} = \hat{w}_j \,| w_t = \hat{w}_i\}$. The transition matrix $\Pi = (\pi_{ij})$ is assumed to be recurrent, irreducible and aperiodic.

A *rational stochastic steady state* for the stochastic model (1) is defined as a set of points $\lambda_1^*, ..., \lambda_K^*$ such that

$$\begin{aligned} \text{if } w_t &= \hat{w}_k, \text{ then } \lambda_t = \lambda_k^*, \text{ where} \\ \lambda_k^* &= \sum_{s=1}^{K} \pi_{ks} G(H(\lambda_s^*, w_s), w_k). \end{aligned} \tag{2}$$

λ_k^* is interpreted as the value of $E_t G(y_{t+1}, w_{t+1})$ when the current state of the shock is k and the value of the endogenous variable is $y_t = H(\lambda_k^*, \hat{w}_k)$ if the exogenous variable has the value $\hat{w}_k$ in the current period t.

We remark that (2) is a natural definition of a stochastic steady state in the current setting. This is because the agents, having seen the current state of the shock in period t, take account of its value and correctly predict the expected value of $G(y_{t+1}, w_{t+1})$ conditionally on the current state with the economy being in the steady state next period, so that y_{t+1} has the alternative values $H(\lambda_s^*, w_s)$, $s = 1, ..., K$.

Model (1) with zero shocks is assumed to have a non-stochastic steady state, which can defined as a solution $\bar{\lambda}$ to the equation

$$\bar{\lambda} = G(H(\bar{\lambda}, 0), 0). \tag{3}$$

[2](Evans and Honkapohja 1998) study a linear model with an exogenous variable that follows a finite Markov chain.

Next, we introduce the notation $\bar{G}_1 = D_1G(H(\bar{\lambda},0),0)$, $\bar{H}_1 = D_1H(\bar{\lambda},0)$, $\|w\| = \max_{1\leq k\leq K}|w_k|$ .and make the following regularity assumption:

Regularity: $\bar{G}_1\bar{H}_1$ is non-zero and $(\bar{G}_1\bar{H}_1)^{-1}$ is not an eigenvalue of Π.

Under the regularity assumption we have existence of stochastic steady states with small Markov chain shocks:

Proposition 1 *Suppose that model (1) has a non-stochastic steady state $\bar{\lambda}$ defined by (3), the Regularity assumption holds for $\bar{\lambda}$. Assume that the transition matrix Π is recurrent, irreducible and aperiodic. Then $\exists\hat{\varepsilon}$ such that $\forall\varepsilon < \hat{\varepsilon}$ model (1) has a stochastic steady state $\lambda_1^*, ..., \lambda_K^*$ defined by equations (2) for Markov chain shocks with transition matrix Π and states $\{\hat{w}_1, ..., \hat{w}_K\}$ that satisfy $\|w\| \leq \varepsilon$.*

Proof. Consider the set of equations (2), which we write in vector form $F(\lambda,w) = 0$, where $\lambda = (\lambda_1, ..., \lambda_K)$ and $w = (w_1, ..., w_K)$. By the implicit function theorem this vector equation defines locally a function $\lambda(w)$ around $w = \mathbf{0} \equiv (0, ..., 0)$ if $\det(D_1F(\bar{\lambda}\mathbf{1},\mathbf{0})) \neq 0$. Here the notation $\bar{\lambda}\mathbf{1} =\bar{\lambda}(\mathbf{1}, ..., \mathbf{1})$ is used. It is easily computed that

