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03-07

Iterated Weak Dominance and Subgame Dominance

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Iterated weak dominance and subgame dominance*

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April 2003

Abstract

In this paper, we consider finite normal form games satisfying *transference of decisionmaker indifference*. We show that any set of strategies surviving k rounds of elimination of *some* weakly dominated strategies can be reduced to a set of strategies equivalent to the set of strategies surviving k rounds of elimination of *all* weakly dominated strategies in every round by (at most k) further rounds of elimination of weakly dominated strategies. The result develops work by Gretlein (Dominance Elimination Procedures on Finite Alternative Games, *Int J Game Theory* 12, 107-113, 1983). We then consider applications and demonstrate how we may obtain a unified approach to the above-mentioned work by Gretlein and recent work by Ewerhart (Iterated Weak Dominance in Strictly Competitive Games of Perfect Information, *J Econ Theory* 107, 474-482, 2002) and Marx and Swinkels (Order Independence for Iterated Weak Dominance, *Games Econ Behav* 18, 219-245, 1997).

Keywords: Weak dominance, weak subgame dominance.
JEL Classification Number: C72.

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*I thank Christian Ewerhart, Elisabeth Hermann Frederiksen, Hans Keiding and Andreas Park for helpful comments and advice. Of course, any errors remain my responsibility.

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1 Introduction

For the class of normal form games where a finite number of players have strict preferences over a finite set of outcomes (utility vectors), Gretlein [6] showed that the set of outcomes resulting from iterative elimination of some weakly dominated strategies contains the set of outcomes that remains from a procedure that removes all weakly dominated strategies in every step until no more strategies can be removed. In this paper, we extend this result and show that any set of strategies surviving k rounds of iterated weak dominance can be reduced to a set of strategies equivalent to the strategies surviving k rounds of elimination of *all* weakly dominated strategies in every round. The reduction can be carried out by at most k further rounds of elimination of weakly dominated strategies (Theorem 1). From this we obtain a strengthened version of Gretlein's result (Corollary 1). Moreover, the result extends to a class of games satisfying *transference of decisionmaker indifference* which is less restrictive than strict preferences over outcomes.

We then consider applications and demonstrate how we may obtain a unified approach to the above-mentioned result by Gretlein and recent work by Ewerhart [2, 4] and by Marx and Swinkels [7].

Our first application is to two-player strictly competitive finite games of perfect information. It is well known (Moulin [9], Gretlein [5, 6]) that these games are dominance solvable in a finite number of steps, and that the outcome is equal to the backward induction outcome. Recently, Ewerhart [2] demonstrated that any chess-like game (a strictly competitive, finite game of perfect information with three possible outcomes) can be solved by two rounds of elimination of weakly dominated strategies. Moreover, he conjectured that the following generalization is true: Any finite, strictly competitive game of perfect information with at most n outcomes is dominance solvable by $n-1$ rounds of elimination of weakly dominated strategies. A proof of this conjecture has now been provided by Ewerhart [4].[1] The proof is complicated by the fact that for an extensive game of perfect information, after one round of elimination of all weakly dominated strategies in the strategic form, the surviving strategies do not necessarily represent the strategic form of any residual extensive game.[2] In other words, the procedure eliminating all weakly dominated strategies in every step does *not* correspond to any

[1] An earlier version of Ewerhart's proof was reported in [3]. A proof has also been reported in independent work by Shimoji [11].

[2] Battigalli [1] provides an example.

procedure removing 'dominated' branches from the game tree. Another difficulty is that 'greedy' elimination of all weakly dominated strategies in every step does not necessarily remove the largest number of strategies from the second round and onwards.

In this paper, we consider iterative elimination of all *weakly subgame dominated strategies*, a procedure that intuitively can be viewed as the removal of certain 'dominated' branches in every step. More precisely, after every step, the remaining strategy set is, up to some redundant strategies, the strategy set of a residual extensive form game where 'weakly dominated subgames' have been removed. We claim that any finite, strictly competitive game of perfect information with at most n outcomes is dominance solvable by $n-1$ rounds of elimination of weakly subgame dominated strategies (Theorem 2), and give a short and elementary proof. By combining the results (Theorem 1 and 2), we also obtain the result by Ewerhart [4] (Corollary 2).

