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Egbert Dierker, Hildegard Dierker and Birgit Grodal

Studiestræde 6, DK-1455 Copenhagen K., Denmark
Tel. +45 35 32 30 82 - Fax +45 35 32 30 00
http://www.econ.ku.dk

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Topics in Mathematical Economics

The participants in a September 2002 Workshop on *Topics in Mathematical Economics* in honor of Birgit Grodal decided to have a series of papers appear on Birgit Grodal's 60'th birthday, June 24, 2003.

The Institute of Economics suggested that the papers became Discussion Papers from the Institute.

The editor of *Economic Theory* offered to consider the papers for a special Festschrift issue of the journal with Karl Vind as Guest Editor.

This paper is one of the many papers sent to the Discussion Paper series.

Most of these papers will later also be published in a special issue of *Economic Theory*.

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Are Incomplete Markets Able to Achieve Minimal Efficiency? *

Egbert Dierker † Hildegard Dierker † Birgit Grodal ‡

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Abstract

We consider economies with incomplete markets, one good per state, two periods, $t = 0, 1$, private ownership of initial endowments, a single firm, and no assets other than shares in this firm. In Dierker, Dierker, Grodal (2002), we give an example of such an economy in which all market equilibria are constrained inefficient. In this paper, we weaken the concept of constrained efficiency by taking away the planner's right to determine consumers' investments. An allocation is called minimally constrained efficient if a planner, who can only determine the production plan and the distribution of consumption at $t = 0$, cannot find a Pareto improvement. We present an example with arbitrarily small income effects in which no market equilibrium is minimally constrained efficient.

JEL Classification: D2, D52, D61, G1

Keywords: Incomplete markets with production, constrained efficiency, Drèze equilibria

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†Institut für Wirtschaftswissenschaften, Universität Wien, Hohenstaufengasse 9, A-1010 Wien, Austria

‡Centre of Industrial Economics and Økonomisk Institut, Københavns Universitet, Studiestræde 6, DK-1455 København, Denmark

1 Introduction

We consider finance economies with production. More precisely, we assume incomplete markets, one good per state, private ownership of initial endowments, production, and two time periods. Due to the incompleteness of markets, shareholders typically disagree about which production decision their firm should take. Drèze (1974) presents a way of resolving the conflict among shareholders by introducing an equilibrium concept that is based on Pareto comparisons with the aim of achieving constrained efficiency. We restrict ourselves to economies with one good per state in order to rule out price effects, which are a well-known cause of constrained inefficiency [cf. Geanakoplos et al. (1990)].

In this paper, we show that the market in such economies may not be able to achieve an allocation that satisfies minimal efficiency requirements as soon as the quasilinear framework is left. This phenomenon is illustrated in economies with only one firm.

The firm has constant returns to scale and makes zero profit. Its state dependent output at $t = 1$ is sold on the asset market in exchange for the corresponding input. When the firm proposes a production ray, consumers choose their optimal investments and this determines their consumption in all states. The firm adjusts its production level to the market clearing scale. The resulting allocation is called a market equilibrium. The set of all allocations the market can achieve consists of all market equilibria corresponding to some production decision of the firm.

A Drèze equilibrium is a market equilibrium with the following property: The (new) shareholders of the firm meet at $t = 0$ after they have chosen their shares optimally. If these shares are held fixed, there is no other production plan such that the shareholders of the firm can achieve a Pareto improvement by adopting that production plan and by making sidepayments at time $t = 0$ to reach unanimity.[1]

Constrained efficiency means that a hypothetical planner cannot find a Pareto improvement by simultaneously choosing the production plan, the shares, and each individual's consumption at $t = 0$. Note that a constrained efficient market equilibrium is a Drèze equilibrium.

An example of an economy with a unique, but constrained inefficient Drèze equilibrium is presented in Dierker, Dierker, and Grodal, henceforth DDG, (2002). This example is driven by the existence of a consumer whose preferences exhibit strong income effects. If there are no income effects, that is to say, if all consumers have quasilinear utility functions, then at least one constrained efficient Drèze

[1] For an extensive treatment of Drèze equilibria in a setting with private ownership of initial endowments, the reader is referred to Magill and Quinzii (1996), chapter 6.

equilibrium exists, since the social surplus is well defined and is maximized at a Drèze equilibrium.

Since the planner, who can implement constrained efficient allocations, is more powerful than the market, we reduce the planner's power substantially and explore whether the planner can still outperform the market. We introduce the following very weak version of constrained efficiency, in which tomorrow's consumption can only be affected by the planner through the choice of the production plan. After the planner has chosen a production plan with input normalized to -1, consumers choose their optimal investments subject to their budget constraints. The firm adjusts production to the market clearing scale. The planner, who is no longer allowed to alter individual consumption at $t = 1$, can only distribute the resources remaining at $t = 0$ after subtracting the input. An allocation is called minimally constrained efficient if the planner, who is subject to these constraints, cannot find a Pareto improvement. It turns out that the example in DDG (2002) is minimally constrained efficient.