$$\begin{aligned} D_1F(\bar{\lambda}\mathbf{1},\mathbf{0}) &= \begin{pmatrix} 1-\pi_{11}\bar{G}_1\bar{H}_1 & -\pi_{12}\bar{G}_1\bar{H}_1 & \cdots & -\pi_{1K}\bar{G}_1\bar{H}_1 \\ -\pi_{21}\bar{G}_1\bar{H}_1 & 1-\pi_{22}\bar{G}_1\bar{H}_1 & \cdots & -\pi_{2K}\bar{G}_1\bar{H}_1 \\ \vdots & \vdots & \ddots & \vdots \\ -\pi_{K1}\bar{G}_1\bar{H}_1 & -\pi_{K2}\bar{G}_1\bar{H}_1 & \cdots & 1-\pi_{KK}\bar{G}_1\bar{H}_1 \end{pmatrix} \\ &= \bar{G}_1\bar{H}_1[(\bar{G}_1\bar{H}_1)^{-1}I - \Pi], \end{aligned}$$

provided $\bar{G}_1\bar{H}_1 \neq 0$. Thus $\det(D_1F(\bar{\lambda}\mathbf{1},\mathbf{0})) \neq 0$ if $(\bar{G}_1\bar{H}_1)^{-1}$ is not an eigenvalue of Π. ■

Proposition 1 shows that, under mild assumptions, there exist steady state solutions to the stochastic model (1). We do not consider existence further since more general existence results, with shocks that are not small, are usually derived for concrete economic models rather than general classes of models such as (1). The asset pricing model with stochastic dividend growth due to (Mehra and Prescott 1985), which is discussed below, is an example of this approach towards existence of equilibria.

3 Convergence of Learning to Steady State

We now formulate the adaptive learning of a stochastic steady state (2) for model (1). Suppose that the agents do not know the steady state values $\lambda_1^*, ..., \lambda_K^*$ but try to infer them from past data. In the past the exogenous shock has been in the different states $k = 1, ...K$ and agents have perceptions that the economy is in an unknown steady state, as defined in (2). A plausible learning rule for this setting is that agents group the data into K different groups conditionally on occurrence of the different values w_k of the exogenous shock and compute the state-contingent averages of the group values of the relevant variable.

Thus let $\lambda_{j,t}$ be the estimate of the steady state value λ_j^* for period $t+1$ when the exogenous variable is in state $\hat{w}_j$ in period t. The temporary equilibrium given the forecasts and the current state $\hat{w}_j$ is then $y_t = H(\lambda_{j,t}, \hat{w}_j)$. Define also the indicator function $\psi_{j,t} = 1$ if $w_t = \hat{w}_j$ and $= 0$ otherwise. The learning rule can be written as

$$\begin{aligned} \lambda_{j,t} &= \lambda_{j,t-1} + t^{-1}\psi_{j,t-1}q_{j,t-1}^{-1}(\sum_{s=1}^{K}\pi_{js}G(H(\lambda_{s,t-1}, w_s), w_j) \qquad (4) \\ &\quad -\lambda_{j,t-1} + u_{t-1}) \\ q_{j,t} &= q_{j,t-1} + t^{-1}(\psi_{j,t-1} - q_{j,t-1}), \qquad (5) \end{aligned}$$

for $j = 1, ..., K$.

The equations of the learning algorithm can be interpreted as follows. $q_{j,t}$ is the fraction of observations through $t-1$ in which the state $\hat{w}_j$ has occurred. (5) is the recursive form for computing the fraction. (4) is the recursive form for computing state contingent averages, except for a small measurement or observation error u_{t-1}. u_{t-1} is assumed to be *iid* with mean 0 and bounded support. (The existence of u_{t-1} is needed only for the instability result.) We remark that the estimates used by agents at time t are based on observations only through period $t-1$.[3] Equation (4) also specifies that $\lambda_{j,t-1}$ is updated only if $w_{t-1} = \hat{w}_j$. We remark that, since the agents formulate the forecasts for next period values for λ_j conditionally on the current state j, they do not use the values λ_i for other states $i \neq j$. Data conditional on j does not provide useful information for estimating the other conditional expectations.

[3] This assumption is often used in the literature. It avoids a simultaneity problem between y_t and the forecasts $E_t^*G(y_{t+1}, w_{t+1})$.

