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Market Clearing and Derivative Pricing

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Birgit Grodal Symposium

Topics in Mathematical Economics

The participants in a September 2002 Workshop on *Topics in Mathematical Economics* in honor of Birgit Grodal decided to have a series of papers appear on Birgit Grodal's 60'th birthday, June 24, 2003.

The Institute of Economics suggested that the papers became Discussion Papers from the Institute.

The editor of *Economic Theory* offered to consider the papers for a special Festschrift issue of the journal with Karl Vind as Guest Editor.

This paper is one of the many papers sent to the Discussion Paper series.

Most of these papers will later also be published in a special issue of *Economic Theory*.

Tillykke Birgit

Troels Østergaard Sørensen	Karl Vind
Head of Institute	Guest Editor

Market Clearing and Derivative Pricing

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Abstract

We develop a method of assigning unique prices to derivative securities, including options, in the continuous-time finance models developed in Raimondo [45] and Anderson and Raimondo [6]. In contrast with the martingale method of valuing options, which cannot distinguish among infinitely many possible option pricing processes for a given underlying securities price process when markets are dynamically incomplete, our option prices are uniquely determined in equilibrium as a function of the underlying economic data and the underlying securities price process; in the single-agent model, this function is given in closed form.

1 Introduction

Assuming that the price of a stock follows a geometric Brownian motion, Black and Scholes [9] developed the Black-Scholes formula for pricing options on that stock. Merton [40] showed that if markets are dynamically complete, there is a trading strategy that replicates the payoff of the option; as a consequence, the Black-Scholes formula for the price of the option can be obtained by arbitrage considerations.

Raimondo [45] proved existence of equilibrium in a continuous-time finance model with one agent; the theorem covers both models with dynamically incomplete as well as dynamically incomplete markets. The existence theorem in that paper had not previously appeared in the literature, although it may be possible to derive it from other results in the literature. The paper made two additional contributions: it obtained the pricing formula in closed form, resulting in specific pricing predictions, and it established methodology that has the potential to prove existence in the multi-agent, dynamically incomplete case; see Anderson and Raimondo [6].

In this paper, we show that we can obtain an explicit formula, in closed form, for pricing options and other derivatives in the same setting as Raimondo [45]. The method extends to the multi-agent model of Anderson and Raimondo [6], except that the option price is not given in closed form, as a function of the stock price and the underlying data. Raimondo [45] showed that the price of a stock in a representative agent model follows generalized geometric Brownian motion only when all three of the following conditions are satisfied:

1. the agent has a CRRA utility function;
2. there is only one stock;
3. there are no endowments in the terminal period, so the only source of wealth in that period is the stock.

In all other cases, the equilibrium price of the stock does not follow geometric Brownian motion, and hence the original Black-Scholes formula for pricing options on the stock does not apply.

The standard technique for pricing options and other derivatives in situations where the underlying securities price process is not geometric Brownian motion is the martingale method; Nielsen [42] provides an excellent exposition of this method. This method was initiated by Harrison and Kreps [24] in the complete markets case, and has since been extended to dynamically incomplete markets. The idea is to construct a martingale whose terminal value is the payoff of the option or other derivative. If markets are dynamically complete, this martingale is uniquely determined by the underlying price process, and determines the pricing of the option by arbitrage. However, if markets are dynamically incomplete, the martingale is not uniquely determined by the underlying securities prices; indeed, the method allows infinitely many different pricing processes for the option.

In this paper, we show how to assign a unique equilibrium price to options and other derivatives in single-agent economies, regardless of whether markets are dynamically complete or incomplete. The derivative price process is uniquely determined by the underlying economic data–the endowment process and utility of the agent, and the dividends of the securities. The equilibrium price is given in closed form, as a function of the underlying data of the economy. The pricing of the securities and derivatives exhibits a number of significant properties; in particular, there are specific correlations among the prices of the underlying securities which are not characteristic of geometric Brownian motion,[1] and the price of an option on one security depends on the price of all securities, not just the price of the security underlying the option.

In this paper and the companion papers cited below, we assert that equilibrium imposes more structure on finance models than that implied by the absence of arbitrage alone. It has been argued that equilibrium has no predictive power on securities and derivatives prices beyond that contained in the absence of arbitrage. The somewhat imprecise argument goes as follows: if prices are free of arbitrage, then under some hypotheses the prices will be martingales with respect to a probability measure mutually absolutely con-

[1] For example, even if the terminal dividends (and hence the terminal prices) of stocks are independent, the prices of the stocks exhibit specific correlations due to wealth effects. If the terminal dividends and prices of securities are correlated, for example if the terminal prices are given by

$$S(T) = e^{S(0)+\mu T+\sigma W}$$

where W is a K-dimensional Brownian motion, $\mu \in \mathbf{R}^N$ and $\sigma \in \mathbf{R}^{N\times K}$ is a matrix, the covariance matrix of the securities at times $t < T$ is not equal to σ. The covariance matrix can be calculated explicitly from the close-form pricing formula.

tinuous with respect to the true probability measure. One can then extract a state-dependent felicity function for a single agent which supports the given arbitrage-free prices as an equilibrium.

Notice that this argument requires that the state-dependent felicity be chosen very carefully to produce the given pricing process. Suppose instead we require that the single agent's utility function be the expected utility generated by some state-*independent* felicity function. In that case, the argument just cited that any arbitrage-free pricing system can be justified as an equilibrium will not hold; state-dependence is essential to the proof. Of course, one might object that state-dependence is commonly observed in practice. It is difficult to argue against this objection, but the implication of this objection is not that one should consider a pricing process justified if it can be supported as equilibrium with respect to a felicity function whose state-dependence is carefully chosen to match the peculiarities of the pricing process. The implication is that the state-dependence should be specified as part of the model, and the pricing process should be required to be an equilibrium with respect to the exogenously given state-dependent utility function. Allowing arbitrary state-dependence is a virtue in a result of the form "for all state-dependent felicity functions, ... ;" is is not a virtue in a result of the form "there exists a state-dependent felicity function ... ."