However, there are economies without any minimally constrained efficient allocations. To show this, we start with a quasilinear economy with three Drèze equilibria. Two of them are surplus maxima and the third is a surplus minimum. Then we perturb the quasilinear utility functions of the example by adding a small term to the utility at $t = 0$. The perturbation does not affect the way in which future consumption streams are ranked, i.e., utility at $t = 1$ is left unchanged. These small perturbations leave the set of Drèze equilibria invariant. However, for arbitrarily small perturbations, all Drèze equilibria, and hence all market equilibria, become minimally constrained inefficient.

The notion of minimal constrained efficiency cannot be weakened further, since the planner should at least retain the possibility of changing the production plan and redistributing total consumption at $t = 0$. We conclude that, even in economies with one good per state, arbitrarily small income effects can make it impossible to select a production plan that achieves a market equilibrium that satisfies at least some weak version of constrained efficiency. The question of how to choose a market equilibrium remains open and is briefly discussed at the end of the paper.

The remainder of the paper is organized as follows. In Section 2 we introduce the framework and present the definitions. In Section 3 we give an example showing that minimally constrained efficient market equilibria need not exist. Section 4 contains concluding remarks.

2 Framework and Definitions

The reason why the market mechanism can be unable to generate allocations that exhibit desirable efficiency properties can be illustrated in a very simple setting. We consider two periods, $t = 0, 1$, and two possible states of nature at $t = 1$, denoted $s = 1$ and $s = 2$, respectively. The unique state at $t = 0$ is included as the state $s = 0$. There is a single good, denoted s, in each state $s = 0, 1, 2$ and there is just one firm. It transforms good 0 into a state dependent output at $t = 1$. We assume that there are no assets other than shares in the firm. The firm has constant returns to scale and makes zero profits. Its technology is given by a family of normalized production plans $(-1, \lambda, 1-\lambda)$. The production set is

$$Y = \{\alpha(-1, \lambda, 1-\lambda) \in \mathbb{R}^3 \mid \alpha \geq 0,\ 0 \leq \underline{\lambda} \leq \lambda \leq \overline{\lambda} \leq 1\}.$$

There are two types of consumers. Ideally, each type is represented by a continuum of mass 1. For convenience, we refer to each continuum of identical consumers as a single consumer denoted $i = 1, 2$. Consumer i has the initial endowment e^i, consumption set $\mathbb{R}^3_+$, and utility function U^i.

If the firm selects the normalized production plan $(-1, \lambda, 1-\lambda)$ and consumer i chooses the investment $\alpha^i \geq 0$ in the firm, the resulting consumption bundle is $e^i + \alpha^i(-1, \lambda, 1-\lambda)$. Consumer i selects α^i so as to maximize utility in the budget set

$$B^i(\lambda) = \{e^i + \alpha^i(-1, \lambda, 1-\lambda) \in \mathbb{R}^3_+ \mid \alpha^i \geq 0\}.$$

If the utility functions are strictly quasiconcave, i's optimal investment $\alpha^i(\lambda)$ is uniquely determined. Agent i consumes $x^i(\lambda) = e^i + \alpha^i(\lambda)(-1, \lambda, 1-\lambda)$, holds shares $\vartheta^i(\lambda) = \alpha^i(\lambda)/(\alpha^1(\lambda) + \alpha^2(\lambda))$, and the firm produces $y(\lambda) = [\alpha^1(\lambda) + \alpha^2(\lambda)](-1, \lambda, 1-\lambda)$.

Definition . *The allocation* $(y(\lambda), x^1(\lambda), x^2(\lambda))$ *is called a* market equilibrium iff

1) $x^i(\lambda) = e^i + \alpha^i(\lambda)(-1, \lambda, 1-\lambda)$, where $\alpha^i(\lambda)$ is i's optimal investment at the production ray λ,
2) $y(\lambda) = [\alpha^1(\lambda) + \alpha^2(\lambda)](-1, \lambda, 1-\lambda) \in Y$.

Market equilibria are the only allocations that the market can achieve. In general, these allocations cannot be Pareto compared and the shareholders face a social choice problem. In order to resolve the problem, Drèze (1974) suggested that shareholders use sidepayments among themselves at $t = 0$ in order to reach unanimity.