The learning rule formulated above is an example of stochastic recursive algorithms (SRA). We employ the techniques for such systems to derive the conditions for convergence of adaptive learning to the stochastic steady state $(\lambda_1^*, ..., \lambda_K^*)$. It turns out that (see below for details) the conditions for local convergence of SRAs can be studied using the local asymptotic stability of the equilibrium of an associated ordinary differential equation (ODE). The stability conditions for latter can in turn be related to those of another ODE:

$$\frac{d\lambda_j}{d\tau} = \sum_{s=1}^{K} \pi_{js} G(H(\lambda_s, \hat{w}_s), \hat{w}_j) - \lambda_j, j = 1, ..., K. \tag{6}$$

Clearly, the steady state $(\lambda_1^*, ..., \lambda_K^*)$ is an equilibrium point of (6). We linearize (6) and introduce the notation $G_{js}^{(1)} = D_1 G(H(\lambda_s^*, \hat{w}_s), \hat{w}_j)$ and $H_s^{(1)} = D_1 H(\lambda_s^*, \hat{w}_s)$. Linearizing (6), we obtain:

Theorem 2 *Assume that no eigenvalue of the matrix*

$$M = \begin{pmatrix} \pi_{11} G_{11}^{(1)} H_1^{(1)} & \pi_{12} G_{12}^{(1)} H_2^{(1)} & ... & \pi_{1K} G_{1K}^{(1)} H_K^{(1)} \\ \pi_{21} G_{21}^{(1)} H_1^{(1)} & \pi_{22} G_{22}^{(1)} H_2^{(1)} & ... & \pi_{2K} G_{2K}^{(1)} H_K^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{K1} G_{K1}^{(1)} H_1^{(1)} & \pi_{K2} G_{K2}^{(1)} H_2^{(1)} & ... & \pi_{KK} G_{KK}^{(1)} H_K^{(1)} \end{pmatrix} \tag{7}$$

has real part equal to 1. *Then*
(i) The rational steady state $(\lambda_1^*, ..., \lambda_K^*)$ *is locally stable under learning if all the eigenvalues of* M *have real parts less than* 1.
(ii) The rational steady state is locally unstable under learning if M *has an eigenvalue with real part greater than* 1.

Proof. The recursive equations for the parameter estimates can be written in the form

$$\theta_t = \theta_{t-1} + t^{-1} \mathcal{H}(\theta_{t-1}, X_t),$$

where $\theta_t = (\lambda_{1,t}, ..., \lambda_{K,t}; q_{1,t}, ..., q_{K,t})'$ is the vector of parameters, $X_t = (\psi_{1,t-1}, ..., \psi_{K,t-1}, u_{t-1})'$ is the vector of state variables, and

$$\begin{aligned} \mathcal{H}_j(\theta_{t-1}, X_t) &= \psi_{j,t-1} q_{i,t-1}^{-1} (\sum_{s=1}^{K} \pi_{js} G(H(\lambda_{s,t-1}, w_s), w_j) - \lambda_{j,t-1} + u_{t-1}) \\ \text{for } j &= 1, ..., K, \text{ and} \\ \mathcal{H}_j(\theta_{t-1}, X_t) &= \psi_{j,t-1} - q_{j,t-1} \\ \text{for } j &= K+1, ...2K. \end{aligned}$$

It is then possible to apply theorems on the convergence of SRAs, see e.g. part II of (Evans and Honkapohja 2001). The assumptions made in Section 2 can easily be shown to imply the required convergence conditions.[4] The results state that, under the appropriate conditions, convergence of these algorithms is governed by the stability of the associated ODE

$$\frac{d\theta}{d\tau} = h(\theta) \text{ where } h(\theta) = \lim_{t\to\infty} E\mathcal{H}(\theta, X_t). \quad (8)$$

To compute the function $h(\theta)$ we first let $\bar{\pi}_1, ..., \bar{\pi}_K$ denote the invariant probabilities of the different states of the Markov chain $\Pi = (\pi_{ij})$. Then it is easily seen that

$$\begin{aligned} h_j(\theta) &= \bar{\pi}_j q_j^{-1}(\sum_{s=1}^{K} \pi_{js} G(H(\lambda_j, \hat{w}_j), \hat{w}_j) - \lambda_j), j = 1, .., K \\ h_j(\theta) &= \bar{\pi}_j - q_j, j = K+1, ..., 2K. \end{aligned}$$

