DISCUSSION PAPERS
Institute of Economics
University of Copenhagen

Papers by Participants in
The Birgit Grodal Symposium, Copenhagen, September 2002

03-19

Monetary Equilibria over an Infinite Horizon

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Birgit Grodal Symposium

Topics in Mathematical Economics

The participants in a September 2002 Workshop on *Topics in Mathematical Economics* in honor of Birgit Grodal decided to have a series of papers appear on Birgit Grodal's 60'th birthday, June 24, 2003.

The Institute of Economics suggested that the papers became Discussion Papers from the Institute.

The editor of *Economic Theory* offered to consider the papers for a special Festschrift issue of the journal with Karl Vind as Guest Editor.

This paper is one of the many papers sent to the Discussion Paper series.

Most of these papers will later also be published in a special issue of *Economic Theory*.

Tillykke Birgit

Troels Østergaard Sørensen	Karl Vind
Head of Institute	Guest Editor

MONETARY EQUILIBRIA
OVER AN INFINITE HORIZON

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Current version: April 24, 2003

ABSTRACT. Money provides liquidity services through a cash-in-advance constraint. The exchange of commodities and assets extends over an infinite horizon under uncertainty and a sequentially complete asset market. Monetary policy sets the path of rates of interest and accommodates the demand for balances. A public authority, inheriting a strictly positive public debt, raises revenue from taxes and seignorage. Competitive equilibria exist, under mild solvency conditions. But, for a fixed path of rates of interest, there is a nontrivial multiplicity of equilibrium paths of prices of commodities. Determinacy requires that, subject to no-arbitrage and in addition to rates of interest, the prices of state-contingent revenues be somehow determined.

KEYWORDS. Money, equilibrium, indeterminacy, monetary policy, fiscal policy.

JEL CLASSIFICATION NUMBERS. D50, E40, E50.

1. INTRODUCTION

1.1. In this paper, we aim to contribute to a general equilibrium theory of monetary economies comparable to the well-developed theory of real economies. We provide conditions for the existence and the determinacy of a competitive equilibrium. These conditions are qualitatively equivalent whether the horizon be finite or infinite.

1.2. The economy extends over time under uncertainty, as represented by a standard event tree. The horizon is infinite and there is no production. We enlarge the canonical Arrow-Debreu paradigm by introducing money balances that facilitate transactions. The transaction technology takes the simple form of the cash-in-advance constraint of Clower [5]. Money balances are supplied by a central bank, which produces these at no cost and lends them at set short-term nominal rates of interest, meeting demand. The profits of the central bank, seignorage, accrue to the public authority.

Elementary securities result in a sequentially complete asset market. These securities are traded at every date-event by individuals and the public authority.

Acknowledgments: We are grateful to Pietro Reichlin, Rabah Amir, Tomoyuki Nakajima, Armando Dominioni and Leo Ferraris for helpful discussions and their reading of preliminary drafts. The usual disclaimer applies. An earlier version was circulated as [4]. *Address for correspondence:* Gaetano Bloise, Dipartimento di Scienze Economiche, 12 Via Andrea Cesalpino, 00161 Roma, Italy.

Consequently, sequential budget constraints can be consolidated into a single intertemporal constraint in terms of present values. The primitives include initial nominal claims and debts held by individuals that, in the aggregate, are the counterpart of an initial public debt held by the public authority.

The public authority covers its initial debt through public revenues, which consist of taxes and/or seignorage. Taxes are lump-sum commodity taxes collected from individuals at predetermined real levels. The public authority distributes its eventual budget surpluses as lump-sum transfers to individuals, while no further instruments are available to correct eventual budget deficits.

1.3. Over a finite horizon with no initial nominal positions and no taxes, Drèze and Polemarchakis [8] prove existence of competitive equilibria, for arbitrarily set nominal rates of interest and price levels at all terminal nodes. Alternatively stated, the overall price level is arbitrary and the variability of short-term rates of inflation is unrestricted. This important indeterminacy feature reflects the intuitive property that the rate of interest at every date-event pins down expected inflation, but not inflation variability. This result parallels conclusions of the discussion of price-levels determinacy in streamlined monetary models, as in Walsh [21], where expected inflation is determinate but realized inflation is still affected by shocks: shocks are there the counterpart of state variability here. The same result holds over an infinite horizon for the special case of no initial nominal positions and no taxes. Clearly, when individuals hold initial nominal positions, to be covered by the value of their endowments, and there is no public debt, their solvency places a lower bound on the overall price level. Qualitative properties of this economy are, thus, identical over a finite and an infinite horizon.

Initial public debt must be met by public revenues from taxes and seignorage. If predetermined tax levels are positive, a suitable lower bound on the overall price level guarantees public solvency. Otherwise, the public authority must rely on seignorage, the yield of which is, roughly speaking, proportional to the overall price level. Public solvency then requires positive nominal rates of interest and positive transactions (hence, demand for money balances), reflecting gains to trade. A condition of gains to trade appears in previous work, notably, in Dubey and Geanokoplos [9, 10]. It imposes that nominal rates of interest do not exceed gains to trade, a rather inocuous requirement for economic substance.

Our main result asserts (A) the existence of equilibria at all overall price levels above a lower bound, provided that conditions on gains to trade, if needed, are satisfied, and (B) the indeterminacy of rates of inflation, up to no-arbitrage conditions. These transparent results rest, among others, on the possibility for the public authority to trade in all nominal assets. If instead the public authority were restricted to trade, say, in nominal bonds only, then public solvency should be reconsidered in all periods of trade. Sufficient conditions to that effect would require suitable public revenues at all dates, leaving conclusions unchanged in their substance.[1]

Our analysis provides insights for the existence and multiplicity of monetary equilibria. Even though we only consider policies that set nominal rates of interest, which offer a great advantage in terms of analytical tractability, our arguments have an apparent counterpart in policies that set the supply of balances, while interest rates clear the money market. Also, contributing to a long debate in monetary

[1]For claims that we do not directly prove here, we refer the reader to [4].

economics, our results do not depend on the horizon of trade, infinite rather than finite. Our results, thus, suggest that more difficult issues could safely be analyzed in the more tractable finite-horizon framework. This remark applies, in particular, to the much needed extension to incomplete asset markets.

1.4. Our work contributes to a well-established tradition on the general equilibrium analysis of monetary economies, different from overlapping generations economies in Samuelson [19], surveyed in Geanakoplos and Polemarchakis [12], or self-insurance economies in Bewley [3]. Though the subject is vast, we only briefly discuss some pieces of literature most directly related to our contribution.

Our specification encompasses one of the possible extensions of Dubey and Geanakoplos [9, 10] to an infinite horizon. Over a single period of trade, they consider the case of a given initial stock of outside money and an additional injection of inside money, which allows for an unambiguous determination of the nominal rate of interest: seignorage revenue should absorb the outside money. Rephrasing their analysis so as to make it comparable with ours, they do not allow for a distribution of public budget surpluses. Consistently, they obtain determinacy of the overall price level and, in the suggested extension, of the variability of rates of inflation, when the public authority is restricted to trade only in safe bonds.

Our description of a monetary economy is similar to that of Grandmont and Younès [13, 14], who study stationary monetary equilibria under a steadily growing supply of balances and in absence of assets other than money balances. Though they provide an analysis in the temporary equilibrium tradition, their work makes precise the relevance of gains to trade for existence of a monetary equilibrium in an intertemporal equilibrium analysis as well. Further down, we explain our understanding of their work and propose a comparison with ours.

Our work, in addition, extends an established body of literature on aggregate monetary economies, which is the current basic paradigm of much monetary economics. In particular, our work faithfully reproduces that of Woodford [23], in the case of heterogenous individuals and multiple commodities, which is in turn similar to cash-in-advance economies with a representative individual of Wilson [22] and Lucas and Stokey [16]. Differently from this literature, our more general formulation provides a framework that is suitable for the study of incomplete asset markets. In this respect, our current research, though preliminary, suggests that such an extension can be accomplished.

Some recent literature (for example, Woodford [23, 24] and Cochrane [6]) has proposed a fiscal theory of price determination. This asserts that the price level is determined so as to balance the initial public debt and public revenues from taxes and seignorage. Without aiming at being exhaustive on this subject, we should remark that we do not obtain analogous conclusions because the public authority can redistribute its eventual budget surpluses.

Finally, our work contributes to a long debate on general equilibrium with incomplete financial markets. For example, Magill and Quinzii [17] have argued that the very notion of nominal assets is a misconception and only real assets should be considered as fruitful for economic analysis. The argument goes further: nominal assets are meaningful only if money is somehow introduced; if money were introduced, however, the real value of money would be determined, roughly speaking, by some sort of quantity theory equations, which would make real any asset initially described as nominal. In this perspective, our conclusions cast doubt of the cogency of the above argument.