A Drèze equilibrium is a market equilibrium in which the production plan of the firm passes the following test: It is impossible for the shareholders to find another production plan and sidepayments at $t = 0$ such that all shareholders are better off if they use their original investment levels and get the sidepayments.[2]

[2] In the usual definition of a Drèze equilibrium, shares ϑ^i, and not the investment levels α^i, are taken as fixed when a production plan is evaluated. The two definitions are equivalent.

More precisely, consider a market equilibrium $(y(\tilde{\lambda}), x^1(\tilde{\lambda}), x^2(\tilde{\lambda}))$ with respect to $\tilde{\lambda}$ and let $\mathcal{I} = \{i \mid \alpha^i(\tilde{\lambda}) > 0\}$. The market equilibrium is a *Drèze equilibrium* if it is impossible to find a normalized production plan $(-1, \lambda, 1-\lambda)$ and a system of sidepayments $(\tau^i)_{i\in\mathcal{I}}$ at $t = 0$ with $\sum_{i\in\mathcal{I}} \tau^i = 0$ such that

$$U^i(e^i + \tau^i(1,0,0) + \alpha^i(\tilde{\lambda})(-1, \lambda, 1-\lambda)) > U^i(x^i(\tilde{\lambda}))$$

for every $i \in \mathcal{I}$. Note that the production plan $(-1, \lambda, 1-\lambda)$ on the left hand side of the above inequality is multiplied by the investment level $\alpha^i(\tilde{\lambda})$ that is optimal at the normalized equilibrium production plan $(-1, \tilde{\lambda}, 1-\tilde{\lambda})$.

We recall the definitions of constrained feasibility and constrained efficiency [cf. Magill and Quinzii (1996)]. A commodity vector $x \in \mathbb{R}^3$ is written as $x = (x_0, x_{\mathbf{1}})$, where $x_0 \in \mathbb{R}$ corresponds to $t = 0$ and $x_{\mathbf{1}} \in \mathbb{R}^2$ corresponds to $t = 1$. An allocation (y, x^1, x^2) is constrained feasible if it can be implemented by a planner who simultaneously determines the production plan $y = (y_0, y_{\mathbf{1}}) \in Y$, the shares ϑ^i of all consumers and who, moreover, freely redistributes good 0. More precisely, the allocation $((y_0, y_{\mathbf{1}}), (x_0^1, x_{\mathbf{1}}^1), (x_0^2, x_{\mathbf{1}}^2)) \in Y \times \mathbb{R}_+^3 \times \mathbb{R}_+^3$ is *constrained feasible* if $x_0^1 + x_0^2 = e_0^1 + e_0^2 + y_0$ and there exist shares $\vartheta^i \geq 0$ such that $x_{\mathbf{1}}^i = \vartheta^i y_{\mathbf{1}}$ for all i and $\sum_i \vartheta^i = 1$. Note that the set of constrained feasible allocations does not depend on how the aggregate endowment of good 0 is distributed across consumers and that it is, in general, larger than the set of market equilibria. A constrained feasible allocation is called *constrained efficient* if there does not exist a Pareto superior constrained feasible allocation.

In searching for constrained efficient market equilibria we can restrict attention to the set of Drèze equilibria since a constrained efficient market equilibrium is a Drèze equilibrium.

In DDG (2002) we present an example of a finance economy with a unique, but constrained inefficient Drèze equilibrium. Thus, the market is unable to achieve constrained efficiency in the example. Therefore, we are led to ask the question of whether the efficiency requirements can be relaxed such that the market can at least achieve an extremely weak form of constrained efficiency.

The planner who can implement constrained efficient allocations is more powerful than the market, since the planner can distribute consumption at $t = 0$ directly and affect consumption at $t = 1$ indirectly by allocating shares to individuals. Clearly, to improve upon a market equilibrium, the planner must be able to compensate the losers of a change of the available asset by reallocating consumption at $t = 0$. Therefore, we cannot deprive the planner of the right to distribute good 0. However, we take away the right to allocate shares. Since the power of a planner who is deprived of this right cannot be further reduced, a constrained feasible allocation is called minimally constrained efficient if it is not possible for a planner who does not possess the right to distribute shares, to Pareto improve upon the allocation.

More precisely, the economy with the weakened planner can be described as follows. First the planner chooses λ. Given λ, each consumer i selects the optimal investment $\alpha^i(\lambda)$ such that the resulting consumption plan $(x_0^i, x_{\mathbf{1}}^i) = e^i + \alpha^i(\lambda)(-1, \lambda, 1-\lambda)$ maximizes i's utility in the budget set associated with i's initial endowment and λ. The planner can redistribute total consumption $\sum_i x_0^i$ at $t = 0$, but cannot affect individual consumption $x_{\mathbf{1}}^i$ at $t = 1$ and λ. That is to say, whenever the planner has chosen λ, the stock market opens and each consumer i chooses $(x_0^i, x_{\mathbf{1}}^i) = e^i + \alpha^i(\lambda)(-1, \lambda, 1-\lambda)$ optimally. Then the stock market is closed and nobody, including the planner, can change $x_{\mathbf{1}}^i$. After the stock market is closed the planner can redistribute good 0.