Our second application is to work by Marx and Swinkels [7].[3] For the class of finite normal form games satisfying the transference of decisionmaker indifference condition, Marx and Swinkels show order independence: Regardless of the order in which weakly dominated strategies are removed, any two *full* reductions are the same up to removal of redundant strategies and renaming of strategies. We round off by formulating this as a corollary of our main result (Corollary 3).

2 Preliminaries

We consider a finite set of players $I = \{1, 2, ..., m\}$. A game in normal form is written $N(S, u)$, where $S_i = \{s_i, s_i', ...\}$ is a finite strategy set for player i, $S = S_1 \times ... \times S_m$, and $u_i : S \to \mathbb{R}$ the *utility function* for player i. Let $s = (s_1, ..., s_m) \in S$ and $u(s) = (u_1(s), ..., u_m(s))$. We write $S_{-i} = S_1 \times ... \times S_{i-1} \times S_{i+1} ... \times S_m$, $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_m)$.

Gretlein (1983) studied the class of games satisfying $u_i(s) = u_i(s') \Rightarrow u(s) = u(s')$ for all $i \in I$, $s, s' \in S$, which he called *strict preferences over outcomes.* Throughout this paper we relax this condition and assume that $u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}) \Rightarrow u(s_i, s_{-i}) = u(s_i', s_{-i})$, for all $i \in I$, $s_i, s_i' \in S_i, s_{-i} \in S_{-i}$. This condition has been referred to as *transference of decisionmaker indifference* (TDI), see Marx and Swinkels [7] for a discussion.

[3] In this paper we restrict attention to finite pure strategy spaces.

A strategy $s_i \in S_i$ is *weakly dominated* by $s'_i \in S_i$ on S_{-i} (or S) if $u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ with at least one strict inequality. Without the latter requirement we say that s'_i is *at least as good as* s_i.

Define $F^k(S) = F^k_1(S) \times ... \times F^k_m(S)$ recursively where $F^k_i(S)$ is the set of strategies in $F^{k-1}_i(S)$ that are not weakly dominated w.r.t. $F^{k-1}_{-i}(S)$, $F^0(S) \equiv F$. Let $L^k(S) = L^k_1(S) \times ... \times L^k_m(S)$ denote a set of strategies where for every i *some* strategies in $L^{k-1}_i(S)$ which are weakly dominated w.r.t. $L^{k-1}_{-i}(S)$ have been removed, $L^0(S) \equiv S$. Let $L^kS = L^k(S)$ and $LS = L^1(S)$.

A game $N(S, u)$ is *dominance solvable* in k steps if the outcomes in $F^k(S)$ are constant.

We say that strategies s_i and s'_i are *equivalent* w.r.t. S_{-i} if $u_i(s_i, s_{-i}) = u_i(s'_i, s_{-i})$ for all $s_{-i} \in S_{-i}$. If s_i and s'_i are equivalent then s_i is *redundant* to s'_i and vice versa.

Let N and $\widetilde{N}$ be normal form games with m players and respective strategy sets S and $\widetilde{S}$ and utility functions u_i and $\widetilde{u}_i, i = 1, ..., m$. Then $\widetilde{N}$ is a *reduction* of N if there exist surjective maps $f_i : S_i \to \widetilde{S}_i$, $i = 1, ..., m$, such that $\widetilde{u}_i(f_1(s_1), ..., f_m(s_m)) = u_i(s)$ for all i and s. We say that N and $\overline{N}$ are *equivalent games,* written $N \sim \overline{N}$, if they have a common reduction.[4] In words, two games are equivalent if they are identical up to removal of redundant strategies and renaming.

We collect some useful technical results below.

Lemma A

1. If $s_i \in L^k_i(S)$ and $s_i \notin L^{k+1}_i(S)$ then there is $s'_i \in L^{k+1}_i(S)$ such that s'_i weakly dominates s_i on $L^k_{-i}(S)$.

2. Let $s_i \in S_i$ and $k > 0$. Then there is $s'_i \in L^k_i(S)$ such that $u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in L^k_{-i}(S)$.