One of the reasons that state-independence appears natural in some models is that the models are partial equilibrium. If a significant portion of household wealth is held in housing, a model that includes stocks but not housing is a partial equilibrium model. Since changes in the value of housing induce wealth effects that alter individuals' willingness to hold stocks, changes in housing values seem, in a stock-only model, to be instances of state-dependent felicity. But in a general equilibrium model which includes both stocks and housing, the state-dependence disappears. In particular, we argue that the relationship between stock pricing and housing can only be properly studied in a general equilibrium model which includes both. More generally, in this and the companion papers, we take the position that all assets and securities should be included in the model, and that felicity functions (and in particular any state-dependence of felicity functions) should be taken as exogenously specified.

This approach has real economic and financial consequences. If one's sole criterion for validating a price process is the absence of arbitrage, then the economic characteristics (endowments, utility functions, and information) of the agent(s) can play no role in determining prices: absence of arbitrage is a property of the pricing process alone, not of the underlying economic data. The pricing process in which stock prices are given by geometric Brownian motion, and option prices by the Black-Scholes formula, is arbitrage-free; hence, if one's sole criterion for price processes is the absence of arbitrage, one can never reject this pricing process on theoretical grounds. If one assumes geometric Brownian motion–Black-Scholes pricing, a specific prediction follows. One can deduce the implied volatility σ of the geometric Brownian motion from the stock price, the strike price of the option, the price of the option, and the time left until the expiration date of the option; the implied volatility σ must be independent of the strike price, so the graph of σ in terms of the strike price must be a horizontal line. Our model provides a specific prediction on the shape of the implied volatility curve in terms of the economic primitives, and it is certainly *not* a horizontal line except in very special cases. Empirically, the implied volatility curve is in the shape of a "smile," with the implied volatility being higher for strike prices well below the current stock price as well as for strike prices well above the current stock price (see Campbell, Lo and McKinley [10]). In future work, we hope to explore assumptions on the underlying primitives which generate a "smile" that matches the empirically-measured implied volatility curve.

Other papers (see for example Heston [25]) have used the martingale method to provide closed-form prices for options outside the geometric Brownian motion context. However, the papers of which we are aware assume a particular stochastic process underlying the stock price. There is no guarantee that these stochastic processes are consistent with market clearing unless one carefully chooses a state-dependent utility function which carefully matches the peculiarities of the given stochastic process; indeed, Raimondo [45] makes it clear that the pricing processes commonly assumed are equilibrium processes only under very special circumstances. Second, the papers of which we are aware either assume dynamically complete markets, or in the case of dynamically incomplete markets provide one example of an arbitrage-free price system; there are infinitely many other arbitrage-free price systems, and there is no guarantee that the arbitrage-free price system they identify is consistent with market clearing with respect to a state-*independent* utility function; as we argue above, any state-dependence of utility should be exogenously specified, not carefully chosen to fit the peculiarities of a particular pricing process one is trying to justify. Third, closed forms solutions are obtained only in special cases.[2]

By contrast, in our paper, the option price is determined uniquely by market clearing. This is true even in situations in which markets are dynamically incomplete. We get the option price in closed form, even with a stochastic interest rate.

In this paper, we get everything in one package. Starting from assumptions about economic primitives, we derive equilibrium stock prices and derivative prices in closed form, whether or not markets are dynamically complete. We are not aware of any other papers that do this.

We emphasize that, in the single-agent case, the option price is given in closed form that can easily be evaluated numerically. The stock pays a dividend $A(\omega, T)$ only at the terminal period T; the dividend equals the terminal value of a geometric Brownian motion $e^{\mu T + \sigma\beta(\omega,T)}$. The equilibrium price of the stock does *not* follow geometric Brownian motion. The price of the stock and the option at times $t < T$ are given explicitly in terms of $\beta(\omega, t)$. Assuming one knows $\beta(\omega, t)$, it is a simple matter to compute the stock and option price numerically. $\beta(\omega, t)$ is measurable with respect to $\mathcal{F}_t$, the σ-algebra of events known at time t. We think of $\beta(\omega, t)$ as encapsulating the information available at time t about the terminal dividend of the stock; $\beta(\omega, t)$ is the conditional expectation, at time t, of $\frac{\ln(A(\omega,T))-\mu T}{\sigma}$. Since the underlying assumption of all continuous-time finance models is that agents know the information in $\mathcal{F}_t$ at time t, $\beta(\omega, t)$ should in principle be observable at time t. We might think of it as the consensus of analysts' projections about the terminal dividend.

As a practical matter, there is a vast amount of data reporting actual stock prices, while the data on analysts' projections is scarcer and harder to interpret. Thus, for empirical work, it is very desirable to calculate the relationship between the stock price and $\beta(\omega, t)$. A sufficient condition to ensure that the relationship between $\beta(\omega, t)$ and $\frac{p_A(\omega,t)}{p_B(\omega,t)}$ is invertible is $-\frac{\varphi_2''(c)}{\varphi_2'(c)} \leq \frac{1}{c}$, i.e. the coefficient of relative risk aversion is everywhere less than or equal to one.[3] This condition is far from necessary, and in future work, we hope to develop much more general conditions which imply invertibility; for exam-

[2]For example, Heston [25] obtains a closed-form solution with a deterministic interest rate, but not with a stochastic interest rate. We obtain a closed form solution with an endogenously determined stochastic interest rate.

[3]We do not need to assume the coefficient of relative risk aversion is constant, only that it is bounded above by one.

ple, if the current value of the individual's holding of a single stock does not make up too large a portion of the individual's expected wealth at period T, invertibility will still hold if the coefficient of relative risk aversion exceeds one but is not too high.