Definition . *A constrained feasible allocation is called* minimally constrained efficient *if there is no Pareto superior allocation* $(\lambda, (c_0^i, x_{\mathbf{1}}^i)_i)$ *satisfying*

(i) $x_{\mathbf{1}}^i = e_{\mathbf{1}}^i + \alpha^i(\lambda)(\lambda, 1-\lambda)$, *where* $\alpha^i(\lambda)$ *is i's optimal investment given* λ,

(ii) $\sum_i c_0^i = \sum_i e_0^i - \sum_i \alpha^i(\lambda)$, *and*

(iii) $\sum_i \alpha^i(\lambda)(-1, \lambda, 1-\lambda) \in Y$.

Condition (i) says that, after the planner has chosen the production ray, individual consumption at $t = 1$ is determined by the market. Condition (ii) states that the planner can redistribute the aggregate consumption $\sum_i e_0^i - \sum_i \alpha^i(\lambda)$ at $t = 0$. Condition (iii) says that the planner adjusts the level of production to the consumers' aggregate investment.

Our method of defining minimal constrained efficiency can, in principle, also be used if there are several goods in each state. In this case, even equilibria with respect to fixed sets of assets are typically constrained inefficient due to price effects. Therefore, Grossman (1977) weakened the definition of constrained efficiency by introducing a central planner with incomplete coordination. We compare our planner with Grossman's. Grossman's planner cannot act simultaneously in different states, but our planner is not even allowed to act in any state other than $s = 0$. At $s = 0$, our planner is, apart from the ability to choose λ, weaker than Grossman's, since shareholdings and individual consumption at $t = 1$ are determined by individual optimization. Our planner can only redistribute the resources at $s = 0$ that are not used for production, whereas Grossman's planner can also allocate shares.

Numerical computation shows that the unique Drèze equilibrium in the example in DDG (2002) is minimally constrained efficient although it is not constrained efficient. This fact can be explained as follows. The example is driven by strong income effects: The optimal investment of the first consumer depends strongly on his wealth and, therefore, on the sidepayment obtained from the planner. However, in the case of minimal efficiency this effect ceases to play a role since individual investments in shares are, by definition, independent of sidepayments. Since the mechanism driving the example in DDG (2002) cannot be used in the case of minimal constrained efficiency, one would like to know whether at least one Drèze equilibrium in a finance economy is minimally constrained efficient.

3 Can the Market Achieve Minimal Constrained Efficiency?

In order to answer this question we proceed as follows. First we present and discuss a quasilinear example. Due to the existence of a representative consumer, a constrained efficient Drèze equilibrium necessarily exists. Then we introduce small income effects by perturbing the example slightly and analyze how the perturbation affects the efficiency properties of the Drèze equilibria.

In the unperturbed example, consumers have quasilinear utilities given by

$$U^1(x_0, x_1, x_2) = x_0 + x_1^{0.6} \quad \text{and}$$
$$U^2(x_0, x_1, x_2) = x_0 + x_2^{0.6},$$

respectively. We assume $\underline{\lambda} = 0.1, \overline{\lambda} = 0.9$ and $e^1 = e^2 = (2, 0, 0)$. It turns out that the economy under consideration has three Drèze equilibria, A, B and C, corresponding to $\lambda_A = 0.1, \lambda_B = 0.5$, and $\lambda_C = 0.9$, respectively.

In the definition of a Drèze equilibrium, shares are kept fixed when shareholders evaluate alternative production plans. In order to gain insight into the consequences of this feature, it is useful to investigate the interior equilibrium B. To do so, we first consider the indirect utility $u^1(2, \lambda)$ that consumer 1 obtains, if the firm chooses the ray λ and if consumer 1 makes the optimal investment $\alpha^1(\lambda) = 0.6(0.6\lambda)^{1.5}$. Since this utility equals $u^1(2, \lambda) = 2 + 0.4(0.6\lambda)^{1.5}$, the function $u^1(2, \cdot)$ is convex. Similarly, the utility level of consumer 2 at λ equals $u^2(2, \lambda) = u^1(2, 1 - \lambda)$ and is convex in λ. As a consequence, shareholders' social surplus associated with the ray λ, $u^1(2, \lambda) + u^2(2, \lambda)$, is convex in λ. Due to the symmetry between $u^1(2, \lambda)$ and $u^2(2, \lambda)$, the social surplus has a critical point at $\lambda_B = 0.5$, which must be a global minimum [see Figure 1].

