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Does Democracy Engender Equality?

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The participants in a September 2002 Workshop on *Topics in Mathematical Economics* in honor of Birgit Grodal decided to have a series of papers appear on Birgit Grodal's 60'th birthday, June 24, 2003.

The Institute of Economics suggested that the papers became Discussion Papers from the Institute.

The editor of *Economic Theory* offered to consider the papers for a special Festschrift issue of the journal with Karl Vind as Guest Editor.

This paper is one of the many papers sent to the Discussion Paper series.

Most of these papers will later also be published in a special issue of *Economic Theory.*

<div align="center">Tillykke Birgit</div>

Troels Østergaard Sørensen	Karl Vind
Head of Institute	Guest Editor

"Does democracy engender equality?"

by

John E. Roemer[*]

§1 Introduction

Among types of political system, the one most identified in contemporary western society with the production of justice is democracy. Even on the political left, democracy has largely replaced socialism as the regime desideratum. Just as those socialists who were dissatisfied with aspects of Soviet society claimed that the Soviet regime was not *real* socialism, so those who continue to be dissatisfied with, for example, the American system, now argue that it is not an instance of real democracy. Real democracy is thought to be a political system in which genuine representation of all citizens – and even justice – is achieved.

The identification of democracy with justice is not simply a practice of many political theorists: perhaps the most important aspect of political transformation in the world in the last fifty years has been the toppling of authoritarian regimes, and their replacement with democratic ones. Just as socialism was a powerful movement in the first half of the twentieth century – by 1950, fully one-third of the world's peoples lived under regimes that described themselves as socialist – so democracy has been the

massively appealing political doctrine in the period since 1960.   And as it was an error of socialists to identify socialism with All Good Things, so now it is an error of democrats to identify democracy with All Good Things[1]. The most common example of this fallacy is when some say that regime *X* cannot be a democracy, because it sustains Bad Thing *Y* (oppression of women, abrogation of civil rights, etc.). But if democracy is defined as a set of political institutions, rather than as an ethos, then the correct approach is to study what those institutions entail. Perhaps, for example, both the oppression of women and its absence can co-exist with democracy.

In this article, I undertake a study of this kind and ask whether democracy, understood as a system of political competition between parties that represent different coalitions of citizens, will engender *economic equality*. I focus upon the role of education as an instrument for reducing the differentials in human capital that would otherwise obtain, and ask whether democracy will entail the long-term equalization of human capital through political decisions concerning educational investment.

We model the following society, one which reproduces itself over many generations. At the initial date, there are households led by adults (parents) characterized by a distribution of human capital, that is, capacities to produce income. Each parent has one child. The human capital the child will have, when next period he has become an adult, is a monotone increasing function of his parent's level of human capital and the

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[1] There are many people who identify democracy with justice. For instance, Adolfo Perez Esquivel, a Nobel Peace Prize laureate, recently said, "The vote does not define democracy. Democracy *means* justice and equality. [Italics added –JER] " (*The Daily Journal* [Caracas], July 12, 2001)

amount that is invested in his education. This relationship is deterministic, and describes the educational production function for all children. Thus, more investment is needed to bring a child from a poor (low human capital) family up to a given level of human capital than a child from a richer family. All parents have the same utility function: a parent cares about her household's consumption (that is, her after-tax income), and the earning power her child will have, as an adult. We will, for simplicity, assume that adults do not value leisure.

Educational finance is, until section 5, purely public. The polity of adults, at each date, must make four political decisions: how much to tax themselves, how to split the tax revenues between a redistributive budget for households' current consumption and the educational (investment) budget, how to partition the budget for redistribution among adults, and how to target the educational budget to investment in particular children, according to their type (that is, their parents' human capital). Once these political decisions are implemented, a distribution of human capital is determined for the next generation. When the present children become adults, characterized by that distribution of human capital, they face the same four political decisions. We wish to study the asymptotic distribution of human capital of this dynamic process.

In the society we have described, a child is characterized by the family into which he is born, for his capacity to transform educational investment into future earning power is determined by his family background, proxied by his parent's human capital. We imagine that the transmission of 'culture' to the child is indicated by the parent's human capital endowment. The child's capacity successfully to absorb educational investment, and transform it into human capital, as a circumstance beyond his control, and so a

society of this kind that wished rapidly to *equalize opportunities* for all children would compensate children from poorer families with more educational investment. Equality of opportunity will be achieved when all adults have the same human capital, for that means, as children in the previous generation, the compensation for disadvantageous circumstances was complete[2]. In the real world, equality of opportunity does not require equalizing outcomes in this way, because people may remain responsible for some aspect of their condition, even after the necessary compensation for disadvantage has been made. But in our model there is no such element of personal responsibility, and so, if we take equality of opportunity as our conception of justice, then justice will have been achieved exactly when the wage-earning capacities of all adults are equal.

One might object that it is here sufficient to equalize (post-fisc) *incomes* for justice. But it may well be the case that individuals derive welfare not only from consumption, but from their human capital, and so we insist that this more demanding condition of human-capital equality is the one of interest. Indeed, if one's human capital is an enabler of self-realization, then it is surely the case that justice would require a concern with levels of human capital in a society, not simply income levels.

We will stipulate a democratic process for solving society's political problems, at each generation. Our question becomes: How close will the asymptotic distribution of human capital engendered by this democratic process be to an equal distribution?

The focus of our model will be on that democratic process. We employ a concept of democratic political equilibrium that takes as data the distribution of preferences of the polity over a given policy space, and produces as output an endogenous partition of the

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[2] See Roemer (1998) for the theory of equality of opportunity, based upon social compensation

polity into two political parties, a policy proposal by each party, and a probability that each party will win the election. We suppose that an election occurs, and the policy of the victorious party is implemented. Our procedure will be to begin with a distribution of adult human capital at date 0, which will determine the distribution of adult preferences at date 0, and initialize the stochastic dynamic process.

Although we have described the political choice as consisting of four independent decisions, we will in fact model the political problem as one on an infinite dimensional policy space. That policy space, denoted T, will consist of pairs of real-valued functions (ψ, r) where $\psi(h)$ is the after-tax household income of an adult with human capital h, and $r(h)$ is the public educational investment in a child from a family where the parent has human capital h. These functions are restricted only to be continuous, to jointly satisfy a budget constraint, and to satisfy an upper and lower bound on their derivatives, when the derivatives exist. Thus, the present analysis marks a substantial technical advance over analyses in political economy that must limit their scope to unidimensional policy spaces, or policy spaces of small dimension. I argue that the advance is not merely technical. It is surely artificial to restrict a democratic polity's choice of policies to ones with simple mathematical properties, such as linearity. Our ability to solve the political problem with no such restrictions means that we are able to model the democratic struggle as *ruthlessly competitive*: no holds, in the sense of unmotivated restrictions on the nature of policy proposals, are barred.

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for disadvantageous circumstances.

Indeed, we show below that the analysis on a policy space of large dimension gives qualitatively different results from the standard, uni-dimensional Hotelling-Downs analysis.

The political equilibrium concept is 'party unanimity Nash equilibrium with endogenous parties (PUNEEP)' In two recent articles, I introduced the concept of 'party unanimity Nash equilibrium (PUNE), (Roemer [1999, 1998]). The extension to 'PUNE with endogenous parties' is introduced in Roemer (2001, Chapter 13). The endogenous-party aspect is grafted from a model of Baron (1993).

It is probably fair to say that most articles in political economy propose a relatively sophisticated model of the economy, and a trivial model of politics (standardly, political equilibrium consists in both parties' proposing the median voter's ideal point, or, more generally, a Condorcet winner in the policy space). Our approach here is just the opposite: the economy is very simple, but the politics are quite complex. Our first justification for this complexity is that, without it, we cannot solve the problem of political equilibrium with multi- and even infinite dimensional policy spaces, when Condorcet winners do not exist. Our second justification, for the problem at hand, is that our focus is upon the workings of democracy, and therefore, a careful articulation of democratic institutions is appropriate. Of course, a more highly articulated model of the economy would also be desirable, if tractability were not sacrificed.

In section 2, the definition of political equilibrium that we will use, and a companion concept of quasi-equilibrium, are presented. In section 3, we characterize the policies in the political equilibria of the model. Section 4 studies the dynamics. Section 5 relaxes the assumption that all educational investment is public. Section 6 analyzes the

dynamics of income distribution in a unidimensional, median-voter model. Section 7

presents some simulations of dynamics. Section 8discusses the results, and concludes.

All proofs of propositions are gathered in the appendix.

§2   Party unanimity Nash equilibrium with endogenous parties (PUNEEP)

In this section, I define PUNEEP and a related concept[3].

Let H be a set of voter types, where $h \in H$ is distributed according a to

probability measure $\mathbf{F}$ in the society in question. Let T be a set of policies. There is a

function $v : T \times H \to \mathbf{R}$ which represents the preferences of types over policies; thus

$v(\cdot, h)$ is the utility function of type h on T. For each h, we assume that $v(\cdot, h)$ is a von

Neumann-Morgenstern utility function for lotteries on T.

Let $t^1, t^2 \in T$ be two policies; we next define $\pi(t^1, t^2)$, the probability that policy t^1

defeats policy t^2. Our datum is a function $\pi^* : [0,1] \to [0,1]$, such that $\pi^*(0) = 0, \pi^*(1) = 1$,

and π^* is strictly increasing on $[0,1]$.

Let $\Omega(t^1, t^2)$ be the set of types who prefer t^1 to t^2 and $I(t^1, t^2)$ be the set of types

who are indifferent between t^1 and t^2. Then we define, *pro tem*[4]:

$$\pi(t^1, t^2) = \pi^*(\mathbf{F}(\Omega(t^1, t^2)) + \frac{1}{2}\mathbf{F}(I(t^1, t^2))). \qquad (2.1)$$

In other words, $\mathbf{F}(\Omega(t^1, t^2)) + \frac{1}{2}\mathbf{F}(I(t^1, t^2))$ is the mass of voters who in principle

will vote for t^1 — but perhaps some voters will make mistakes or perhaps $\mathbf{F}$ is measured

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[3] For a more relaxed and carefully motivated presentation of PUNE and PUNEEP, see Roemer (2001, Chapters 8 and 13).

[4] To be modified in Remark 2, below.

imperfectly. Equation (2.1) says that the probability that t^1 defeats t^2 is an increasing function of the 'expected' vote for t^1.

A *party structure* is a partition of H into two elements. We specialize, now, to the case $H = \mathbf{R}_+$, and further specialize by requiring that both elements in a party structure be intervals: thus a party structure is characterized by a pivotal type h^*, with $L = [0,h^*)$ and $R = [h^*,\infty)$. We call the two parties Left (L) and Right (R).

Associated with a party is a utility function, which is the average of its members utility functions.

Thus

$$
\left.
\begin{aligned}
v_L(t) &= \int_0^{h*} v(t,h)d\mathbf{F}(h) \\[2mm]
v_R(t) &= \int_{h*}^{\infty} v(t,h)d\mathbf{F}(h)
\end{aligned}
\right\}
\tag{2.2}
$$

(We drop a multiplicative constant.) The utility functions $v(\cdot,h)$ are assumed to be cardinally measurable and unit comparable (CUC), so that averaging them makes sense.

All parties contain three factions of decision makers: opportunists, reformists, and militants. (These factions are *not* to be identified with particular citizen types. They are, if you will, of measure zero in the population.) Each faction possesses a real-valued *payoff function* defined on $T \times T$. The payoff functions of the three factions in Left are defined by:

$$
\left.
\begin{aligned}
{}^L\Pi^{\mathrm{Opp}}\!\left(t^1,t^2\right) &= \pi\!\left(t^1,t^2\right) \\[2mm]
{}^L\Pi^{\mathrm{Ref}}\!\left(t^1,t^2\right) &= \pi\!\left(t^1,t^2\right)v_L\!\left(t^1\right) + \left(1 - \pi\!\left(t^1,t^2\right)\right)v_L\!\left(t^2\right) \\[2mm]
{}^L\Pi^{\mathrm{Mil}}\!\left(t^1,t^2\right) &= v_L\!\left(t^1\right)
\end{aligned}
\right\}
\tag{2.3}
$$

with an analogous definition for Right's three factions. The three factions are interested, respectively, in winning (opportunists), party-member welfare (reformists), and publicity (militants).

Definition 1. A *party unanimity Nash equilibrium with endogenous parties* (PUNEEP) is a party structure (L,R) given by $L = [0,h^*)$ and $R = [h^*,\infty)$ with $h^* > 0$, and a pair of policies $t^L, t^R \in T$ such that

(A)   there is no policy $t \in T$ such that

$${}^L\Pi^J(t,t^R) \geq {}^L\Pi^J(t^L,t^R), \text{ for } J = O,R,M$$

with at least one of these inequalities strict;

(B)   there is no policy $t \in T$ such that

$${}^R\Pi^J(t^L,t) \geq {}^R\Pi^J(t^L,t^R), \text{ for } J = O,R,M$$

with at least one of these inequalities strict;

(C) $$\begin{aligned} h \in L &\Rightarrow v(t^L,h) \geq v(t^R,h) \\ h \in R &\Rightarrow v(t^R,h) \geq v(t^L,h) \end{aligned}.$$

The three payoff functions of a parties' factions each represent a complete order on $T \times T$. Their intersection represents a quasi–order on $T \times T$. A PUNEEP is a Nash equilibrium of the game played by these two quasi–orders, with the additional requirement (C). Requirement (C) was initially proposed by Baron (1993), in another context, as modeling the stability of a party structure.

Remark 1. It is easily shown that the reformists are gratuitous in definition 1. That is, if we eliminate the reformist factions, we do not alter the set of equilibria. But notice, once this is done, we never need mention *expected* utility, since only the reformists calculate

that. It thus suffices that $\{v(\cdot,h)|h \in H\}$ be a profile of CUC utility functions (i.e., they need not represent preferences over lotteries).