The latter set of differential equations is independent of the former and is clearly globally stable with $q_j \to \bar{\pi}_j$. It follows that $(\lambda_j^*, \bar{\pi}_j), j = 1, ..., K$, is a locally asymptotically stable equilibrium point of the associated differential equation (8), provided λ_j^* is locally asymptotically stable equilibrium point for the "small" ODE (6). Result (i) follows.

To prove result (ii) we remark that the conditions for a standard instability result for SRAs are also satisfied.[5] ■

These results provide simple conditions to evaluate whether the steady state is stable under learning.[6] The relevant conditions are obtained by studying the "small" ODE (6). This relationship establishes that a concept known as expectational stability (or E-stability) is the key condition leading to stability under learning.[7] (Evans and Honkapohja 2001) provide an extensive discussion of this connection between convergence of adaptive learning and E-stability for different kinds of models.

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[4] The arguments on pp. 68-69 of (Evans and Honkapohja 1998) can be applied here to show that conditions for general algorithms are satisfied for the model under study.

[5] The formal details are analogous to those in (Evans and Honkapohja 1994), Proof of Proposition 5.2.

[6] For brevity we refrain from the details on the sense of convergence. See, Chapters 6 and 7 of (Evans and Honkapohja 2001) for a discussion.

[7] An REE is defined to be E-stable if it is a locally asymptotically stable equilibrium point of an ODE between given forecasts and the resulting temporary equilibrium. In model (1) the ODE defining E-stability is just (6).

4 The Case of Small Shocks

It is important to note that the derivatives in matrix M in Theorem 2 are evaluated at different points $(\lambda_j^*, \hat{w}_j)$. Thus it is first necessary to compute the steady state values $(\lambda_j^*, \hat{w}_j)$. However, if the random shocks are small in the sense that the different values $\hat{w}_j$ are near a constant value, say, zero then an important further result can be obtained. We now take up this issue.

Letting $w = (w_1, ..., w_K)$, suppose that the states of the shock process satisfy $\|w\| \leq \varepsilon$ for small $\varepsilon > 0$. As in Section 2 we can consider the stochastic steady state $\lambda_j^*, j = 1, ..., K$, to be a function of w. By continuity, the non-stochastic steady state $\bar{\lambda}$ is a limiting value, so that $\lambda_j^*(w) \to \bar{\lambda}$ for all j as $\|w\| \to 0$ and the steady state values $\lambda_j^*(w)$ satisfy $\left|\lambda_j^*(w) - \bar{\lambda}\right| < \delta, j = 1, ..., K$, for some $\delta > 0$. Also denote by $M(w)$ the matrix (7) in Theorem 2 when the elements are viewed as functions of w. By continuity of eigenvalues we have $\lim_{\|w\| \to 0} M(w) \to \bar{G}_1 \bar{H}_1 \Pi$.

Under the regularity assumption $\bar{G}_1 \bar{H}_1 \neq 1$ and thus we have:

Proposition 3 *Consider a given transition probability matrix* Π, *a stochastic steady state* $\lambda_j^*(w), j = 1, ..., K$ *and the corresponding non-stochastic steady state* $\bar{\lambda}$.
(i) If $\left|\bar{G}_1 \bar{H}_1\right| < 1$ *for the non-stochastic steady state, then there exists* $\bar{\varepsilon} > 0$ *such that the stochastic steady state is E-stable for economies when the different states of exogenous shock satisfy* $\|w\| < \bar{\varepsilon}$.
(ii) The stochastic steady state for economies with an Markovian exogenous variable satisfying $\|w\| < \varepsilon$ *for* ε *sufficiently small, is stable under learning only if* $\bar{G}_1 \bar{H}_1 \leq 1$ *for the limit non-stochastic steady state.*[8]