Observe that the situation changes drastically if the shareholders are deprived of the possibility of adjusting their shares, or, equivalently, their investment levels, when λ_B is tested against some alternative λ. Consider consumer 1 who wants to choose the investment level $\alpha^1(\lambda)$ in proportion to $\lambda^{1.5}$. If α^1 is now taken as fixed at its value at $\lambda_B = 0.5$, then the utility reached at ray λ equals $\tilde{u}^1(2, \lambda) = c_0 + c_1\lambda^{0.6}$ with $c_1 > 0$, whereas the indirect utility with share adjustment is a function of the type $u^1(2, \lambda) = c_0' + c_1'\lambda^{1.5}$ with $c_1' > 0$. Thus, by disregarding how consumer 1's individual investment level $\alpha^1(\lambda)$ varies with λ, the originally convex function $u^1(2, \cdot)$ is turned into a concave function $\tilde{u}^1(2, \cdot)$. As a consequence, $\tilde{u}^1(2, \cdot) + \tilde{u}^2(2, \cdot)$ is a concave function and the critical point $\lambda = 0.5$ becomes a maximum. For this reason, λ_B yields a Drèze equilibrium. Clearly, the utility sum $\tilde{u}^1(2, \cdot) + \tilde{u}^2(2, \cdot)$ constructed by fixing the shares does not represent owners' welfare at alternative production rays correctly.

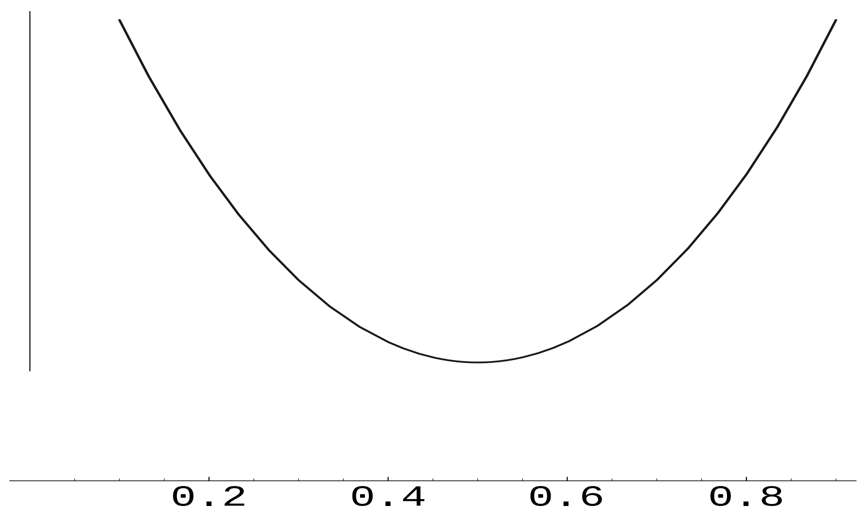

Figure 1: Surplus minimum at the Drèze equilibrium $\lambda_B = 0.5$

At the Drèze equilibria A and C, consumers' social surplus is maximized. Hence, A and C are constrained efficient.

Now we perturb the quasilinear example by altering the utility derived from consumption at $t = 0$ without changing the utility obtained from consumption at $t = 1$. In particular, the utility functions stay additively separable after perturbation. Let

$$U_a^1(x_0, x_1, x_2) = x_0 + ax_0^2 + x_1^{0.6} \quad \text{and} \quad U_a^2(x_0, x_1, x_2) = x_0 + ax_0^2 + x_2^{0.6}\,, \quad (1)$$

where $0 < a \leq 0.1$. It is easy to show that i's utility function is quasiconcave in the relevant range.

As in the unperturbed example, the production ray varies in the interval $[0.1, 0.9]$ and there are three Drèze equilibria corresponding to $\lambda_A = 0.1$, $\lambda_B = 0.5$, and $\lambda_C = 0.9$, respectively. However, the boundary equilibria are no longer constrained efficient for any $a > 0$. Moreover, the boundary equilibria are not even minimally constrained efficient.

Proposition . *For arbitrarily small $a > 0$, no market equilibrium associated with some ray λ is minimally constrained efficient.*

Proof. Consider any ray λ and the corresponding market equilibrium allocation. Clearly, the equilibrium corresponding to $\lambda_B = 0.5$ is not minimally constrained efficient. Therefore, let $\lambda \neq 0.5$. We show that the production ray $1-\lambda$, together with a suitable reallocation of consumption at $t = 0$ is preferred to λ by both types of consumers. Due to symmetry we can assume $\lambda < 0.5$.

