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Menu Auctions with Demand Uncertainty

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Topics in Mathematical Economics

The participants in a September 2002 Workshop on *Topics in Mathematical Economics* in honor of Birgit Grodal decided to have a series of papers appear on Birgit Grodal's 60'th birthday, June 24, 2003.

The Institute of Economics suggested that the papers became Discussion Papers from the Institute.

The editor of *Economic Theory* offered to consider the papers for a special Festschrift issue of the journal with Karl Vind as Guest Editor.

This paper is one of the many papers sent to the Discussion Paper series.

Most of these papers will later also be published in a special issue of *Economic Theory*.

Tillykke Birgit

Troels Østergaard Sørensen	Karl Vind
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Menu Auctions with Demand Uncertainty[1]

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Abstract

We apply a Bernheim-Whinston (1986) type mechanism to a situation where a single buyer with uncertain demand wishes to buy from a small number of suppliers. We let suppliers bid a payment contingent on own quantity supplied, and another payment contingent on the realization of total demand. We show that there is a unique equilibrium which is also efficient. This equilibrium is equivalent to the one under the 'truthful bids' restriction used in the model without uncertainty in Bernheim-Whinston (1986).

Keywords: Procurement, Efficient Auctions, Multi Unit Auctions, Uniqueness.

1 Introduction

The menu auction mechanism suggested by Bernheim-Whinston (1986) is particularly relevant in situations where a single buyer wishes to buy from a (small) number of potential suppliers. The buyer could be a firm or a local or central government, and the good or service in question could range from care for the elderly, snow removal, garbage collection to electricity generation. The Bernheim-Whinston mechanism is a tractable and reasonable one. It has a single agent on one side of the market and this agent is assumed to behave in his own interests and maximize his surplus as one would expect authorities to do when outsourcing. Nevertheless, it has two difficulties. First, there are many equilibria in the Bernheim-Whinston game of which only one (which occurs when bids are restricted to differ from true total costs only by a constant) is efficient. Second, demand is assumed to be certain while it seems inherently uncertain in the situations we have in mind: the exact level of services needed or electricity required depends on random elements. Therefore, contracts or bids will need to be contingent on the outcome of such randomness.[1]

Despite these drawbacks a large literature applies this model together with the restriction on bids (called the "truthful bids" restriction from here on) that makes equilibrium unique. Grossman and Helpman (1994), (1995) (and a literature spawned by them) develop models of political economy and lobbying using this framework.[2] Kr-

[1]The supply function bidding mechanism studied by Klemperer and Meyer (1989), and before them Robson (1981), could be used to deal with such randomness. However, equilibria in supply functions are not efficient unless suppliers are identical.

[2]Earlier literature using the common agency/menu auction model to study lobbying and political influence includes Spiller (1990) and Tranæs (1993).

ishna and Tranæs (2002) use it to study multi unit auctions with many bidders, when bidder valuations take on a number of different shapes. Anton and Yao (1989, 1992) study a single-buyer-two-supplier version. However, their results do not generalize to more than two suppliers.

In this paper we show that taking care of the second difficulty successfully overcomes the first as well. Introducing demand uncertainty into the model of Bernheim-Whinston (1986) has a dramatic influence on the set of equilibria; with demand certainty almost any allocation is a (Nash) equilibrium, while with uncertain demand there exists a unique Nash equilibrium. Moreover, this equilibrium is *ex-post* efficient and for each realization of demand it coincides with the equilibrium under the "truthful-bid" restriction used in Bernheim-Whinston (1986).

The single-buyer-many-sellers game we study in this paper goes like this: a number of suppliers submit bid functions simultaneously. Next total demand is realized. After this the buyer allocates demand to each bidder so as to minimize his acquisition costs. Each supplier is paid according to his bid, that is, if he is asked to supply x units he is paid the amount of money he demanded in return for x units delivered. Note that this is not the case in efficient mechanisms like the Clark-Groves-Vickrey mechanism where your payment is determined by everyone else's bid.[3] The difference from the Bernheim-Whinston mechanism arises from demand being uncertain. To accommodate this uncertainty we allow firms to bid two functions: a fixed payment function which depends on the realization of total demand and a variable payment

[3] After Vickrey (1961), Clark (1971), and Groves (1973).

function that depends on the firm's supply.

Our results suggest that if demand is uncertain our scheme can be used to ensure that competition, even among a small number of suppliers, results in efficiency. In equilibrium suppliers ask for their costs as the variable payment and ask for as much as they can as the fixed payment, keeping in mind that if they ask for too much, they will not be used at all. The fixed payment that suppliers ask for in equilibrium is exactly their *social contributions,* that is, the additional costs which the society, behaving optimally, would have had to incur had that particular supplier not existed. Given these payment requests, demand is allocated across suppliers so that their marginal costs are equalized and production is efficient. As each supplier obtains his social contribution as profits, externalities resulting from investment and entry decisions would be internalized and optimal under such mechanism, which is a further merit to the scheme.[4]

In Section 2 we set up the model and in Section 3 we prove the main results. Section 4 discusses our basic assumptions and considers generalizations. Section 5 concludes.

2 The Model

There is one buyer and m potential suppliers. The set of potential suppliers is denoted by M. Each supplier $i \in M$ faces production costs $C_i(q_i)$ with $C_i(0) = 0$. Suppliers first simultaneously submit their bids. Following this, the state is realized. This

[4] In Krishna and Tranæs (2001) we discuss the applicability of this mechanism to deregulated markets like wholesale electricity markets.

determines the level of completely inelastic demand, n, which is a random variable with strictly positive density everywhere on the support $[0, N)$. If $N = \infty$, we say that demand has *full support*. Finally, demand is allocated across the suppliers to minimize acquisition costs.

All players have complete information about the model as well as each others costs and are profit maximizers. A strategy for Supplier i is a function $B_i(q_i, n)$, specifying the total payment requested as a function of the quantity supplied q_i, and n. We restrict this to consist of a "variable" payment $T_i(q_i)$, solely depending on the quantity supplied q_i and with $T_i(0) = 0$; and a "fixed" payment $S_i(n)$, which is independent of the quantity supplied but contingent on the total purchase n, by the buyer; $S_i(n)$ is paid only to those chosen to supply. Hence, $B_i(q_i, n) = T_i(q_i) + S_i(n)$. The variable payment is assumed to be a twice continuously differentiable function mapping quantity into revenue: $T_i : (-\infty, \infty) \to (-\infty, \infty)$.[5] The fixed payment is just any function: $S_i : [0, \infty) \to (-\infty, \infty)$, which maps total quantity into revenue.