1.5. The paper is organized as follows: In section 2, we describe the economy and our assumptions on fundamentals, trades and the conduct of monetary and fiscal policies. In section 4, we argue that standard arguments on the consolidation of sequential constraints apply, as well, to our monetary economy. In section 5, we present our results on the existence and multiplicity of equilibria under alternative policy regimes. In sections 6 and 7, after some observations on the welfare properties of monetary equilibria, we perform a duality analysis using a notion of supportability that plays the role of efficiency in the standard Welfare Theorems. Further observations follow as a conclusion. All proofs are collected in the appendix.

2. The Economy

2.1. For a countable set, $\mathcal{A}$, the space of all real(-valued) maps on $\mathcal{A}$, $\ell(\mathcal{A})$, is an ordered vector space. Vector subspaces are the spaces of all bounded, $\ell_\infty(\mathcal{A})$, and summable, $\ell_1(\mathcal{A})$, real maps on $\mathcal{A}$, endowed with their respective norms. A vector, x, is positive (strictly positive, uniformly strictly positive) if, for every α in $\mathcal{A}$, $x_\alpha \geq 0$ ($x_\alpha > 0$, $x_\alpha \geq \epsilon > 0$). It is decomposed into a positive, $x^+ \geq 0$, and a negative, $x^- \geq 0$, part, so that $x = x^+ - x^-$. The positive cone of an ordered vector space consists of all its positive vectors.[2]

2.2. Time and the resolution of uncertainty are described by an event-tree, a countable set, $\mathcal{S}$, endowed with a (partial) order, $\succeq$. For every date-event, σ, an element of $\mathcal{S}$, t_σ denotes its date. The unique initial date-event is ϕ, with $t_\phi = 0$. For a given date-event, σ, $\sigma_+ = \{\tau \succ \sigma : t_\tau = t_\sigma + 1\}$ denotes the set of its immediate successors, a finite set; $\mathcal{S}_\sigma = \{\tau \in \mathcal{S} : \tau \succeq \sigma\}$ the set of all its (weak) successors, a subtree; $\mathcal{S}^t = \{\sigma \in \mathcal{S} : 0 \leq t_\sigma \leq t\}$ the set all date-events up to date t; $\mathcal{S}_t = \{\sigma \in \mathcal{S} : t_\sigma = t\}$ the set all date-events at date t.[3]

2.3. Markets are sequentially open for commodities, assets and balances that are numéraire. At every date-event, there is a finite set, $\mathcal{N}$, of tradable commodities, which are perfectly divisible and perishable. The *commodity space* coincides with the space of all bounded real maps on $\mathcal{S} \times \mathcal{N}$ and prices of commodities, p, are a positive real map on $\mathcal{S} \times \mathcal{N}$.

The asset market is sequentially complete. A portfolio, holdings of assets, is described by its payoffs across date-events, v, a real map on $\mathcal{S}$. Prices of assets are *state prices*, the prices of revenues across date-events, a, a strictly positive real map on $\mathcal{S}$, normalized so that $a_\phi = 1$. At a date-event, a portfolio, with payoffs $(v_\tau : \tau \in \sigma_+)$ across its immediate successors, has market value

$$a_\sigma^{-1} \sum_{\tau \in \sigma_+} a_\tau v_\tau.$$

State prices correspond to prices of (implicit) elementary Arrow securities.

[2] For details, see Aliprantis and Border [1]. It should be remarked that, throughout the paper, we use the term 'positive' ('negative') to mean 'greater than or equal to zero' ('less than or equal to zero').

[3] The construction is standard: see, Magill and Quinzii [18] or Santos and Woodford [20].

At given state prices, one-period *nominal rates of interest*, r, a positive real map on $\mathcal{S}$, are implicitly defined by the equations

$$r_\sigma = \frac{a_\sigma}{\sum_{\tau \in \sigma_+} a_\tau} - 1 \geq 0.$$

Indeed, since balances are storable, no-arbitrage requires that nominal rates of interest be positive.

2.4. There is a finite set of individuals. An individual, i, is described by preferences, $\succeq^i$, over the *consumption space*, the positive cone of the commodity space, and an endowment, e^i, of commodities, an element of the consumption space. We make two common assumptions on preferences and endowments of commodities.

(P) Preferences. *For every individual, preferences, $\succeq^i$, are continuous in the (relative) Mackey topology, convex and strictly monotone.*

(E) Endowments. *The endowment of every individual, e^i, is uniformly strictly positive.*

Continuity of preferences in the Mackey topology, introduced in Bewley [2], is a strong requirement.[4] In particular, it implies that the individual is impatient: sufficiently distant modifications of consumption plans do not reverse the order of preference. Uniform impatience across individuals would be a stronger requirement. The much stronger assumption of a uniform rate of impatience across date-events, in some recent literature on incomplete asset markets over an infinite horizon (Hernández and Santos [15] and Magill and Quinzii [18]), is not needed here.[5]

2.5. A public authority (or a government, or a central bank) conducts monetary, fiscal, transfers and portfolio policies. The supply of balances, m, a positive real map on $\mathcal{S}$, is contingent on dates and information. A *monetary policy* consists of setting nominal rates of interest, r, and supplying balances so as to accommodate demand. Although our analysis could be adapted to cope with all arbitrarily set nominal rates of interest, we impose a restriction that facilitates presentation.

(M) Monetary Policy. *Monetary policy, r, is bounded.*

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[4]It encompasses, for example, preferences that are represented by an additively separable utility function,

$$\sum_{\sigma \in \mathcal{S}} \mu_\sigma \beta^{t_\sigma} u^i \left(x^i_\sigma \right),$$

where μ_σ is the probability of σ, $0 < \beta < 1$ is the discount factor, and u^i is a bounded, continuous, increasing, concave real map on $\ell(\mathcal{N})$ with $u^i(0) = 0$.

[5]Alternatively, we could include unbounded maps in the commodity space and require preferences to be continuous in the product topology. By re-scaling units of measurement of different commodities, which does not affect continuity in the product topology, one can always suppose that the aggregate endowment is bounded. For the purposes of equilibrium theory, it is not necessary to consider individual consumptions that exceed this bound, though this may be contrary to the spirit of competitive equilibrium, since individuals might, indeed, contemplate unbounded consumption plans. In this direction, continuity in the product topology restricted to consumption plans uniformly bounded by the aggregate endowment is equivalent to continuity in the Mackey topology; this amounts to the impatience of individuals exceeding the rate of growth of the aggregate endowment. The stronger assumption of continuity in the product topology, as in Geanakoplos and Polemarchakis [12], keeps separate restrictions on preferences and restrictions on endowments.

A *fiscal policy* is represented by lump-sum commodity taxes, $\big(\ldots, g^i, \ldots\big)$, that is, bounded positive real maps on $\mathcal{S} \times \mathcal{N}$. In the aggregate, taxes are $g = \sum_i g^i$. We restrict fiscal policy so as to avoid problems of solvency, when individuals hold initial nominal debts, and to carry out a limit argument in the proof of existence of equilibria.

(F) Fiscal Policy. *Fiscal policy, $\big(\ldots, g^i, \ldots\big)$, is positive and such that the net endowment of every individual, $e^i - g^i$, is uniformly strictly positive.*

A *transfers policy* is represented by lump-sum nominal transfers to individuals, $\big(\ldots, h^i, \ldots\big)$, positive real maps on $\mathcal{S}$. In the aggregate, transfers are $h = \sum_i h^i$. To avoid additional redistributive indeterminacy, we assume that aggregate transfers are assigned to individuals according to given shares, $\big(\ldots, \zeta^i, \ldots\big)$, that is, $h^i = \zeta^i h$.

(T) Transfers Policy. *Transfers policy, $\big(\ldots, h^i, \ldots\big)$, is positive and such that the aggregate transfers, h, are distributed to individuals according to given shares, $\big(\ldots, \zeta^i, \ldots\big) \geq 0$, with $\sum_i \zeta^i = 1$.*

Trade in assets by the public authority is described by public liabilities, w, a real map on $\mathcal{S}$, with a given initial value, $w_\phi = \delta$. The initial public liability, δ, corresponds to initial nominal claims, $\big(\ldots, \delta^i, \ldots\big)$, of individuals, that is, $\delta = \sum_i \delta^i$. To simplify the presentation, at no loss of realism, we assume that there is a strictly positive initial public liability.

(L) Initial Public Liability. *The initial public liability, δ, is strictly positive.*

The public authority is subject, at every date-event, to a sequential budget constraint,

$$\left(\frac{r_\sigma}{1 + r_\sigma}\right) m_\sigma + a_\sigma^{-1} \sum_{\tau \in \sigma_+} a_\tau w_\tau = w_\sigma + h_\sigma - \left(\frac{1}{1 + r_\sigma}\right) p_\sigma \cdot g_\sigma.$$

A *portfolio policy* sets portfolio composition, but not its magnitude, at every date-event. It is represented by Θ, a real map on $\mathcal{S}$, and imposes the additional restriction that $(w_\tau : \tau \in \sigma_+)$ belong to the span of $(\Theta_\tau : \tau \in \sigma_+)$.