Remark 2. It is now convenient to alter the convention on how voters who are indifferent between the policies they face, vote, in the presence of parties. When parties are present, we will say that a voter who is indifferent between policies votes for the policy of his party. (Recall that each citizen is a member of one party: this is part of the description of a political environment.) Thus, formally, we now *revise* the definition of π to:

$$\pi(t^L,t^R) = \pi^*(\mathbf{F}(\Omega(t^L t^R)) + \mathbf{F}(L \cap I(t^L,t^R))). \qquad (2.1')$$

Remark 3. In Roemer (2001, Chapter 8), it is shown that if sufficient convexity is present, then every PUNEEP can be viewed as the outcome of generalized Nash *bargaining* between the militant and opportunist factions of each party, given the other party's proposal. There is, in general, a two dimensional manifold of PUNEEP. Each one is characterized by specifying the relative bargaining strengths of the two active factions in each party − thus, two positive numbers. Thus, parties compete with each other à la Nash *equilibrium*, while internal factions bargain with each other à la Nash *bargaining*. The PUNEEP concept thus owes its origins doubly to John Nash.

We now further specialize to the case that **F** has a continuous, strictly increasing distribution function, F, on $\mathbf{R}_+$.

Remark 4. There is another story (besides the three-faction story) that leads to the same equilibrium concept. Suppose there are *two* factions of decision makers in a party -- the opportunists, and those whom we may call the *guardians*. The opportunists (as above) wish to maximize the probability of their party's winning. The guardians insist that the

policy chosen guarantee at least some given level of utility to their constituents: the

guardians in party *L*, for instance, require that the inequality $v_L(t^L) \geq k_L$ be satisfied, for

some number k_L. Subject to this constraint, and facing the opposition's policy, the

opportunists choose a policy to maximize the probability of victory.

The reader can easily observe that the guardians are behaving just like the

militants, in the original formulation, and it therefore follows from *Remark* 1 that

PUNEEPs are exactly equilibria of this model, as well.

The opportunist-guardian story is really one of *bounded rationality*. There are no

longer any reformists or any militants. Reformism, after all, requires sophisticated

preferences, ones which can rank not only policies, but lotteries over policies. Hence,

the story of this *Remark* may be a more appealing way to 'rationalize' the PUNEEP

concept. The multiplicity of PUNEEPs is now seen to be associated with various values

of the ordered pair (k_L, k_R). The two-dimensionality of the equilibrium manifold is again

apparent.

We next define an auxiliary notion that is useful in the analysis.

Definition 2. A *quasi-PUNE* is an ordered pair $(h^*, y) \in H \times \mathbf{R}$ and a pair of policies

$t^L, t^R \in T$, such that $v(t^L, h^*) = y = v(t^R, h^*)$ and:

2A. t^L solves

$$\max \int_0^{h^*} v(t,h)\, d\mathbf{F}(h)$$

subject to

$$t \in T$$

$$h \in [0, h^*) \Rightarrow v(t,h) \geq v(t^R, h) \tag{L0}$$

$$v(t,h^*) \geq y \qquad \text{(L1)}$$

2B. t^R solves

$$\max \int_{h^*}^{\infty} v(t,h)\, d\mathbf{F}(h)$$

subject to

$$t \in T$$

$$h \in [h^*,\infty) \Rightarrow v(t,h) \geq v(t^L,h) \qquad \text{(R0)}$$

$$v(t,h^*) \geq y \qquad \text{(R1)}$$

2C.   Constraints (L1) and (R1) bind at t^L and t^R respectively.

We have:

Proposition 1. Let v be continuous in h. If (t^L,t^R,h^*) is a PUNEEP, then (t^L,t^R,h^*,y) is

a quasi-PUNE, with $y = v(t^L,h^*)$.

Proof: See appendix.

The converse of Proposition 1 is not true: there may be quasi–PUNEs that are not

PUNEEPs. For if (t^L,t^R,h^*,y) is a quasi-PUNE, it is possible that there exists a policy t

which improves the payoff of both Left's militants and opportunists, by assembling a set

of voters who favor t over t^R that is disconnected and does not contain h^*.

We can give an alternative, conceptual definition of quasi-PUNE, which may be

helpful. Consider the definition of PUNEEP (definition 1); let (h^*,t^L,t^R) be a candidate

PUNEEP. Consider the opportunists in Left who are 'testing' the policy t^L: they are

searching for a policy that gives them a greater probability of victory than $\pi(t^L,t^R)$ and

delivers at least the same average utility to L's members as does t^L. If they find such a

policy, then (h^*,t^L,t^R) is not a PUNEEP. Indeed, there may be such a policy, call it t^*,

and the set of types that favor t^* over t^R might not include L.   This would still show that

(h^*, t^L, t^R) is not a PUNEEP. Now suppose that we *restrict* the opportunists of L to

looking for policies t that increase their probability of victory when played against t^R, and

do not lower the average utility of L, *and such that* the coalition that favors t over t^R

includes L.   (We similarly require the analogous additional condition of Right's

opportunists.) If (h^*, t^L, t^R) survives this stronger deviation test, then it is (precisely) a

quasi-PUNE. Because the test for quasi-PUNE is stricter than the test for PUNEEP, the

set of quasi-PUNEs includes the set of PUNEEPs.

We can now give a preview of our strategy. In our politico-economic

environment, we can *fully characterize* the set of quasi-PUNEs: the important and useful

fact is that we can compute quasi-PUNEs without recourse to fixed point theorems, using

only optimization methods. We will further note that the set of PUNEEPs is a

non−empty subset of the set of quasi-PUNEs. We then conduct our dynamic analysis

assuming that each generation's political equilibrium is some quasi-PUNE. Whatever we

conclude will hold *a fortiori* for societies whose political equilibria are genuine ones, that

is, PUNEEPs. In this manner we avoid ever having to solve the intractable problem of

characterizing precisely the set of PUNEEPs.

§3   Equilibrium at one date

Throughout this section, we analyze the society at one date.

A. The politico−economic environment

(i) Preferences

A typical society, in our problem, consists of a continuum of adult types, each characterized by her human capital h, where h is distributed according to a probability measure $\mathbf{F}$, whose mean is denoted μ, and whose support is the positive real line. We denote the distribution function (CDF) of $\mathbf{F}$ by F. Each adult has one child. Adults care about their household's consumption, and their child's (future) human capital.

We assume:

$$u(x, h') = \log x + \gamma \log h', \tag{3.1}$$

where x is the household's consumption, or after-tax income, and h' is the child's (future) human capital. Zero consumption is minimal household consumption. Note there is no preference for leisure[5].

(ii)   Technology

If r is invested in the education of a child whose parent is of type h then the child's future human capital will be

$$h' = \alpha h^b r^c \tag{3.2}$$

where α, b, c are positive constants[6].

b is the elasticity of child's human capital w.r.t. parental human capital and c is the elasticity of child's human capital w.r.t. educational investment. Think of the influence of the parent's human capital on the child's human capital as operating through

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[5] As I noted in the introduction, we may think of adults as caring about their own human capital as well as their income. But since their human capital is fixed by the time they make decisions, we need not include the utility it generates in the utility function. A utility function of this type is used elsewhere -- see, for example, Banerjee and Newman (1993) and Zalor and Zeira (1993).

[6] A production function of this type is employed in Becker and Tomes (1986), who use the term 'familial cultural infrastructure,' which I shorten to 'family culture.'

family culture, or perhaps neighborhood effects (if neighborhoods are income-

segregated).   Bénabou (in press) uses a relationship like (3.2), and gives some weak

evidence that $b+c<1$; we will, however, study the case $b+c=1$, for reasons explained

below, and even the case $b+c>1$. We refer to these cases as ones of decreasing ,

constant, and increasing returns to scale, respectively.

If an adult of type h works at her full potential then her (pre-tax) earnings are h.

Thus human capital is measured in units of income-earning capacity.

(iii)   The policy space

Let C be the space of continuous functions on the domain $\mathbf{R}_+$. A *policy* is a pair

of functions $(\psi, r) \in C \times C$ such that

$$\int_0^\infty (\psi(h)) + r(h)) \, d\mathbf{F}(h) \leq \mu \qquad (3.3a)$$
$$\text{for all } h, \; \psi'(h) + r'(h) \geq 0, \qquad (3.3b) \qquad ,$$
$$\psi'(h) + r'(h) \leq 1, \qquad (3.3c)$$

where 'prime' indicates derivative, and the inequalities are meant to hold where the

derivatives exist. The interpretation is that $\psi(h)$ is the after-tax income of an adult of

type h, and $r(h)$ is the public educational investment in a child from an h-family. We

call $\psi(h) + r(h) \equiv X(h)$ the *total resource bundle* allocated to an h household, so (3.3b,c)

restrict the rates at which the total resource bundle changes with h. We may think of

(3.3b,c) *social norms*, as they are not motivated by political competition or incentive

compatibility considerations[7].

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[7] There is a natural incentive compatibility condition, that adult utility be non-decreasing

in h, so that no adult would have an incentive to work at a lower income-earning capacity

than her true capacity. The local version of this condition is:

Thus, our policy space is $T = \{(\psi, r) \in C \times C \mid (3.3a,b,c) \text{ hold}\}.\backslash$

Our results will be sensitive to (3.3c), at least. We must put upper and lower

bounds on the derivatives of the total resource function X; without out, there will be no

equilibria in our model with continuous policies. We justify placing the upper bound of

one (in (3.3c)) on X' with the observation that, in a laissez-faire regime, with no

taxation, we have $X(h) = h$, and so $X'(h) = 1$. Thus the laissez-faire policy is on a

boundary of our policy space, and assumption (3.3c) says that distributional policy

should not be more regressive than laissez-faire. (Compare with the ubiquitous

Hotelling-Downs model, with affine income tax policies, in which the marginal tax rate is

restricted to be in the interval [0,1]. This is the analogue of our constraints (3.3b and c).)

We assume that all educational investment is public[8]. A policy (ψ, r) specifies

decisions on the four political problems described in Section 1.

Thus the indirect utility function $v{:}T \times H \to \mathbf{R}$ is given by

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$$\frac{\psi'(h)}{\psi(h)} + \gamma c \frac{r'(h)}{r(h)} \geq 0. \quad (3.5)$$

Some might prefer to substitute (3.5) for (3.3b) in the model, but doing so renders the

analysis below much more difficult: it converts what will be a convex optimization

problem on an infinite-dimensional space to a difficult, non-convex problem. In the

interests of simplicity, and not diffusing attention from our main concern, we use (3.3b)

in lieu of (3.5). We conjecture, however, that the results we report would remain the

same if (3.3b) were replaced with (3.5).

[8] This assumption is relaxed below in section 5.

$$v(\psi, r, h) = \log \psi(h) + \gamma \log \alpha h^b r(h)^c$$
$$= \log \psi(h) + \gamma \log \alpha h^b + \gamma c \log r(h) \qquad (3.4)$$
$$\neq \log \psi(h) + \gamma c \log r(h)$$

where, in the last line of (3.4), we have dropped a gratuitous constant term.

B. Quasi-PUNEs

For a given point $(h^*, y) \in \mathbf{R}_+ \times \mathbf{R}$, consider the following two programs:

$$
\left.
\begin{array}{l}
\displaystyle \max_{(\psi, r) \in C^2} \quad \int_0^{h^*} \log(\psi(h) + \gamma c \log r(h)) \, d\mathbf{F}(h) \\[2mm]
\text{s.t.} \\[1mm]
0 \le \psi'(h) + r'(h) \le 1 \qquad\qquad (3.61) \\[2mm]
\displaystyle \int_0^{\infty} (\psi(h) + r(h)) \, d\mathbf{F}(h) \le \mu \qquad (3.62) \\[2mm]
\log \psi(h^*) + \gamma c \log r(h^*) \ge y \qquad (3.63)
\end{array}
\right\} \qquad (3.6)
$$

$$
\left.
\begin{array}{l}
\displaystyle \max_{(\psi, r) \in C^2} \quad \int_{h^*}^{\infty} \log(\psi(h) + \gamma c \log r(h)) \, d\mathbf{F}(h) \\[2mm]
\text{s.t.} \\[1mm]
0 \le \psi'(h) + r'(h) \le 1 \qquad\qquad (3.71) \\[2mm]
\displaystyle \int_0^{\infty} (\psi(h) + r(h)) \, d\mathbf{F}(h) \le \mu \qquad (3.72) \\[2mm]
\log \psi(h^*) + \gamma c \log r(h^*) \ge y \qquad (3.73)
\end{array}
\right\} \qquad (3.7)
$$

Let (h^*, y) be such that solutions (ψ^L, r^L) and (ψ^R, r^R) exist to (3.6) and (3.7), respectively, and such that inequalities (3.63) and (3.73) bind at the solutions. We will show that the following hold:

1. $0 \le h \le h^* \Rightarrow v(\psi^L, r^L, h) \ge v(\psi^R, r^R, h)$, and
2. $h^* \le h \le \infty \Rightarrow v(\psi^R, r^R, h) \ge v(\psi^L, r^L, h)$.

It will follow that (ψ^L, r^L) and (ψ^R, r^R) constitute a quasi-PUNE at (h^*, y), and

that solutions of programs (3.6) and (3.7) at which (3.63) and (3.73) bind comprise

precisely the quasi-PUNEs for our problem.