We make the following assumptions throughout:

Assumption 1 $C_i(\cdot)$ is twice continuously differentiable, with $C_i'(0) = 0$, $C_i'(\cdot) > 0$, and $0 < C_i''(\cdot) < \infty$ for $q_i > 0$.

Assumption 2 $0 < T_i'(q_i) < \infty$ for $q_i > 0$.

In Section 4 we discuss the importance of our assumptions and consider generalizations.

[5] Letting suppliers specify a variable payment in case they supply a negative quantity is of no importance to our results, but it makes the variable payment functions everywhere differentiable and this simplifies the analysis.

3 Existence and Uniqueness

In this section we present our main results. We are going to make extensive use of the concept of a supplier's social contribution defined below. Let $q = (q_1, ..., q_m)$ and

$$\begin{aligned} C^{\min}(q) &= \min_q \sum_i C_i(q_i) \\ s.t. \quad \sum_i q_i &= n \text{ and } q_i \geq 0 \text{ for all } i \in M \end{aligned}$$

and

$$\begin{aligned} C^{\min}(q, -j) &= \min_q \sum_i C_i(q_i) \\ s.t. \quad \sum_{i \neq j} q_i &= n \text{ and } q_i \geq 0 \text{ for all } i \in M. \end{aligned}$$

Then supplier j's social contribution $S_j^{sc}(n)$ is defined as

$$S_j^{sc}(n) = C^{\min}(q, -j) - C^{\min}(q),$$

and thus it measures the decrease in the minimized costs associated with including supplier j in providing the n units. We can now state our main theorem. We consider only pure strategy Nash equilibria.

Theorem 1 *Assume that n has full support, and for all $i, T_i'(0) = 0$. Then there exists a unique equilibrium which consists of each supplier asking for a variable payment which equals his production costs, $C_i(q_i)$, and a fixed payment which equals $S_i^{sc}(n)$ for all n. This equilibrium is ex-post efficient.*

Proof.

The proof has two parts. In part 1, we show that $\{B_i^*(q_i,n)\}_{i\in M}$, where $B_i^*(q_i,n) = C_i(q_i)+\ S_i^{sc}(n)$, is an equilibrium. In part 2 we show it is the unique equilibrium.

Part 1. Assume that all suppliers but j, M_{-j}, bid $B_i^*(q_i,n) = C_i(q_i)+\ S_i^{sc}(n)$. We label this profile $\{B_i^*(q_i,n)\}_{i\neq j}$. We check that it is a best reply for supplier j to bid $C_j(q_j)+\ S_j^{sc}(n)$ as well. For each realization of the random variable n, the buyer allocates the n units to the different suppliers so as to minimize his total payment. Thus, the buyer chooses a vector $q = (q_1, ..., q_m)$ to solve the problem,

$$\underset{q}{Min} \sum_i (T_i(q_i) + S_i(n))$$

$$\text{s.t.} \quad \sum_i q_i = n,\ q_i \geq 0,\ i = 1, .., m.$$

Note that if $T_i'(q_i)$ became vertical at some point, this problem would not have a solution and if $C_i'(\cdot)$ ever did, it might be impossible for a firm to produce certain output levels. Assumption 1 and 2 exclude these situations.

In order to show that $B_j^*(q_j,n)$ is a best reply against $\{B_i^*(q_i,n)\}_{i\neq j}$ we first construct $j's$ residual marginal revenue curve.

(1) We derive the highest payment Supplier j can get from the buyer for any quantity q_j supplied in state n, given $\{B_i^*(q_i,n)\}_{i\neq j}$. We name this payment $P_j(q_j; n, B_{i\neq j}^*)$ and show that it is a concave function.

(2) We characterize the profit maximizing supply $q_j^*(n)$ given $\{B_i^*(q_i,n)\}_{i\neq j}$, which is the q_j that maximizes $P_j(q_j; n, B_{i\neq j}^*) - C_j(q_j)$, and the maximal profits that can be obtained by Supplier j. $P'(q_j; n, B_{i\neq j}^*)$ can be interpreted as supplier $j's$ residual marginal revenue curve.

(3) Finally, we show that bidding $B_j^*(q_j, n)$ ensures that Supplier j gets to supply $q_j^*(n)$ units and obtain these maximized profits, given that all other suppliers bid $\{B_i^*(q_i, n)\}_{i \neq j}$. In effect we show that $B_j^*(q_j, n)$ is a best response to $B_{i \neq j}^*$ even if only j was allowed to make variable bids which were state contingent. Thus it is certainly a best response when his strategies are limited to the form assumed here.

Consider (1). $P_j(q_j; n, B_{i \neq j}^*)$ is the difference in minimized costs to the buyer of purchasing all n units demanded, versus $n - q_j$ of the n units demanded, from the suppliers in M_{-j}. This is because the alternative to buying q_j from Supplier j is buying it from the remaining suppliers. In order to purchase $n - q_j$ units at minimum costs from the suppliers in M_{-j}, the buyer solves the problem

$$\underset{q_{-j}}{Min} \sum_{i \neq j} \left(C_i(q_i) + S_i^{sc}(n) \right) \tag{1}$$

subject to

$$\sum_{i \neq j} q_i + q_j = n, \;\; q_i \geq 0 \; for \; i \neq j.$$

where q_{-j} is the allocation vector for all suppliers but j. Let the value function for this problem be denoted $R_{-j}(q_j, n; B_{i \neq j}^*)$. It is the minimum cost of buying $n - q_j$ units from all suppliers but j when n units are demanded in total. Similarly, $R_{-j}(0, n; B_{i \neq j}^*)$ gives the minimized cost of obtaining n units from the suppliers in M_{-j}. Note that $R'_{-j}(n; n, B_{i \neq j}^*) = 0$ as marginal costs, and hence the variable bids, emanate from the origin.

Thus $P_j(q_j; n, B_{i \neq j}^*) = R_{-j}(0; n, B_{i \neq j}^*) \; - R_{-j}(q_j; n, B_{i \neq j}^*)$. This defines the highest total payment Supplier j can get from selling q_j units in state n given $\{B_i^*(q_i, n)\}_{i \neq j}$.