(P) Portfolio Policy. *At every date-event, σ, the composition of the public portfolio, $(\Theta_\tau : \tau \in \sigma_+)$, is positive and non-zero.*

Our representation of public policies incorporates a minimal requirement of consistency. Indeed, arbitrarily set policies determine, through the sequential public budget constraint, the evolution of public liabilities, for all given prices and demands of balances. For every date-event, σ, public liability at every immediately following date-event, τ, is given by

$$w_\tau = \left(\frac{\Theta_\tau}{\sum_{\xi \in \sigma_+} a_\xi \Theta_\xi}\right) \left(a_\sigma w_\sigma + a_\sigma h_\sigma - \left(\frac{1}{1 + r_\sigma}\right) a_\sigma p_\sigma \cdot g_\sigma - \left(\frac{r_\sigma}{1 + r_\sigma}\right) a_\sigma m_\sigma\right).$$

However, a given policy may not satisfy an inter-temporal public budget constraint at all arbitrary prices and demanded balances.

Remark 1. Consolidated budget constraints are consistent with a specification of mutually independent monetary and fiscal authorities. Suppose that v and u represent, respectively, the liabilities of the monetary and fiscal authorities. The central bank is an institution which issues balances and runs balanced accounts:

outstanding money, m, is matched by claims on individuals or the government, that is, $v = 0$. The bank is owned by individuals or the government and distributes dividends, d, to share-holders: the imposition of balanced accounts implies, at every date-event, that

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) m_\sigma = d_\sigma.$$

Consistently, the government is subject, at every date-event, to a sequential budget constraint,

$$\left(1 - \sum_i \xi^i\right) d_\sigma + a_\sigma^{-1} \sum_{\tau \in \sigma_+} a_\tau u_\tau = u_\sigma + h_\sigma - \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot g_\sigma,$$

where $\left(\ldots, \xi^i, \ldots\right) \geq 0$ and $\left(1 - \sum_i \xi^i\right) \geq 0$ are, respectively, the shares of individuals and the government in the central bank. □

2.6. The constraints that an individual faces, at every date-event, are a budget constraint,

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) m_\sigma^i + a_\sigma^{-1} \sum_{\tau \in \sigma^+} a_\tau w_\tau^i + p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right) \leq w_\sigma^i + h_\sigma^i - \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot g^i,$$

a liquidity constraint,

$$p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right)^- - m_\sigma^i \leq 0,$$

and a solvency constraint,

$$-a_\sigma^{-1} \sum_{\tau \in \mathcal{S}_\sigma} a_\tau \left(h_\tau^i - \left(\frac{1}{1+r_\tau}\right) p_\tau \cdot \left(e_\tau^i - g_\tau^i\right)\right) \leq w_\sigma^i,$$

where initial nominal claims are predetermined by the condition $w_\phi^i = \delta^i$. In the budget constraint, the nominal interest rate represents the opportunity cost of holding wealth in liquid form. Equivalently, here, of collecting proceeds of sales with a one-period lag.

Remark 2. Our constraints coincide with those of Woodford [23] and, in a finite horizon, with those of Dubey and Geanakoplos [10]. Liquidity constraints correspond to cash-in-advance. At every date, after information is acquired, σ, an individual has nominal claims w_σ^i and receives a transfer h_σ^i. He purchases a portfolio, with payoffs $\left(v_\tau^i : \tau \in \sigma_+\right)$, and balances n_σ^i so as to satisfy the constraint

$$n_\sigma^i + a_\sigma^{-1} \sum_{\tau \in \sigma_+} a_\tau v_\tau^i \leq w_\sigma^i + h_\sigma^i.$$

He employs balances for the purchase of commodities, according to the constraint

$$p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right)^+ - n_\sigma^i \leq 0,$$

receives balances from the sale of goods and pays off commodity taxes g_σ^i. The end of period amount of balances, before paying taxes, is, therefore,

$$m_\sigma^i = n_\sigma^i - p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right)^+ + p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right)^-.$$

At the following date, after taxes are paid to the public authority and information is revealed, τ, nominal claims amount to $w_\tau^i = v_\tau^i + m_\sigma^i - p_\sigma \cdot g_\sigma^i$. □

Remark 3. Solvency constraints serve to eliminate Ponzi schemes. They are equivalent to the restriction that an individual can incur any amount of nominal debt that can be repayed in finite time. The value of the endowment in commodities at a date-event is taxed at the nominal interest rate, since revenues from sales are carried over in the form of balances that do not earn interest. □

For an individual, a plan consists of a consumption plan, x^i, balances, m^i, and asset holdings, w^i. The sequential budget set is the set of all consumption plans which satisfy the sequential budget, liquidity and solvency constraints, for some balances and asset holdings, given initial nominal claims.

3. Equilibrium

A *monetary equilibrium* (or, simply, an *equilibrium*) consists of monetary, fiscal, portfolio and transfers policies, plans for individuals, prices of commodities and state prices (consistent with set nominal rates of interest) such that:

(a) the plan of every individual, i, is optimal subject to sequential budget, liquidity and solvency constraints, given initial nominal claims, δ^i;

(b) at every date-event, σ, market clearing is achieved in markets of commodities,

$$\sum_i x^i_\sigma = \sum_i e^i_\sigma,$$

and assets,

$$\sum_i w^i_\sigma = w_\sigma,$$

where public liabilities, w, satisfy public sequential budget constraints at balances demanded by individuals, $m = \sum_i m^i$, given initial public liability, δ.

In our analysis, monetary and fiscal policies are set exogenously and independently of any intertemporal public budget constraint. Transfers policy is, instead, treated as determined endogenously subject to the restrictions imposed by our assumptions. Portfolio policy is studied as endogenously determined, though some remarks clarify how conclusions vary in the case of an exogenously set portfolio policy.

4. Consolidation

Since the asset market is complete, the sequence of budget constraints faced by an individual reduces to a single constraint at the initial date-event.

Lemma 4.1. *At equilibrium, present value prices of commodities, ap, are a summable real map on $\mathcal{S} \times \mathcal{N}$.*

At equilibrium, therefore, the intertemporal budget constraint of an individual,

$$\sum_{\sigma\in\mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m^i_\sigma + \sum_{\sigma\in\mathcal{S}} a_\sigma p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) \quad \leq$$

$$\delta^i + \sum_{\sigma\in\mathcal{S}} a_\sigma h^i_\sigma - \sum_{\sigma\in\mathcal{S}} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g^i_\sigma,$$

is well-defined.

Lemma 4.2. *At equilibrium, a consumption plan is attainable under sequential budget, liquidity and solvency constraints if and only if it is attainable under the unique intertemporal budget constraint and sequential liquidity constraints. Optimality of a consumption plan requires that the intertemporal budget constraint be satisfied with equality and, at every date-event,*

$$\left(\frac{r_\sigma}{1+r_\sigma}\right)\left(p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^- - m^i_\sigma\right) = 0.$$

The transversality condition takes the form

$$\lim_{t\to\infty} \sum_{\sigma\in\mathcal{S}_t} a_\sigma w^i_\sigma = 0.$$

As the liquidity constraint is binding whenever the nominal rate of interest is strictly positive, the intertemporal budget constraint of an individual reduces to

$$\sum_{\sigma\in\mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) \pi_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^- + \sum_{\sigma\in\mathcal{S}} \pi_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) \quad \leq$$

$$\delta^i + \zeta^i \sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma - \sum_{\sigma\in\mathcal{S}} \left(\frac{1}{1+r_\sigma}\right) \pi_\sigma \cdot g^i_\sigma,$$

where $\pi = ap$ are present value prices of commodities, a summable positive real map on $\mathcal{S}\times\mathcal{N}$. At equilibrium, aggregation across individuals yields

$$\sum_{\sigma\in\mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) \pi_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- + \sum_{\sigma\in\mathcal{S}} \left(\frac{1}{1+r_\sigma}\right) \pi_\sigma \cdot g_\sigma = \delta + \sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma,$$

which is the intertemporal public budget 'constraint'. It is clear that state prices are of no allocative relevance, unless transfers policy is arbitrarily set.

5. Existence and Determinacy

5.1. Under standard assumptions on endowments and preferences, equilibrium exists, for given monetary and fiscal policies, when public and private solvency are guaranteed.

At equilibrium, the overall public revenue must (weakly) exceed the initial public liability. While the latter is a given nominal magnitude, the source of revenue is real, either taxes or seignorage. If taxes or seignorage are positive, any high enough overall price level suffices to balance an intertemporal public budget, possibly by the exhaustion of budget surplus through transfers. In order that seignorage be positive, however, gains to trade are to be so high as to sustain a positive demand for balances. The assumption below guarantees an overall positive public revenue at the outset.

(R) Positive public revenue. *Policies guarantee public revenue if either fiscal policy is strictly positive or monetary policy is strictly positive and the initial allocation, $\left(\ldots, e^i, \ldots\right)$, is weakly Pareto dominated by an alternative allocation, $\left(\ldots, x^i, \ldots\right)$, that satisfies, at every date-event,*

$$\sum_i x^i_\sigma + \left(\frac{r_\sigma}{1+r_\sigma}\right) \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- \leq \sum_i e^i_\sigma.$$

Although individuals might hold initial nominal debts, their solvency can be ensured at all high enough price levels, though it might fail at low overall price levels.