Our first task is to characterize the set

$\Gamma = \{(h^*, y) \in \mathbf{R}_+ \times \mathbf{R} \mid \text{solutions to (3.6) and (3.7) exist at which (3.63) and (3.73) bind}\}.$

Note that $T = \{(\psi, r) \in C \times C \mid \text{(3.61) and (3.62) hold}\}.$ For fixed h^*, consider the

following three programs:

$$\left.\begin{array}{c} \max \int_0^{h^*} v(\psi, r; h)\, d\mathbf{F}(h) \\ (\psi, r) \in T \end{array}\right\} \tag{3.8}$$

$$\left.\begin{array}{c} \max \int_{h^*}^{\infty} v(\psi, r; h)\, d\mathbf{F}(h) \\ (\psi, r) \in T \end{array}\right\} \tag{3.9}$$

$$\left.\begin{array}{c} \max v(\psi, r; h^*) \\ (\psi, r) \in T \end{array}\right\} \tag{3.10}$$

Let their solutions be denoted τ^L, τ^R, and τ^*, respectively, where $\tau = (\psi, r)$ is the generic

policy. Let $y^*(h^*)$ be the value of program (3.10), i.e.

$$y^*(h^*) = v(\tau^*, h^*),$$

and define

$$y^L(h^*) = v(\tau^L, h^*), \qquad y^R(h^*) = v(\tau^R, h^*).$$

We have:

Proposition 2. For h^* given, $(h^*, y) \in \Gamma$ iff

$$\max[y^L(h^*), y^R(h^*)] \le y \le y^*(h^*), \tag{3.11}\textbackslash$$

or $\qquad \Gamma = \left\{(h^*, y) \big| \max[y^L(h^*), y^R(h^*)] \le y \le y^*(h^*)\right\}.$

Proof: See appendix.

The virtue of the quasi-PUNE notion is now evident: we can characterize the set Γ, and thus the set of quasi-PUNEs, by solving the three optimization programs (3.8), (3.9), and (3.10). No fixed-point machinery is needed to do this.

We next solve these three programs.

Proposition 3.   Let (ψ, r) *be a solution to* (3.6), (3.7), (3.8), (3.9), *or* (3.10).

Then

$$r(h) = \gamma c\,\psi(h)\ = \frac{\gamma c}{1 + \gamma c} X(h). \tag{3.12}$$

Lemma. Let X be the total resource dedicated to household h. Then the household's optimal distribution of X between consumption ψ *and educational investment r is*

$$\psi = \frac{1}{1 + \gamma c} X$$

$$r = \frac{\gamma c}{1 + \gamma c} X.$$

Proof: The household would choose consumption ψ to maximize its utility, which leads immediately to the claim.   ∎

Proof of Proposition 3: See appendix.

Remark 5. If we replaced social norm (3.3b) with incentive compatibility (3.5),

Proposition 3, although probably true, is much more difficult to prove. It is for this

reason that we employ (3.3b).

Recall that $X(h) = \psi(h) + r(h)$. By substituting from (3.12), we can reduce

(3.8), (3.9), and (3.10) to the following three programs:

$$
\left.
\begin{aligned}
&\max_{X \in C} \quad \int_0^{h^*} \log X(h)\, d\mathbf{F}(h) \\
&\text{s.t.} \\
&\int_0^\infty X(h)\, d\mathbf{F}(h) \le \mu \\
&0 \le X'(h) \le 1.
\end{aligned}
\right\}
\tag{3.8a}
$$

$$
\left.
\begin{aligned}
&\max_{X \in C} \quad \int_{h^*}^\infty \log X(h)\, d\mathbf{F}(h) \\
&\text{s.t.} \\
&\int_0^\infty X(h)\, d\mathbf{F}(h) \le \mu \\
&0 \le X'(h) \le 1,
\end{aligned}
\right\}
\tag{3.9a}
$$

and

$$
\left.
\begin{aligned}
&\max_{X \in C} \quad \log X(h^*) \\
&\text{s.t.} \\
&\int_0^\infty X(h)\, d\mathbf{F}(h) \le \mu \\
&0 \le X'(h) \le 1.
\end{aligned}
\right\}
\tag{3.10a}
$$

In other words, Proposition 3 enables us replace optimization problems on $C \times C$ with

optimization problems on C.

We have:

Proposition 4.

a. *The solution to* (3.8a) *is*

$$X^L_{h^*}(h) \equiv \mu;$$

b. *The solution to* (3.9a) is

$$X^R_{h^*}(h) = \begin{cases} x + h, \text{ for } h \le y \\ x + y, \text{ for } h > y \end{cases}$$

where (x,y) solves the following two simultaneous equations:

$$x + \int_0^y h dF(h) + y(1 - F(y)) = \mu \qquad (3.13a)$$

$$\int_{h^*}^y \frac{dF(h)}{h + x} = \frac{F(y)}{x + y} \quad . \qquad (3.13b)$$

We have $x > 0$. The solution is illustrated in Figure 1.

c. *The solution to* (3.10a) *is illustrated in Figure 1. It is given by*

$$X_{h^*}(h) = \begin{cases} X^*_0 + h, & \text{if} & 0 \le h \le h^* \\ X^*_0 + h^* & \text{if} & h > h^* \end{cases},$$

where X^*_0 *is the solution of the equation*

$$X^*_0 + \int_0^{h^*} h \, d\mathbf{F}(h) + h^*(1 - \mathbf{F}(h^*)) = \mu. \qquad (3.14)$$

The solution is illustrated in Figure 1.

Proof: See appendix.

Here is an intuitive argument for part of Proposition 4.

Consider program (3.8a). The *benefit* to household h is $\log X(h)$; the *cost* (to the optimizer) of supplying household h is $X(h)$; hence the benefit-cost ratio, $\dfrac{\log X(h)}{X(h)}$, is non-increasing in h, because $X'(h) \geq 0$ is required. So the optimizer should give as much of the resource as possible to low h: front-loading, so to speak. The binding constraint is $X'(h) \geq 0$: so the planner allocates $X(h) \equiv \mu$.

Second, consider (3.10a). Clearly it is a waste to give any resource to $h > h^*$, so we must have

$$X(h) = X(h^*) \text{ for } h > h^*.$$

Now the optimizer wants to minimize what goes to $[0,h^*)$, conditional upon reaching a high value at h^*, so X should descend rapidly (at rate 1) to the left of h^*. The stated function X_{h^*} makes the value at h^* as large as possible.

Claim (3.9a) is harder to motivate, and so we do not do so.

In like manner using Proposition 3, we can reduce programs (3.6) and (3.7) to:

$$
\left.
\begin{aligned}
&\max_{X \in C} \quad \int_0^{h^*} \log X(h)\, d\mathbf{F}(h) \\
&\text{s.t.} \\
&0 \leq X'(h) \leq 1 \\
&\int_0^{\infty} X(h)\, d\mathbf{F}(h) \leq \mu \\
&\log X(h^*) \geq y, \qquad\quad (3.63a)
\end{aligned}
\right\}
\qquad (3.6a)
$$

and

$$\left.\begin{array}{l} \max_{X \in C} \int_{h^*}^{\infty} \log X(h) d\mathbf{F}(h) \\ \text{s.t.} \\ 0 \le X(h) \le 1 \\ \int_0^{\infty} X(h) d\mathbf{F}(h) \le \mu \\ \log X(h^*) \ge y, \qquad (3.73a) \end{array}\right\} \qquad (3.7a)$$

where $\quad y = \log(1 + \gamma c) + \dfrac{y - \gamma c \log \gamma c}{1 + \gamma c}$.

Of course, the analogous result to Proposition 2 holds, that is:

Proposition 2a. Let $\Gamma = \{(h^*, y) \in \mathbf{R}_+ \times \mathbf{R} \mid$ (3.6a) and (3.7a) have solutions at which

(3.63a) *and* (3.73a) *bind*$\}$. *Define*

$$y^L(h^*) = \log X_{h^*}^L(h^*)$$
$$y^R(h^*) = \log X_{h^*}^R(h^*)$$
$$y^*(h^*) = \log X_{h^*}(h^*)$$

Then

$$\max [y^L(h^*), y^R(h^*)] \le y \le y^*(h^*). \qquad (3.11a)$$

Conversely, if (3.11a) *holds, then* $(h^*, y) \in \Gamma$.

Proof: As in Proposition 2.

Γ is our parameterization of the set of quasi-PUNEs associated with the 'reduced'

problem, where we work with the total-resource bundle function, *X.* From consideration

of the three programs (3.8), (3.9), and (3.10), it is clear that the interval of admissible

values y is non-empty for every h^*.

In figure 2, we illustrate the manifold Γ . Although the picture is not accurate, it

is the case that quasi-PUNEs exist for every $h^*>0$.

We next derive what the quasi-PUNE looks like at $(h^*, y) \in \Gamma$.

Proposition 5. Let $(h^*, y) \in \Gamma$. Then:

a.   *The solution to* (3.6a) *is illustrated in Figure* 3. *It is defined by:*

$$X^L(h) = \begin{cases} X_0^L & , \quad 0 \le h \le h_L \\ X_0^L + (h - h_L) & , \quad h_L \le h \le h^* \\ e^y & , \quad h > h^* \end{cases}$$

where (X_0^L, h_L) is the simultaneous solution of the two equations:

$$\log(X_0^L + (h^* - h_L)) = y, \qquad (3.15a)$$

$$X_0^L + \int_{h_L}^{h^*} (h - h_L) \, d\mathbf{F}(h) + (1 - F(h^*))(h^* - h_L) = \mu. \qquad (3.15b)$$

We have $X_0^L > 0$.

b.   *The solution of* (3.7a) *is illustrated in Figure* 3.   *It is defined by:*

$$X^R(h) = \begin{cases} X_0^R + h, & 0 \le h \le h_R \\ X_0^R + h_R, & h > h_R \end{cases}$$

where (X_0^R, h_R) *is the simultaneous solution of*:

$$\log(X_0^R + h^*) = y, \tag{3.15c}$$

$$X_0^R + \int_0^{h_R} h \, d\mathbf{F}(h) + (1 - F(h_R))h_R = \mu. \tag{3.15d}$$

We have $X_0^R > 0$.

Proof: See appendix.

Propositions 4 and 5 completely characterize the manifold of quasi-PUNEs.

We have one item left to check: that every member of each party weakly prefers her party's policy to the other party's policy. This claim is easy to verify. Indeed, from figure 3 we see that the total resource bundle functions of the two parties coincide on the interval $[h_L, h^*]$ of types, and indeed, each member of a party weakly favors her party's policy to the other's.

The two educational investment functions are just multiples of the functions graphed in Figure 3, for according to Proposition 3, if $X(\cdot)$ is proposed by either party in a quasi-PUNE, then

$$r(h) = \frac{\gamma c}{1 + \gamma c} X(h), \text{ and}$$

$$\psi(h) = \frac{1}{1 + \gamma c} X(h).$$

From Figure 3, we see that, any quasi-PUNE, the Left policy is more egalitarian than the Right policy. Furthermore, each is more egalitarian than the laissez-faire policy, a fact we state as:

Proposition 6. Let b+c=1. Let $h^2 < h^1$ be two levels of human capital, and let h^{iJ} be the human capital level of the child of h^i, for i=1,2 in regime J, where J can be L,R, or lf, standing for Left victory, Right victory, or laissez-faire. (The L and R policies are associated with a given quasi-PUNE.) *Then:*

$$\frac{h^{1L}}{h^{2L}} \le \frac{h^{1R}}{h^{2R}} < \frac{h^{1lf}}{h^{2lf}} = \frac{h^1}{h^2}.$$

Proof: See appendix.

Proposition 6 tells us that, under either Left or Right victory in a democracy, there is an equalizing effect on the dispersion of human capital from one generation to the next, when there are constant returns to scale. If $b+c < 1$, the equalizing effects of Left and Right policies are only magnified.

C.   Existence of PUNEEP

We know every PUNEEP is a quasi-PUNE. We now show that the set of PUNEEPs is non-empty. To do so, we compute the PUNEEP where each party plays the ideal policy of its militants.

Let h^* be any type, and let $L = [0, h^*)$ and $R = [h^*, \infty)$. Let each party play the ideal policy of its militants. We have denoted these policies X_{h*}^L and X_{h*}^R. This is clearly a PUNE because the militant factions will not deviate to any other policy. It will,

however, generally not satisfy the endogenous party constraint [Definition 1, part (C)]. It *will* satisfy that constraint exactly when $X_{h*}^L(h^*) = \mu = X_{h*}^R(h^*) = x(h^*) + h^*$, where $x(h^*)$ is the number x that solves equation (3.13b). Thus, there is a PUNEEP where both parties play the ideal policies of their militant factions at h^* when the triple (x,y,h^*) solves equations (3.13a), (3.13b), and

$$x + h^* = \mu,$$

simultaneously.

It would be distracting to show that such a solution exists for any F. Without proving this, we simply display the solution for F taken as the lognormal distribution with mean 40 and median 30: it is

$$h^* = 39.3739, \quad x(h^*) = .626149, \quad y(h^*) = 171.38.$$

Unfortunately, this is the only PUNEEP whose existence is easy to prove, because it is trivial to observe that neither parties' militants will accept any deviation.

Hence the set of PUNEEP is non-empty. I conjecture that the set of PUNEEPs is indeed a 2-manifold in the set of quasi-PUNEs, but verifying that conjecture is beyond this paper's scope.

§4   Democratic dynamics

A. Introduction

We now imagine a sequence of overlapping generations, at dates $t = 0, 1, ...$ The probability distribution of adult wages at date 0 is $\mathbf{F}^0$. Political competition is organized over the questions of taxation and educational investment, and a PUNEEP is realized, inducing a policy lottery. One party wins the election, and its educational investment policy is implemented, giving rise to a distribution of wages at date 1, $\mathbf{F}^1$. This process continues forever, inducing a sequence $\{\mathbf{F}^t\}$ of wage distributions. We are interested in properties of the asymptotic distribution of human capital.