We will use $P_j(q_j; n, B^*_{i\neq j})$ to derive Supplier j's profit maximizing supply in each state given $\{B^*_i(q_i, n)\}_{i\neq j}$. Given our assumptions on costs, there is a well defined unique solution to the above problem and by the Berge maximum theorem, $R_{-j}(q_j, n; B^*_{i\neq j})$ is continuous in q_j. In addition, by the envelope theorem, $R'_{-j}(q_j, n; B^*_{i\neq j}) = \frac{\partial R_{-j}(q_j, n; B^*_{i\neq j})}{\partial q_j}$ < 0 equals the fall in cost when one less unit is purchased by the buyer from all suppliers but j, which equals their common marginal cost. Also, $R''_{-j}(q_j, n; B^*_{i\neq j}) > 0$ as marginal costs are increasing. Hence, $P'_j(q_j; n, B^*_{i\neq j}) = -R'_{-j}(q_j, n; B^*_{i\neq j}) > 0$ and $P''_j(q_j; n, B^*_{i\neq j}) = -R''_{-j}(q_j, n; B^*_{i\neq j}) < 0$. Thus $P_j(q_j; n; B^*_{i\neq j})$ is increasing and concave in q_j and $P'_j(n; n, B^*_{i\neq j}) = -R'_{-j}(n; n, B^*_{i\neq j}) = 0$.

In Figure 1, if the origin for j is at the left, and for all others is at the right, $P'_j(0; n, B^*_{i\neq j})$ is represented by the height at the intersection of the curve representing the horizontal sum of the marginal costs of all other included suppliers but j, with the vertical axis, or O_jD, as the n'th unit is purchased at the common marginal cost of the remaining included suppliers. Of course, $P'_j(q_j; n, B^*_{i\neq j})$ traces out the entire curve DO_{-j} as q_j rises from zero. The concavity of $P_j(q_j; n, B^*_{i\neq j})$ is reflected in the fact that DO_{-j} is downward sloping. Note that this means that DO_{-j} also traces out $-R'_{-j}(q_j; n, B^*_{i\neq j})$. $P'_j(n; n, B^*_{i\neq j}) = 0$ ensures that Firm $j's$ residual marginal revenue is anchored at O_{-j}. It is worth noting that as q_j rises, $n - q_j$ falls so that the set of included suppliers for the given payments offered by suppliers will shrink. At points where the set of suppliers shrinks, $P_j(q_j; n, B^*_{i\neq j})$ will have a kink. Since $P(q_j; n, B^*_{i\neq j})$ is concave given the payments offered are B^*, $P'_j(q_j; n, B^*_{i\neq j})$ will have a vertical drop at such points.

When a supplier is added, it is because the savings from paying his lower marginal bid are just enough to cover his fixed bid. Thus, fixed payments are incorporated in the area below $P_i'(.)$. Of course, at $q_i = n$, it is worth dropping the last other supplier and his fixed payments are not captured by the area under $P_i'(.)$. Notice that for this reason, with 2 suppliers, $P_i'(.)$ only captured variable bids of the other supplier.

Outputs in the candidate equilibrium are all positive by construction. Later on we will show that all suppliers must be included in any equilibrium. A consequence of this is that in equilibrium such vertical drops occur only to the right of the equilibrium level of q_j and so are not relevant. For this reason we do not even draw these drops in our figures. Of course, when there are only two suppliers this is not an issue.

Consider (2). What quantity should Supplier j aim for in each state and what are the highest profits he can obtain state by state?

Supplier j maximizes his maximal available profit,

$$\Pi_j(q_j; n, B^*_{i\neq j}) = P_j(q_j; n, B^*_{i\neq j}) - C_j(q_j).$$

Let $q_j^*(n)$ denote the value of q_j that maximizes $\Pi_j(q_j; n, B^*_{i\neq j})$. As $P_j(q_j; n, B^*_{i\neq j})$ is increasing and concave in q_j and $C_j(q_j)$ is assumed to be strictly convex, $\Pi_j(q_j; n, B^*_{i\neq j})$ is strictly concave. Hence, for each realization of n there is a unique $q_j^*(n)$ that maximizes $\Pi_j(q_j; n, B^*_{i\neq j})$. Thus, $q_j^*(n)$ is such that

$$C_j'(q_j^*(n)) = P_j'(q_j^*(n); n, B^*_{i\neq j}), \tag{2}$$

and $\Pi_j(q_j^*(n); n, B^*_{i\neq j})$ is the maximized profit available to Supplier j given $\{B_i^*(q_i, n)\}_{i\neq j}$. Recall that as $P_j'(q_j; n, B^*_{i\neq j}) = -R'_{-j}(q_j, n; B^*_{i\neq j})$ (which in turn equals $C_j'(q_i(n))$ or the marginal cost of any included supplier), $P_j'(q_j; n, B^*_{i\neq j})$ is the horizontal sum of the marginal bids of the suppliers M_{-j} , which are their marginal costs by assumption, or $O_{-j}D$ in Figure 1. Thus $q_j^*(n)$ is given by the intersection of $j's$ marginal cost curve, O_jC with $O_{-j}D$, and $\Pi_j(q_j^*(n); n, B^*_{i\neq j})$ is the area between $O_{-j}D$ and O_jC up to the equilibrium output. Thus, $\Pi_j(q_j^*(n); n, B^*_{i\neq j}) = S_j^{sc}(n)$, which is suppliers j's social contribution.

Consider (3). Can Supplier j get to supply $q_j^*(n)$ and earn $\Pi_j(q_j^*(n); n, B^*_{i\neq j})$ using only the restricted functions allowed, given $\{B_i^*(q_i, n)\}_{i\neq j}$? The function B_j^* does the trick. Bidding a variable component of $C_j(q_j)$ leaves $P_j(q_j^*(n); n, B^*_{i\neq j}) - C_j(q_j^*(n)) = \Pi_j(q_j^*(n); n, B^*_{i\neq j})$ still available to be retained for Supplier j. To appropriate it, j sets $S_j(n) = \Pi_j(q_j^*(n); n, B^*_{i\neq j})$. Thus, with $\Pi_j(q_j^*(n); n, B^*_{i\neq j}) = S_j^{sc}(n)$ it is certainly a best reply for Supplier j to bid $B_j^*(q_j, n) = C_j(q_j) + S_j^{sc}(n)$ given that all other suppliers bid $\{B_i^*(q_i, n)\}_{i\neq j}$.

Since Supplier j was chosen arbitrarily we have shown that $\{B_i^*(q_i, n)\}_{i\in M}$ is an equilibrium which completes the proof of Part 1.

Part 2. We need to show that $\{B_i^*(q_i, n)\}_{i\in M}$ is the unique equilibrium. First we show that the equilibrium allocation must be an interior one (Lemma 1). Second, we show the variable bids need to be the supplier's true costs (Lemma 2 and 3). Finally, given this, we show that each supplier's fixed payment needs to be his social contribution.

Let $S_j^{mc}(n; B)$ be the marginal contribution of j in state n when the m suppliers bid B. It defines the highest fixed payment j can ask for without being excluded by the buyer. In an interior solution, it is depicted by the area between $P_j'(q_j; n, B_{i \neq j})$ and $T_j'(q_j)$ up to $q_j(n; B)$, the supply obtained from j by the buyer given B and n. Recall that marginal contributions would equal social contributions if all suppliers bid their true marginal costs as their variable payments.