Proposition 5.1 (Equilibrium)**.** *For given monetary and fiscal policies that guarantee public revenue, there exists a continuum of equilibria, indexed by the overall price level, up to a lower bound, that ensures solvency of every individual and positivity of transfers, and by arbitrarily chosen state prices, up to consistency with nominal rates of interest. The indeterminacy associated with state prices is purely nominal, with no real effects.*

The overall price level has real effects because it redistributes wealth across individuals. State prices have no real effects because markets are sequentially complete, allowing full inflation insurance.

If also portfolio policy is set exogenously, the argument needs only minor modifications that reflect the need for public solvency at all date-events. In particular, a positive public revenue must accrue at all date-events, which is ensured (if needed) by suitable stronger conditions in terms of gains to trade. In this case, a lower bound on the overall price level guarantees solvency of every individual and positivity of transfers at all date-events, while state prices remain indeterminate, up to consistency with nominal rates of interest.

5.2. Different formulations require adaptations of the existence and multiplicity result. We shall consider the two polar cases of unrestricted transfers and no transfers.

If negative transfers are allowed, an equilibrium exists. Also indeterminacy occurs both in the overall price level, up to a lower bound which now only ensures private solvency, and in state prices, up to consistency with nominal rates of interest. This is true whether portfolio policy be pegged exogenously or not. Unrestricted transfers policies, in fact, capture what is called the Ricardian regime in some literature: by means of negative transfers, public revenue, which now includes negative transfers, balances public liabilities at all date-events.

The other extreme case is that of no transfers policy at all, which is the framework that is privileged by the fiscal theory of the price level. An equilibrium exists provided that no individual holds an initial debt, since the overall price level now serves to balance the initial public liability and the present value of public revenue at the outset, so it cannot be used to guarantee private solvency. Such a non-Ricardian regime, as usually named in the literature, results in the determinacy of the overall price level and, if also a portfolio policy is set exogenously, of state prices.

5.3. Conditions for public solvency parallel those on gains to trade of Dubey and Geanakoplos [9, 10] over a finite horizon: though they consider the case of a given supply of balances, that is equivalent to a pegging of the nominal rate of interest.[6] In order to guarantee existence of a stationary equilibrium, Grandmont and Younès [13] state sufficient conditions on gains to trade in terms of impatience. This seemingly puzzling result can be accommodated in our formulation as follows in the case, to simplify, of common impatience across individuals and constant supply of

[6] In their construction, an exogenous money supply, in addition to what they call outside balances, uniquely pegs a nominal rate of interest consistent with a monetary equilibrium.

balances. In a stationary equilibrium, prices of commodities are constant and the nominal rate of interest coincides with the real rate of interest, that is, with the common rate of impatience. If the initial allocation is not Pareto efficient, there are gains to trade at all low enough nominal rates of interest. Hence, there are gains to trade at all low enough rates of impatience.[7]

6. Efficiency

Neither of the Welfare Theorems holds in a monetary economy under strictly positive nominal rates of interest: (a) equilibrium allocations, in general, fail to be Pareto efficient; (b) Pareto efficient allocations cannot, in general, be sustained as equilibrium allocations (though they can under suitable redistributions of endowments of commodities). The Pareto inefficiency follows from the wedge driven by the cash-in-advance constraint between buying and selling prices of commodities. More importantly, one can construct robust examples of economies exhibiting Pareto-ranked equilibria at given nominal rates of interest.

To clarify our last claim, we provide a simple example without aiming at being exhaustive. There are two individuals and two physical commodities. Let the nominal rate of interest, $r > 0$, be given. Assuming a common rate of time preference across individuals, we can treat a stationary infinite horizon economy as a simple one-period economy. Individual 1's preferences are represented by $x_1^1 + (1+r)^{-1} x_2^1$ and endowments are $(0,1)$. Individual 2's preferences are represented by $(1+r)^{-1} x_1^2 + x_2^2$ and endowments are $(1,0)$. A symmetric allocation is represented by $0 \leq \theta \leq 1$, with consumptions $x_\theta^1 = (\theta, 1-\theta)$ and $x_\theta^2 = (1-\theta, \theta)$. The strictly positive amount of public debt is equally distributed across the two individuals. It is simple to verify that, for every $0 < \theta \leq 1$, $\left(x_\theta^1, x_\theta^2\right)$ is an equilibrium with prices π_θ proportional to $(1,1)$. There is thus a continuum of equilibria ranking from the no-trade to the symmetric Pareto-efficient allocation. Notice that all such equilibria involve no transfers and can be indexed by the overall price level, up to a lower bound.

In the simpler framework where the public authority reduces to a bank, owned by individuals and conducting monetary policy, our model is formally equivalent to that of an Arrow-Debreu economy where (i) some of the commodities are money balances used for transactions purposes; (ii) the single firm is a bank, owned by individuals, producing the balances at no cost, and maximizing profits by supplying these balances at fixed prices subject to quantity constraints. It is well-known that the two Welfare Theorems fail to hold in the presence of price rigidities and quantity constraints (*e.g.*, Drèze and Müller [7]). That general result applies to the monetary economy just described, which is a special case of the economy with public authority considered here.

The concept of constrained efficiency suitable for monetary economies is not evident. The next section, which may be seen as a digression, is devoted to a related duality property.

7. Duality

We carry out a duality analysis using an auxiliary notion of supportability. One may be interested in such an analysis for two reasons: (a) it allows us to provide a

[7] Grandmont and Younès [13] do not consider bonds or nominal rates of interest. The argument, however, goes through if nominal rates of interest are replaced by corresponding shadow prices.

characterization of equilibria without any explicit reference to prices; (b) it gives a better understanding of the displacement from Pareto efficiency caused by liquidity constraints. Throughout, nominal rates of interest are considered as given and are bounded.

An allocation, $(\ldots, x^i, \ldots)$, is *feasible* if, for every date-event,

$$\sum_i x^i_\sigma - \sum_i e^i_\sigma \leq 0.$$

It is said to be *supportable* (respectively, *weakly supportable*) if it is not Pareto dominated (respectively, weakly Pareto dominated) by any allocation, $(\ldots, z^i, \ldots)$, which satisfies, at every date-event,

$$\sum_i z^i_\sigma + \sum_i \left(\frac{r_\sigma}{1+r_\sigma}\right) \left(z^i_\sigma - e^i_\sigma\right)^- \leq \sum_i e^i_\sigma + \sum_i \left(\frac{r_\sigma}{1+r_\sigma}\right) \left(x^i_\sigma - e^i_\sigma\right)^- .$$

Notice that supportability is defined for given nominal rates of interest *and* endowments of commodities. The latter seems unavoidable if the notion is to be suitable for a duality analysis.

Clearly, every supportable feasible allocation is also weakly supportable. The converse is true as well since preferences are continuous and strictly monotone.

Lemma 7.1. *A feasible allocation is supportable if and only if it is weakly supportable. Moreover, it is supportable only if, at every date-event,*

$$\sum_i x^i_\sigma - \sum_i e^i_\sigma = 0.$$

We now establish a variation on the First Welfare Theorem.

Proposition 7.1 (First Pseudo-Welfare Theorem)**.** *Every equilibrium allocation is supportable.*

We say that present value prices of commodities, π, *support* a feasible allocation, $(\ldots, x^i, \ldots)$, whenever, for every individual, $z^i \succeq^i x^i$ implies

$$\sum_{\sigma \in \mathcal{S}} \pi_\sigma \cdot \left(z^i_\sigma - x^i_\sigma\right) \geq \sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) \pi_\sigma \cdot \left(\left(x^i_\sigma - e^i_\sigma\right)^- - \left(z^i_\sigma - e^i_\sigma\right)^-\right).$$

If an allocation is supported by some present value prices of commodities, consumption plans of individuals are (weakly) optimal at those prices.

We can now present a formulation of the Second Welfare Theorem.

Proposition 7.2 (Second Pseudo-Welfare Theorem)**.** *Every supportable feasible allocation is supported by some (non-zero) present value prices of commodities.*

Supportability can be interpreted as the absence of re-trading benefits if trade were physically costly, as pointed out by Dubey and Geanakoplos [9, 10].

Proposition 7.3 (Gains to Trade)**.** *A supportable feasible allocation,* $(\ldots, x^i, \ldots)$*, is not Pareto dominated by an allocation,* $(\ldots, z^i, \ldots)$*, which satisfies, at every date-event,*

$$\sum_i z^i_\sigma + \sum_i \left(\frac{r_\sigma}{1+r_\sigma}\right) \left(z^i_\sigma - x^i_\sigma\right)^- \leq \sum_i x^i_\sigma.$$

A supportable feasible allocation coincides with a Pareto-efficient no-trade allocation of an economy with redistributed initial endowments and a costly trading, or marketing, technology, as in Foley [11]. In such an economy, competitive firms, or intermediaries, produce (net) trades across individuals using a linear technology, which involves some destruction of resources (in our cases, such costs correspond to liquidity costs). It should be clear, however, that an equilibrium allocation does not, in general, correspond to a Pareto efficient allocation of an economy where transactions involve real costs, since liquidity costs have in fact no real counterpart.