Agent i consumes $x^i(\lambda) \in B^i(\lambda)$ when the ray λ is chosen. If λ is replaced by $1-\lambda$, agent i consumes $x^i(1-\lambda)$ and achieves the utility level $U_a^i(x^i(1-\lambda)) \neq$

$U_a^i(x^i(\lambda))$. Let τ^i be the amount of good 0 required in addition to $x^i(1-\lambda)$ in order to let i achieve the original utility level $U_a^i(x^i(\lambda))$. More precisely,

$$U_a^1(x^1(1-\lambda)+\tau^1(1,0,0)) = U_a^1(x^1(\lambda)) \tag{2}$$

and

$$U_a^2(x^2(1-\lambda)+\tau^2(1,0,0)) = U_a^2(x^2(\lambda))\,. \tag{3}$$

Since $\lambda < 0.5 < 1-\lambda$, we have $\tau^1 < 0$ and $\tau^2 > 0$. Moreover, by symmetry, $x_0^1(\lambda) = x_0^2(1-\lambda)$ and

$$U_a^1(x^1(\lambda)) = U_a^2(x^2(1-\lambda))\,, \qquad U_a^2(x^2(\lambda)) = U_a^1(x^1(1-\lambda))\,. \tag{4}$$

We add (2) and (3), use symmetry and the utility specifications (1), and obtain

$$(\tau^1+\tau^2) + a((\tau^1)^2 + (\tau^2)^2) + 2a(\tau^1 x_0^1(1-\lambda) + \tau^2 x_0^1(\lambda)) = 0\,. \tag{5}$$

Since calculation of consumer 1's optimal shares yields that the demand for good zero is strictly decreasing, we have $x_0^1(\lambda) > x_0^1(1-\lambda) > 0$.

Assume that $\tau^1+\tau^2 \geq 0$ and, hence, $\tau^2 \geq |\tau^1|$. Then $\tau^2 x_0^1(\lambda)) > |\tau^1 x_0^1(1-\lambda)|$. Therefore, the left hand side of (5) must be strictly positive for every $a > 0$, which is a contradiction. We conclude that $\tau^1+\tau^2 < 0$. Hence, the equilibrium corresponding to λ is not minimally constrained efficient. □

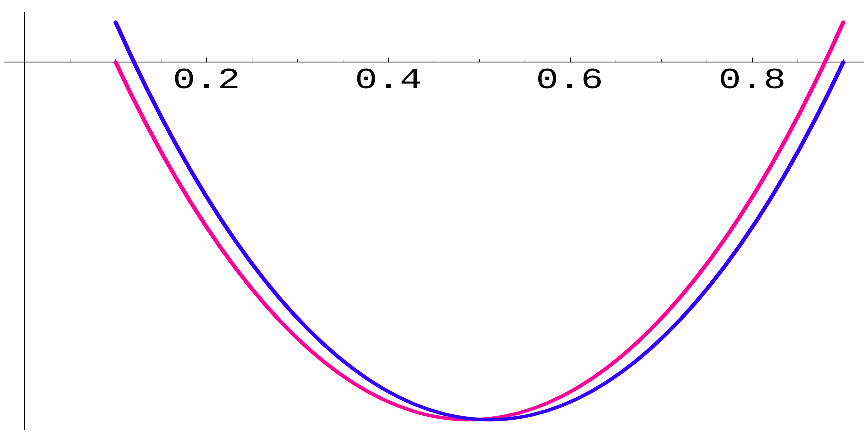

Figure 2: Intersecting total "saving" functions

Figure 2 illustrates the argument. Take the equilibrium at 0.1 and consider the sidepayment $\tau^1(\lambda)$ necessary to keep consumer 1 at the utility level $U_a^1(x^1(0.1))$ if the ray 0.1 is replaced by the ray λ. That is to say, $\tau^1(\lambda)$ is given by

$$U_a^1(x^1(\lambda)+\tau^1(\lambda)(1,0,0)) = U_a^1(x^1(0.1))\,.$$

Let $\tau^2(\lambda)$ be defined in a similar way. Thus, $\tau^1(\lambda) + \tau^2(\lambda)$ specifies the total amount of compensation required to maintain the utility levels achieved at 0.1. The relationship to Figure 1 becomes clearer if the compensation is replaced by $-(\tau^1(\lambda) + \tau^2(\lambda))$, which is the amount of good 0 that can be saved at λ while keeping consumer i on the utility level $U_a^i(x^i(0.1))$. This total "saving" function becomes positive at $\lambda = 0.9$, which indicates that the equilibrium with respect to $\lambda = 0.1$ is not minimally constrained efficient. A similar saving function can be defined if the other boundary $\lambda = 0.9$ is taken as the reference point. If a goes to 0, both curves in Figure 2 approach the social surplus curve depicted in Figure 1 (up to a constant).