Notice that in an interior solution, given what other suppliers bid, each supplier can do no better than bid his marginal cost as his variable payment and his state dependent marginal contribution as his fixed payment. This strategy obtains the state by state maximum for each supplier. This is easy to see as the function $P_i(.)$ is continuous, since it is a value function. Hence $P_i(.) - C_i(.)$ is also continuous and attains its maximum somewhere in $[0, n]$ independent of whatever peculiar bids are offered by other suppliers. If this is an interior maximum these maximized profits can be attained by bidding as suggested above. Notice that $P_i'(.)$ is the horizontal sum of marginal bids of included suppliers in M_{-i}. Also, that $P_i'(.)$ acts like supplier i's marginal revenue curve. We will make extensive use of these facts in showing uniqueness below.

Lemma 1 *In any equilibrium, all suppliers supply strictly positive quantities for all* $n > 0$.

Proof. See the Appendix.

Next we establish that variable payments are necessarily equal to variable costs. First we show this is so at zero.

Lemma 2 *In equilibrium,* $T_i'(0) = C_i'(0) = 0$ *for all* i.

Proof. See the Appendix.

Thus, $T_i'(q_i)$ originates from zero. We have assumed that $T_i(0) = 0$ and are now ready to show that in any equilibrium all suppliers offer their true costs as their variable payment when suppliers are required to choose between strictly positive but finite marginal variable payments. Hence, while we do allow suppliers to have downwards sloping marginal variable payments as well as flat segments, this can not happen for $q_i = 0$ as $T_i'(q_i) = 0$ as just shown in Lemma 2. Finally,

Lemma 3 *In equilibrium $T_i'(q_i) = C_i'(q_i)$ for all q_i and all $i \in M$.*

Proof. See the Appendix.

Having established that suppliers need bid their true costs in equilibrium, we check that there are no equilibria where one or more suppliers, for some realization of n, ask for a fixed payment different from their social contribution while bidding their true costs as their variable payment.

When all suppliers bid $T_j(q_j) = C_j(q_j)$, then the only candidate for an equilibrium fixed payment profile is that each supplier asks for his social contribution as his fixed payment so that $S_j(n) = S_j^{sc}(n)$ for each supplier j and state n. The allocation is interior and $T_j(q_j) = C_j(q_j)$ for all j by Lemma 1 to 3, and hence the allocation is unique given our assumptions on the costs functions. Suppose that $m = 2$. Both suppliers will ask for $C_1(q_1)$ and $C_2(q_2)$ as variable payments. They will not ask for less than $S_1^{sc}(n)$ and $S_2^{sc}(n)$ in equilibrium as they can obtain $S_i^{sc}(n)$ no matter what other suppliers bid for their fixed payment. If both suppliers ask for more, on the other hand, only one of them will serve the buyer; the one that asks for the greater increment will be eliminated from consideration. Even if both ask for the same increment, it is best to eliminate one of them. If, for example, Supplier 2 asks for his costs as his variable payment and as fixed payment asks for $S_2^{sc}(n)+\epsilon$, and Supplier 1 asks for his costs and a fixed payment of $S_1^{sc}(n)+\ \epsilon$, then if the buyer buys from *only* one supplier he pays less than if he buys from both; he saves ϵ. Hence, the buyer only buys from one of the suppliers. This, however, means that the other will cut his fixed payment request in order to be the chosen one, and so no positive ϵ can be maintained in equilibrium. The extension from 2 to m suppliers is trivial and requires no extra arguments.

This completes our proof of Theorem 1. ■

Remark 1 *If we replace the assumptions that $C_i'(0) = 0$ and $T_i'(.) > 0$ by $C_i'(0) = k > 0$ and $T_i'(.) > k$ for all i the proof above goes through when 0 is replaced by k in the relevant places. See Section 4.2 for more on this.*

In this manner, uncertainty in n serves to pin down the variable components and then inclusion constraints pin down the fixed components of the strategies. When there is no uncertainty the model is a special case (Bernheim and Whinston, 1986) and we obtain the expected multiplicity of equilibria. Bernheim and Whinston obtain predictive power in this game by restricting bids to what they call "truthful bids", namely those where the bidder is indifferent between all allocations made to him so that the bidder is free of regrets ex-post. The problem is that in the absence of well defined trembles to the game their restriction remains hard to motivate. However, for a given n, the outcome in their setup is identical to that in our setup. Hence, our results can be seen as providing a rationale for their refinement.

Note that our equilibrium is not in dominant strategies. If, for example, the only other supplier asks for a huge bonus in each state and bids his marginal cost as his marginal variable payment, it will be optimal to ask for a huge bonus too. However, this could not be an equilibrium as some supplier will be eliminated from consideration by the buyer and the eliminated supplier will find it in his interest to reduce his bid to be considered. In equilibrium, supplier earnings exceed variable costs by the amount of their social contribution. This social contribution is large if marginal costs rise steeply with output and if the number of suppliers falls. In this event, our scheme could be expensive to implement in practice.

3.1 Private Values Implementation

So far we have made the assumption of complete information. As a result firms are assumed to know not only their own costs, but those of all others. Under the implementation scheme used so far, firms cannot offer their social contribution if they do not know the costs of all other firms. We now show that Theorem 1 remains valid even if firms only know their own costs, if we alter the implementation scheme slightly.

The altered implementation scheme has three rounds. First, all suppliers announce variable payment functions, $T_i(q_i)$, simultaneously. Second, after being informed about the bids of the first round, the suppliers bid a fixed payment vector $S_i(n)$. Finally, the state, n, is realized and the buyer chooses the allocation so as to minimize the costs for acquiring the n units. This procedure implements the allocation and payments of

Theorem 1 as a subgame perfect equilibrium assuming firms know their own costs and know that other suppliers have cost functions which belong to the class of functions described in Assumption 1.

Theorem 2 *Assume that cost functions are private information, that n has full support, and that for all $i, T_i'(0) = 0$. Then the sequential bidding game implements the allocation and payments of Theorem 1 as a unique subgame perfect equilibrium.*

Proof: In Appendix.

We now take a closer look at three assumptions made so far in our analysis.

4 Revisiting the Standing Assumptions

In this part we show what goes wrong if marginal payments requested are not strictly positive as in Assumption 2, and what happens if marginal costs at zero are not zero as in Assumption 1, but positive. Then we turn to the role played by the full support assumption.