8. Remarks

8.1. Various contributions over the last decad (*inter alia*, Drèze and Polemarchakis [8] and Dubey and Geanakoplos [9, 10]) have pointed out that finite time is suitable to meaningfully address issues of monetary analysis. Our current work is intended to confirm this view. Remarkably, the finite-horizon model provides a tractable disaggregate framework for a short-term analysis of, for instance, financial markets and nominal price rigidities.

8.2. Throughout our analysis, we have maintained the assumption of a sequentially complete asset market. This has allowed for a focus only on balances needed for transaction purposes. A sequentially incomplete asset market would enrich our analysis in a number of ways and, in particular, it would make the variability of inflation rates of real allocative relevance.

8.3. Our analysis points at a limited relevance of the fiscal theory of the price level (Woodford [23, 24] and Cochrane [6]). Differently from the framework of that theory, we only assume that eventual public budget surpluses are distributed to individuals through transfers. This seems innocuous and, yet, dramatically changes the conclusions on the determinacy of prices.

8.4. As a final remark, we observe that, in an economy with heterogeneous individuals, the occurrence of sunspot fluctuations need not be related to indeterminacy. Under interest rate pegging, extrinsic uncertainty might still affect the real allocation of resources at equilibrium even though nominal rates of interest are not contingent at all.[8] Nominal rates of interest put, however, bounds on the variability of consumptions across sunspot states.

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Proofs

Proof of Lemma 4.1. Solvency constraints imply that

$$\sum_{\tau \in \mathcal{S}_\sigma} a_\tau \left(h^i_\tau + \left(\frac{1}{1+r_\tau} \right) p_\tau \cdot \left(e^i_\tau - g^i_\tau \right) \right)$$

takes finite value at every non-initial date-event and, hence, at every date-event. By assumptions (T) and (F),

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1+r_\tau} \right) a_\tau p_\tau \cdot \left(e^i_\tau - g^i_\tau \right)$$

is finite. Hence, by assumptions (M) and (F), the claim easily follows. □

Proof of Lemma 4.2. Suppose that a plan $\left(x^i, m^i, w^i\right)$ satisfies sequential budget, liquidity and solvency constraints, given initial nominal claims. Multiplication of the sequential budget constraints by a_σ and summation over $\mathcal{S}^t$ yield

$$\begin{aligned} \sum_{\sigma\in\mathcal{S}_{t+1}} a_\sigma w_\sigma^i + \sum_{\sigma\in\mathcal{S}^t}\left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m_\sigma^i + \sum_{\sigma\in\mathcal{S}^t} a_\sigma p_\sigma \cdot x_\sigma^i \quad &\leq \\ \delta^i + \sum_{\sigma\in\mathcal{S}^t} a_\sigma h_\sigma^i + \sum_{\sigma\in\mathcal{S}^t} a_\sigma p_\sigma \cdot e_\sigma^i - \sum_{\sigma\in\mathcal{S}^t}\left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g_\sigma^i. \end{aligned}$$

The solvency constraint at every date-event, then, implies

$$\begin{aligned} \sum_{\sigma\in\mathcal{S}^t}\left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m_\sigma^i + \sum_{\sigma\in\mathcal{S}^t} a_\sigma p_\sigma \cdot x_\sigma^i - \sum_{\sigma\in\mathcal{S}^t}\left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot e_\sigma^i \quad &\leq \\ \delta^i + \sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma^i + \sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot e_\sigma^i - \sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g_\sigma^i. \end{aligned}$$

Since the left-hand side is bounded, the first term is non-decreasing and the other two terms converge, taking the limit as $t \to \infty$ implies

$$\begin{aligned} \sum_{\sigma\in\mathcal{S}}\left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m_\sigma^i + \sum_{\sigma\in\mathcal{S}} a_\sigma p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right) \quad &\leq \\ \delta^i + \sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma^i - \sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g_\sigma^i. \end{aligned}$$

Therefore, $\left(x^i, m^i\right)$ satisfy the inter-temporal budget constraint and sequential liquidity constraints.

Conversely, suppose that a plan $\left(x^i, m^i\right)$ satisfies the intertemporal budget constraint and sequential liquidity constraints and define w^i, at all non-initial date-events, by

$$\begin{aligned} a_\sigma w_\sigma^i \quad = \quad & \sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{r_\tau}{1+r_\tau}\right) a_\tau m_\tau^i + \sum_{\tau\in\mathcal{S}_\sigma} a_\tau p_\tau \cdot \left(x_\tau^i - e_\tau^i\right) \\ & + \sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{1}{1+r_\tau}\right) a_\tau p_\tau \cdot g_\tau^i - \sum_{\tau\in\mathcal{S}_\sigma} a_\tau h_\tau^i. \end{aligned}$$

Solvency constraints are satisfied, since liquidity constraints imply that

$$\begin{aligned} -a_\sigma^{-1}\sum_{\tau\in\mathcal{S}}\left(\frac{1}{1+r_\tau}\right) a_\tau p_\tau \cdot e_\tau^i \quad &\leq \\ -a_\sigma^{-1}\sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{1}{1+r_\tau}\right) a_\tau p_\tau \cdot \left(x_\tau^i - e_\tau^i\right)^- + a_\sigma^{-1}\sum_{\tau\in\mathcal{S}_\sigma} a_\tau p_\tau \cdot \left(x_\tau^i - e_\tau^i\right)^+ \quad &\leq \\ w_\sigma^i + a_\sigma^{-1}\sum_{\tau\in\mathcal{S}_\sigma} a_\tau h_\tau^i - a_\sigma^{-1}\sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{1}{1+r_\tau}\right) a_\tau p_\tau^i \cdot g_\tau^i. \end{aligned}$$

To see that sequential budget constraints are satisfied as well, observe that, at every non-initial date-event, the definition of w^i implies that

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) m_\sigma^i + a_\sigma^{-1}\sum_{\tau\in\sigma_+} a_\tau w_\tau^i + p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right) = w_\sigma^i + h_\sigma^i - \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot g_\sigma^i.$$

At the initial date-event, the intertemporal budget constraint and the definition of w^i imply that

$$\left(\frac{r_\phi}{1+r_\phi}\right) m^i_\phi + \sum_{\sigma \in \phi^+} a_\sigma w^i_\sigma + p_\phi \cdot \left(x^i_\phi - e^i_\phi\right) \leq \delta^i + h^i_\phi - \left(\frac{1}{1+r_\phi}\right) p_\phi \cdot g^i_\phi.$$

At an optimal plan, the intertemporal budget constraint must hold with equality since preferences are strictly monotone. Moreover, it is clear that the liquidity constraint is non-binding only if the nominal rate of interest is zero.

Concerning transversality, a plan satisfies solvency constraints only if

$$\liminf \sum_{\sigma \in \mathcal{S}_t} a_\sigma w^i_\sigma \geq 0.$$

It, then, suffices to show that a plan is maximal only if

$$\limsup \sum_{\sigma \in \mathcal{S}_t} a_\sigma w^i_\sigma \leq 0.$$

If not, then, for infinitely many dates, n, and some $\epsilon > 0$,

$$\begin{aligned} \epsilon + \sum_{\sigma \in \mathcal{S}^n} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m^i_\sigma + \sum_{\sigma \in \mathcal{S}^n} a_\sigma p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) \quad &\leq \\ \delta^i + \sum_{\sigma \in \mathcal{S}^n} a_\sigma h^i_\sigma - \sum_{\sigma \in \mathcal{S}^n} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g^i_\sigma. \end{aligned}$$

From the limit, since all series must converge, it follows that

$$\begin{aligned} \sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m^i_\sigma + \sum_{\sigma \in \mathcal{S}} a_\sigma p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) \quad &< \\ \delta^i + \sum_{\sigma \in \mathcal{S}} a_\sigma h^i_\sigma - \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g^i_\sigma, \end{aligned}$$

which violates optimality. □

Proof of Proposition 5.1. The proof is organized as follows. First (A), we introduce a notion of abstract equilibrium, which allows for the determination of present value prices of commodities independently of state prices. Second (B), we show that an abstract equilibrium exists in every truncated economy for every low enough overall price level. Third (C), we prove that the limit of truncated equilibria is an abstract equilibrium of the infinite-horizon economy. Fourth (D), we show that, if the overall price level is high enough, then transfers are positive. Last (E), we argue that there is a multiplicity of state prices compatible with a given abstract equilibrium allocation. □