3. The relation $\sim$ is an equivalence relation on the set of finite normal form games, and when $N(S, u) \sim N(\overline{S}, \overline{u})$ then $N(F^k(S), u|F^k(S)) \sim N(F^k(\overline{S}), \overline{u}|F^k(\overline{S})), k > 0$.

Proof: For A.1, see Gretlein [6]. For A.2, Ewerhart [3] or [4] provides a proof for the elimination procedure F in the two-player case, but the proof applies to any elimination procedure L in the m-player case by a change of notation. For A.3, see Ewerhart [3] for a proof in the two-player case for $k = 1$ that easily generalizes to the m-player case and for any k. □

[4]This terminology is used by Ewerhart [3].

Note that if two strategies s_i and s'_i are not equivalent w.r.t. S, and if $\widetilde{S}$ is a reduction of S with surjective maps f_i, then $f_i(s_i) \neq f_i(s'_i)$. Thus, for each player i there are at least as many strategies in $\widetilde{S}$ as there are classes of equivalent strategies in S. We say that a reduction $\widetilde{S}$ of S is *minimal* if for each player i the number of strategies in $\widetilde{S}$ is equal to the number of equivalence classes of strategies for player i in S_i. It can be shown that for any game a minimal reduction exists, and for any two equivalent games there is a common reduction which is minimal for both games.

Following Gretlein, if $R_i \subseteq T_i$ for all i we say that R is in T. Further, we write $R\alpha T$ if i) R is in T, and ii) for all i and all $t_i \in T_i$ there is $r_i \in R_i$ such that r_i is at least as good as t_i on T. Moreover, $R\beta T$ if i) R is in T, and ii) for all i and all $t_i \in T_i$ there is $r_i \in R_i$ such that r_i is equivalent to t_i on T. We define R^k and T^k recursively: $R^{k+1} = FR^k \cap FT^k, T^{k+1} = R^k \cap FT^k$ with $R^0 \equiv R$ and $T^0 \equiv T$.

Below we highlight some useful results developed by Gretlein within the proofs of his Lemma 4 and Theorem 1. Gretlein consider games with strict preferences over outcomes, but his results are valid for games satisfying TDI.[5]

Lemma B

1. *If $R\beta T$ then $F^k R\beta F^k T$ for all k.*
2. *If $R\alpha T$ then $R^k \alpha T^k$ for all k.*
3. *$R^{k+1}\beta FR^k$ and $T^{k+1}\beta FT^k$ for all k.*

Proof: Gretlein obtains B.1, B.2, and B.3 in [6] p. 112, at line 3, 15, and 16 respectively. □

For later reference, we define a two-player, strictly competitive extensive-form game G of perfect information and its strategic form. Following Ewerhart [2], we write

$$G = (X, x^0, \alpha, \iota, \omega),$$

where X is a finite set of *nodes*, $x^0 \in X$ is the *initial node*, $\alpha : X \backslash \{x^0\} \to X$ is the *anterior node function*, $Z = X \backslash \alpha(X)$ denotes the set of *terminal nodes*, $\iota : X \backslash Z \to \{1, 2\}$ denotes the *player function*, and $\omega_i : Z \to \mathbb{R}$ denotes the *outcome function*. G is *strictly competitive* if $\omega_i(z) < \omega_i(z') \Rightarrow \omega_j(z) > \omega_j(z')$, $i \neq j$.

[5] In fact, Greitlein uses only the TDI property implied by strict preferences over outcomes in his proofs.

Node x *precedes* node x' if there is a non-negative integer H and a sequence $\{x_0, ..., x_H\}$ such that $x = x_0, x' = x_H$ and $\alpha(x_h) = x_{h-1}$ for $h = 1, ..., H$. $X(x) = \{x' \mid x \text{ precedes } x'\}$. The initial node x^0 precedes any $x \in X$. Define a *subgame* rooted in the node x^1 by

$$G(x^1) = (X(x^1), x^1, \alpha^1, \iota|_{X(x^1)\backslash Z^1}, \omega|_{Z^1}),$$

where $\alpha^1 = \alpha|_{X(x^1)\backslash Z}$ and $Z^1 = X(x^1)\backslash\alpha^1(X(x^1))$. Assume w.l.o.g. that if $x \in X\backslash x^0$ and $\iota(x) = i$ then $\iota(\alpha(x)) \neq i$.