The nonexistence of constrained efficient and minimally constrained market equilibria is caused by the following facts. First, the example is built upon a nonconvexity. In the unperturbed, quasilinear example, the nonconvexity can be described as follows. The amount of good 0 initially available in the economy just suffices to maintain the utility profile $(u^1(2, 0.1),\ u^2(2, 0.1))$ reached at the boundary point $\lambda = 0.1$, if the other boundary point $\lambda = 0.9$ is chosen. However, if the firm implements any ray λ strictly between 0.1 and 0.9, this amount is insufficient. Second, as soon as the perturbation parameter a becomes positive, the graphs of the two saving functions intersect each other. To maintain the profile $(u_a^1(2, 0.1),\ u_a^2(2, 0.1))$ at $\lambda = 0.9$, one can dispense with a positive amount of good 0. A similar statement holds if the two boundary points are interchanged [cf. Figure 2]. These two features cannot be ruled out in general. Therefore, one cannot expect the market to be able to achieve minimally constrained efficient outcomes.[3]

The allocations attainable by the market depend on the initial allocation of endowments. To obtain a situation in which a constrained efficient market equilibrium exists in the perturbed example, a lump sum redistribution of initial endowments is required. Markets do not perform such redistributions and thus, are less powerful than even the very weak planner discussed in the context of minimal constrained efficiency. The importance of the initially determined distribution of wealth in nonconvex environments was first pointed out by Guesnerie (1975) in the framework of complete markets and nonconvex production sets. Guesnerie showed that all marginal cost pricing equilibria can be inefficient, even though Pareto efficiency requires prices to equal marginal costs.

[3] It has been emphasized in the literature on compensation criteria à la Hicks and Kaldor that intersecting utility possibility frontiers often entail inconsistent policy recommendations [see, e.g., Gravel (2001)].

4 Concluding Remarks

We have seen that shareholders' social surplus can reach its minimum at a Drèze equilibrium if all shareholders have quasilinear utilities. This is due to the fact that the definition of a Drèze equilibrium only takes welfare changes of first order into account. Thus, no distinction is made between an interior maximum and any other critical point.

In the quasilinear case, a constrained efficient Drèze equilibrium exists. Therefore, it is tempting to refine the Drèze equilibria in order to rule out constrained inefficient allocations. However, our example shows that this endeavor can fail to provide any solution as soon as one deviates from the quasilinear setting: Arbitrarily small income effects render all market equilibria constrained inefficient.

Moreover, even if the efficiency requirements are substantially reduced, they can remain unfulfilled at every market equilibrium in a finance economy. In our example the stock market cannot even achieve a minimally constrained efficient outcome if the quasilinear setting is abandoned. Hence, the existence of a constrained efficient equilibrium in the quasilinear economy should be viewed as an artifact lacking any robustness.

Clearly, there are economies in which the problem does not arise. For example, Drèze equilibria are constrained efficient if there is only one firm and if every consumer's indirect utility function is quasiconcave. This function describes the maximum amount of utility the consumer can derive from a production decision at different levels of wealth at $t = 0$. The indirect utility functions underlying Figure 1 are not quasiconcave. This is due to the fact that the specification of the direct utility functions U^i makes optimal shareholdings sufficiently sensitive to changes in the production ray.[4] Since the indirect utility depends on how the optimal number of shares that an individual holds varies with the asset span and individual wealth at $t = 0$, it is, unless attention is restricted to particularly simple examples, quite difficult to state economically meaningful conditions ensuring the quasiconcavity of indirect utility functions. We do not think that imposing restrictions on consumers' characteristics presents a promising approach to overcome the problem of nonexistence of constrained efficient market equilibria.

It has been suggested to us to use lotteries instead of deterministic allocations. A similar approach has been successfully applied in other settings. Cole and Prescott (1997), for instance, use random allocations to analyze equilibria in economies with clubs. Club membership is indivisible and lotteries are used to restore convexity. Lotteries have also been used to overcome the nonconvexity of the set of feasible allocations in economies with adverse selection. In that case, the nonconvexity is due to individual incentive constraints and eliminated

[4] If the power 0.6 in the definition of U^i is replaced by a number below 0.5, quasiconcavity of the indirect utility function u^i results.

by introducing random allocations; see Prescott and Townsend (1984). In this paper, the difficulty is not due to a nonconvexity on the individual level but to a pure public good problem.