A. Abstract Equilibrium. Let X^i be the consumption space of individual i, the positive cone of $\ell_\infty(\mathcal{S} \times \mathcal{N})$, and Π the space of normalized present value prices of commodities, the subset of the positive cone of $\ell_1(\mathcal{S} \times \mathcal{N})$ satisfying the normalization $\|\pi\|_1 = 1$. For (π, x) in $\ell_1(\mathcal{S} \times \mathcal{N}) \times \ell_\infty(\mathcal{S} \times \mathcal{N})$, $\pi \cdot x = \sum_{\sigma \in \mathcal{S}} \pi_\sigma \cdot x_\sigma$ denotes the duality operation.[9]

[9]For a real map, x, on $\mathcal{S}$ and a real map, z, on $\mathcal{S} \times \mathcal{N}$, $xz = zx$ is the real map on $\mathcal{S} \times \mathcal{N}$ obtained by point-wise product, $(\ldots, x_\sigma z_\sigma, \ldots) = (\ldots, z_\sigma x_\sigma, \ldots)$. Moreover, we use

$$\left(\frac{1}{1+r}\right) = \left(\ldots, \left(\frac{1}{1+r_\sigma}\right), \ldots\right) \quad \text{and} \quad \left(\frac{r}{1+r}\right) = \left(\ldots, \left(\frac{r_\sigma}{1+r_\sigma}\right), \ldots\right)$$

We consider the following abstract notion of equilibrium: it consists of present value prices of commodities, π, an allocation, $(\ldots, x^i, \ldots)$, an aggregate transfer, β, and a positive index for (the reciprocal of) the overall price level, μ, such that:

(a) market clearing is achieved,

$$\sum_i x^i - \sum_i e^i = 0;$$

(b) for every individual,

$$z^i \succ^i x^i \text{ implies } \pi \cdot \left(z^i - x^i\right) + \left(\frac{r}{1+r}\right) \pi \cdot \left(\left(z^i - e^i\right)^- - \left(x^i - e^i\right)^-\right) > 0$$

and

$$\pi \cdot \left(x^i - e^i\right) + \left(\frac{r}{1+r}\right) \pi \cdot \left(x^i - e^i\right)^- = \mu \left(\delta^i + \zeta^i \beta\right) - \left(\frac{1}{1+r}\right) \pi \cdot g^i.$$

To offset the redundancy stemming from the choice of the unit of account, we add the normalization $\pi \in \Pi$. Notice that, in an abstract equilibrium, $\mu = 0$ is allowed. $\square$

B. Truncated Abstract Equilibrium. Suppose that all vector spaces are of finite dimension, which corresponds to a truncated economy. We show that an abstract equilibrium exists for some transfer β when individuals possibly hold initial debt positions. Choose any positive μ small enough for non-emptiness of the budget constraint of every individual evaluated at $\beta = -\delta$ and at all normalized present value prices of commodities (this can be done by interiority assumptions). Consider the space of all

$$f = \left(\left(\ldots, x^i, \ldots\right), \pi, \beta\right) \in \cdots \times X^i \times \cdots \times \Pi \times B = F,$$

where X^i is the consumption space of individual i, Π is the space of normalized present value prices and $B = \{\beta \geq -\delta\}$. A correspondence $\hat{f} \twoheadrightarrow \bar{f}$ is defined by:

(a) $\bar{x}^i$ is an optimal choice subject to

$$\hat{\pi} \cdot \left(x^i - e^i\right) + \left(\frac{r}{1+r}\right) \hat{\pi} \cdot \left(x^i - e^i\right)^- \leq \mu \left(\delta^i + \zeta^i \hat{\beta}\right) - \left(\frac{1}{1+r}\right) \hat{\pi} \cdot g^i;$$

(b) $\bar{\beta}$ solves

$$\left(\frac{r}{1+r}\right) \hat{\pi} \cdot \sum_i \left(\hat{x}^i - e^i\right)^- + \left(\frac{1}{1+r}\right) \hat{\pi} \cdot g = \mu \left(\delta + \beta\right);$$

(c) $\bar{\pi}$ maximizes

$$\pi \cdot \sum_i \left(\hat{x}^i - e^i\right).$$

A fixed point exists and it can be shown to be an abstract equilibrium of the truncated economy (Drèze and Polemarchakis [8]). Therefore, in a truncated economy, an abstract equilibrium exists for all arbitrarily chosen positive μ small enough. $\square$

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for notational convenience.

C. Limit Argument. We now make truncation explicit. For a vector, x, in $\ell(\mathcal{S}\times\mathcal{N})$, let x^t denote its truncation at t. That is, $x^t_\sigma = x_\sigma$, if $0 \le t_\sigma \le t$, and $x^t_\sigma = 0$, otherwise. A t-truncated economy is constructed as follows: preferences on the consumption space, X^i, the positive cone of $\ell_\infty(\mathcal{S}\times\mathcal{N})$, are recovered using $x^i \succeq^i_t z^i$ if and only if $x^{i,t} + \left(e^i - e^{i,t}\right) \succeq^i z^{i,t} + \left(e^i - e^{i,t}\right)$; truncated present value prices of commodities are elements of $\Pi_t = \{\pi \in \Pi : \pi^t = \pi\}$.

Consider a sequence of abstract equilibria of t-truncated economies: allocations are $\left(\ldots, x^i_t, \ldots\right)$, present value prices of commodities are π_t, transfers are β_t and the constant index for the overall price level is μ. To simplify, write $\alpha^i_t = \mu\left(\delta^i + \zeta^i\beta_t\right)$. Notice that, in every truncated economy, one can assume that

$$\alpha^i_t + \left(\frac{1}{1+r}\right)\pi_t \cdot \left(e^i - g^i\right) \ge \epsilon > 0,$$

where ϵ does not depend on the truncation for given μ. This, indeed, follows from interiority assumptions and boundedness of nominal rates of interest. We refer to such inequalities as solvency conditions.

Letting

$$\varphi_t = (\ldots, \varphi_t(\sigma), \ldots) = \left(\ldots, \left(\frac{1}{1+r(\sigma)}\right)\pi_t(\sigma), \ldots\right),$$

π_t and φ_t can be viewed as elements of $ba(\mathcal{S}\times\mathcal{N})$, the norm dual of $\ell_\infty(\mathcal{S}\times\mathcal{N})$ consisting of all finitely additive set functions on $\mathcal{S}\times\mathcal{N}$ and endowed with the norm $\|\cdot\|_{ba}$ (the norm of total variation). Let $\sigma(ba, \ell_\infty)$ denote the weak* topology of $ba(\mathcal{S}\times\mathcal{N})$. Since

$$\|\varphi_t\|_1 = \|\varphi_t\|_{ba} \le \|\pi_t\|_{ba} = \|\pi_t\|_1 = 1$$

and since, by Alaoglu Theorem, the unit sphere in $ba(\mathcal{S}\times\mathcal{N})$ is $\sigma(ba, \ell_\infty)$ compact, without loss of generality, $\{\pi_t\}$ and $\{\varphi_t\}$ converge to π and φ, respectively, in the $\sigma(ba, \ell_\infty)$ topology. Moreover, both π and φ, as well as $\pi - \varphi$, are positive elements of $ba(\mathcal{S}\times\mathcal{N})$ and $0 < \|\varphi\|_{ba} \le \|\pi\|_{ba} = 1$.

By Tychonov Theorem, without loss of generality, every $\left\{x^i_t\right\}$ converges to x^i in the product topology. Since the product and the Mackey topology coincide on bounded subsets of $\ell_\infty(\mathcal{S}\times\mathcal{N})$, it follows that every $\left\{x^i_t\right\}$ converges to x^i in the Mackey topology.

As $\{\beta_t\}$ can be assumed to be bounded, without loss of generality, it converges to β. Defining $\alpha^i = \mu\left(\delta^i + \zeta^i\beta\right)$, it follows that every α^i_t converges to α^i.

We now show that, for every positive μ small enough, there is an abstract equilibrium. The proof, which is presented in a sequence of steps (C.1-C.4), uses standard arguments. □

C.1. Decomposition. Since π (φ) is a positive linear functional, it follows from the Yosida-Hewitt Theorem that there is a unique decomposition $\pi = \pi_f + \pi_b$ $(\varphi = \varphi_f + \varphi_b)$, where π_f (φ_f) is a positive functional in $\ell_1(\mathcal{S}\times\mathcal{N})$, the Mackey-topology dual of $\ell_\infty(\mathcal{S}\times\mathcal{N})$, and π_b (φ_b) is a positive finitely additive measure (a pure charge) vanishing on all vectors having only a finite number of non-zero components. □

C.2. $z^i \succeq^i x^i$ implies

$$\varphi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ \ge \alpha^i + \varphi \cdot \left(z^i - e^i\right)^-.$$

For a strictly positive real number, λ, $z^i + \lambda e^i \succ_t^i x_t^i$ for all t large enough, which implies that

$$\varphi_t \cdot g^i + \pi_t \cdot \left(z^i - (1-\lambda)\, e^i\right)^+ \geq \alpha_t^i + \varphi_t \cdot \left(z^i - (1-\lambda)\, e^i\right)^- .$$