Proof. (i) Since Π is a Markov matrix, 1 is an eigenvalue of Π and all eigenvalues have modulus ≤ 1. Thus, as $\|w\| \to 0$, the limits of the eigenvalues of $M(w)$ are equal to $\bar{G}_1 \bar{H}_1 \mu$, where μ is an eigenvalue of Π. If $\left|\bar{G}_1 \bar{H}_1\right| < 1$, the eigenvalues of $M(w)$ are sufficiently close to the values $\bar{G}_1 \bar{H}_1 \mu$, which then satisfy $\left|\bar{G}_1 \bar{H}_1 \mu\right| < 1$.

(ii) Suppose to the contrary that $\bar{G}_1 \bar{H}_1 > 1$. By continuity, at least one eigenvalue of $M(w)$ is greater than one for all $\|w\|$ sufficiently small, which implies instability under learning by Theorem 2. ∎

[8] We remark that $\bar{G}_1 \bar{H}_1 < 1$ is the E-stability condition for the non-stochastic model, see (Evans and Honkapohja 1995).

Proposition 3 can be seen as the basis for the common practice whereby one linearizes the model (1) at the non-stochastic steady state and studies the resulting linear model. Since a finite Markov chain can be written in an autoregressive form, see p.679 of (Hamilton 1994)), the linear model can be written, after centering, as

$$y_t = AE_t^* y_{t+1} + Bw_{t+1} + Cw_t, \tag{9}$$

$$w_{t+1} = Pw_t + v_t, \tag{10}$$

where $A = \bar{G}_1 \bar{H}_1$, $B = \bar{G}_2 \bar{H}_1$ and $C = \bar{H}_2$. This can be analyzed further in the usual way. For example, a sufficient condition for stability under learning is $|A| < 1$, while $A < 1$ is a necessary condition.[9]

5 Application I: Asset Pricing

The standard model of asset pricing, see e.g. Chapter 10 of (Ljungqvist and Sargent 2000) for an exposition, leads to an Euler equation for the price p_t of an asset paying a dividend d_t in each period:

$$p_t u'(d_t) = \beta E_t^* p_{t+1} u'(d_{t+1}) + \beta E_t^* d_{t+1} u'(d_{t+1}). \tag{11}$$

Here $u(.)$ is the utility function of the representative consumer and β is the discount factor. We do not assume that expectations are necessarily always rational, as indicated by * in the expectations.

Case 1. (Asset pricing with stationary dividends) Following (Ljungqvist and Sargent 2000), Example 2, we first consider the case where dividends assume a finite set of values $\{\hat{d}_1, ..., \hat{d}_K\}$ and they evolve according to a finite Markov chain with transition probabilities $\pi_{ij} = \Pr\left\{ d_{t+1} = \hat{d}_j \middle| d_t = \hat{d}_i \right\}$. For simplicity, this Markov chain is assumed to be known to the representative agent. At time t the agent needs to predict the ex dividend asset price p_{t+1} for the period.

Equivalently, we can assume that the agent make prediction of the quantity $E_t^* p_{t+1} u'(d_{t+1})$. Letting $y_t = p_t u'(d_t)$ and $w_{t+1} = d_{t+1} u'(d_{t+1})$ the model (11) becomes

$$y_t = \beta E_t^* (y_{t+1} + w_{t+1}),$$

[9] (Evans and Honkapohja 1998) study stability conditions for learning in the special case $B = 0, C = 1$. These special assumptions are inconsequential for the stability and instability results.

which is a special case of (1) when $H(y,w) = \beta y$ and $G(y,w) = y + w$. We can directly apply the general results and conclude that the REE is E-stable and hence locally stable under adaptive learning.