For any non-terminal node $x \in X\backslash Z$, $A_x = \{x' \mid \alpha(x') = x\}$ is the set of *actions* available at node x. An *action profile* is a tuple $s = (a_x)_{x\in X\backslash Z}$ where $a_x \in A_x$. Any action profile s uniquely determines a *path* $p(s) = (x_0, ..., x_H)$ where $x_0 = x^0$ and $x_H = z(s)$ is a terminal node, and abusing notation we write $\omega(s) = \omega(z(s))$.

The strategic form $N(G)$ of G is the normal form game with strategy set $S_i = \prod_{x\in X\backslash Z, \iota(x)=i} A_x$ and utility function $u_i(s) = \omega_i(z(s))$, $i = 1, 2$. A *strategy* $s_i \in S_i$ then specifies a move at every node for which the player has the move.

The *value* $v(x) = (v_1(x), v_2(x))$ of $x \in X$ is the backward induction outcome of $G(x)$.[6]

3 Iterated weak dominance

In the following, consider a normal form game $N(S, u)$. If R is in S, R induces a game $N(R, u|R)$ and we occasionally refer to R as a game. For R and T in S, we write $R \sim T$ if $N(R, u|R) \sim N(T, u|T)$.

We proceed with the following five lemmas.

Lemma 1 *Let R and T be in S.*

1. *If $R\beta T$ then $R \sim T$.*

2. *If R in T and $R \sim T$ then $R\beta T$.*

3. *If $R\beta T\beta P$ then $R\beta P$.*

[6]For extensive games with strict preferences over outcomes, any subgame perfect equilibrium outcome yields the backward induction outcome (cf. e.g. Osborne and Rubinstein [10], Chapter 6).

4. If $R\alpha T\beta P$ then $R\alpha P$.

Proof: 1.1: We show that R is a reduction of T. Let $f_i : T_i \to R_i$ be such that $f_i(s_i)$ is equivalent to s_i w.r.t. T and such that $f_i(s_i) = s_i$ if $s_i \in R_i$. Thus $f_i : T_i \to R_i$ is surjective. By TDI $u(f_1(s_1), s_2, ..., s_m) = u(s)$ for all $s \in T$. Generally for some $h \in \{2, ..., m\}$ by TDI if $u(f_1(s_1), ..., f_{h-1}(s_{h-1}), s_h, ..., s_m) = u(s)$ for all s, then $u(f_1(s_1), ..., f_h(s_h), s_{h+1}, ..., s_m) = u(s)$ for all s. Thus by induction on h we have $u(f_1(s_1), ..., f_m(s_m)) = u(s)$ for all s. Since R is a reduction if itself, R and T have a common reduction, i.e. $R \sim T$.

1.2: Let R be in T. We show that [not $R\beta T$] $\Rightarrow$ [not $R \sim T$]. If [not $R\beta T$] then there is i and $t_i \in T_i$ such that there is no $r_i \in R_i$ where t_i and r_i are equivalent w.r.t. T. Now, suppose there is a minimal common reduction $\widetilde{S}$. Then the number of strategies in $\widetilde{S}_i$ is equal to the number of equivalence classes in R_i w.r.t. R. If two strategies in R_i are not equivalent w.r.t. R then they are not equivalent w.r.t. T. Thus the number of equivalence classes in the restriction R_i of T_i w.r.t. T is as least as large as the number of equivalence classes in R_i w.r.t. R. Then since t_i is not equivalent to any strategy in R_i w.r.t. T there must be at least one extra equivalence class in T_i w.r.t. T, contradiction that $\widetilde{S}$ is a minimal common reduction.

1.3: Since R is in T and T is in P, R is in P. Moreover, since $R \sim T$ and $T \sim P$ then by Lemma $A.3$ $R \sim P$ implying by 1.2 that $R\beta P$.