Although this paper only explicitly discusses the representative agent model of Raimondo [45], the method extends readily to the multi-agent model of Anderson and Raimondo [6]. The principal difference is that the formula for the option price depends on the terminal wealths of the agents; since these terminal wealths are not given in closed form, neither is the option price. One of the key lessons of Anderson and Raimondo [6] is that *equilibrium* prices of the basic securities, as well as derivative securities, necessarily depend on the terminal wealths of the agents. These terminal wealths depend on the whole history of prices, not just the terminal prices. As a consequence, the current price of an option is not determined as a function of the current prices of the securities, let alone a function of the current price of the stock on which it is written; instead, the current *equilibrium* price of an option depends on the entire history of securities prices up to that time. As a result, the usual formula for the trading strategy that replicates an option does not apply, even when markets are dynamically complete. We shall explore in future work whether the methods we have developed allow one to construct replication strategies for equilibrium securities prices.

As in Raimondo [45], our method makes use of nonstandard analysis, and in particular the nonstandard theory of stochastic processess as developed in Anderson [1], Keisler [32], and Lindström [33, 34, 35, 36]. These methods of nonstandard stochastic analysis have previously been applied to the theory of option pricing in Cutland, Kopp and Willinger [13, 14, 15, 16, 17, 18, 19]. Those papers primarily concern convergence of discrete versions of options to continuous-time versions, and their methods can likely be used to establish convergence results for the pricing formulas considered in this paper.

In nonstandard analysis, a hyperfinite set is an infinite set which possesses all the formal properties of finite sets; in particular, the Radner [43] and Duffie and Shafer [21, 22] existence results ensure that a hyperfinite incomplete markets economy has an equilibrium. We begin with a standard continuous-time model, construct a nonstandard hyperfinite economy, obtain an equilibrium for the hyperfinite economy, then use the nonstandard theory of stochastic processes to induce an equilibrium in the standard continuous-time model. For further comments on the methodology, see Raimondo [45].

2 The Model

The model we consider is essentially the same as that in Raimondo [45], except for the following changes. The payout of the stocks in the terminal period is given by the terminal value of a geometric generalized Brownian motion. Thus, there are d independent Brownian motions $\beta_1, \ldots, \beta_d$, and J stocks ($J \leq d$) with $A_j(\omega, T) = e^{\mu_j T + \sigma_j \beta_j}$ with $\mu_j \geq 0$ and $\sigma_j \geq 0$ (in Raimondo [45], $\mu_j = 0$ and $\sigma_j = 1$). We add M derivative securities $D_m (1 \leq m \leq M)$. D_m is in zero net supply. The payoff of D_m is given by

$$\begin{aligned} D_m(\omega, t) &= 0 \text{ if } t < T \\ D_m(\omega, T) &= G_m(A_1(\omega, T), \cdots, A_J(\omega, T)) \end{aligned}$$

for some continuous function $G_m : \mathbf{R}^J \to \mathbf{R}$ satisfying $|G_m(x)| \leq \max\{S_m, |x|^r\}$ for some $S_m \in \mathbf{R}$ and $r \in \mathbf{R}_+$. The definitions of pricing process, trading strategy, and budget sets are adapted in the obvious way to take into account the availability of the derivative securities; the market clearing condition at equilibrium requires that the agent hold zero units of each D_m at the terminal period T and at almost all times $t \in [0, T)$. As in Raimondo [45], we use $e^{\mu T + \sigma \beta(\omega, t) + \sigma \sqrt{T-t} x}$ to denote the vector

$$\left(e^{\mu_1 T + \sigma_1 \beta_1(\omega, t) + \sigma_1 \sqrt{T-t} x_1}, \ldots, e^{\mu_J T + \sigma_J \beta_J(\omega, t) + \sigma_J \sqrt{T-t} x_J} \right)$$

Theorem 2.1 *There is a standard probability space $(\Omega, \mathcal{F}, \mu)$, a filtration $\mathcal{F}_t$, and a d-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_d)$ such that the continuous time finance model just described has an equilibrium. The pricing process is given by*

$$\begin{aligned} p_{A_j}(\omega, t) &= e^{\mu_j t + \sigma_j \beta_j(\omega, t)} \int_{-\infty}^{\infty} \varphi_2' \left(F(t, \omega, x) \right) e^{\mu_j (T-t) + \sigma_j \sqrt{T-t} x_j} d\Phi(x) \\ p_B(\omega, t) &= \int_{-\infty}^{\infty} \varphi_2' \left(F(t, \omega, x) \right) d\Phi(x) \\ p_C(\omega, t) &= \varphi_1'(1) \text{ for } t < T \\ p_C(\omega, T) &= \varphi_2' \left(F(T, \omega, 0) \right) \\ p_{D_m}(\omega, t) &= \int_{-\infty}^{\infty} \varphi_2' \left(F(t, \omega, x) \right) G_m \left(e^{\mu T + \sigma \beta(\omega, t) + \sigma \sqrt{T-t} x} \right) d\Phi(x) \end{aligned} \quad (1)$$

where

$$F(t, \omega, x) = \rho \left(\beta(\omega, t) + \sqrt{T-t} x \right) + \mathbf{1}_J \cdot \left(e^{\mu T + \sigma \beta(\omega, t) + \sigma \sqrt{T-t} x} \right)$$

and Φ is the cumulative distribution function of the standard d-dimensional normal.

Before we turn to the proof, it may be helpful to give several examples of models included in Theorem 2.1 and the pricing processes that prevail in those models.