Over time, it is not reasonable to suppose that α remains constant. We therefore denote its time-dated value by α^t. Indeed, α includes the effect of capital stock on wages, which is not modeled explicitly. As we are assuming that policy has no effect on labor supply, we similarly assume that it has no effect on capital investment; thus, we may take the sequence $\{\alpha^t\}$ to be exogenously given.

Let μ^t be the average human capital at date t. The coefficient of variation (CV) of $\mathbf{F}^t$ is

$$\eta^t = \left(\int (\frac{h}{\mu^t} - 1)^2 \, d\mathbf{F}^t(h) \right)^{\frac{1}{2}}. \tag{4.1}$$

We are, in particular, interested in the limit of $\{\eta^t\}$. Does it exist, and if so, is it positive or zero? If it is zero, we say that the distribution of human capital *converges to equality.* (In that case, given any pair of dynasties, the ratio of levels of human capital of their representatives tends to one with time.)

B. Laissez-faire

Under laissez-faire, $X(h) = h$. The optimizing parent divides her income between household consumption and investment as follows, from the lemma:

$$r(h) = \frac{\gamma c}{1 + \gamma c} X(h), \quad \psi(h) = \frac{1}{1 + \gamma c} X(h).$$

Consequently, her child has human capital

$$h' = \alpha(\frac{\gamma c}{1 + \gamma c})^c h^{b+c}.$$

Suppose that $b + c = 1$. Then it follows that the distribution of human capital tomorrow is identical to the distribution today: all human capitals are multiplied by a constant. Consequently, the coefficient of variation of human capital is constant across time.

If $b + c < 1$, then it follows immediately from the above formula that the ratio of human capitals in any two dynasties approaches one over time, and hence the coefficient of variation of human capital approaches zero.

If $b+c > 1$, then the ratio of human capitals in any two dynasties approaches infinity, and the coefficient of variation explodes.

C. Democracy

We will first work with an altered sequence of distributions, normalized to maintain the mean constantly at μ^0. Define the distribution function

$$F^t(h) = F^t(\frac{\mu^0}{\mu^t} h), \tag{4.2}$$

and let $\mathbf{F}^t$ be the associated probability measure. Then the mean of $\mathbf{F}^t$ is μ^0. Since $\mathbf{F}^t$ has the same coefficient of variation as $\mathbf{F}^t$, it will suffice to study the coefficients of variation of the sequence $\{\mathbf{F}^t\}$.

Proposition 7. Let $b + c \leq 1$. *Then*

(*a*) *then the distribution function* F^{t+1} *cuts the distribution function* F^t *once from below. That is,*

$$(\exists h')(0 < h < h' \Rightarrow F^{t+1}(h) < F^t(h) \text{ and } h > h' \Rightarrow F^{t+1}(h) > F^t(h)).$$

(b) *The sequence* $\{\eta^t\}$ *is monotone decreasing, and hence converges.*

Proof. See appendix.

We have noted that if $b+c < 1$, then, under laissez-faire, the distribution of human capital converges to equality. It is easy to show that the same thing happens under democracy, regardless which quasi-PUNE is realized at each date: this follows from the argument in Proposition 6, which shows that the coefficient of variation under democracy decreases at least as fast as under laissez-faire.   Thus to compare the performance of democracy and laissez-faire, with regard to equality, for the case $b+c < 1$, would require comparing speeds of convergence, a delicate undertaking.   Rather than attempting this, we will study, instead, the asymptotic properties of the democratic regime under the assumption that $b+c=1$, for we know that under laissez-faire, there is no change in the CV over time in this case. Thus, laissez-faire provides a clean bench-mark in the case of constant returns.

We will study the asymptotic distribution of human capital for certain special sequences of quasi-PUNEs. Fix a type *h*>0*, and define the sequence *A(h*)* as those quasi-PUNEs which lie on the lower boundary of the manifold Γ^t at each date *t,* and the pivotal type h_t^* which demarcates the partition of the type space into the two parties is the *t*[th] descendent of *h*.* Let *B(h*)* be that sequence of quasi-PUNEs which, at each date *t,*

lie on upper boundary of the manifold Γ^t, and where h_t^* is the tth descendent of h^*. See

figure 2.

Theorem 1. *Let b+c=1. For any h*>0,* the limit CV of the distribution of human

capital for the sequence A(h*) is zero, and the limit CV of distribution of human capital

for the sequence B(h*) is positive.

The sequence $B(h^*)$ is associated with quasi-PUNEs at which both parties play the ideal

policy of type h_t^* : these are quasi-PUNEs where the opportunists in the two parties are

all-powerful. To see this, note that the upper envelope of the manifold of quasi-PUNEs,

illustrated in Figure 2, is associated with pairs (h^*, y) at which the right-hand inequality

of (3.11a) is binding. This means that both parties are playing the ideal policy of type h^*:

the parties are as far as possible from satisfying their militants.   In the sequence $A(h^*)$, in

contrast, the militants in the two parties are powerful: in at least one party, at those quasi-

PUNEs, the party plays the ideal policy of its militants, and in the other party, the

militants are as strong as they can be. These quasi-PUNEs are associated with points

(h^*, y) at which the left-hand inequality in (3.11a) is binding.

Note that, if F^t is the CDF of the distribution of types at date *t,* for either the

sequence $A(h^*)$ or $B(h^*)$, then $F^t(h_t^*)$ is constant for all *t,* because the members of a

dynasty occupy the same rank in their respective type-distributions forever. Since the

probability of Left victory at a quasi-PUNE in either of these sequences is $\pi^*(F^t(h_t^*))$,

this probability is a positive constant over time, in the open interval (0,1). Therefore, in

both sequences of quasi-PUNEs, each of Left and Right win elections an infinite number of times. We use this fact below.

Proof of Theorem: See appendix.

§5 Topping Off

We have assumed until now that educational funding is purely public. But winning publicly financed education has been itself a significant victory of democracy. So it would have been more convincing to begin with the supposition that education could be privately or publicly financed.

First, note that at any quasi-PUNE, under our assumptions, no household will desire to top off public education with additional private education, because every quasi-PUNE partitions the household's total resource bundle just as the optimizing household would. So there will be no demand for further private education at these equilibria.

Now suppose that it is not assumed, initially, that education will be publicly financed. Thus, a party may propose a policy (ψ, r) assuming that citizens will top off privately, if $r(h) < \dfrac{\gamma c}{1 + \gamma c} X(h)$. Thus, the h-household facing the policy $(\psi(h), r(h))$ solves for its private educational investment, which we denote $r^P(h)$:

$$\underset{r^P}{Max} \log(\psi(h) - r^P) + \gamma c \log(r(h) + r^P).$$

The solution is

$$r^P(h) = \frac{\gamma c \psi(h) - r(h)}{1 + \gamma c}.$$

Then

$$\psi(h) - r^P(h) = \frac{X(h)}{1 + \gamma c}, \text{and}$$

$$r^P(h) + r(h) = \frac{\gamma c X(h)}{1 + \gamma c}.$$

Without loss of generality, we may therefore write the household's indirect utility

function as

$$v(\psi, r; h) = \log \frac{X(h)}{1 + \gamma c} + \gamma c \log \frac{\gamma c X(h)}{1 + \gamma c} \cong \log X(h).$$

Now each party takes account of the fact that its members will top off, if need be, and so

we may write the program of the Left party (for instance) at h^* as:

$$\underset{z(\cdot)}{Max} \int_0^{h^*} \log X(h) \, d\mathbf{F}(h)$$

$$s.t. \quad 0 \le X'(h) \le 1$$

$$\int_0^\infty X(h) \, d\mathbf{F}(h) \le \mu$$

$$\log X(h^*) \ge y^*.$$

It is thus clear that the set of quasi-PUNEs where private financing of education is not

precluded is isomorphic to the set of quasi-PUNEs where only public financing is

possible. It is a matter of indifference whether education is publicly funded or whether

households finance some or all education privately: the children receive identical

educational investments in both cases.

In other words, our model is not constructed to elucidate why publicly financed

education is an almost ubiquitous institution of advanced democracies.

§6 Democratic dynamics with a unidimensional policy space

In this section we study the dynamics of the distribution of human capital under

'median voter politics,' in order to compare what occurs to the results of section 4.

To this end, suppose that the political problem is to choose a total-resource-bundle function $X(h)$, where X is an *affine* function satisfying constraints:

$$0 \leq X'(h) \leq 1$$

$$\int X(h)\,dF(h) = \mu.$$

It is understood that the partition of X into consumption and educational investment will be given by:

$$\psi(h) = \frac{1}{1 + \gamma c} X(h)$$

$$r(h) = \frac{\gamma c}{1 + \gamma c} X(h).$$

Using the budget constraint we may write:

$$X(h) = ah + (1-a)\mu,$$

and thus the policy space is the unidimensional interval $a \in [0,1]$. The indirect utility function of voter h on the policy space is

$$v(a;h) = Log(ah + (1-a)\mu),$$

which is single-peaked (in a). Consequently, there is a unique Condorcet winner, the ideal policy of the voter with median human capital. It follows that the political equilibrium under Hotelling-Downs politics[9] is that both candidates (or parties) propose the policy:

$$a = \begin{cases} 1, & \text{if } m > \mu \\ 0, & \text{if } m < \mu \end{cases}$$

where m is the median of F.

Suppose $b+c = 1$. If $m > \mu$, then the political equilibrium is laissez-faire, and it follows that the coefficient of variation is constant forever, since the distribution simply reproduces itself forever (*modulo* a constant multiplicative factor).

On the other hand, if $m^0 < \mu^0$ (we are now time-dating the distributions), then

$r(h) = \dfrac{\gamma c}{1 + \gamma c} \mu^0$ is the investment function at the Condorcet winner at date 0. It follows that the human capital of the child of h will be:

$$S(h) = k_0 h^b,$$

where $k_0 = \alpha(\dfrac{\gamma c}{1 + \gamma c} \mu^0)^c$. At date 1, then, the mean of the new distribution is

$$\mu^1 = k_0 \int h^b dF^0(h),$$

and its median is:

$$m^1 = k_0 m^{0^b},$$

since the median of the new distribution is the child of the median of the earlier distribution.

If $m^1 < \mu^1$ then again the Condorcet winner is the policy $X(h) = \mu^1$, and

$S_2(h) = k_1 h^{b^2}$ is the human capital of the grandchild of h.

It therefore follows that the coefficient of variation of human capital converges to zero if and only if $m^t < \mu^t$ for all t. For if it ever happens that $m^t > \mu^t$, then the CV is constant from that time onwards. If $m^t < \mu^t$ is always true, it is obvious from the formula for $S_t(h)$ that the ratio of human capitals of any two dynasties approaches unity.

We now have:

Theorem 2. *Let b+c=1. Let F be the date* 0 *distribution of h, with median m and mean*

μ. *Under median-voter politics, on the unidimensional policy space, the* CV *of the*

distribution of human capital converges to zero if and only if

$$\log m \leq \int \log h \, dF(h).$$

Proof: See appendix.

The condition "$\log m \leq \int \log h \, dF(h)$" is stronger than the condition "$m \leq \mu$". (Just

note that $\log m \leq \int \log h \, dF(h) \Rightarrow m \leq \exp(\int \log h \, dF(h)) \Rightarrow m \leq \int h \, dF(h) = \mu$, where the

last implication follows from Jensen's inequality [for convex functions].) But the

converse direction is generally false. So the critical inequality for the theorem is one that

can be interpreted as *strong positive skewness* of the distribution F (because '$m<\mu$' is

commonly called *positive skewness).*

It is easy to see that if $b+c<1$, then the CV converges to zero always. Moreover,

if $b+c>1$ and $b<1$, then the argument of Theorem 2 shows that the CV converges to zero

if and only if strong positive skewness holds; when it does not hold, the CV explodes.

§7 Simulations

We here report the results of several simulations. In the first economy simulated,

$b=c=0.5=\gamma$, and we begin with the lognormal distribution whose mean and median are

40 and 30, respectively -- this looks like the distribution of income in the US, in

thousands of dollars, in the early 1990s. We take as the quasi-PUNE realized the one

with pivot h^* at the median of the distribution at each date (which means each party

represents exactly one-half the population) and which is located half-way between the

upper and lower boundaries in the manifold of quasi-PUNEs – thus, we attempt to capture a political system where the opportunists and militants each have some bargaining power inside the parties.

Some results are presenting in Table 1 and Figure 5. At each date, each party wins with probability one-half; there is a different sequence of *L-R* victories in the nine simulations displayed in the table, depending on the realization of this random variable. We see that the CV of the distribution of human capital appears to converge very rapidly to zero in the six generations of our simulations. Theorem 1, however, does not tell us whether, in fact, convergence to zero occurs, because the PUNEs we study are not on the lower boundary of the manifold.

Thus, we see that the convergence to equality – or at least to very low levels of inequality -- which occurs at certain quasi-PUNEs, even when $b+c =1$, is very dramatic in the model.   Ruthlessly competitive politics are radically different from laissez-faire.

As a second illustration of the equalizing power of political competition, I ran simulations for the economy with $b=c= 0.75$ -- that is, with increasing returns to scale. (I began with same lognormal distribution of human capital.) The results for two six-generation societies are presented in Table 2. We see that the CV decreases over time, although not so dramatically. But after six generations of laissez-faire, the CV of the distribution of human capital is 1.03×10^{11} ! In the light of the evident effect of increasing returns, political competition, as here formulated, is strongly equalizing.

§ 8   Discussion

We summarize the dynamics of human capital in Table 3.