Case 2. (Asset pricing with dividend growth) (Mehra and Prescott 1985) formulate a finite state model of dividend growth by assuming that the growth of dividends follows a finite Markov chain. In other words, we assume that

$$d_{t+1} = \mu_t d_t,$$

where μ_t is a finite (K-state) Markov chain with a transition matrix Π. Assume also that the utility function of the representative consumer exhibits constant relative risk aversion, so that

$$u(d) = \frac{d^{1-\gamma}}{1-\gamma}, \gamma > 0.$$

The price dividend ratio p_t/d_t can be shown, see p.240 of (Ljungqvist and Sargent 2000), to satisfy the equation

$$\frac{p_t}{d_t} = E_t^* \left[\beta(\mu_{t+1})^{1-\gamma} (\frac{p_{t+1}}{d_{t+1}} + 1) \right], \tag{12}$$

where again we have allowed for the possibility that expectations may not always be rational.

Letting $y_t = p_t/d_t$ and $w_t = \mu_t$, it is easily seen that (12) is a special case of model (1) with $H(y,w) = y$ and $G(y,w) = \beta(w)^{1-\gamma}(y+1)$. The E-stability of the REE is governed by the matrix

$$\begin{pmatrix} \pi_{11}\beta(w_1)^{1-\gamma} & \pi_{12}\beta(w_2)^{1-\gamma} & \cdots & \pi_{1K}\beta(w_K)^{1-\gamma} \\ \pi_{21}\beta(w_1)^{1-\gamma} & \pi_{22}\beta(w_2)^{1-\gamma} & \cdots & \pi_{2K}\beta(w_K)^{1-\gamma} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{K1}\beta(w_1)^{1-\gamma} & \pi_{K2}\beta(w_2)^{1-\gamma} & \cdots & \pi_{KK}\beta(w_K)^{1-\gamma} \end{pmatrix} \tag{13}$$

and E-stability requires all of the eigenvalues of matrix (13) must have real parts less than one. The model is no longer linear and in principle the stability of the REE under learning could be affected by the nonlinearity. However, instability under learning can be ruled out using a further argument exploited by (Mehra and Prescott 1985) to ensure uniqueness of the REE. They impose the further condition that all eigenvalues of the matrix (13) should be inside the unit circle. This determinacy condition clearly implies that the unique REE is also E-stable.

We summarize the analysis in the following proposition:

Proposition 4 *The stochastic steady state in the standard asset pricing model is stable under learning when (i) the dividends follows a finite Markov chain and also (ii) when the growth rate of dividends follows a finite Markov chain and the eigenvalues of (13) have real parts less than one.*

We emphasize that stability of the REE under learning could in principle be affected by the size of the Markov chain shocks in Case 2 of the asset pricing model. (Case 1 is a linear model in which the magnitude of the shocks does not matter for stability.) The assumption used by (Mehra and Prescott 1985) to ensure existence and uniqueness of REE is, however, sufficient also to rule out instability of the REE under learning.[10]

6 Application II: Overlapping Generations

As a second application of our results we consider the standard overlapping generations model (the so-called Samuelson model) with productivity shocks. This model was suggested by (Evans and Honkapohja 1995), Section 6 and we generalize their analysis to the case where the shock are a Markov chain rather than *iid*. The analysis will illustrate that the size of the shocks is important for the stability of the steady state under learning.

In the Samuelson model it is assumed that each generation lives for two periods. They work when young and consume only when old. Thus the utility function of the representative consumer born in period t takes the form

$$U(c_{t+1}) - V(n_t),$$

where c_{t+1} denotes his consumption in period $t+1$ (when the consumer is old) and n_t denotes labor supply in period t (when he is young). The utility function for consumption $U(c)$ is assumed to be strictly concave while the disutility of labor supply $V(n)$ is taken to be strictly convex. Both are

[10] We remark that uniqueness (or determinacy) of REE does not in general imply stability under learning for all models, see Parts III and IV of (Evans and Honkapohja 2001) for a detailed discussion.

assumed to be twice continuously differentiable. Output is assumed to be perishable and the production function of the consumer is

$$q_t = n_t + \mu_t,$$

where the additive productivity shock μ_t is taken to be nonnegative random variable. It is observable when time t decisions are made. We assume that μ_t follows a finite Markov chain that is recurrent, irreducible and aperiodic.