Example 2.2 There is one Brownian motion and a single stock based on it, one bond, and a European call option on the stock, with strike price $\bar{A}$; thus, we are in the case $J = d = K = 1$. The agent has no endowment in the terminal period T, so ρ is identically zero. In period T, the agent has the CRRA utility function $\phi_2(c) = \sqrt{c}$. Markets are dynamically complete. Observe that

$$\begin{aligned}
\int_a^b e^{\alpha x} d\Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{\alpha x} e^{-x^2/2} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_a^b e^{(2\alpha x - x^2)/2} dx \\
&= \frac{e^{\alpha^2/2}}{\sqrt{2\pi}} \int_a^b e^{(-\alpha^2 + 2\alpha x - x^2)/2} dx \\
&= \frac{e^{\alpha^2/2}}{\sqrt{2\pi}} \int_a^b e^{-(x-\alpha)^2/2} dx \\
&= \frac{e^{\alpha^2/2}}{\sqrt{2\pi}} \int_{a-\alpha}^{b-\alpha} e^{-x^2/2} dx \\
&= e^{\alpha^2/2} \left(\Phi(b-\alpha) - \Phi(a-\alpha) \right)
\end{aligned}$$

The pricing process is

$$\begin{aligned}
p_A(\omega, t) &= e^{\mu T + \sigma\beta(\omega,t)} \int_{-\infty}^{\infty} \varphi_2' \left(F(t, \omega, x) \right) e^{\sigma\sqrt{T-t}x} d\Phi(x) \\
&= e^{\mu T + \sigma\beta(\omega,t)} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{e^{\mu T + \sigma\beta(\omega,t) + \sigma\sqrt{T-t}x}}} e^{\sigma\sqrt{T-t}x} d\Phi(x) \\
&= \frac{e^{(\mu T + \sigma\beta(\omega,t))/2}}{2} \int_{-\infty}^{\infty} e^{\sigma\sqrt{T-t}x/2} d\Phi(x) \\
&= \frac{e^{(\mu T + \sigma\beta(\omega,t))/2 + \sigma^2(T-t)/8}}{2} \\
p_B(\omega, t) &= \int_{-\infty}^{\infty} \varphi_2' \left(F(t, \omega, x) \right) d\Phi(x) \\
&= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{e^{\mu t + \sigma\beta(\omega,t) + \sigma\sqrt{T-t}x}}} d\Phi(x)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2e^{(\mu T+\sigma\beta(\omega,t))/2}} \int_{-\infty}^{\infty} e^{-\sigma\sqrt{T-t}x/2} d\Phi(x) \\
&= \frac{e^{-(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2} \\
\frac{p_A(\omega,t)}{p_B(\omega,t)} &= \frac{\frac{e^{(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}}{\frac{e^{-(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}} \\
&= e^{\mu T+\sigma\beta(\omega,t)} \\
p_D(\omega,t) &= \int_{-\infty}^{\infty} \varphi_2'\left(F(t,\omega,x)\right) G\left(e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}\right) d\Phi(x) \\
&= \int_{-\infty}^{\infty} \frac{\max\{0, e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x} - \bar{A}\}}{2\sqrt{e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}}} d\Phi(x) \\
&= \int_{\frac{\ln \bar{A}-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}}^{\infty} \frac{e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}}{2\sqrt{e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}}} d\Phi(x) \\
&\quad -\bar{A}\int_{\frac{\ln \bar{A}-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}}^{\infty} \frac{1}{2\sqrt{e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}}} d\Phi(x) \\
&= \frac{1}{2}\int_{\frac{\ln \bar{A}-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}}^{\infty} \sqrt{e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}} d\Phi(x) \\
&\quad -\frac{\bar{A}}{2e^{\mu T+\sigma\beta(\omega,t)/2}} \int_{\frac{\ln \bar{A}-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}}^{\infty} e^{-\sigma\sqrt{T-t}x/2} d\Phi(x) \\
&= \frac{e^{(\mu T+\sigma\beta(\omega,t))/2}}{2} \int_{\frac{\ln \bar{A}-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}}^{\infty} e^{\sigma\sqrt{T-t}x/2} d\Phi(x) \\
&\quad -\frac{\bar{A}e^{-(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(-\frac{\sigma\sqrt{T-t}}{2} - \frac{\ln \bar{A}-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}\right) \\
&= \frac{e^{(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(\frac{\sigma\sqrt{T-t}}{2} - \frac{\ln \bar{A}-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}\right) \\
&\quad -\frac{\bar{A}e^{-(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(-\frac{\sigma\sqrt{T-t}}{2} - \frac{\ln \bar{A}-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}\right) \\
&= \frac{e^{(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(\frac{\ln(e^{\mu T+\sigma\beta(\omega,t)}/\bar{A}) + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right) \\
&\quad -\frac{\bar{A}e^{-(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(\frac{\ln(e^{\mu T+\sigma\beta(\omega,t)}/\bar{A}) - \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{p_D(\omega,t)}{p_B(\omega,t)} &= e^{\mu T+\sigma\beta(\omega,t)}\Phi\left(\frac{\ln(e^{\mu T+\sigma\beta(\omega,t)}/\bar{A})+\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right) \\
&\quad -\bar{A}\Phi\left(\frac{\ln(e^{\mu T+\sigma\beta(\omega,t)}/\bar{A})-\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right) \\
&= \frac{p_A(\omega,t)}{p_B(\omega,t)}\Phi\left(\frac{\ln(p_A(\omega,t)/p_B(\omega,t)\bar{A})+\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right) \\
&\quad -\bar{A}\Phi\left(\frac{\ln(p_A(\omega,t)/p_B(\omega,t)\bar{A})-\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)
\end{aligned}
$$

Note that the ratio $\frac{p_D(\omega,t)}{p_B(\omega,t)}$ follows the standard Black-Scholes formula. Note also that the ratio $\frac{p_A(\omega,t)}{p_B(\omega,t)}$ follows geometric Brownian motion. The reader may be surprised to see that the term μT rather than μt appears in the formula, i.e. $\frac{p_A(\omega,t)}{p_B(\omega,t)}$ follows a geometric Brownian motion with no drift. The reason is that the factor $e^{\mu T}$ in the payoff of the stock at time T just multiplies the value of the stock at all time periods by a constant factor. If we modified our utility function to incorporate time-discounting, identical drift factors would appear in p_A and p_B, so the ratio would be unchanged. We could obtain a nonzero drift in an infinite horizon model in which the agent discounts, and dividends are paid at all times and follow a geometric Brownian motion with drift.