Taking the limit, one obtains

$$\varphi \cdot g^i + \pi \cdot \left(z^i - (1-\lambda)\, e^i\right)^+ \geq \alpha^i + \varphi \cdot \left(z^i - (1-\lambda)\, e^i\right)^- .$$

As lattice operations are continuous in the norm topology and π and φ are norm-continuous linear functionals, letting λ go to zero, the claim is proven. $\square$

C.3. $z^i \succ^i x^i$ implies

$$\varphi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ > \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- .$$

Continuity of preferences implies that $\lambda z^i \succ^i x^i$ for some $0 < \lambda < 1$. Since

$$\varphi \cdot g^i + \lambda \pi \cdot \left(z^i - e^i\right) + (\pi - \varphi) \cdot \left(\lambda z^i - e^i\right)^- \geq \alpha^i + (1-\lambda)\, \pi \cdot e^i$$

and

$$\lambda \left(z^i - e^i\right)^- + (1-\lambda)\, e^i \geq \left(\lambda z^i - e^i\right)^- ,$$

one obtains

$$\begin{aligned} \varphi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ \quad &\geq \quad \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- + \left(\frac{1-\lambda}{\lambda}\right) \left(\varphi \cdot \left(e^i - g^i\right) + \alpha^i\right) \\ &\geq \quad \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- + \left(\frac{1-\lambda}{\lambda}\right) \epsilon, \end{aligned}$$

where the last inequality follows from solvency conditions. $\square$

C.4. $\pi_b = \varphi_b = 0$ and

$$\varphi \cdot g^i + \pi \cdot \left(x^i - e^i\right)^+ = \alpha^i + \varphi \cdot \left(x^i - e^i\right)^- .$$

For a vector u in $\ell\,(\mathcal{S} \times \mathcal{N})$,

$$\varphi_t \cdot u = \pi_t \cdot \left(\left(\frac{1}{1+r}\right) u\right)$$

holds at every t and, hence, in the limit,

$$\varphi \cdot u = \pi \cdot \left(\left(\frac{1}{1+r}\right) u\right).$$

Using truncations, one can show that

$$\varphi_b \cdot u = \pi_b \cdot \left(\left(\frac{1}{1+r}\right) u\right).$$

It follows that $\pi_b = 0$ if and only if $\varphi_b = 0$.

Suppose that $\pi_b > 0$, so that, by interiority assumptions, $\varphi_b \cdot \left(e^i - g^i\right) = \xi > 0$. Since $x^{i,t} + \lambda e^{i,t} \succ^i x^i$ for all t large enough and all strictly positive real numbers,

λ,

$$\begin{aligned}
\varphi \cdot g^i + \pi \cdot \left(\left(x^i - e^i\right)^+\right)^t + \lambda \pi \cdot e^{i,t} &\geq \\
\varphi \cdot g^i + \pi \cdot \left(x^{i,t} + \lambda e^{i,t} - e^i\right)^+ &\geq \\
\alpha^i + \varphi \cdot \left(x^{i,t} + \lambda e^{i,t} - e^i\right)^- &\geq \\
\alpha^i + \varphi \cdot \left(\left(x^i - e^i\right)^-\right)^t - \lambda \varphi \cdot e^{i,t} + \varphi \cdot \left(e^i - e^{i,t}\right).
\end{aligned}$$

In the limit, one obtains

$$\begin{aligned}
\varphi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i\right)^+ + \lambda \left(\pi_f - \varphi_f\right) \cdot e^i &\geq \\
\alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \varphi_b \cdot \left(e^i - g^i\right) &\geq \\
\alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \xi.
\end{aligned}$$

Thus,

$$\varphi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i\right)^+ \geq \alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \xi.$$

To prove equality, notice that, for all $0 \leq s \leq t$,

$$\varphi_t \cdot g^i + \pi_t \cdot \left(\left(x_t^i - e^i\right)^+\right)^s \leq \alpha^i + \varphi_t \cdot \left(\left(x_t^i - e^i\right)^-\right)^s + \varphi_t \cdot \left(e^i - e^{i,s}\right).$$

Therefore, in the limit,

$$\varphi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i\right)^+ \leq \alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \xi.$$

Summing over individuals,

$$\varphi_f \cdot g + \left(\pi_f - \varphi_f\right) \cdot \sum_i \left(x^i - e^i\right)^- > \sum_i \alpha^i.$$

Observe that, for all $0 \leq s \leq t$,

$$\varphi_t \cdot g^s + \left(\pi_t - \varphi_t\right) \cdot \left(\sum_i \left(x_t^i - e^i\right)^-\right)^s \leq \sum_i \alpha^i.$$

In the limit,

$$\varphi_f \cdot g + \left(\pi_f - \varphi_f\right) \cdot \sum_i \left(x^i - e^i\right)^- \leq \sum_i \alpha^i,$$

a contradiction to the previous reverse strict inequality. □

C.5. Limit is an abstract equilibrium. By point-wise limits, one obtains

$$\varphi = \left(\frac{1}{1+r}\right) \pi,$$

thus proving the claim. □

D. Positivity of Transfers. We show now that, for every small enough $\mu > 0$, there is an abstract equilibrium with associated transfer $\beta_\mu \geq 0$. Suppose that, letting μ vanish, there is a sequence of abstract equilibria with associated transfers $-\delta \leq \beta_\mu \leq 0$. One can show that, possibly using subsequences, the limit is also an abstract equilibrium with $\mu = 0$, which implies

$$\left(\frac{r}{1+r}\right) \pi \cdot \sum_i \left(x^i - e^i\right)^- + \left(\frac{1}{1+r}\right) \pi \cdot g = 0.$$

Since present value prices of commodities are strictly positive in every abstract equilibrium, one is to assume that fiscal policy is zero, for, otherwise, a contradiction would emerge. If monetary policy is strictly positive, however, the limit allocation, $\left(\ldots, x^i, \ldots\right)$, does not involve trade, that is, it coincides with the initial allocation, $\left(\ldots, e^i, \ldots\right)$. By the condition on trade at equilibrium, there exists an allocation, $\left(\ldots, z^i, \ldots\right)$, which weakly Pareto dominates the initial allocation, $\left(\ldots, e^i, \ldots\right)$, and satisfies

$$\sum_i z^i + \left(\frac{r}{1+r}\right) \sum_i \left(z^i - e^i\right)^- \leq \sum_i e^i.$$

By the optimality of plans, it follows that, for every individual,

$$\begin{aligned} \pi \cdot \left(z^i - e^i + \left(\frac{r}{1+r}\right)\left(z^i - e^i\right)^-\right) &= \\ \pi \cdot \left(z^i - e^i\right) + \left(\frac{r}{1+r}\right) \pi \cdot \left(z^i - e^i\right)^- &> 0, \end{aligned}$$

which, summing over individuals, implies

$$\pi \cdot \left(\sum_i z^i - \sum_i e^i + \left(\frac{r}{1+r}\right) \sum_i \left(z^i - e^i\right)^-\right) > 0,$$

a contradiction. Therefore, as μ vanishes, associated transfers are strictly positive, that is, $\beta_\mu > 0$. □

E. State Prices. Consider any abstract equilibrium with tansfers $\beta \geq 0$ and, without loss of generality, assume that $\mu = 1$. Let u be the positive element of $\ell\left(\mathcal{S}\right)$ defined, at every date-event, by

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1+r_\tau}\right) \pi_\tau \cdot g_\tau + \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1+r_\tau}\right) \pi_\tau \cdot \sum_i \left(x^i_\tau - e^i_\tau\right)^- = u_\tau.$$

For arbitrarily selected state prices, a, consistent with nominal rates of interest, r, one sets, at the initial date-event, ϕ, $h_\phi = \beta$ and, at every non-initial date-event, σ, $h_\sigma = 0$ and $a_\sigma w_\sigma = u_\sigma$. It is easily verified that such a construnction fulfills public sequential budget constraints for some portfolio policy, Θ. □

Proof of Lemma 7.1. Let $\left(\ldots, x^i, \ldots\right)$ be a weakly supportable feasible allocation and suppose that there is an allocation $\left(\ldots, z^i, \ldots\right)$ which Pareto dominates $\left(\ldots, x^i, \ldots\right)$ and satisfies adapted feasibility. In particular, assume that $z^j \succ^j x^j$, so that $\theta = \left(z^j - x^j\right)^+ > 0$ by the strict monotonicity of preferences. For a positive small real number α, define

$$y^j = z^j - \alpha\theta = x^i + (1-\alpha)\left(z^j - x^j\right)^+ - \left(z^j - x^j\right)^- \geq 0$$

and

$$y^i = z^i + \alpha\left(n-1\right)^{-1}\left(\frac{1}{1+r}\right)\theta > z^i,$$

where n is the number of individuals. The number α can be chosen so small as to satisfy $y^i \succ^i x^i$, for all individuals, because preferences are Mackey-continuous and strictly monotone. Notice that

$$\sum_i y^i = \sum_i z^i - \alpha\theta + \alpha\left(\frac{1}{1+r}\right)\theta = \sum_i z^i - \alpha\left(\frac{r}{1+r}\right)\theta$$

and

$$\sum_i \left(\frac{r}{1+r}\right)\left(y^i - e^i\right)^- \le \sum_i \left(\frac{r}{1+r}\right)\left(z^i - e^i\right)^- + \alpha\left(\frac{r}{1+r}\right)\theta.$$

This contradicts the fact that $\left(\ldots, x^i, \ldots\right)$ is a supportable feasible allocation.