[table 3 here]

The case $b+c=1$ offers the cleanest way of comparing the performance of the three regimes. In our hypothetical laissez-faire society, the coefficient of variation(CV) of the distribution of human capital stays constant over time, when the educational production function exhibits constant returns to scale. Under democracy, the CV of that distribution decreases monotonically; whether or not it decreases to zero depends, in our model, on the nature of intra-party struggle. If militants (or guardians) are relatively powerful, then the limit distribution is one of perfect equality. If the opportunists are relatively powerful, it is not. One might paraphrase by saying that, to the extent that democratic politics are ideological then democracy engenders equality, but if democratic politics become dominated by political opportunists then equality is not achieved in the limit. If there are decreasing returns to scale ($b+c <1$), then democracy and laissez-faire both produce equality in the long-run, and we suggest (without proof) that our results translate into statements about relative speeds of convergence to equality.

Our conclusion is very different from what Hotelling-Downs politics produces, on a unidimensional policy space. When $b+c=1$, convergence to equality occurs, when political competition occurs with respect to the natural unidimensional policy space, if and only if the initial distribution of human capital is 'strongly positively skewed.' This result differs in two ways from the result under democracy. First, the conclusion of Theorem 1 is independent of the initial distribution, and second, on the large policy space, convergence to equality *never* occurs when the opportunists are the dominant factions in the parties. (This is, after all, the analogue to Hotelling-Downs politics, where each party desires only to maximize its vote fraction.) We propose that this difference

between the models shows the pitfalls of the unidimensional Hotelling-Downs analysis, and underscores our point that large-dimensional policy spaces are not simply a mathematical nicety: rather, a serious misspecification occurs if we model politics in a unidimensional way, when, in reality, policies are not so restricted.

We offer an intuition to explain the different results that we attain on the uni- and infinite-dimensional policy spaces. When the opportunists dominate party politics, equilibrium entails that both parties propose the ideal policy of the *median* (in the unidimensional case) or the *pivot* (in the infinite dimensional case) voter. In the unidimensional case, if the median is less than the mean human capital, then the median voter's ideal policy is in fact progressive -- that is, very good for the poor. (It entails equal investment in all children.) This is a consequence of the fact that there are very few degrees of policy freedom, so that doing well by the median voter requires, willy-nilly, doing well by the poor. But in the infinite dimensional case, doing well by the pivot voter *requires* a regressive policy for the poor! That is to say, on the infinite dimensional policy space, the interests of the poor and the pivot diverge, something which is not true on the unidimensional policy space.

However, when the militants dominate party politics, then the policy is progressive for the poor in the Left party, and this delivers ultimate convergence to equality.

It is also noteworthy that, although our policy space is infinite dimensional, the policies proposed by each party in political equilibrium in fact lie in two two-dimensional spaces of functions. First, note that once the total-resource function $X(\cdot)$ is specified, then we know the two functions ψ and $r;$ second, note that once we know the two

coordinates $(h^*, X(h^*))$, then we can immediately compute both the Right and the Left

total-resource function, given the budget constraint. So each party, in the end, chooses

its policy in a two dimensional subspace of *T,* parameterized by the pivot type and the

total resource she receives.   In real democracies, we almost always observe piece-wise

linear tax policies, often with more pieces than the policies parties propose in our

model[10]. We can generate more pieces on the tax policies (in our model) by including

more factions in the parties than the ones here postulated.

In other words, the relative simplicity of observed income tax schedules is

consistent with our model's assumption, that parties are choosing policies from an

infinite dimensional policy space.

Why have we not observed more rapid convergence to equality of wages in

advanced democracies?   Besides the fact (if it is one) that politics are dominated by

opportunists, a number of reasons can be suggested -- reasons which take the form of

divergence from the premises of our model. These include:

- random talent or effort

- technological shocks

- imperfectly representative democracy

- elastic labor supply

- non-economic issues

- multi-party politics.

---

[10] Every OECD country except Germany has a piece-wise linear income tax schedule. Germany
uses a smooth polynomial.

We take these up in turn.

In the model, the human capital of the child is a deterministic function of the human capital of the parent and educational investment. In reality, children from families with similar parents differ according to their talent and their effort, which we could model by inserting a stochastic multiplicative term in the educational production function. Doing this does not complicate the analysis very much. Statements about convergence to equality in the deterministic model become translated into analogous statements about the *non-persistence* of the effect of initial conditions on the wages of distant descendents. That is, in the model with stochastic talent, in the sequence of PUNEs that lie on the lower boundary of the manifold, it is the case that there is eventually no influence of the wage of the initial parent (Eve) on the wages of members of her dynasty, whereas, for the sequence of PUNEs on the upper boundary of the manifold, there is persistence. Thus, 'convergence to equality' becomes translated into 'equality of opportunity,' in the sense that the socio-economic status of an individual's family origin eventually has no influence on his own level of human capital.

In our model, the only kind of technological change allowed was neutral, in the sense of time-dating the constant α^t. In reality, technological change is often non-neutral. This is the case with the shock to the educational production function in the US and UK in the last twenty years. Clearly, if non-neutral technological change is historically important, it can upset the process of convergence to equality.

Our model assumes that every citizen is a member of a party, and that parties aggregate the interests of their members in an unbiased way: these two premises constitute the assumption of 'perfectly representative democracy.' There are very few

countries today where an approximation of these premises holds, and , in every case, this ideal has only existed for at most two generations.

Because we have assumed an inelastic labor supply, parties in the model put forth policies which involve 100% marginal tax rates on certain intervals of the income distribution. With elastic labor supply, this will not happen, and convergence to equality will be retarded, if not eliminated.

Non-economic issues, especially concerning racial and ethnic questions, are politically salient in many countries, and these issues can retard redistribution. Thus, poor natives (or whites) of a country may vote for the party of the Right because that party opposes immigration (or income redistribution to minorities). There is reason to believe that American racism can explain a large part of the difference in the degrees of redistribution between the US and Europe. (See Alesina, Glaeser, and Sacerdote [2001], and Lee and Roemer [2002].) But, in our model, citizens are presumed to have only economic interests.

Finally, our model assumes only two parties. We offer no analysis of the case when the number of parties is endogenous.

On the other hand, there are features of reality, not present in the model, that render reality *more* prone to equalization than the model: principally, citizens do have some degree of 'altruism,' which is to say, concern for the children of others. Even without altruism, because education is to some degree a public good -- parents want the children of others to be educated because that will enhance the welfare of their children—convergence to equality will be at least as rapid as in our model.

Thus, our model should be viewed as asking the question: In an ideal type of democracy, where citizens are interested only in their own dynasty, and education is a private good, will competitive politics induce equalization of human capital? Clearly this is only the beginning of a thorough analysis.

In the United States, funding for public education of h-households does increase with h: this is accomplished through the linking of educational finance with the local property tax base. In the political equilibria of our model, this is the case—that is, both parties propose policies $r(h)$ that are everywhere non-decreasing in h, and increasing in h in some intervals. In many European countries, equal public educational investment in children of all backgrounds is closer to the truth. Section 5 tells us that, in these equilibria, rich parents will top off the public investment in their children. I conjecture that, at least in the Nordic countries, this topping-off does not occur. We may understand this as the consequence of the operation of another social norm – not one we have modeled here. There is, however, an alternative explanation, that the education of other people's children is a public good.

We conclude with two final comments about the nature of democracy as here modeled. First, our results on dynamics center on the opportunist-ideological controversy in democratic politics. We believe this conflict is an essential feature of political representation, and hence defend the importance that it is given in the PUNE model. Secondly, the militants, reformists, and guardians are utilitarian, in the sense that they employ an objective function which is the average welfare of the party's constituents. It is key for our results that the poor be represented in one of the parties. Suppose that the poor were not represented in either party. Then even the militants in

Left would propose policies giving the poor as little as possible, consistent with giving the pivot voter his required utility. Both parties would propose regressive educational policies for the poor, and convergence to equality would never occur. It is probably the case that the poor can be *underrepresented* in Left, without losing convergence to equality, but they must be represented at least to some positive degree. From a practical political viewpoint, this may be the most demanding condition of our model.

<div align="center">Appendix:Proofs of propositions</div>

Proof of Proposition 1:

Let (t^L, t^R, h^*) be a PUNEEP with $h^* > 0$, $L = [0, h^*)$, and $R = [h^*, \infty)$. By Remark 2,

$\pi(t^L, t^R) = \pi^*(\mathbf{F}(L))$ and $0 < \pi^*(\mathbf{F}(L)) < 1$ by definition of π^* and the fact that F is strictly increasing on $\mathbf{R}_+$. By Condition 1A of PUNEEP, there is no policy t that gives Left's militants a higher payoff than they receive at t^L *and* gives a higher probability of victory against t^R. In particular, there is no policy t that gives Left's militants a higher payoff than at t^L and such that

$$h \in [0, h^*) \Rightarrow v(t, h) \geq v(t^R, h),$$

and

$$v(t, h^*) > y,$$

for if there were, than, by continuity of v in h there would be an interval $[h^*, h^* + \varepsilon)$ such that

$$h \in [h^*, h^* + \varepsilon) \Rightarrow v(t, h) > v(t^R, h).$$

It would then follow that at least the set of voters $L \cup [h^*, h^* + \varepsilon)$ would favor t

and so a higher probability of victory could be achieved for Left at no cost to her

militants.

It therefore follows that statement 2A of definition 2 is true, and that $(L1)$ binds.

In like manner, statement 2B of definition 2 is true and $(R1)$ binds, which

concludes the proof. ■

Proof of Proposition 2:

1.   Suppose $y > y^*(h^*)$. Then there is no feasible solution to (3.6) or (3.7), for (3.63)

will never hold on *T*. Thus we must have $y \le y^*(h^*)$ if $(h^*, y) \in \Gamma$.

2.   Suppose $y < y^L(h^*)$. Then constraint (3.63) is not binding at the solution to (3.6),

since the solution to (3.8) is indeed the solution to (3.6). Similarly, if $y < y^R(h^*)$,

then constraint (3.73) is not binding at the solution to (3.7). Thus $(h^*, y) \in \Gamma$

implies $y \ge \max[(y^L(h^*), y^R(h^*)]$.

3.   Conversely, if (3.11) holds, then the opportunity sets of (3.6) and (3.7) are non-

empty, and at the optimal solutions, (3.63) and (3.73) must bind, because

$y \ge \max[y^L(h^*), y^R(h^*)]$.

4.   The proof thus far ignores the compactness issue - whether non-emptiness of the

opportunity sets for programs (3.6) and (3.7) implies the attainment of (optimal)

solutions. We shall show below that if (3.11) holds, solutions are indeed attained.

■

Proof of Proposition 3:

Let (ψ, r) be a solution to program (3.6), and suppose that the claim were false. Let

$X(h) = \psi(h) + r(h)$, and define

$$\psi(h) = \frac{X(h)}{1 + \gamma c},$$
$$r(h) = \frac{\gamma c \, X(h)}{1 + \gamma c}.$$

It is straightforward to check that $(\psi, r) \in T$. Furthermore, each household receives the

same total resource at (ψ, r) and at (ψ, r). But according to the lemma, for every h,

$(\psi(h), r(h))$ is the optimal way for household h to allocate the total resource assigned to

it between consumption and education. Therefore the objective function of (3.6)

increases if we substitute (ψ, r) for (ψ, r), a contradiction. (To be precise, the argument

shows that (ψ, r) must equal (ψ, r) except possibly on a set of **F**-measure zero.

Continuity then completes the argument.)   ■

Proof of Proposition 4^{11}:

We prove part (b). Proofs of the other parts are somewhat simpler, and of the same

character. Our task is to solve for the ideal policy of the Right militants:

---

[11] We construct a proof that is 'elementary,' in the sense of not requiring any knowledge of optimal control theory or the calculus of variations.

$$\underset{\psi}{Max} \int_{h*}^{\infty} Log\ X(h)dF(h)$$

$$s.t. \quad 0 \le X'(h) \le 1$$

$$\int_{0}^{\infty} X(h)dF(h) \le \mu.$$

The solution is shown in the figure 1, where (x, y) is the simultaneous solution of the

following two equations:

$$x + \int_{0}^{y} hdF(h) + y(1 - F(y)) = \mu \qquad \text{(A1)}$$

$$\int_{h*}^{y} \frac{dF(h)}{h + x} = \frac{F(y)}{x + y} \quad . \qquad \text{(A2)}$$

(A1) says that the function X integrates to μ , as required; (A2) fixes a particular pair (x,y).

Denote this policy by $X*$.

Define the function

$$\Delta(\varepsilon) = \int_{h*}^{\infty} Log(X*(h) + \varepsilon g(h))dF(h) + \int_{0}^{y} \lambda(h)(1 - (X*(h) + \varepsilon g(h))')dh + \delta(\mu - \int_{0}^{\infty} (X*(h) + \varepsilon g(h))dF(h)).$$

I will produce a non-negative function λ and a positive number δ such that, for any

function g, Δ is maximized at $\varepsilon=0$.   In particular, it follow that $\Delta(0) \ge \Delta(1)$.The second

and third terms on the r.h.s of the definition of Δ vanish at $\varepsilon=0$.   This will thus imply that

$$\int_{h*}^{\infty} LogX*(h)dF(h) \ge \int_{h*}^{\infty} Log(X*(h) + g(h))dF(h),$$

for any variation g, proving the claim.

Note that Δ is a concave function[12]. It therefore suffices to show that $\Delta'(0) = 0$.

Define λ and δ as follows, where f is the density of F and (x,y) are defined above:

(i) $\lambda(0)=0$,

(ii) $\lambda'(h) = \delta f(h)$ on $[0,h^*]$,

(iii) $\lambda'(h) = \delta f(h) - \dfrac{f(h)}{h+x}$ on $[h^*, y]$,

(iv) $\lambda(y) = 0$, and

(v) $\delta = \dfrac{1}{x+y}$.