Example 2.3 In this example, we modify Example 2.2 by changing the utility function in period T to the CARA function $\phi_2(c) = -e^{-\alpha c}$, for $\alpha > 0$. For simplicity, we also take $\mu_1 = 0$ and $\sigma_1 = 1$.

$$
\begin{aligned}
p_D(\omega,t) &= \int_{-\infty}^{\infty} \varphi_2'\left(F(t,\omega,x)\right) G\left(e^{\beta(\omega,t)+\sigma\sqrt{T-t}x}\right) d\Phi(x) \\
&= \int_{-\infty}^{\infty} \alpha e^{-\alpha e^{\beta(\omega,t)+\sqrt{T-t}x}} \max\left\{e^{\beta(\omega,t)+\sqrt{T-t}x}-\bar{A},0\right\} d\Phi(x) \\
&= \int_{\frac{\ln\bar{A}-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} \alpha e^{-\alpha e^{\beta(\omega,t)+\sqrt{T-t}x}} e^{\beta(\omega,t)+\sqrt{T-t}x} d\Phi(x) \\
&\quad -\bar{A}\int_{\frac{\ln\bar{A}-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} \alpha e^{-\alpha e^{\beta(\omega,t)+\sqrt{T-t}x}} d\Phi(x) \\
&= \int_{\frac{\ln\bar{A}-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} \alpha\left(\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}e^{(n+1)\beta(\omega,t)}e^{(n+1)\sqrt{T-t}x}\right) d\Phi(x)
\end{aligned}
$$

$$
\begin{aligned}
& -\bar{A}\int_{\frac{\ln \bar{A}-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} \alpha \left(\sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} e^{n\beta(\omega,t)} e^{n\sqrt{T-t}x} \right) d\Phi(x) \\
= \; & \alpha \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} e^{(n+1)\beta(\omega,t)} \int_{\frac{\ln \bar{A}-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} e^{(n+1)\sqrt{T-t}x} d\Phi(x) \\
& -\bar{A}\alpha \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} e^{n\beta(\omega,t)} \int_{\frac{\ln \bar{A}-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} e^{n\sqrt{T-t}x} d\Phi(x) \\
= \; & \alpha \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} e^{(n+1)\beta(\omega,t)} e^{\frac{(n+1)^2(T-t)}{2}} \Phi\left(\frac{(n+1)\sqrt{T-t}}{2} - \frac{\ln \bar{A}-\beta(\omega,t)}{\sqrt{T-t}} \right) \\
& -\bar{A}\alpha \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} e^{n\beta(\omega,t)} \; e^{\frac{n^2(T-t)}{2}} \Phi\left(\frac{n\sqrt{T-t}}{2} - \frac{\ln \bar{A}-\beta(\omega,t)}{\sqrt{T-t}} \right) \\
= \; & \alpha \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} e^{n\beta(\omega,t)} \left[e^{\frac{(n+1)^2(T-t)}{2}} e^{\beta(\omega,t)} \Phi(z_n) - \bar{A} e^{\frac{n^2(T-t)}{2}} \Phi\left(z_n - \frac{\sqrt{T-t}}{2} \right) \right] \\
p_A(\omega,t) \; = \; & \alpha \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} e^{(n+1)\beta(\omega,t)} e^{\frac{(n+1)^2(T-t)}{2}} \\
p_B(\omega,t) \; = \; & \alpha \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} e^{n\beta(\omega,t)} e^{\frac{n^2(T-t)}{2}}
\end{aligned}
$$

where

$$
z_n = \frac{\ln e^{\beta(\omega,t)}/\bar{A}}{\sqrt{T-t}} + \frac{(n+1)\sqrt{T-t}}{2}
$$

Example 2.4 In this example, we modify Example 2.2 by adding a stochastic endowment in the terminal period T. We take $J = K = 1$, $d = 2$, and $e(\omega,T) = e^{\beta_2(\omega,T)}$, so the endowment is given by the terminal value of an exponential Browian motion for which there is no corresponding stock. Markets are dynamically incomplete. The pricing formula becomes

$$
\begin{aligned}
p_{A_1}(\omega,t) &= e^{\mu_1 T+\sigma_1\beta_1(\omega,t)} \int_{-\infty}^{\infty} \varphi_2'\left(F(t,\omega,x)\right) e^{\sigma_1\sqrt{T-t}x_1} d\Phi(x) \\
&= e^{\mu_1 T+\sigma_1\beta_1(\omega,t)} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{F(t,\omega,x)}} e^{\sigma_1\sqrt{T-t}x_1} d\Phi(x) \\
p_B(\omega,t) &= \int_{-\infty}^{\infty} \varphi_2'\left(F(t,\omega,x)\right) d\Phi(x) \\
&= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{F(t,\omega,x)}} d\Phi(x)
\end{aligned}
$$

$$\frac{p_A(\omega,t)}{p_B(\omega,t)} = \frac{e^{\mu_1 T+\sigma_1\beta_1(\omega,t)}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{F(t,\omega,x)}}e^{\sigma\sqrt{T-t}x_1}d\Phi(x)}{\int_{-\infty}^{\infty}\frac{1}{2\sqrt{F(t,\omega,x)}}d\Phi(x)}$$

$$p_D(\omega,t) = \int_{-\infty}^{\infty}\varphi_2'\left(F(t,\omega,x)\right)G\left(e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}\right)d\Phi(x)$$