4.1 Positive Marginal Payments

The assumption $T'(.) > 0$ for $q_i > 0$ is needed for uniqueness. Consider the following counter example if this assumption is not met and marginal payments are allowed to be negative. Let there be two suppliers with the marginal costs of Supplier 1 being above those of Supplier 2 for each quantity q. Suppose that Supplier 2 requests a downward sloping marginal payment function starting at his origin, while Supplier

1 offers a flat one of zero. Suppose that $S_1(n) = C_1(n)$ while $S_2(n) = C_1(n) - C_2(n) + (C_2(n) - T_2(n))$ which is equal to his cost advantage in producing n plus the variable losses he has incurred. This leaves Supplier 2 with total profits of $S_2(n) + (T_2(n) - C_2(n)) = C_1(n) - C_2(n)$ or his cost advantage in making n. The buyer is indifferent between buying from either supplier as his total price is $C_1(n)$ from either. Note that neither supplier can do better so that this is an equilibrium. If 2 asks for higher profits, the buyer would switch to Supplier 1 who offers to sell n for a payment equal to his costs. Supplier 1 cannot supply the whole n units for a lower total price than 2 and he cannot make it worthwhile for the buyer to buy some of the units from him as his marginal cost lies above Supplier $2'$s marginal payment function. A similar example can be constructed if marginal payments are allowed to be negative but increasing in quantity.

4.2 Marginal Costs Emanate from Origin

The assumption $C_i'(0) = 0$ also relates to uniqueness. It is made for convenience given that we permit asymmetries across firms. Costs are always assumed to be strictly convex. The assumption that for all i, $C_i'(0) = 0$ and for $q_i > 0$, $T_i'(q_i) > 0$ can be easily replaced by $C_i'(0) = k$, and for $q_i > 0$, $T_i'(q_i) > k$ as mentioned in the proof of Theorem 1. However, allowing $C_i'(0) = k_i$ creates a problem when we restrict bids in an analogous manner to $T_i'(q_i) > k_i$ for $q_i > 0$. Consider the following two supplier example where Supplier 1 has a higher intercept, k_1, for marginal cost than does Supplier 2 as $k_2 = 0$. Then the following is an equilibrium. Supplier 1 offers a flat marginal payment function just a bit above k_1. Supplier 2 offers a marginal payment function which lies below the lowest point of Supplier $1'$s variable bid for all q; for example he could bid one that is increasing and asymptotic to a line below k_1. As this is so, the buyer will always choose one or the other supplier and thus the suppliers can be thought of as competing over who supplies all the n units. The supplier with the lower cost will be the one chosen by the buyer and can make his cost advantage in profits. Thus the suppliers ask for the fixed payments $S_1(n) = C_1(n) - T_1(n)$ and $S_2(n) = C_1(n) - T_2(n)$ in this equilibrium.

Sufficient conditions for uniqueness in allocations when marginal cost intercepts can differ are that for all $q_i > 0$, $T_i'(q_i) > 0$, and as q_i goes to ∞, so does $T'(q_i)$. This ensures that, at least for very large n, Supplier 1 cannot be excluded as his marginal cost at zero must lie below the marginal bid of Supplier 2 for the last unit. This breaks the equilibrium above. We still have a bit of a problem as all points on the marginal payment function need not be uniquely determined by variations in n when the intercept of the marginal payment requested differ. For small n, the supplier with the lowest marginal costs will have his maximum marginal costs lie below the minimum marginal cost of all others, so that he is the sole supplier in equilibrium. This supplier, will then be indifferent between combinations of fixed and variable bids which give him equal profits. The allocation remains efficient and the equilibrium profits unique. Note that when intercepts of the marginal cost curves differ not all suppliers supply in all states. As n rises, suppliers enter in sequence as their marginal costs at zero are met.

4.3 Bounded Support

What if the uncertainty has bounded support, that is, if n has strictly positive density only on $[0, N]$ and $N < \infty$. In this case suppliers may not care about what they bid in certain regions as the allocation never enters these regions and there can be multiplicity as each ones bid does affect the bids of others. Nevertheless, the equilibrium *allocation* will still be uniquely determined in this case and equal to be the efficient one. However, when the support of the uncertainty is bounded, fixed payments need not equal social contributions.

Assume that n has strictly positive density only on $[0, N]$ with $N < \infty$. In order to describe the set of equilibria in this case we need some further notation. Let $\{B_i^*(q_i, n)\}_{i \in M}$ be the equilibrium profile where all suppliers ask for their true costs as their variable payment, and let $\left\{B_j^*(q_i, n)\right\}_{j \neq i}$ be the profile where we have removed supplier $i's$ strategy. Then by $P_i(q_i; n, B_{j\neq i}^*)$ we denote the highest payment Supplier i can get from the buyer for any quantity q_i supplied in state n, given $\left\{B_j^*(q_j, n)\right\}_{j \neq i}$.[6] Then we obtain the following results.

Theorem 3 *Assume n has bounded support with $N < \infty$. Then there exists a unique equilibrium allocation $(q_i^*(n))_{i \in M}$ for each $n \in [0, N]$. It consists of each supplier i asking for a fixed payment which equals his state dependent marginal contribution $S_i^{mc}(n; B^*)$, and a variable payment which equals his production costs $C_i(q_i)$ for $q_i \in [0, q_i^*(N)]$ and equals any function $T_i(\cdot)$ with $P_i'(q_i; n, B_{j\neq i}^*) \leq T_i'(q_i) \leq C_i'(q_i)$ over the interval $[q_i^*(N), N]$. In this equilibrium, marginal costs are equalized across suppliers*

[6] The function $P_i(\cdot)$ is defined formally in the proof of Theorem 1.

in each state and thus output is provided efficiently.

Proof: In Appendix.

5 Conclusion

In this paper we have demonstrated the applicability of the Bernheim-Whinston (1986) framework to situations were one buyer wishes to buy an uncertain number of units from a small number of sellers. Adding uncertainty about total demand to the original model of Bernheim-Whinston (1986) improves predictability by providing a unique equilibrium. This equilibrium is furthermore ex-post efficient and coincides (for each realization of total demand) with the truthful-bid equilibrium of Bernheim-Whinston (1986).

In Krishna and Tranæs (2001) we compare our scheme to a Clark-Groves-Vickrey mechanism, which delivers the same outcome as a dominant strategy equilibrium. We argue that a reason why such mechanisms are not widely used in practice is that they provide incentives to form coalitions between the buyer and some seller(s) with the object of colluding. Our scheme limits the extent of vulnerability to collusion between the buyer and seller(s) because unlike the Clark-Groves-Vickrey mechanism suppliers are paid their own bid and it is not possible for one supplier to influence the payment to other suppliers for units they are going to deliver anyway. Both schemes provide the same incentives for *suppliers* to collude and raise their joint surplus in equilibrium at the cost of the buyer.