1.4: Clearly R is in P. By $T\beta P$ for any $p_i \in P_i$ there is $t_i \in T_i$ equivalent to p_i on P_{-i}. Moreover, by $R\alpha T$, there is $r_i \in R_i$ at least as good as t_i on T_{-i}. Now, assume that r_i is not at least as good as p_i on $P_{-i} \backslash T_{-i}$. Let $\overline{p}_{-i} \in P_{-i} \backslash T_{-i}$ be such that $u_i(r_i, \overline{p}_{-i}) < u_i(t_i, \overline{p}_{-i}) = u_i(p_i, \overline{p}_{-i})$. Moreover, for all $j \neq i$ let $\overline{t}_j \in T_j$ be a strategy equivalent to $\overline{p}_j$ w.r.t. P, $\overline{t}_{-i} = (\overline{t}_1, ..., \overline{t}_{i-1}, \overline{t}_{i+1}, ..., \overline{t}_m) \in T_{-i}$. Then $u_i(r_i, \overline{t}_{-i}) = u_i(r_i, \overline{p}_{-i}) < u_i(t_i, \overline{p}_{-i}) = u_i(t_i, \overline{t}_{-i})$, a contradiction. □

Lemma 2 *Let $R\alpha T$. Then $R^k\beta F^k R$ and $T^k\beta F^k T$ for all k.*

Proof: We prove the first claim by induction on k. The claim is trivial for $k = 0$. Assume that $R^k\beta F^k R$ for some $k \geq 0$. Then by Lemma B.1 we have that $FR^k\beta F^{k+1}R$. By Lemma B.3 $R^{k+1}\beta FR^k\beta F^{k+1}R$. Thus by Lemma 1.3 we have $R^{k+1}\beta F^{k+1}R$.

The proof of the second claim is similar. □

Lemma 3 *Let $R\alpha T$. Then $F^k R \sim R^k\alpha F^k T$.*

Proof: If $R\alpha T$ then by Lemma B.2 $R^k\alpha T^k$ and by Lemma 2 $T^k\beta F^kT$ thus by Lemma 1.4 we have $R^k\alpha F^kT$. Hence, by Lemma 2 and Lemma 1.1, $F^kR \sim R^k\alpha F^kT$. $\square$

Lemma 4 *Let $\overline{A} \sim B$ and let $\overline{A}$ be obtained from A by one round of elimination of some weakly dominated strategies. Then if $C \sim A$ there is $\overline{C}$ obtained from C by one round of elimination of some weakly dominated strategies such that $\overline{C} \sim B$.*

Proof: By Lemma A.3 if $\overline{A} \sim \overline{C}$ and $\overline{A} \sim B$ then $B \sim \overline{C}$. Thus it is sufficient to show that if $A \sim C$ and $\overline{A}$ is obtained from A by one round of elimination of some weakly dominated strategies then there is $\overline{C}$ obtained from C by one round of elimination of some weakly dominated strategies such that $\overline{A} \sim \overline{C}$.

For this, let $N(\widetilde{S}, \widetilde{u})$ be a minimal common reduction of A and C (note that $\widetilde{S}$ is equivalent to A and C). Moreover, assume that either all or no strategies within an equivalence class are removed when obtaining $\overline{A}$ from A. We can assume this w.l.o.g. since if some but not all strategies are removed within an equivalence class, then we consider another set of strategies $\widehat{A} \supseteq \overline{A}$ obtained by leaving all strategies within an equivalence class w.r.t. A if at least one member survives in $\overline{A}$. Then C is equivalent to $\widehat{A}$ if and only if C is equivalent to $\overline{A}$ and we can replace $\overline{A}$, with $\widehat{A}$.

Let $f = (f_1, ..., f_m)$ and $g = (g_1, ..., g_m)$ be surjective maps from A and C respectively to $\widetilde{S}$ that gives a common reduction. Let $\overline{S}_i = f_i(\overline{A}_i) \subseteq \widetilde{S}_i$ be the strategies in $\widetilde{S}_i$ which are the image of the strategies that survive in A_i. Let $\overline{S} = \overline{S}_1 \times \cdots \times \overline{S}_m$. Then $\overline{A}$ is equivalent to $\overline{S}$, since $\overline{S}$ is a reduction of $\overline{A}$ via surjective maps $f_i|\overline{A}_i$, $i = 1, ..., m$ and since $\widetilde{u}_i(f_1(s_1), ..., f_m(s_m)) = u_i(s)$ for all i and $s \in A$ then $\widetilde{u}_i|\overline{S}_1(f_1|\overline{A}_1(s_1), ..., f_m|\overline{A}_m(s_m)) = u|\overline{A}_i(s)$ for all i and $s \in \overline{A}$. Now, let $\overline{C}_i \equiv g_i^{-1}(\overline{S}_i)$ for all i. Then $\overline{C} \sim \overline{S}$ and thus $\overline{C} \sim \overline{A}$. Moreover, since the strategies in $A_i \backslash \overline{A}_i$ are weakly dominated w.r.t. A, the strategies in $\widetilde{S}_i \backslash \overline{S}_i$ are weakly dominated w.r.t. $\widetilde{S}$ for all i. Hence the strategies in $C_i \backslash \overline{C}_i$ are weakly dominated w.r.t. C for all i. $\square$