We must show that (i)-(iv) are consistent, given (v). Note that $\lambda' \geq 0$ on $[0,h^*]$ from (ii) and $\lambda' \leq 0$ on $[h^*,y]$ from (iii) and the definition of y. From (i), (ii) and (v):

$$\lambda(h^*) = \int_0^{h^*} \delta f(h)dh = \frac{F(h^*)}{x+y}.$$

From (iii):

$$\lambda(y) - \lambda(h^*) = \int_{h^*}^{y} f(h)(\frac{1}{x+y} - \frac{1}{h+x})dh.$$

Therefore (iv) is true if

$$0 - \frac{F(h^*)}{x+y} = \int_{h^*}^{y}(\frac{1}{x+y} - \frac{1}{h+x})f(h)dh = \frac{F(y) - F(h^*)}{x+y} - \int_{h^*}^{y}\frac{dF(h)}{h+x},$$

which is true if and only if:

$$\int_{h^*}^{y}\frac{dF(h)}{h+x} = \frac{F(y)}{x+y}.$$

But the last equation is true by definition of (x,y).

---

[12] Here is where we exploit the fact that the optimization program is convex.

Thus the function λ is well-defined and non-negative on its domain, as required.

We now differentiate Δ, where g is an arbitrary, differentiable function:

$$\Delta'(0) = \int_{h*}^{\infty} \frac{g(h)dF(h)}{X*(h)} - \int_0^y \lambda(h)g'(h) - \delta\int_0^{\infty} g(h)dF(h)$$

$$= \int_{h*}^y \frac{g(h)dF(h)}{h+x} + \int_y^{\infty} \frac{g(h)dF(h)}{x+y} + \int_0^{h*} \lambda'(h)g(h)dh + \int_{h*}^y \lambda'(h)g(h)dh - \lambda(h)g(h)\Big|_0^y -$$

$$\delta\int_0^{h*} g(h)dF(h) - \delta\int_{h*}^{\infty} g(h)dF(h)$$

where we used integration by parts,

$$=$$

$$\int_{h*}^y (\frac{f(h)}{h+x} + \lambda'(h))g(h)dh + (\frac{1}{x+y} - \delta)\int_y^{\infty} g(h)dF(h) - \delta\int_{h*}^y g(h)dF(h) + \int_0^{h*} (\lambda'(h) - \delta f(h))g(h)dh -$$

$$\lambda(y)g(y) + \lambda(0)g(0)$$

$$= \int_{h*}^y (\frac{f(h)}{h+x} + \lambda'(h) - \delta f(h))g(h)dh + (\frac{1}{x+y} - \delta)\int_y^{\infty} g(h)dF(h) + \int_0^{h*} (\lambda'(h) - \delta f(h))g(h)dh + 0$$

$$= \text{[by definition of } \lambda' \text{ and } \delta] \quad 0,$$

as was to be shown.

We finally prove that $x>0$. Using integration by parts:

$$\int_0^y h\, dF(h) = hF(h)\Big|_0^y - \int_0^y F(h)dh = yF(h) - \int_0^y F(h)dh.$$

Hence, (3.13a) reduces to

$$x = \mu + \int_0^y F(h)dh - y = \mu + \int_0^y (1 - F(h))dh = \int_y^\infty (1 - F(h))dh,$$

where the last step uses the well-known fact that $\mu = \int_0^\infty (1 - F(h))dh$. Since F's support is

the positive real line, we have that $x > 0$ if $y < \infty$. Suppose $y = \infty$. Then (3.13b) becomes

$$\int_{h*}^\infty \frac{dF(h)}{h} = 0,$$

a contradiction. Therefore $x > 0$. ■

Proof of Proposition 5:

The proof that the optimal policies are as stated follows the template of the proof of

Proposition 4(b), presented above. We will not go through those analogous

constructions. We do prove here that $X_0^R > 0$, and hence, that $X_0^L > 0$. From (3.11a), we

have that

$$Exp(y) \geq Exp(y^R(h^*)) = X_{h*}^R(h^*) = x + h^*,$$

where the last part follows from Proposition 4(b). Therefore, since $X^R(h)$ dominates

$X_{h*}^R(h)$ on $[0, h^*]$, we have $X^R(0) = X_0^R \geq x$. Now use the fact that $x > 0$ (Prop.4).

■

Proof of Proposition 6:

1. We may view figure 3 as a graph of the two educational investment functions,

$r^L(\cdot)$ and $r^R(\cdot)$. Observe this geometric fact: Any chord on the graph of either function

cuts the vertical axis above the origin, when extended.   (It is crucially important that the graphs of these two functions cut the vertical axis above the origin.)

2. Consider the R policy and let the equation of the chord connecting $(h^1, r^R(h^1))$ and $(h^2, r^R(h^2))$ be denoted $r = mh + d$: we know that $m \geq 0$ and $d > 0$. Therefore:

$$\frac{h^{1R}}{h^{2R}} = \frac{\alpha h^{1^b} r^R(h^1)^c}{\alpha h^{2^b} r^R(h^2)^c} = \frac{h^{1^b}}{h^{2^b}} \frac{(mh^1 + d)^c}{(mh^2 + d)^c} < \frac{h^{1^b}}{h^{2^b}} \frac{h^{1^c}}{h^{2^c}}$$

because $\dfrac{mh^1 + d}{mh^2 + d} < \dfrac{h^1}{h^2}$.   Therefore

$$\frac{h^{1R}}{h^{2R}} < \frac{h^1}{h^2},$$

because $b + c = 1$.   The rest of the claim is straight-forward.   ∎

Proof of Proposition 7:

Part (a). Let (ψ, r) be the PUNEEP at date t. Since the mapping $h \rightarrow \alpha^t h^b r(h)^c$ is strictly monotone increasing, parents and children occupy the same ranks in their respective wage distributions, that is:

$$\forall h \qquad F^{t+1}(\alpha^t h^b r(h)^c) = F^t(h) \tag{4.3}$$

Hence, from (4.2):

$$\forall h \qquad F^{t+1}(\frac{\mu^{t+1}}{\mu^0} \alpha^t h^b r(h)^c) = F^t(\frac{\mu^t}{\mu^0} h) \tag{4.4}$$

Let $\theta: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be defined by:

$$\forall h \in \mathbf{R}_+ \quad \frac{\mu^t}{\mu^0} h \to \frac{\mu^{t+1}}{\mu^0} \alpha^t h^b r(h)^c.$$

Then we may rewrite (4.4) as $F^{t+1}(h) = F^t(\theta^{-1}(h))$, and so $F^{t+1}(h) \gtreqless F^t(h)$ as

$$F^t(\theta^{-1}(h)) \gtreqless F^t(h) \text{ as } \theta^{-1}(h) \gtreqless h \text{ as } h \gtreqless \theta(h) \text{ as } h \lesseqgtr \frac{\mu^{t+1}}{\mu^t} \alpha^t h^b r(h)^c \text{ as } \frac{h^{1-b}}{r(h)^c} \gtreqless \alpha^* \equiv \frac{\mu^{t+1}}{\mu^t} \alpha^t.$$

We next argue that the function $\zeta(h) = \dfrac{h^{1-b}}{r(h)^c}$ is strictly increasing on $\mathbf{R}_+$, taking

on values from zero to "infinity," which means that

$$(\exists h') \big| 0 \le h < h' \Rightarrow \zeta(h) < \alpha^* \text{ and } h > h' \Rightarrow \zeta(h) > \alpha^*).$$

This will prove part (a).

Suppose Left won the election at date t. The graph of $r(h)$ is a multiplicative

constant of the graph of X^L pictured in Figure 3. Obviously $\zeta(h)$ is strictly increasing on

the intervals $[0, h_L)$ and $[h^*, \infty)$, where r is constant. On the interval $[h_L, h^*]$, we have

$r(h) = \beta_0 + \dfrac{\gamma c}{1 + \gamma c} h$, where $\beta_0 > 0$. Therefore on this interval

$$\zeta(h) = \frac{h^{1-b}}{(\beta_0 + \dfrac{\gamma c}{1 + \gamma c} h)^c}.$$

Therefore we have

$$\frac{d}{dh} \log \zeta(h) = \frac{1-b}{h} - \frac{\gamma c^2}{(1 + \gamma c)\beta_0 + \gamma ch}$$

and so

$$\frac{d}{dh} \log \zeta(h) > 0 \Leftrightarrow \frac{1-b}{c} > \frac{\gamma ch}{(1 + \gamma c)\beta_0 + \gamma ch}. \tag{4.5}$$

Since $\beta_0 > 0$, the r.h.s. of the last inequality is smaller than unity, and hence $\zeta(h)$ is

strictly increasing on $[h_L, h^*]$ if $\dfrac{1-b}{c} \geq 1$. But this means $b + c \leq 1$, which is our premise.

Now suppose that Right won the election at date t. Again consult figure 3.

Exactly, the same kind of argument shows that ζ is strictly increasing.

Part (b). Since the sequence $\left\{\mathbf{F}^t\right\}$ is mean-preserving and F^{t+1} cuts F^t once from below,

we have that $\mathbf{F}^{t+1}$ second-order stochastic dominates $\mathbf{F}^t$. It therefore follows that the

sequence of CVs is monotone decreasing, and therefore converges[13]. ∎

Proof of Theorem 1:

1. We prove the second claim first. Fix $h^* > 0$; without loss of generality, normalize by

setting $h^* = 1$. At date 0, *both* parties play the policy defined in Proposition 4, part (c).

We shall, at each date, renormalize so that the descendents of h^* always have one unit of

human capital -- that is, we divide all human capitals by the level of human capital of the

contemporaneous member of the h^* dynasty. This does not affect coefficients of

variation.

Therefore, at date 1, denoting the human capital of the child of h by $S_1(h)$, we

have:

$$S_1(h)^{\frac{1}{c}} = \frac{h^{\frac{b}{c}}(h + X_0^*)}{1(1 + X_0^*)}, \text{ for } h \leq 1, \qquad (4.9)$$

where X_0^* is defined by (3.14). Eqn. (4.9) follows directly from Proposition 4(c), and our

normalization procedure.

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[13] We can prove that the sequence of distribution functions F^t converges weakly to a limit

distribution. But all we need in what follows is the convergence of the sequence of CVs.

2. Denote the distribution of human capital at date t in the sequence $B(h^*)$ by F^t. Then (Prop.4(c)) we have that the total resource bundle function at date t is

$$X^t(h) = \begin{cases} X_t^* + h, & 0 \le h \le 1 \\ X_t^* + 1, & h > 1, \end{cases}$$

where X_t^* is defined by

$$X_t^* + \int_0^1 h \, dF^t(h) + (1 - F^t(1)) = \mu^t, \tag{4.10}$$

where μ^t is the mean of distribution F^t. Thus:

$$S_2(h)^{\frac{1}{c}} = \frac{S_1(h)^{\frac{b}{c}}(S_1(h) + X_1^*)}{1 + X_1^*}, \text{ for } h \le 1$$

$$= \frac{h^{\frac{b}{c}}(h + X_0^*)}{(1 + X_0^*)(1 + X_1^*)} + \frac{X_1^* S_1(h)^{\frac{b}{c}}}{1 + X_1^*},$$

using (4.9), where $S_2(h)$ is the (normalized) human capital of the grandchild of h. By induction, for the Tth descendent we have:

$$\text{for } h \le 1, \quad S_T(h)^{\frac{1}{c}} = \frac{h^{\frac{b}{c}}(h + X_0^*)}{\prod_{j=0}^{T-1}(1 + X_j^*)} + \sum_{t=1}^{T-1} \lambda_t S_t(h)^{\frac{b}{c}}, \tag{4.11}$$

where $\lambda_t = \dfrac{X_t^*}{\prod_{j=t}^{T-1}(1 + X_j^*)}$, $t = 1, ..., T-1$.

3. Let $0 < h^2 < h^1 < 1$ be two levels of human capital at date 0. If the product

$\Delta = \prod_{j=0}^{\infty}(1 + X_j^*)$ converges, then from (4.11), it follows that $S_\infty(h^2)^{\frac{1}{c}} < S_\infty(h^1)^{\frac{1}{c}}$, and so the

CV of F^t does not converge to zero, because the ratio of human capitals of pairs of dynasties does not converge to unity.

4. (the key step) Thus, to prove the claim, we need only show convergence of the infinite

product Δ. Integrating by parts, note that:

$$\int_0^1 h\,dF^t(h) = F^t(1) - \int_0^1 F^t(h)dh,$$

and so from (4.10) we deduce:

$$X_t^* = \int_1^\infty (1 - F^t(h))dh; \qquad\qquad (4.12)$$

that is, X_t^* is the area 'above' the CDF on the interval $[1,\infty)$. (Use the fact that

$$\mu^t = \int_0^\infty (1 - F^t(h))dh\,.) \quad \text{By definition of } F^1 \text{ we have:}$$

$$X_1^* = \int_1^\infty (1 - F^1(h))dh = \int_1^\infty (1 - F^1(S_1(h)))dS_1(h) =$$

$$\int_1^\infty (1 - F^0(h))dS_1(h),$$

because $F^0(h) = F^1(S_1(h))$ (i.e., members of a dynasty occupy the same rank in their

respective distributions). For $h>1$ we have:

$$S_1(h)^{\frac{b}{c}} = \frac{h^{\frac{b}{c}}(1 + X_0^*)}{1 + X_0^*} = h^{\frac{b}{c}} \qquad \text{(from Prop.4(c)),}$$

and so $\dfrac{dS_1(h)}{dh} = bh^{b-1}$. Therefore, continuing the above expansion:

$$X_1^* = \int_1^\infty (1 - F^0(h))bh^{b-1}dh \le b\int_1^\infty (1 - F^0(h))dh.$$

By induction, it follows that:

$$X_t^* \le b^t \int_1^\infty (1 - F^0(h))dh.$$

Therefore $\sum X_t^*$ converges, and, in particular, $X_t^* \to 0$. But note that

$$\log \Delta = \sum \log(1 + X_t^*),$$

which converges iff $\sum X_t^*$ converges, because for X_t^* near zero, $\log(1 + X_t^*) \cong X_t^*$.