Concerning the second statement, suppose that $\left(\ldots, x^i, \ldots\right)$ is a supportable feasible allocation and

$$-\theta = \sum_i \left(x^i - e^i\right) < 0.$$

Define, for each individual, $z^i = x^i + n^{-1}\theta$, where n is the number of individuals, and notice that, by the strict monotonicity of preferences, $z^i \succ^i x^i$. Summing over individuals, we have that

$$\sum_i z^i + \sum_i \left(\frac{r}{1+r}\right)\left(z^i - e^i\right)^- \le \theta + \sum_i x^i + \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-,$$

which contradicts supportability. $\square$

Proof of Proposition 7.1. To obtain a contradiction, assume $\left(\ldots, x^i, \ldots\right)$ is an equilibrium allocation and is not weakly supportable. Therefore, there is a weakly Pareto improving allocation, $\left(\ldots, z^i, \ldots\right)$, which satisfies adapted feasibility. Equilibrium implies

$$\pi \cdot \left(z^i - x^i\right) > \left(\frac{r}{1+r}\right)\pi \cdot \left(\left(x^i - e^i\right)^- - \left(z^i - e^i\right)^-\right)$$

and, summing over all individuals,

$$\pi \cdot \left(\sum_i z^i + \sum_i \left(\frac{r}{1+r}\right)\left(z^i - e^i\right)^- - \sum_i e^i - \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-\right) > 0.$$

Since π is a positive linear functional, a contradiction is obtained. $\square$

Proof of Proposition 7.2. Because of Lemma 7.1, one can assume that $\left(\ldots, x^i, \ldots\right)$ is a weakly supportable allocation. For every individual, define

$$F^i = \left\{z^i + \left(\frac{r}{1+r}\right)\left(z^i - e^i\right)^- \in X^i : z^i \in X^i \text{ and } z^i \succ^i x^i\right\}.$$

Clearly, by the strict monotonicity of preferences, F^i has a nonempty interior in the norm topology. We then show that every F^i is convex.

Consider

$$f_0^i = z_0^i + \left(\frac{r}{1+r}\right)\left(z_0^i - e^i\right)^-, \qquad \text{with } z_0^i \succ^i x^i,$$
$$f_1^i = z_1^i + \left(\frac{r}{1+r}\right)\left(z_1^i - e^i\right)^-, \qquad \text{with } z_1^i \succ^i x^i.$$

For every $0 < \lambda < 1$, define $f_\lambda^i = (1-\lambda) f_0^i + \lambda f_1^i$ and $z_\lambda^i = f_\lambda^i - r\left(f_\lambda^i - e^i\right)^-$, so that

$$z_\lambda^i + \left(\frac{r}{1+r}\right)\left(z_\lambda^i - e^i\right)^- = z_\lambda^i + r\left(f_\lambda^i - e^i\right)^- = f_\lambda^i.$$

Since

$$\begin{aligned}
z^i_\lambda &= \\
f^i_\lambda - r\left(f^i_\lambda - e^i\right)^- &\geq \\
f^i_\lambda - (1-\lambda)\, r\left(f^i_0 - e^i\right)^- - \lambda r\left(f^i_1 - e^i\right)^- &= \\
f^i_\lambda - (1-\lambda)\left(\frac{r}{1+r}\right)\left(z^i_0 - e^i\right)^- - \lambda\left(\frac{r}{1+r}\right)\left(z^i_1 - e^i\right)^- &= \\
(1-\lambda)\, z^i_0 + \lambda z^i_1 &\geq 0,
\end{aligned}$$

it follows that z^i_λ belongs to the consumption space X^i. Convexity and monotonicity of preferences then guarantee that $z^i_\lambda \succeq^i (1-\lambda)\, z^i_0 + \lambda z^i_1 \succ^i x^i$, thus implying that f^i_λ is an element of F^i.

Consider the convex set

$$F = \sum_i F^i - \sum_i x^i - \left(\frac{r}{1+r}\right)\sum_i \left(x^i - e^i\right)^-$$

and notice that $0 \notin F$, since $\left(\ldots, x^i, \ldots\right)$ is a weakly supportable feasible allocation. One can then apply the Separating Hyperplane Theorem, which gives a norm-continuous non-zero linear functional π on $\ell_\infty\left(\mathcal{S}\times\mathcal{N}\right)$ such that, for all f in F, $\pi\cdot f \geq 0$. Since F contains the positive cone, π is a positive functional. Therefore, the Yosida-Hewitt Decomposition Theorem allows one to write $\pi = \pi_f + \pi_b$, where π_f is a norm-continuous positive linear functional on $\ell_\infty\left(\mathcal{S}\times\mathcal{N}\right)$ admitting a sequence representation (thus, a Mackey-continuous positive linear functional on $\ell_\infty\left(\mathcal{S}\times\mathcal{N}\right)$) and π_b is a positive purely finitely additive measure. We show that π_f is non-zero and separates.

Fix any f in F. There is an allocation, $\left(\ldots, z^i, \ldots\right)$, weakly Pareto-improving upon $\left(\ldots, x^i, \ldots\right)$, such that

$$f = \sum_i z^i + \sum_i \left(\frac{r}{1+r}\right)\left(z^i - e^i\right)^- - \sum_i x^i - \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-$$

Mackey-continuity of preferences implies that $z^{i,t} + \left(e^i - e^{i,t}\right) \succ^i x^i$, for t large enough, and, therefore,

$$\sum_i \left(z^i - e^i\right)^t + \sum_i \left(\frac{r}{1+r}\right)\left(\left(z^i - e^i\right)^-\right)^t - \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-$$

is also an element of F. It follows that

$$\begin{aligned}
\pi_f \cdot \left(\sum_i \left(z^i - x^i\right)^t + \sum_i \left(\frac{r}{1+r}\right)\left(\left(z^i - e^i\right)^-\right)^t - \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-\right) \\
\geq\ \pi_b\left(\sum_i \left(\frac{r}{1+r}\right)\pi\cdot\left(x^i - e^i\right)^-\right)\ \geq\ 0.
\end{aligned}$$

Taking the limit and using the Mackey continuity of π_f, one establishes that $\pi_f \cdot f \geq 0$.

Suppose now that $\pi_b = 0$. Since, for each t large enough, $\left(x^i + e^i\right)^t \succ^i x^i$, it follows that

$$\sum_i \left(x^i + e^i\right)^t + \sum_i \left(\frac{r}{1+r}\right)\left(e^i - e^{i,t}\right) - \sum_i x^i - \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^- \quad =$$

$$\sum_i x^{i,t} + \sum_i \left(\frac{1}{1+r}\right) e^{i,t} - \sum_i \left(\frac{1}{1+r}\right) e^i - \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-$$

is an element of F. Separation, therefore, gives

$$0 \geq \pi_b \cdot \left(\sum_i \left(\frac{1}{1+r}\right) e^i + \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-\right) > 0,$$

where the last strict inequality follows from interiority assumptions and boundedness of nominal rates of interest. By contradiction, this proves that $\pi_f > 0$.

Fix an individual j and suppose that $z^j \succeq^j x^j$. Define, for all other individuals, i, $z^i = x^i$. Observe that, for all strictly positive real numbers λ,

$$f_\lambda = z^j + \lambda \sum_i e^i + \left(\frac{r}{1+r}\right)\sum_i \left(z^i - (1-\lambda)\, e^i\right)^- - x^j - \left(\frac{r}{1+r}\right)\sum_i \left(x^i - e^i\right)^-$$

is an element of F. As f_λ converges to f_0 in the Mackey-topology (lattice operations are Mackey-continuous) and π_f is a Mackey-continuous linear functional, one obtains

$$\pi_f \cdot f_0 = \pi \cdot \left(z^j - x^j\right) + \left(\frac{r}{1+r}\right)\pi \cdot \left(\left(z^j - e^j\right)^- - \left(x^j - e^j\right)^-\right) \geq 0,$$

thus establishing the claim. □

Proof of Proposition 7.3. Suppose that an alternative allocation $\left(\ldots, z^i, \ldots\right)$ Pareto-dominates $\left(\ldots, x^i, \ldots\right)$ and satisfies the inequality in the Proposition. Since

$$\left(z^i - e^i\right)^- \leq \left(z^i - x^i\right)^- + \left(x^i - e^i\right)^-$$

and r is positive, it follows that

$$\sum_i z^i + \sum_i \left(\frac{r}{1+r}\right)\left(\left(z^i - e^i\right)^- - \left(x^i - e^i\right)^-\right) \leq \sum_i x^i = \sum_i e^i,$$

which contradicts supportability. □

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