Lemma 5 *Let $\overline{A}^i$ be obtained from A^i by one round of elimination of some weakly dominated strategies, and let $A^{i-1} \sim \overline{A}^i$, $i = 1, ..., k$. Then there is $T \sim A^0$ where T is obtained from A^k in at most k rounds of elimination of some weakly dominated strategies.*

Proof: By induction on k. For $k = 1$, the claim holds with $T = \overline{A}^1$. Assume that the claim holds for some $k > 0$. Now, let $\overline{A}^i$ be obtained from A^i by one round of elimination of some weakly dominated strategies, $A^{i-1} \sim \overline{A}^i$, $i = 1, ..., k+1$. By the induction hypothesis, there is $T \sim A^1$ where T is obtained from A^{k+1} by at most k rounds of elimination of weakly dominated strategies. Then by Lemma 4 there is $\overline{T}$ obtained from T by one round of elimination of weakly dominated strategies such that $\overline{T} \sim A^0$, and since $\overline{T}$ is obtained from A^{k+1} in at most $k+1$ rounds of elimination of weakly dominated strategies the claim holds for $k+1$. □

Theorem 1 *For any k and $L^k(S)$ there is $T \sim F^k(S)$, where T is obtained from $L^k(S)$ by at most k rounds of elimination of weakly dominated strategies.*

Proof: Fix k. Then $FL^{h-1}(S)\alpha L^h(S)$, for all $h = 1, ..., k$. By Lemma 3 then $F^{k-h+1}L^{h-1}(S) \sim A[h]$ where $A[h]$ is obtained from $F^{k-h}L^h(S)$ by one round of elimination of some weakly dominated strategies. Hence by Lemma 5 there is $T \sim F^k(S)$ where T is obtained from $L^k(S)$ by at most k rounds of elimination of weakly dominated strategies. □

Gretlein [6] shows (in his Theorem 2) that $\lim_{k\to\infty} u(F^k(S)) \subseteq \lim_{k\to\infty} u(L^k(S))$. By Theorem 1 above, we obtain the following strengthened version.

Corollary 1 *$u(F^k(S)) \subseteq u(L^k(S))$ for all $k > 0$.*

4 Application I: Strictly competitive games of perfect information

We now consider two-player, strictly competitive games of perfect information. Let $x \in X \backslash x^0$, and let player i be the player called to move at $\alpha(x)$. $G(x)$ is a *weakly dominated subgame* if

$$v_i(\alpha(x)) \geq \max_{z \in Z \cap X(x)} \omega_i(z) \text{ and } v_i(\alpha(x)) > v_i(x).$$

Thus, a proper subgame is weakly dominated if the highest possible outcome within the subgame is not higher than, and the value of the subgame is lower than, the value of the subgame arising from the anterior node (for the player called to move at the anterior node leading to the subgame). We then say that a strategy $s_i \in S_i$ is *weakly subgame dominated* on S_j if there exists $s_j \in S_j$, such that $p(s_i, s_j)$ reaches a weakly dominated subgame. Let $E^k(S) = E_1^k(S) \times E_2^k(S)$ be the set of strategies not subgame dominated on $E^{k-1}(S) = E_1^{k-1}(S) \times E_2^{k-1}(S)$, $E^0(S) \equiv S$. G is *subgame dominance solvable* in k steps if the outcomes in $E^k(S)$ are constant.