6 Appendix

Proof of Lemma 1.

The key to the proof of Lemma 1 lies in realizing that even if other suppliers make strange marginal bids, Supplier i can, state by state, attain the largest area between $P_i'(.)$ and $C_i'(0)$. Neither $P_i'(.)$ nor $C_i'(0)$ are affected by i's bid so that the maximum available profits for Supplier i state by state are given and attainable by asking for marginal costs as the variable payment and marginal contributions state by state as fixed payments as long as Supplier i is included. Note that $P_i'(0, n, B_{-i}) > 0$ since $T_j'(q_j) > 0$ for all $q_j > 0$. As $C_i'(0) = 0$, $P_i'(0, n, B_{-i}) > C_i'(0)$ and this makes it impossible to exclude i in any state as he can always ask for his marginal costs and a small fixed payment and do better.

Think of Figure 1 in making the argument. Suppose that suppliers bid the profile B. Supplier j is excluded in some states and thus earns zero profits there. As $T_i'(q_i) > 0$ for all $q_i > 0$ and all i by assumption, it is ensured that $P_j'(q_1; n, B_{-j}) = -R_{-j}'(q_j, n, B_{-j}) = T_k'(q_k(q_j; n, B_{-j})) > 0$ for all suppliers, k, chosen to supply by the buyers. This ensures that $P_j'(q_j; n, B_{-j}) > 0$ for all q_j such that $n > q_j \geq 0$. Thus, the point where $P_j'(q_j; n, B_{-j})$ intersects the vertical line at O_j or the point D, must be positive for all $n > 0$. But now as Supplier j's marginal costs are strictly increasing and start from O_j they must lie below $P_j'(q_j; n, B_{-j})$ for small q_j. Hence, Supplier j can always get included and make positive profits by bidding $T_j'(q_j) = C_j'(q_j)$ and asking for positive fixed payments equal to his marginal contribution. This equals the area between $P_j'(q_j; n, B_{-j})$ and $C_j'(q_j)$ up to their intersection. As his marginal contribution is positive, this gives j positive profits. Hence, in equilibrium, Supplier j or any other supplier can not be excluded for any realization of n.[7] One might worry that gains in some realizations are offset or more than offset by losses in other realizations. However this is not an issue as this strategy which gives higher profits in a state where j is otherwise excluded, also attains the maximal profits in all states where j was included. ■

Proof of Lemma 2.

There are only two possibilities if this is not true. Either all suppliers offer $T_i'(0) > C_i'(0) = 0$ or only some do.[8] In either case, using slightly different arguments, we show that this could not be an equilibrium.

Suppose that all suppliers bid such that $T_i'(0) > C_i'(0) = 0$ so that by continuity, their marginal payments remain strictly positive for small supplies. But since $C_i'(0) = 0$, this means that there exists an $\epsilon > 0$, such that all suppliers offer marginal payments which exceed their marginal costs for all output levels up to ϵ as depicted in Figure 2. That is, $T_i'(q_i) > C_i'(q_i)$ for $x \in [0; \epsilon)$. Since in equilibrium, say with allocation $\overline{q}$, $P_i'(\overline{q}_i; n, \cdot) = T_i'(\overline{q}_i)$ for all included suppliers, and as $T_i'(0) > 0$, $P_i'(n; n, \cdot) > 0$. This implies that there exists $\epsilon' > 0$ small enough such that for each supplier i, $P_i'(x; \epsilon', \cdot)$ $> C_i'(x)$ for $x \in [0; \epsilon')$. For any realization $n \in [0; \epsilon')$ therefore, it must be that no more

[7] One might think that if other suppliers offer a negative fixed payment, that is a bribe to the buyer for choosing them, then it may be possible for them to keep Supplier j out. However, this is not possible as the buyer gets the fixed payment as long as they are included even if he reduces their supply.

[8] $T_i'(0) < 0$ is ruled out by assumption.

than one supplier supplies as each supplier who supplies something, wants to supply everything, since his residual marginal revenue $P_i'(\cdot)$ exceeds his marginal costs. But we have already shown (in Lemma 1) that this is not possible in equilibrium so that we have a contradiction.

Suppose that some suppliers $Z \subset M$ bid such that $T_i'(0) > C_i'(0)$ while the rest of them, $Y = M \backslash Z$, bid such that $T_i'(0) = C_i'(0)$. Now as in equilibrium, all suppliers are included for each value of n, their marginal payments offered in equilibrium must be equalized. In particular this holds for the suppliers in Y who offer $T_j'(0) = C_j'(0)$. As n approaches zero, by continuity, their allocation and their marginal bids in equilibrium must approach zero as $C_i'(0) = 0$ for all i. Thus, there exists an $\epsilon > 0$ small enough such that the marginal payment requested by the suppliers $j \in Y$, in equilibrium, must lie everywhere below the lowest $T_i'(0)$ for $i \in Z$. This is depicted in Figure 3 where $T_1'(0) = T_2'(0) = 0 < T_4'(0) < T_3'(0)$ and ϵ is chosen so that $T_1'(x)$ and $T_2'(x)$ lie below $T_4'(0)$ for all $x < \epsilon$. Again by continuity of $T_i'(\cdot)$ there exists $\epsilon' \in (0, \epsilon)$ such that for all $x \in [0, \epsilon']$, $T_i'(x)$ for all $i \in Y$ lies below $T_i'(x)$ for all $i \in Z$, see Figure 3. But this means that the marginal payment requested by any supplier in Z lies above the marginal payment requested by any supplier in Y. This means that we do not have an interior solution and someone is excluded. This is again in contradiction with Lemma 1.■

Proof of Lemma 3.

By Lemma 2, $T_i'(0) = 0$. By assumption we have that $T_i'(q_i) > 0$ for all $q_i > 0$.

Together these imply that for all $n > 0$, $P_i(q_i; n, \cdot)$ is a well defined function over $[0, n]$ with $P_i'(0; n, \cdot) > 0$ and $P_i'(n; n, \cdot) = 0$.

Now assume by way of contradiction that there exists an equilibrium in which one or more of the suppliers bid such that $T_i'(x) \neq C_i'(x)$ over some interval. Let the strategy profile be $\left\{\widetilde{B}_i(q_i, n)\right\}_{i \in M}$ and let the equilibrium allocation vector be $\widetilde{q}(n)$. As all suppliers are in and the buyer is choosing his suppliers to minimize his costs it must be that $P_i'(\widetilde{q}_i; n, \widetilde{B}) = T_i'(\widetilde{q}_i)$, for all i, in equilibrium. Also $P_i'(\widetilde{q}_i; n, \widetilde{B}) = C_i'(\widetilde{q}_i)$ as the suppliers are maximizing their profits. Thus, $P_i'(\tilde{q}_i(n); n, \widetilde{B}) = T_i'(\widetilde{q}_i(n)) = C_i'(\widetilde{q}_i(n))$.