$$= \int_{-\infty}^{\infty}\frac{\max\{0,e^{\mu_1 T+\sigma_1\beta_1(\omega,t)+\sigma_1\sqrt{T-t}x_1}-\bar{A}\}}{2\sqrt{F(t,\omega,x)}}d\Phi(x)$$

$$\frac{p_D(\omega,t)}{p_B(\omega,t)} = \frac{\int_{-\infty}^{\infty}\frac{\max\{0,e^{\mu_1 T+\sigma_1\beta_1(\omega,t)+\sigma_1\sqrt{T-t}x_1}-\bar{A}\}}{2\sqrt{F(t,\omega,x)}}d\Phi(x)}{\int_{-\infty}^{\infty}\frac{1}{2\sqrt{F(t,\omega,x)}}d\Phi(x)}$$

where

$$F(t,\omega,x) = e^{\beta_2(\omega,t)+\sqrt{T-t}x_2}+\left(e^{\mu_1 T+\sigma_1\beta_1(\omega,t)+\sigma_1\sqrt{T-t}x_1}\right)$$

Notice that the inclusion of any nonzero endowment in the final period, whether stochastic or deterministic, precludes the simplification that was possible in Example 2.2. In particular, $\frac{p_{A_1}}{p_B}$ is *not* geometric Brownian motion and the price of the option is *not* given by the conventional Black-Scholes formula. Note also that the price of the stock A_1 and the option D_1 depend not only on the Brownian motion β_1 that determines the final stock dividend, but also the Brownian motion β_2, which is independent of the dividends of the stock.

Note that the option price is given in closed form that can easily be evaluated numerically. Assuming one knows $\beta(\omega,t)$, one simply has to do two numerical integrations with respect to the Gaussian distribution; it is not necessary to simulate a random walk process. Alternatively, one can compute the stock price $p_{A_1}(\omega,t)/p_B(\omega,t)$ numerically as a function of $\beta(\omega,t)$. One can then compute the option price as a function of the p_A/p_B, the strike price of the option, and $T-t$, given the utility function and the final period endowment.

Example 2.5 Suppose we modify Example 2.2 by eliminating the bond. Thus, we consider a model in which there is one stock, no bond, and a European call option on the stock. This does not fall under Theorem 2.1 as stated, but essentially the same argument guarantees that equilibrium exists.[4] The equilibrium trading strategy prescribes that the agent holds

[4]The only change, other than notation, is that in the construction of the consumption

zero units of the bond at all (ω, t); accordingly, the equilibrium consumptions and stock and option holdings in this example are the same as those in Example 2.2. However, in this example, there is no market for the bond. If we consider the market with the stock but not the bond or the option present, markets are dynamically incomplete and there is no trading strategy involving only the stock which will replicate the option. It is not possible to price the option by a pure arbitrage argument on the stock; equivalently, the martingale method identifies infinitely many possible pricing processes. Nonetheless, in the market with the stock and the option both present, the option has a unique *equilibrium* price. The prices are exactly the same as in Example 2.2. The pricing formula is

$$\begin{aligned} p_A(\omega,t) &= \frac{e^{(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2} \\ p_D(\omega,t) &= \frac{e^{(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(\frac{\ln(e^{\mu T+\sigma\beta(\omega,t)}/\bar{A})+\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right) \\ &\quad -\frac{\bar{A}e^{-(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(\frac{\ln(e^{\mu T+\sigma\beta(\omega,t)}/\bar{A})-\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

where Φ is the cumulative distribution function of the standard 1-dimensional normal.

3 Proof

Up to now, all of our definitions and results have been stated without any reference to nonstandard analysis. Our proof makes extensive use of nonstandard analysis, in particular Anderson's construction of Brownian Motion and the Itô Integral ([1]) and Lindström's extension of that construction to stochastic integrals with respect to L^2 martingales [33, 34, 35, 36]. It is beyond the scope of this paper to develop these methods; Anderson [3] and Hurd and Loeb [31] are suitable references.

We construct our probability space, filtration and Brownian Motion following Anderson's construction [1]. Specifically, we construct a hyperfinite

plan $\hat{\hat{c}}$ and trading strategy $\hat{\hat{y}}$, one must buy our sell units of the stock rather than units of the bond.

economy as in Raimondo [45], with the following changes to incorporate the derivatives $D_1, \ldots, D_M$:

1. For all $\omega \in \hat{\Omega}$, define $\hat{A}(\omega, t) = \hat{B}(\omega, t) = \hat{D}(\omega, t) = 0$ for all $t < \hat{T}$, $\hat{A}(\omega, \hat{T}) = e^{\mu \hat{T} + \sigma \hat{\beta}(\omega, \hat{T})}$ (i.e. $\hat{A}_j(\omega, \hat{T}) = e^{\mu_j \hat{T} + \sigma_j \hat{\beta}_j(\omega, \hat{T})}$, $j = 1, \ldots, J$), $\hat{B}(\omega, \hat{T}) = 1$, and $\hat{D}(\omega, \hat{T}) = {}^*G\left(e^{\mu \hat{T} + \sigma \beta(\omega, \hat{T})}\right)$, i.e. $\hat{D}_m(\omega, \hat{T}) = {}^*G_m\left(e^{\mu \hat{T} + \sigma \beta(\omega, \hat{T})}\right) = {}^*G_m\left(e^{\mu_1 \hat{T} + \sigma_1 \beta_1(\omega, \hat{T})}, \ldots, e^{\mu_J \hat{T} + \sigma_J \beta_J(\omega, \hat{T})}\right)$ $(m = 1, \ldots, M)$. Define $A(\omega, t) = B(\omega, t) = D(\omega, t) = 0$ for $T \in [0, T)$, $A(\omega, T) = e^{\mu T + \sigma \beta(\omega, T)}$, $B(\omega, T) = 1$, $D(\omega, T) = G\left(e^{\mu T + \sigma \beta(\omega, T)}\right)$. Note that $A(\omega, T) = {}^\circ\hat{A}(\omega, \hat{T})$ and $D(\omega, T) = {}^\circ\hat{D}(\omega, \hat{T})$ for μ-almost all ω.