In our framework, random allocations could be introduced by making the production decision stochastic and letting consumers choose their investments contingent on the realization. More specifically, consider the set $U = \{(u^1, u^2) \leq (u^1(2, \lambda), u^2(2, \lambda)) \mid \lambda \in [0.1, 0.9]\}$ of vectors that are below a utility profile attained at some market equilibrium in the quasilinear example. The set U is nonconvex. Let the production ray become random and consumers have von Neumann-Morgenstern utility functions. If consumers are allowed to choose their investments after they have learned the realization of λ the set U is convexified. More precisely, the convex hull of U is generated by the two profiles $(u^1(2, \lambda), u^2(2, \lambda))$ associated with the boundary equilibria $\lambda = 0.1$ and $\lambda = 0.9$. In comparison to the deterministic market equilibrium at $\lambda = 0.5$, both consumers are better off in expectation if the firm chooses a symmetric lottery over $\lambda = 0.1$ and $\lambda = 0.9$.

In the example the procedure corresponds to the introduction of a veil of ignorance. Before the lottery takes place it is not known whose favorite production ray will be realized. This ex ante viewpoint is appropriate for certain fairness considerations, but appears unnatural in the analysis of the efficiency of equilibria in economies with incomplete markets. The introduction of lotteries does not provide a genuine extension of the Arrow-Debreu-McKenzie theory to the case of incomplete markets. Furthermore, introducing lotteries amounts to making markets more complete. Our goal, however, is to analyze efficiency issues in a model with a given, small set of assets.[5] Since the introduction of lotteries over production plans is difficult to justify on economic grounds and since it changes the nature of the underlying problem in an essential way, we do not think that the use of lotteries lends itself to the present framework.

Majority voting presents another way to overcome the social choice problems faced by shareholders. For properties of corporate control by majority voting, see DeMarzo (1993) and Geraats and Haller (1998). Apart from problems such as equilibrium existence, agenda control, non sincere voting etc., the following point deserves attention. Since the voting outcome depends on power, it need not reflect welfare properly. The point is easily understood in the context of the quasilinear example in Section 3. To break ties, a third quasilinear consumer with arbitrarily small weight is introduced, whose utility increases if the ray λ approaches 0.5. This additional consumer becomes the median voter. Due to symmetry, majority voting leads to $\lambda = 0.5$ if every shareholder has one vote. Moreover, it is not difficult to modify the example such that voting according to

[5] In addition, even if the set of market equilibria is convexified, it differs substantially from the set of constrained efficient allocations. We do not see how one would obtain an analogue to the first welfare theorem.

the one share-one vote rule yields the same outcome. The median voter, although of arbitrarily small weight, has overwhelming power. The median voter's optimal choice, though, is the welfare minimum. Thus, majority voting should be seen as a modelling device that is better suited for positive than for normative purposes.

Instead of examining whether a proposed production plan can be unanimously improved upon after sidepayments are made, one can compare the gains, expressed in units of good 0, that are obtained from any production plan in comparison to a given reference point. In the perturbed quasilinear example the point of zero production, that is to say, the allocation (e^1, e^2) of initial endowments, can be used for reference. Consumer i's surplus $S^i(\lambda)$ is given by the amount of good 0 consumer i needs in excess of e^i to obtain the same utility level as if the firm chose the ray λ. The total surplus $\sum_i S^i(\lambda)$ associated with some market equilibrium can then be maximized. In the perturbed quasilinear example the maximum is taken at both boundary points $\lambda = 0.1$ and $\lambda = 0.9$. Thus, the same outcome as in the quasilinear case is obtained.

A major advantage of this approach lies in the fact that it relies on the maximization of continuous functions rather than maximization of incomplete, intransitive, and nonconvex relations. The surplus maximum is characterized as follows: It presents the minimum amount of good 0 needed in the absence of the firm in order to be able to compensate all consumers such that they can attain every utility profile that is induced by some production decision. Clearly, this type of surplus maximization, which is motivated by the lack of constrained efficient market equilibria, does not aim at achieving constrained efficiency and its theoretical foundation remains controversial.

The surplus function described above can be viewed as a particular social welfare function. To overcome the problem of the nonexistence of constrained or even minimally constrained efficient market equilibria, one might also resort to any other social welfare function. However, it is a priori unclear which welfare function is particularly well suited for this purpose.[6]

A less radical procedure suggesting itself in the perturbed quasilinear example is the choice of the boundary equilibria $\lambda = 0.1$ or $\lambda = 0.9$ on the basis that they are "less inefficient" than, say, $\lambda = 0.5$. To define the degree of inefficiency, interpersonal utility comparisons are not required.

The last three approaches provide welfare oriented methods that may be used to overcome the problem presented in this paper. In each case a particular function is optimized. These approaches require a large amount of information and are far more complex than the usual profit maximization in General Equilibrium Theory with complete markets. They would change the character of the theory considerably.

[6] In another context involving lotteries, Dhillon and Mertens (1999) argue in favor of relative utilitarianism.

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