Therefore $\Delta < \infty$, as we set out to prove.

5. We next consider the sequence $A(h^*)$. The lower boundary of the manifold Γ^t

consists of two segments. On the first segment, the Left party plays the ideal policy of its

militants which is the constant function $X^t(h) = \mu^t$. On the second segment, the Right

party plays the ideal policy of its militants, given by Prop 4(b). If the sequence $A(h^*)$

spends an infinite number of periods on the first segment, then, since the Left wins an

infinite number of times in that subsequence, the CV converges to zero, because the ratio

of human capitals in any two dynasties approaches unity. We therefore assume, w.l.o.g.,

that the quasi-PUNEs in the sequence $A(h^*)$ always lie on the second segment.

Thus, at date t, the Right plays the policy

$$X_t^R(h) = \begin{cases} x_t + h, & h \le y_t \\ x_t + y_t, & h > y_t \end{cases}$$

where (x_t, y_t) are defined by the time-dated versions of equations (3.13a, 3.13b), and the

Left plays the policy

$$X_t^L(h) = \begin{cases} x_t + h_t^L, & h \le h_t^L \\ x_t + h, & h_t^L < h \le h_t^* \\ x_t + h_t^*, & h > h_t^*, \end{cases}$$

where h_t^L is defined by

$$x_t + h_t^L F(h_t^L) + \int_{h_t^L}^{h^{*t}} h\, dF^t(h) + (1 - F^t(h^{*t}))h^{*t} = \mu^t.$$

6. We next observe that, to show the CVs of $\{F^t\}$ converges to zero, it suffices to show

that the CVs would converge to zero if Left won every election in the sequence *A(h*).*

To see this, suppose that the actual sequence of Left/Right victories and associated

policies is

$$X_0^L, X_1^R, X_2^R, X_3^L \ldots \tag{i}$$

Now replace the Right policies in this sequence with the laissez-faire policy, which is

$$X^{lf}(h) \equiv h;$$

thus:

$$X_0^L, X^{lf}, X^{lf}, X_3^L \ldots \tag{ii}$$

The laissez-faire policy leaves the CV unchanged. Thus, the limit CV of (ii) is the limit

CV of :

$$X_0^L, X_3^L \ldots \tag{iii}$$

that is, of the sequence of Left policies. But the limit CV of (i) is surely no larger than

the limit CV of (ii), because the right policy at every date *reduces* the CV (see Prop. 6).

So if the limit CV of (iii) is zero, so is the limit CV of (i), *a fortiori*.

7. Therefore it suffices to show that the limit CV of a sequence of Left victories is zero.

8. Consider the graph of the CDF of a distribution function at some date, illustrated in

Figure 4, with various areas labeled. Integrating (3.13a) by parts shows that

$$x = \int_y^\infty (1 - F(h)) dh;$$

that is, area $D = x$. The mean of the distribution, μ, is the area above the CDF; that is:

$$\mu = D + A + H + G + J + B + I, \text{ and}$$
$$h^* = H + G + E + J + B + I + C.$$

Therefore $\mu - (x + h^*) = A - (C + E)$.

But (4.14) says $\mu - (x + h^*) = B - E$, and so, reverting to the time-dated notation:

$$A^t = B^t + C^t. \qquad\qquad (4.15)$$

9. We next note that $A^t + D^t \to 0$ with t; this employs the same argument as in step 4

above, because X_t^L is always a constant function for $h > h_t^*$. Therefore, $A^t \to 0$, and so

from (4.15), $\lim B^t = \lim C^t = 0$.

10. Denote $r \equiv F^t(h_t^*)$, so by definition, area $B^t = h_t^L(r - F^t(h_t^L))$. By step 9, since

$B^t \to 0$, we have that either $h_t^L \to 0$ or $F^t(h_t^L) \to r$. We claim that h_t^L does not approach

zero as a limit. Recall that $x_t \to 0$ (since $x_t = \int_{y_t}^{\infty} (1 - F^t(h))dh < \int_{1}^{\infty} (1 - F^t(h))dh \to 0$).

Integrating (3.13b) by parts gives:

$$\int_{h_t^*}^{y_t} \frac{F^t(h)}{(x_t + h)^2} dh = \frac{F^t(h_t^*)}{x_t + h_t^*},$$

and therefore $\lim_{t \to \infty} \int_{1}^{y_t} \frac{F^t(h)}{h^2} dh = r.$

But $\int_{1}^{y_t} \frac{F^t(h)}{h^2} dh < \int_{1}^{y_t} \frac{1}{h^2} dh = 1 - \frac{1}{y_t}$, and so $\lim y_t \geq \frac{1}{1 - r} > 1$. Letting $y* = \lim y_t$, we

therefore have that in the limit the total resource bundle function proposal by Right, in the

sequence $A(h*)$ is:

$$X_\infty^R(h) = \begin{cases} h, & \text{for } h \leq y* \\ y*, & \text{for } h > y*. \end{cases}$$

If $h_t^L \to 0$, then in the limit the total resource bundle function proposal of Left is:

$$X_\infty^L(h) = \begin{cases} h, & \text{for } h \leq 1 \\ 1, & \text{for } h > 1. \end{cases}$$

Now both X_t^L and X_t^R must integrate to μ^t, which we next show gives a contradiction:

$$0 = \lim \int_0^\infty (X_t^R(h) - X_t^L(h)) dF^t(h) =$$

$$\lim(\int_1^{y*} h\, dF^t(h) + (1 - F^t(y*))(y*-1)) = \lim\{hF^t(h)\big|_1^{y*} - \int_1^{y*} F^t(h)\, dh + (1 - F^t(y*))(y*-1)\}$$

$$= (y*-1) + \lim(F^t(y*) - r - \int_1^{y*} F^t(h)\, dh). \qquad (4.16)$$

Since $y*>1$, $\lim F^t(y*) = 1$. (Recall , Left policies squeeze all $h>1$ eventually to 1.) And

$$\lim \int_1^{y*} F^t(h)\, dh \le \int_1^{y*} dh = y*-1.$$

Therefore, (4.16) says that

$$0 = (y*-1) + (1-r) - (y*-1)^-,$$

where the last term on the r.h.s. is a number no larger than $y*-1$. But this is impossible.

The contradiction demonstrates that $\lim h_t^L \ne 0$, and so it follows that $\lim F^t(h_t^L) = r$.

11. There are now two cases to consider: either

   (a) 1 is not a limit point of $\{h_t^L\}$, or

   (b) 1 is a limit point of $\{h_t^L\}$.

In case (b), for large t, the policy X_t^L is arbitrarily close to giving all types the same total resources, and so almost the same is spent on the education of all children, and so the limit CV of the human-capital distribution is zero. We will therefore complete the proof of the theorem by showing that case (a) cannot occur.

   Suppose, then, case (a). Then for large t, h_t^L is bounded away from 1 from below. In particular, there exist types $h^2 < h^1 < 1$ such that, for all large t, $h_t^L < h^2$. For such t, we have:

$$S_t(h^i)^{\frac{1}{c}} = \frac{h^{i\frac{b}{c}}(h^i + x_t)}{1 + x_t}, \quad i = 1,2.$$

It follows, again by application of the argumentation of step 4, that in the limit, the human capital possessed by the descendents of h^2 and h^1 are different, that is:

$$S_\infty(h^2) < S_\infty(h^1).$$

Thus, there is an interval of measure $F^0(h^1) - F^0(h^2)$ that lies between the largest limit point of the sequence $\{h_t^L\}$ and 1. This is impossible, since $\lim F^t(h_t^L) = \lim F^t(1) = r.$ The contradiction establishes that, indeed, case (b) holds, which proves the theorem. ∎

Proof of Theorem 2:

The argument preceding the statement of the theorem in the text shows that convergence to equality occurs if and only if, for all *t,* the median m^t is less than the mean μ^t. (Equality of the mean and median is a singularity that we ignore.) In period *t,* if the median has been less than the mean in all prior periods, the median and mean of the distribution are given by

$$km^{b^t} \quad \text{and} \quad k\int h^{b^t} dF(h),$$

respectively, for some constant *k.* We wish, therefore, to know if:

$$m^r < \int h^r dF(h) \tag{6.1}$$

where $r = b^t$; in other words, whether

$$m < \left(\int h^r dF(h)\right)^{\frac{1}{r}} \equiv M_r(F). \tag{6.2}$$

Now $b<1$ implies that r decreases to zero as t gets large. It is well-known that, as r decreases, so does $M_r(F)$ (see Hardy, Littlewood, and Polya [1964] Theorem 192, p. 143), and further, that

$$\lim_{r\to 0} M_r(F) = \exp(\int \log h \, dF(h)).$$

(Hardy *et al*, Theorem 187, p, 139). Therefore, (6.2) is true for all $t \geq 0$ iff

$m \leq \exp(\int \log h \, dF(h))$, or , in other words iff

$$\log m \leq \int \log h \, dF(h).$$

This is therefore precisely the condition in which the CV of the distribution of human capital tends to zero.   ■

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Figure 1 The policies of Proposition 4(b) and 4(c)

total resource
X(h*)

boundary
where
opportunists
dominate

B(h*)

boundary
where
militants
dominate

Γ   *A(h*)*

h *

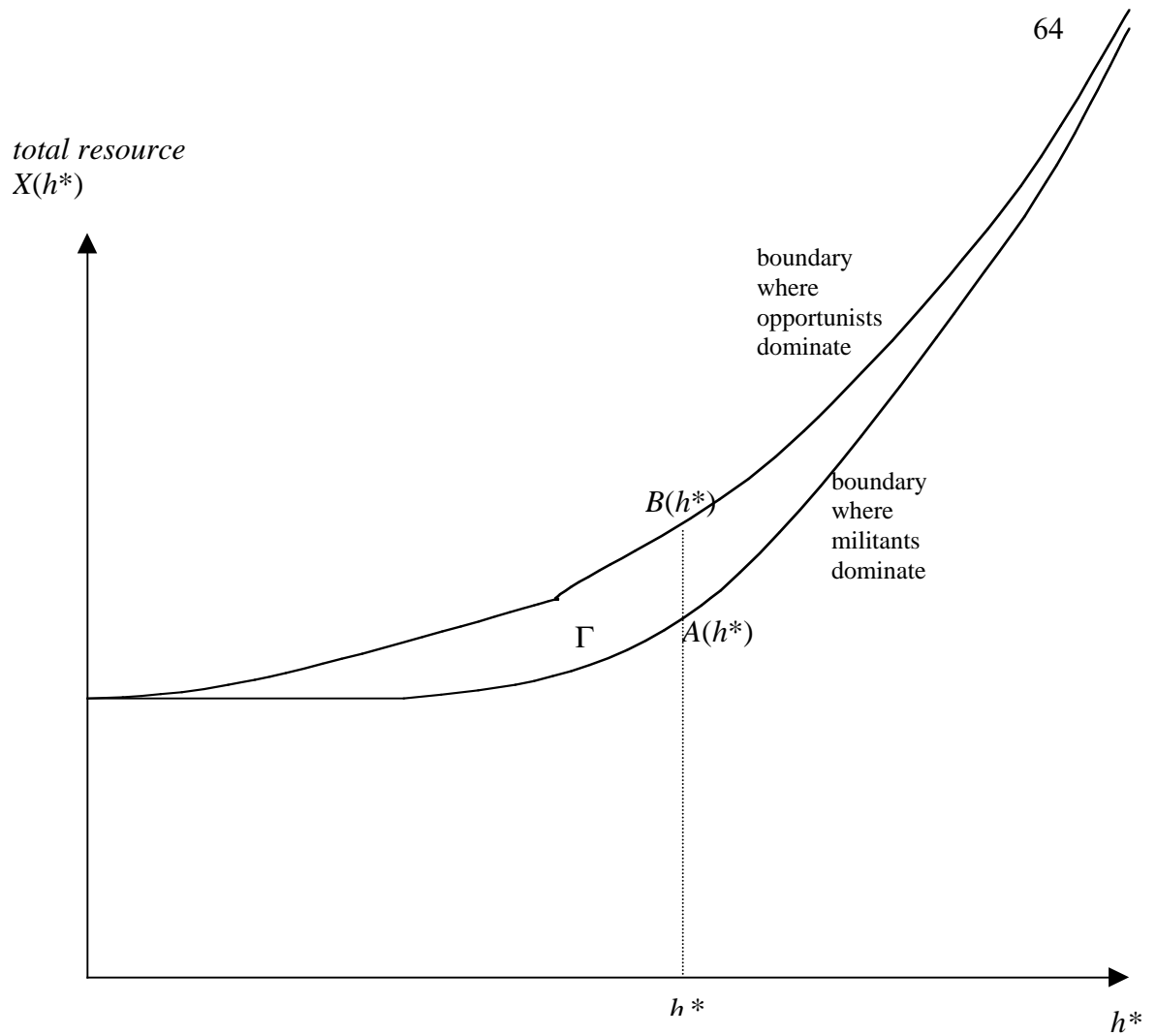*h**
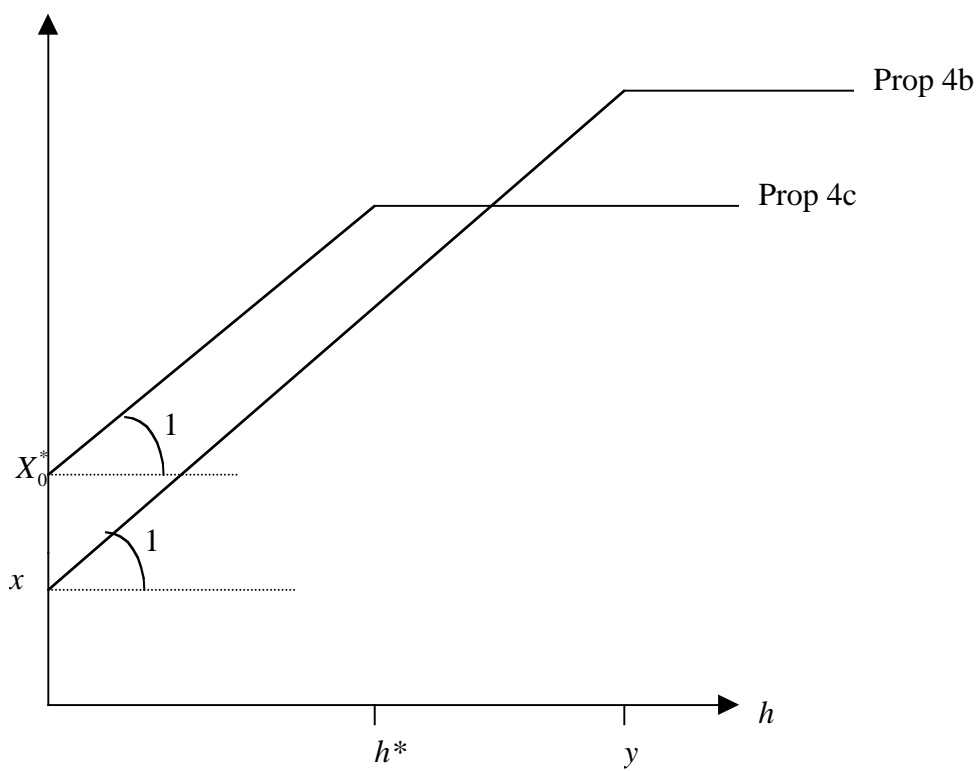
Figure 2   The manifold of quasi-PUNEs