2. A security price is an internal function $\hat{p} = (\hat{p}_A, \hat{p}_B, \hat{p}_D) : \mathcal{T} \times \hat{\Omega} \to {}^*\mathbf{R}_+^J \times {}^*\mathbf{R}_+ \times {}^*\mathbf{R}_+^K$ such that $\hat{p}(t, \cdot)$ is $\hat{\mathcal{F}}_t$-measurable. A consumption price is an internal function $\hat{p}_C : \mathcal{T} \times \hat{\Omega} \to {}^*\mathbf{R}_+$.

3. A trading strategy is $\hat{z} = (\hat{z}_A, \hat{z}_B, \hat{z}_D) : \mathcal{T} \times \hat{\Omega} \to {}^*\mathbf{R}^J \times {}^*\mathbf{R} \times {}^*\mathbf{R}^D$ which satisfies the short-sale constraint

$$\hat{z}(\omega, t) \geq ((-M, \ldots, -M), -M, (-M, \ldots, M))$$

for all t, ω and such that $\hat{z}(t, \cdot)$ is $\hat{\mathcal{F}}_t$-measurable.

4. An equilibrium for the economy is a security price process $\hat{p}$, a consumption price process $\hat{p}_C$, a trading strategy $\hat{z}$ and a consumption plan $\hat{c}$ which lies in the demand set so that the securities and goods markets clear, i.e. for all ω

$$\begin{aligned} \hat{z}_A(\omega, t) &= \mathbf{1} \text{ for all } t \in \mathcal{T} \\ \hat{z}_B(\omega, t) &= 0 \text{ for all } t \in \mathcal{T} \\ \hat{z}_D(\omega, t) &= 0 \text{ for all } t \in \mathcal{T} \\ \hat{c}(\omega, t) &= 1 \text{ for all } t < \hat{T} \\ \hat{c}(\omega, \hat{T}) &= \hat{e}(\omega, \hat{T}) + \mathbf{1}_J \cdot e^{\hat{\beta}(\omega, \hat{T})} \end{aligned}$$

Theorem 3.1 *The hyperfinite economy just described has an equilibrium.*

The pricing process is given by

$$\begin{aligned}
\hat{p}_{A_j}(\omega,t) &= e^{\mu_j t+\sigma\hat{\beta}_j(\omega,t)} \int_{{}^*\mathbf{R}} {}^*\varphi_2'\left(\hat{F}(t,\omega,x)\right) e^{\mu_j(T-t)+\sigma_j\sqrt{\hat{T}-t}x_j} d\hat{\Phi}(x)\\
\hat{p}_B(\omega,t) &= \int_{{}^*\mathbf{R}} {}^*\varphi_2'\left(\hat{F}(t,\omega,x)\right) d\hat{\Phi}(x)\\
\hat{p}_{D_m}(\omega,t) &= \int_{{}^*\mathbf{R}} {}^*\varphi_2'\left(\hat{F}(t,\omega,x)\right) {}^*G_m\left(e^{\mu\hat{T}+\sigma\hat{\beta}(\omega,t)+\sigma\sqrt{\hat{T}-t}x}\right) d\hat{\Phi}(x)\\
\hat{p}_C(\omega,t) &= \varphi_1'(1) \text{ for } t<T\\
\hat{p}_C(\omega,\hat{T}) &= {}^*\varphi_2'(\hat{F}(\hat{T},\omega,0))
\end{aligned}$$

where

$$\hat{F}(t,\omega,x) = {}^*\rho\left(\hat{\beta}(\omega,t)+\sqrt{\hat{T}-t}x\right) + \mathbf{1}_J\cdot\left(e^{\mu\hat{T}+\sigma\hat{\beta}(\omega,t)+\sigma\sqrt{\hat{T}-t}x}\right)$$

and $\hat{\Phi}$ is the cumulative distribution function of the d-dimensional normalized binomial distribution, each of whose components is distributed as

$$\sqrt{\frac{4\Delta T}{\hat{T}-t}}\,{}^*b\left(\left(\frac{\hat{T}-t}{\Delta T},\frac{1}{2}\right)-\frac{\hat{T}-t}{2\Delta T}\right)$$

Proof: The proof is essentially identical to the proof of Theorem 3.1 of Raimondo [45]. ∎

Proposition 3.2 . *Suppose $a \in \mathrm{ns}({}^*\mathbf{R}^d)$. Let*

$$f_a(x) = e^{a\cdot x} = e^{a_1x_1+\cdots+a_dx_d}$$

Then $f_a \in SL^1({}^*\mathbf{R}^d, d\hat{\Phi})$.

Proof: See Raimondo [45] ∎

Theorem 3.3 *Suppose that $\varphi_2'(c) = O(1/c^r)$ as $c \to 0$ and $|G_m(x)| \le |x|^r$, for some $r \in \mathbf{R}$. Then for μ-almost all ω, the equilibrium pricing process satisfies*

$$\begin{aligned}
{}^\circ\left(\hat{p}_A(\omega,t)\right) &= e^{\mu t+\sigma\beta(\omega,{}^\circ t)} \int_{-\infty}^{\infty} \varphi_2'\left(F({}^\circ t,\omega,x)\right) e^{\mu(T-t)+\sigma\sqrt{T-{}^\circ t}x} d\Phi(x)\\
{}^\circ\left(\hat{p}_B(\omega,t)\right) &= \int_{-\infty}^{\infty} \varphi_2'\left(F({}^\circ t,\omega,x)\right) d\Phi(x)\\
{}^\circ\left(\hat{p}_C(\omega,t)\right) &= \varphi_1'(1)\\
{}^\circ\left(\hat{p}_C(\omega,\hat{T})\right) &= \varphi_2'\left(F(T,\omega,0)\right)\\
{}^\circ\left(\hat{p}_D(\omega,t)\right) &= \int_{-\infty}^{\infty} \varphi_2'\left(F({}^\circ t,\omega,x)\right) G\left(e^{\mu T+\sigma\beta(\omega,{}^\circ t)+\sigma\sqrt{T-{}^\circ t}x}\right) d\Phi(x)
\end{aligned}$$

for all $t \in \mathcal{T}$, where Φ is the cumulative distribution function of the standard d-dimensional normal.