The budget constraints of the consumer are

$$\begin{aligned} p_t q_t &= M_t, \\ p_{t+1} c_{t+1} &= M_t, \end{aligned}$$

where p_t is the price of output and M_t denotes his savings in the form of money. It is assumed that there is a constant stock of nominal money in the economy. As is well-known, utility maximization and market clearing yield the equation

$$(n_t + \mu_t)V'(n_t) = E_t^*[(n_{t+1} + \mu_{t+1})U'(n_{t+1} + \mu_{t+1})], \tag{14}$$

which characterizes the (interior) temporary equilibria when the consumers have the expectations $E_t^*(.)$.

The model (14) can be cast in the general framework (1) as follows. First, we define $y_t = n_t$, $w_t = \mu_t - E(\mu)$, where $E(\mu)$ is the mean of the invariant distribution of μ_t and set

$$G(y, w) = (y + w + E(\mu))U'(y + w + E(\mu)).$$

Second, note that the left hand (14) is strictly increasing in n_t, so that (14) can be solved for n_t. Then $y = H(x, w)$ is implicitly defined by the equation

$$x = (y + w + E(\mu))V'(y).$$

With these definitions the basic results, Theorem 2 and Proposition 3 can be applied.

We are interested in studying whether the size of the shocks can affect the stability of the stochastic steady state of the model. To show this we assume that utility functions are isoelastic

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}, V(n) = \frac{n^{1+\varepsilon}}{1+\varepsilon},$$

in which case (14) takes the form

$$(n_t + \mu_t)n_t^{\varepsilon} = E_t^*[(n_{t+1} + \mu_{t+1})^{1-\sigma}].$$

We also let $\mu_t = E(\mu) + v_t$, where $v_t \in \{v_1, v_2\}$ with transition probability matrix Π.

We select the following numerical values for the parameters: $\sigma = 4$, $\varepsilon = 1$, $E(\mu) = 0.6$ leading to a non-stochastic steady state at 0.645 and this steady state is stable under steady state learning, see p.201 of (Evans and Honkapohja 1995). Note that the non-stochastic steady state is also a limiting case, with $v_1, v_2 \to 0$ for any transition matrix Π, of the stochastic steady state when the Markov chain shocks are present. We now specify the values $\pi_{11} = \pi_{22} = 0.1$ for the transition matrix and $v_1 = 0.05$. We then vary the value of v_2 and consider E-stability of the stochastic steady state when the value of v_2 is increased. Table 1 reports the signs of the trace and determinant of the matrix $M - I$ for different values of v_2.[11]

v_2	0.1	0.15	0.2	0.25	0.3	0.35
Tr	$-$	$-$	$-$	$-$	$-$	$-$
det	$+$	$+$	$+$	$+$	$-$	$-$

Table 1: E-stability of the stochastic steady state

Table 1 illustrates how the steady state becomes unstable when the size of the shock becomes larger. As might have been anticipated on the basis of Theorem 2 and Proposition 3, there are models in which the steady state REE is stable under learning when shocks are small but is not stable with large enough shocks. We summarize the finding in

Remark 5 *In stochastic nonlinear models stability under learning of a steady state can be affected by the magnitude of the random shocks. A steady state that is stable under learning when shocks are small can become unstable if the shocks are sufficiently large.*

[11]In this example M is a 2×2 matrix and we can consider stability by computing the trace and determinant of $M - I$.

7 Concluding Remarks

In this paper we have extended the earlier results of stability under learning of steady states in stochastic nonlinear models to an important case, where the exogenous shocks are no longer *iid* and are instead correlated over time. Explicit local stability and instability results in terms of the underlying economic framework were obtained when the shocks follow a finite state Markov chain. We also discussed the significance of small vs. large shocks and the linearization of the model.

The assumption that the Markov chain is finite is restrictive, though this case has been fairly often used in applications. The assumption enables the definition of steady states and formulation of adaptive learning, so that agents realistically estimate the values of a finite number of parameters.

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