This is not possible as n varies, given that n has full support, unless $T_i'(q_i) = C_i'(q_i)$ for all i and all n. Why? As $T_i'(\widetilde{q}_i(n)) = C_i'(\widetilde{q}_i(n))$ we know that the horizontal sum of $T_i'(.)$ and of $C_i'(.)$ over all i must intersect and do so at the quantity n. This is easiest to see by constructing a figure, left to the reader, which has as a base of length n. The horizontal sum of the $T_i'(.)$ over all suppliers in M is denoted by $T_M'(n)$. It must intersect the horizontal sum of the $C_i'(.)$ over the set of all suppliers M, which is denoted by $C_M'(n)$, somewhere on the right hand axis. This height gives the common value of the marginal variable payment $P_i'(.)$ for all i. Changing n changes the size of the box. If $T_i'(.) \neq C_i'(.)$ for some i, then there are two possibilities. Either $C_M'(n) \neq T_M'(n)$ so that there is an aggregate discrepancy for some n. For such n there must be some i for whom $T_i'(\widetilde{q}_i(n)) \neq C_i'(\widetilde{q}_i(n))$ and this contradicts the assumption that $\widetilde{q}(n)$ is an equilibrium allocation. Alternatively, $C_M'(n) = T_M'(n)$ for all n, but for suppliers in $K \subset M$ we have $T_i'(\widetilde{q}_i(n)) \neq C_i'(\widetilde{q}_i(n))$ in such a way that the discrepancies in $C_i'(.)$ and $T_i'(.)$ cancel out in the aggregate. Let these discrepancies occur in state n'

that is for allocation $\widetilde{q}(n')$. In this case, consider any one supplier, j, from K. By construction, $C_j'(\widetilde{q}_j(n')) \neq T_j'(\widetilde{q}_j(n'))$ which immediately gives a contradiction since both have to be equal to $P_j'(\widetilde{q}_j(n'))$; if $C_j'(\widetilde{q}_j(n')) \neq P_j'(\widetilde{q}_j(n'))$ Supplier j can do better and if $T_j'(\widetilde{q}_j(n')) \neq P_j'(\widetilde{q}_j(n'))$ the buyer can do better. Thus, we have a contradiction with $\left\{\widetilde{B}_i(q_i, n)\right\}_{i\in M}$ being an equilibrium. ■

Proof of Theorem 2: The theorem is proved in three steps.

1. In equilibrium, all suppliers supply strictly positive quantities for all $n > 0$. That it is never optimal for the buyer to exclude a supplier or a set of suppliers follows directly from the assumption that marginal variable payments are strictly upwards sloping starting from zero and Lemma 1.

2. Suppose the bids in the initial round were a profile $T_i^*(q_i)_{i\in M}$ meeting the assumptions made. Then there exists a unique equilibrium in the proceeding subgame in which all suppliers ask for their marginal contributions given $T_i^*(q_i)_{i\in M}$. The argument for this is in the proof of Theorem 1.

3. Thus, in the first round, suppliers know that they will bid, and eventually get, their *marginal contribution,* state by state, in the second round. So what will they choose to bid in the first round? It follows from the assumption $0 > T_i''(q_i) > 0$ that $P_i'(q_i; n, T_{j\neq i}^*)$ is strictly downwards sloping for all i and from the assumption that $T_i'(0) = 0$ that $P_i'(n; n, T_{j\neq i}^*) = 0$. So the best bid is the one that maximizes a suppliers marginal contribution plus variable payment, state by state. The unique bid that does this is a suppliers true variable cost function and this is the unique best bid independent of the bids made by the supplier's opponents.

When all suppliers announce their true costs in the first round it follows that marginal contributions equal social contributions in the second round which concludes the proof. ■

Finally, we prove Theorem 3 as a corollary to Theorem 1.

Proof of Theorem 3:

If n has finite support, then it is straightforward to check that the set of strategy profiles given in Theorem 3 all form equilibria. But why are there not other equilibria as well? The fact that all suppliers ask for variable payments $T_i(q_i)$ equal to their production costs $C_i(q_i)$ for $q_i \in [0, q_i^*(N)]$ follows from Lemma 3. But this does not pin down the rest of $T_i(\cdot)$ which will affect the fixed payments asked for in equilibrium when n has finite support. As established above, in equilibrium, suppliers have to ask for their state dependent marginal contribution $S_i^{mc}(n)$, which will depend on $P_i(q_i; n, B_{j\neq i}^{**})$, where $\{B_i^{**}(q_i, n)\}_{i\in M}$ is the equilibrium in question. As long as a supplier asks for $T_i'(q_i)$ above $P_i'(q_i; n, B_{j\neq i}^{*})$, it will not be called upon to supply for all possible realizations and so there is no cost of such a deviation. Asking for $T_i'(q_i) > C_i'(q_i)$ can never be part of an equilibrium. If it were part of an equilibrium, the marginal contributions of other firms would exceed their social contributions. However, if these firms asked for their marginal contributions as their fixed payment, it would be in the interest of firm i to shade its marginal bids downwards by a little bit on the units above $q_i^*(N)$ (since it would make a marginal profit by supplying these units) and this would cause the buyer to exclude someone which we show can never happen in equilibrium, which is a contradiction. Thus, a supplier is willing to ask for any $T_i'(q_i)$

such that $P_i'(q_i; n, B^*_{j\neq i}) \leq T_i'(q_i) \leq C_i'(q_i)$. Note that this means that the marginal contributions of suppliers in equilibrium can be less than their social contributions. If this occurs, then entry and investment are reduced below their optimal levels. ■

References

[1] Anton, J.J., Yao, A.: Split awards, procurement, and innovation, Rand Journal of Economics **20(4)**, 538-552 (1989).

[2] Anton, J.J., Yao, A.: Coordination in split awards auctions, Quarterly Journal of Economics **107(2)**, 681-707 (1992).

[3] Bernheim, B.D., Whinston, M.: Menu auctions, resource allocation, and economic influence, Quarterly Journal of Economics **101**, 1-31 (1986).

[4] Clark, E.: Multipart pricing of public goods, Public Choice **8**, 19-33 (1971).

[5] Grossman, G., Helpman, E.: Protection for sale, American Economic Review **84**, 833-850 (1994).

[6] Grossman, G., Helpman, E.: Trade wars and trade talks, Journal of Political Economy **103(4)**, 675-708 (1995).