Proof: The proofs for p_A and p_B are essentially the same as in Raimondo [45]. Anderson [1] showed that, for almost all ω, ${}^\circ\hat{\beta}(\omega, t) = \beta(\omega, {}^\circ t) \in \mathbf{R}$ for all $t \in \mathcal{T}$; fix any such ω. Note that, since ρ is continuous, $\hat{F}(t, \omega, x) \simeq F({}^\circ t, \omega, {}^\circ x)$ for all $t \in \mathcal{T}$. Note that $\varphi_2'(c)$ is decreasing and there exists $\gamma \in \mathbf{R}$ such that $\varphi_2'(c) \leq \frac{\gamma}{c^r}$ for all $c \in (0, 1]$. Since the d-dimensional binomial distribution $\hat{\Phi}$ converges in distribution to the d-dimensional normal distribution Φ, Anderson and Rashid [4] shows that $L(d\hat{\Phi})\mathrm{st}^{-1} = d\Phi$, where $L(d\hat{\Phi})$ is the Loeb measure generated by $d\hat{\Phi}$. Thus, if $J \geq 1$,

$$
\begin{aligned}
&\left| {}^*\varphi_2' \left(\hat{F}(t, \omega, x) \right) {}^*G_m \left(e^{\mu \hat{T} + \sigma \hat{\beta}(\omega, t) + \sigma \sqrt{\hat{T} - t} x} \right) \right| \\
&\leq \varphi_2'(1) \left| {}^*G_m \left(e^{\mu \hat{T} + \sigma \hat{\beta}(\omega, t) + \sigma \sqrt{\hat{T} - t} x} \right) \right| + \gamma \left(\mathbf{1}_J \cdot \left(e^{\hat{\beta}(\omega, t)} e^{\sqrt{\hat{T}} x} \right) \right)^{-r} \left| {}^*G_m \left(e^{\hat{\beta}(\omega, t) + \sqrt{\hat{T} - t} x} \right) \right| \\
&\leq \varphi_2'(1) e^{r \hat{\beta}(\omega, t)} \left| e^{\sqrt{\hat{T}} x} \right|^r + \gamma e^{-r \hat{\beta}(\omega, t)} S_m \left(\mathbf{1}_J \cdot \left(e^{\sqrt{\hat{T}} x} \right) \right)^{-r} \\
&\in SL^1(d\hat{\Phi})
\end{aligned}
$$

by Proposition 3.2; the argument in case $J = 0$, using the lower bound $\rho(x) \geq e^{\alpha \cdot x}$ for some $\alpha \in \mathbf{R}^d$, is similar. Therefore,

$$
\begin{aligned}
&\int_{{}^*\mathbf{R}} {}^*\varphi_2' \left(\hat{F}(t, \omega, x) \right) {}^*G_m \left(e^{\mu \hat{T} + \sigma \hat{\beta}(\omega, t) + \sigma \sqrt{\hat{T} - t} x} \right) d\hat{\Phi} \\
&\simeq \int_{{}^*\mathbf{R}} \varphi_2' \left({}^\circ \hat{F}(t, \omega, x) \right) G_m \left(e^{\mu T + \sigma \beta(\omega, {}^\circ t) + \sigma \sqrt{T - {}^\circ t}\, {}^\circ x} \right) L(d\hat{\Phi}) \\
&= \int_{\mathbf{R}} \varphi_2' \left(F({}^\circ t, \omega, x) \right) G_m \left(e^{\mu T + \sigma \beta(\omega, {}^\circ t) + \sigma \sqrt{T - {}^\circ t} x} \right) L(d\hat{\Phi}) \mathrm{st}^{-1} \\
&= \int_{\mathbf{R}} \varphi_2' \left(F({}^\circ t, \omega, x) \right) G_m \left(e^{\mu T + \sigma \beta(\omega, {}^\circ t) + \sigma \sqrt{T - {}^\circ t} x} \right) d\Phi
\end{aligned}
$$

■

Theorem 3.4 *$\hat{p}_A$, $\hat{p}_B$ and $\hat{p}_D$ are internal almost surely S-continuous SL^2 martingales with respect to the internal filtration $\{\hat{\mathcal{F}}_t\}$. If we define*

$$
\begin{aligned}
p_A(\omega, t) &= {}^\circ \hat{p}_A(\omega, \hat{t}) \\
p_B(\omega, t) &= {}^\circ \hat{p}_B(\omega, \hat{t}) \\
p_D(\omega, t) &= {}^\circ \hat{p}_D(\omega, \hat{t})
\end{aligned}
$$

for $t \in [0, T]$, then p_A and p_B are almost surely continuous square integrable martingales with respect to the filtration $\{\mathcal{F}_t\}$.

Proof: The proof for $\hat{p}_A$ and p_A is given in Raimondo [45]. The proofs for $\hat{p}_B$, p_B, $\hat{p}_D$ and p_D are similar. ■

Proof of Theorem 2.1: The proof is essentially identical to the proof of Theorem 2.1 in Raimondo [45]. ■

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