Figure 3 Left (bold) and Right policies in a quasi-PUNE (Prop. 5)
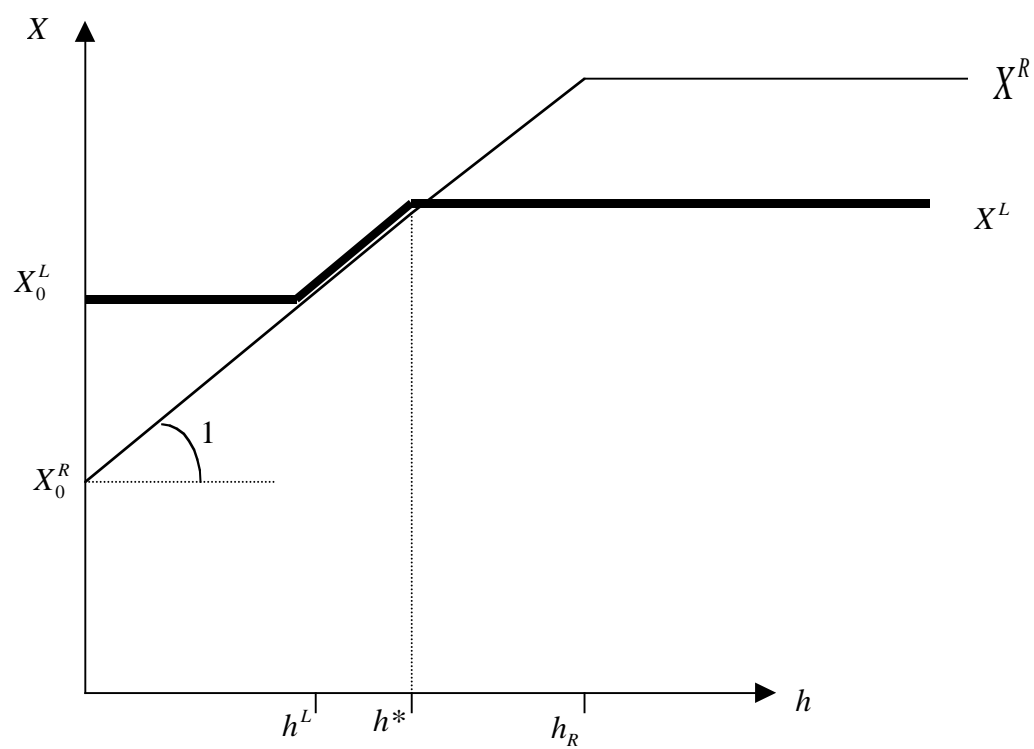
Figure 4   Aid in the proof of main theorem
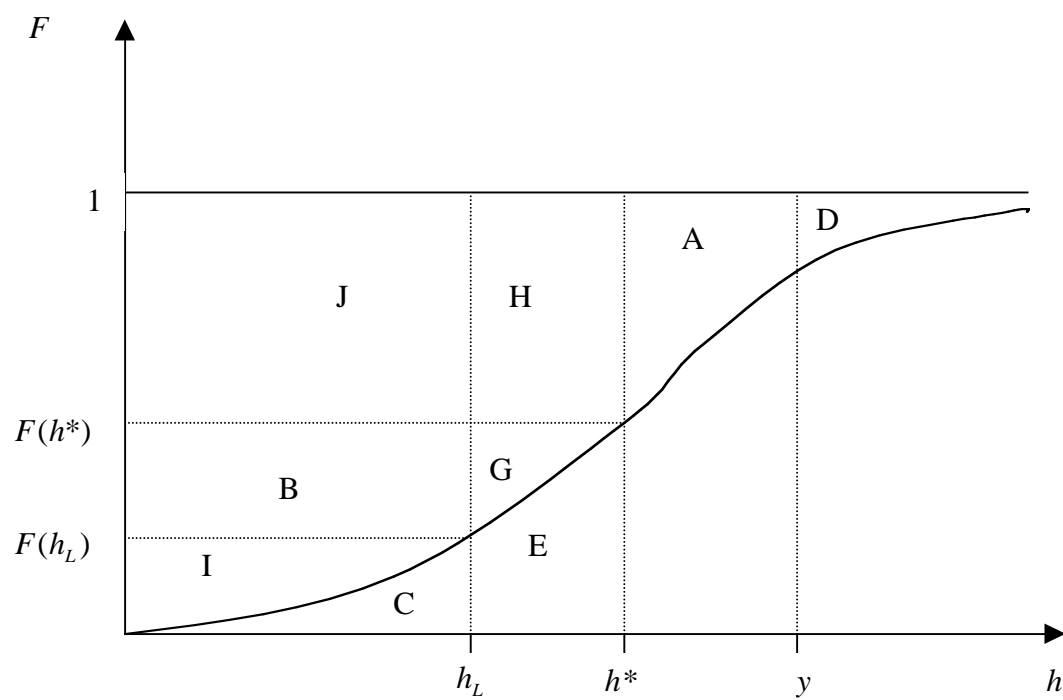
Figure 5a   CDF of human capital after five Right victories: simulation

This is generation 5 after five Right victories
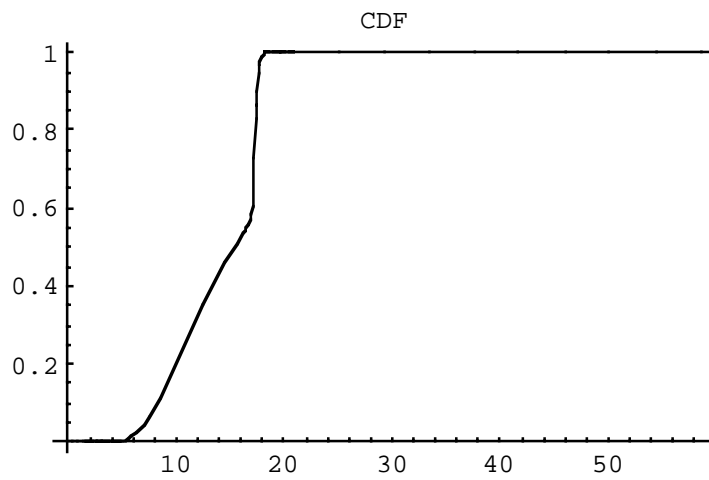
Figure 5b CDF of human capital after five Left victories: simulation

This is generation 5 after five Left victories
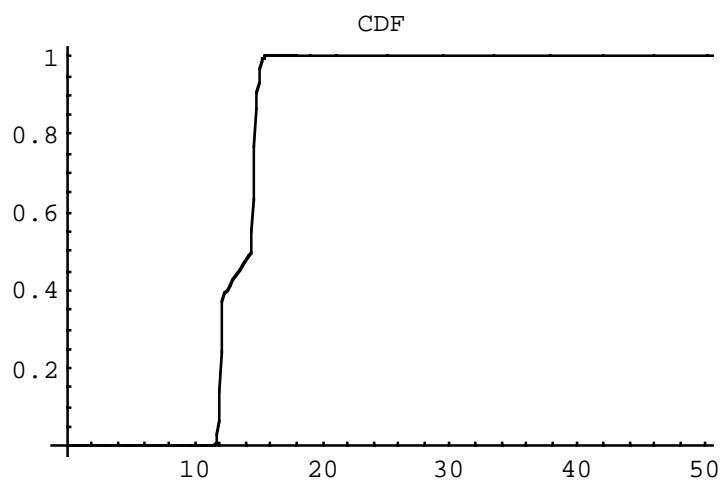
	Laissez-faire (LF)	Democracy	Hotelling-Downs politics
$b+c < 1$	$CV \rightarrow 0$	$CV \rightarrow 0$, faster than LF	$CV \rightarrow 0$
$b+c = 1$	constant CV	Theorem 1	Theorem 2
$b+c > 1$	CV explodes	CV decreases to 0.4 (simulation)	if $b<1$: strong skew $\Rightarrow CV \rightarrow 0$; otherwise, CV explodes

Table 3: Behavior of coefficient of variation of human capital by regime type

Table 1   Coefficients of variation in six-generation simulations when $b=c=0.5$

gen	mean	median	cvar	winner
0	40.	30.	0.777778	Right
1	31.4903	29.4005	0.238905	Right
2	25.5745	25.8911	0.133496	Left
3	20.6967	21.9691	0.0552047	Right
4	16.9311	18.2861	0.045739	Right
5	13.854	15.1532	0.040081	Right
6	11.3373	12.5166	0.0363719	none

gen	mean	median	cvar	winner
0	40.	30.	0.777778	Right
1	31.4903	29.4005	0.238905	Left
2	24.4413	25.8911	0.111755	Right
3	19.9764	21.7406	0.0876068	Left
4	16.026	18.1286	0.0647746	Right
5	13.1141	14.9711	0.0598669	Left
6	10.8325	12.3349	0.0400345	none

gen	mean	median	cvar	winner
0	40.	30.	0.777778	Right
1	31.4903	29.4005	0.238905	Left
2	24.4413	25.8911	0.111755	Left
3	19.401	21.7406	0.0756528	Left
4	15.6823	18.0099	0.0605814	Left
5	12.8274	14.8266	0.0492044	Right
6	10.4978	12.1695	0.0478734	none

gen	mean	median	cvar	winner
0	40.	30.	0.777778	Right
1	31.4903	29.4005	0.238905	Right
2	25.5745	25.8911	0.133496	Left
3	20.0601	21.9691	0.0825377	Right
4	16.4099	18.2861	0.0712898	Left
5	13.3159	15.1532	0.0535628	Left
6	10.9721	12.4772	0.0378218	none

gen	mean	median	cvar	winner
0	40.	30.	0.777778	Left
1	30.1078	29.4005	0.209562	Left
2	23.3832	25.3509	0.104315	Left
3	18.6632	21.1867	0.0740434	Right
4	15.2701	17.5126	0.0672418	Left
5	12.5011	14.4383	0.051507	Left
6	10.3844	11.8592	0.0346534	none

gen	mean	median	cvar	winner
0	40.	30.	0.777778	Left
1	30.1078	29.4005	0.209562	Left
2	23.3832	25.3509	0.104315	Right
3	19.1164	21.1867	0.0841093	Left
4	15.4042	17.6068	0.0640802	Right
5	12.6056	14.5167	0.0599651	Left
6	10.4662	11.9462	0.038533	none

gen	mean	median	cvar	winner
0	40.	30.	0.777778	Left
1	30.1078	29.4005	0.209562	Right
2	24.5092	25.3509	0.127354	Right
3	20.0248	21.4133	0.0967003	Right
4	16.3769	17.9313	0.0802714	Left
5	13.2326	14.9242	0.0565481	Left
6	10.8889	12.3143	0.0382226	none

gen	mean	median	cvar	winner
0	40.	30.	0.777778	Right
1	31.4903	29.4005	0.238905	Left
2	24.4413	25.8911	0.111755	Left
3	19.401	21.7406	0.0756528	Left
4	15.6823	18.0099	0.0605814	Right
5	12.8334	14.8266	0.0574073	Right
6	10.5026	12.1876	0.0552878	none

gen	mean	median	cvar	winner
0	40.	30.	0.777778	Right
1	31.4903	29.4005	0.238905	Left
2	24.4413	25.8911	0.111755	Left
3	19.401	21.7406	0.0756528	Right
4	15.8728	18.0099	0.0675324	Left
5	12.9463	14.8723	0.0520871	Right
6	10.5949	12.2253	0.0500408	none

Table 2 Coefficients of variation when $b=c=0.75$

gen	mean	median	cvar	winner
0	40.	30.	0.777778	Left
1	46.3957	40.6303	0.431824	Right
2	62.1499	57.4749	0.397763	Left
3	92.1395	94.2662	0.293562	Right
4	178.626	186.659	0.389374	Right
5	497.265	525.59	0.50656	none

gen	mean	median	cvar	winner
0	40.	30.	0.777778	Left
1	46.3957	40.6303	0.431824	Right
2	62.1499	57.4749	0.397763	Left
3	92.1395	94.2662	0.293562	Left
4	168.301	186.659	0.245359	Right
5	439.997	508.532	0.358216	none