[7] Groves, T.: Incentives in teams, Econometrica **41**, 617-631 (1973).

[8] Klemperer, P., Meyer, M.: Supply function equilibrium in oligopoly with uncertainty, Econometrica **57(6)**, 1283-1298 (1989).

[9] Krishna, K., Tranæs, T.: Providing uncertain quantities efficiently, mimeo, Penn State University (2001).

[10] Krishna, K., Tranæs, T.: Allocating multiple units, Economic Theory **20**, 733-750 (2002).

[11] Robson, A.J.: Implicit oligopolistic collusion in destroyed by uncertainty, Economics Letters **7**, 75-80 (1981).

[12] Spiller, P.: Politicians, interest groups, and regulators: a multiple-principals agency theory of regulation (or let them be bribed), Journal of Law and Economics **33**, 65-101 (1990)

[13] Tranæs, T.: Decision-making by an authority under influence, Economics Letters **43,** 35-39 (1993).

[14] Vickrey, W.: Counterspeculation, auctions, and competitive sealed tenders, Journal of Finance **16,** 8-37 (1961).

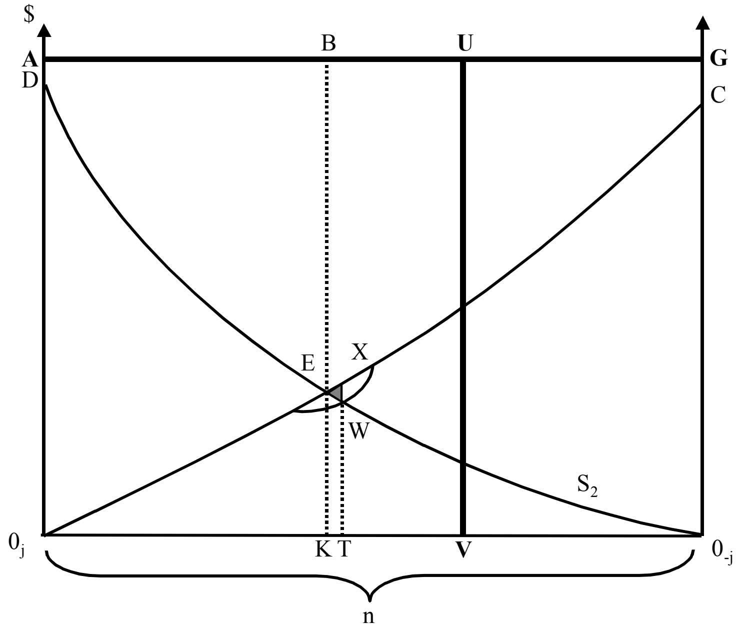

FIGURE 1. COMPETITION IN SUPPLY.

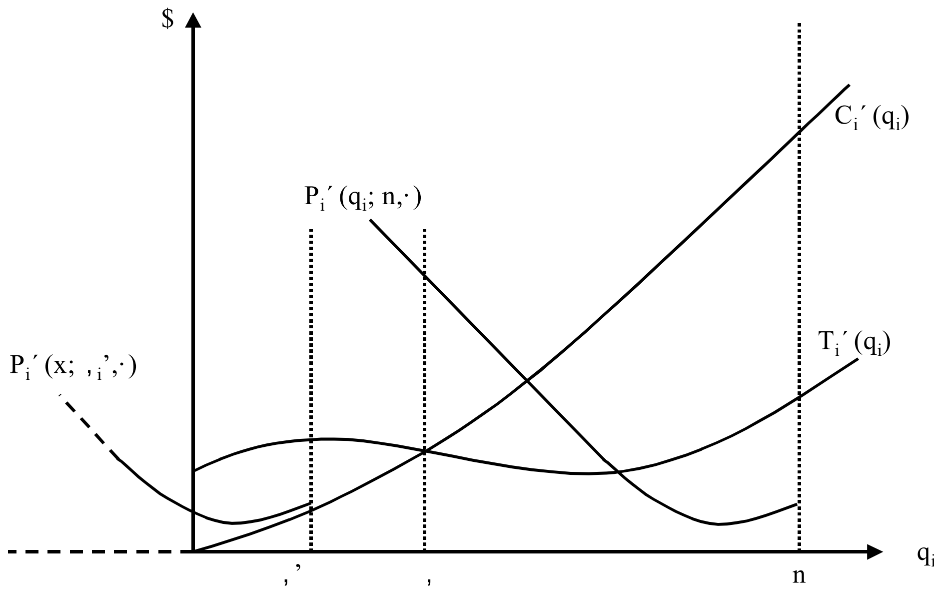

FIGURE 2. COUNTER EXAMPLE TO $T_i'(0) > 0$ FOR ALL i.

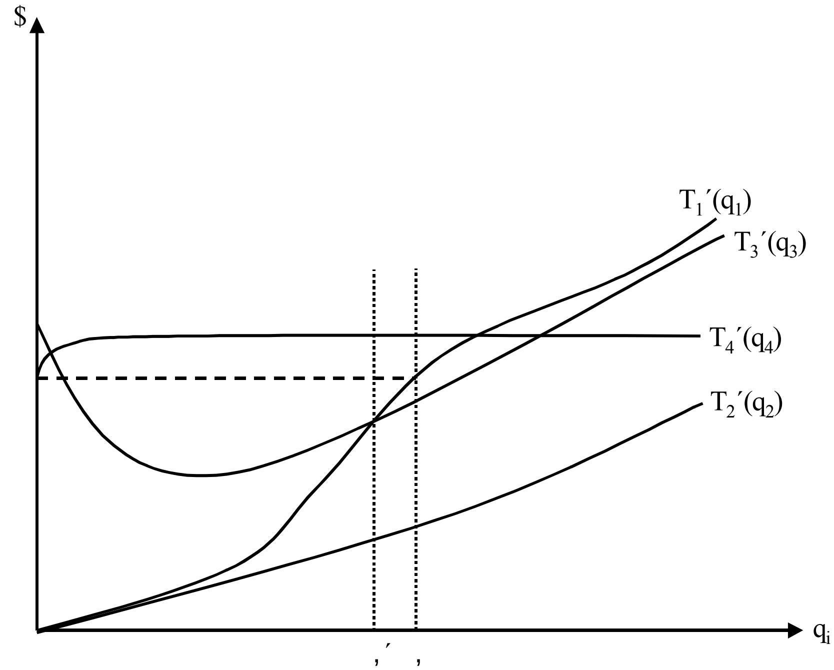

FIGURE 3. COUNTER EXAMPLE TO $T_i'(0) > 0$ FOR SOME i.