Weak subgame dominance is stronger than weak dominance since a weakly subgame dominated strategy s_i is weakly dominated, for example by a strategy s_i' which consists of a maxmin strategy at subgames beginning at nodes leading to dominated subgames and equal to s_i elsewhere. On the other hand, a weakly dominated strategy does not necessarily lead the outcome path to a weakly dominated subgame.

With the definitions in place, we may then find an upper bound for the number of steps necessary to solve a game of perfect information, removing all dominated subgames at each step.

Theorem 2 *Let G be a finite, strictly competitive game of perfect information with at most n outcomes. Then $N(G)$ is subgame dominance solvable in $n-1$ steps.*

We proceed with the following two lemmas.

Lemma 6 *Let G be a finite, strictly competitive game of perfect information with strategy set S. Let $x \in X$ and consider the subgame $G(x)$. Assume that $v_i(x) = \max_{z \in Z \cap X(x)} \omega_i(z)$ for a player i. Let $s \in S$ be a strategy leading the path through x and $\omega_i(s) < v_i(x)$. Then s_i is a weakly subgame dominated strategy.*

Proof: Let $p(s) = (x_0, ..., x_H)$ be the path determined by s. Let $\overline{h} = \max \{h \mid v_i(x_h) = v_i(x), 0 \leq h \leq H\}$. As $v_i(x_{\overline{h}}) > v_i(x_{\overline{h}+1})$, player i is called to move at node $x_{\overline{h}}$. Moreover, $v_i(x_{\overline{h}}) \geq \max_{z \in Z \cap X(x_{\overline{h}+1})} \omega_i(z)$. Thus $G(x_{\overline{h}+1})$ is a weakly dominated subgame and s_i is a weakly subgame dominated strategy. □

Lemma 7 *Let G be a finite, strictly competitive game of perfect information with strategy set S. Assume that $v_i(G) < \max_{z \in Z} \omega_i(z)$ for a player i, and let $s \in S$ be a strategy where $\omega_i(s) = \max_{z \in Z} \omega_i(z)$. Then s is eliminated after two rounds of elimination of weakly subgame dominated strategies.*

Proof: Let $p(s) = (x_0, ..., x_H)$, and $\overline{h} = \max\{h \mid v_i(x_h) \neq \omega_i(s_i, s_j), 0 \leq h \leq H\}$. As $v_i(x_{\overline{h}}) < v_i(x_{\overline{h}+1})$, player j is called to move at $x_{\overline{h}}$. In the subgame $G(x_{\overline{h}+1})$, player i has a strategy that ensures the highest possible outcome within this subgame, that is $v_i(x_{\overline{h}+1}) = \max_{z \in Z \cap X(x_{\overline{h}+1})} \omega_i(z)$. From Lemma 6, after one round of elimination of weakly subgame dominated strategies all remaining strategy pairs reaching the subgame $G(x_{\overline{h}+1})$ yield outcome $\max_{z \in Z \cap X(x_{\overline{h}+1})} \omega_i(z)$ to player i. If $s \in E^1(S)$ then, since $v_j(x_{\overline{h}}) > \max_{z(s') \in Z \cap X(x_{\overline{h}+1}), s' \in E^1(S)} \omega_j(z(s'))$, s_j is a weakly subgame dominated strategy on $E^1(S)$. □

Note that if a strategy is weakly subgame dominated w.r.t. a particular subgame, then all other strategies leading the path to the same dominated subgame for some strategy of the opponent are also subgame dominated. Therefore, if a strategy points to a weakly dominated subgame at some node then is must either be a weakly subgame dominated strategy *or* the node is not reached by any possible strategy of the opponent and is therefore equivalent to another strategy that does not point to the weakly dominated subgame. Thus, up to the removal of redundant strategies, the strategies surviving elimination of weakly subgame dominated strategies are the strategy set of a residual extensive game where all weakly dominated subgames have been removed. We may now complete the proof of Theorem 2.

Proof of Theorem 2: Apply Lemma 7 $\min\{v_1(x^0)-1, n-v_1(x^0)\}$ times, first on G and then sequentially on the residual games where all weakly dominated subgames have been removed. Then apply Lemma 6 once (with $x = x^0$) if necessary and we obtain by Lemma A.3 that G is subgame dominance solvable in at most $n - 1$ steps. □

Combining Theorem 1 and 2 we also obtain the result by Ewerhart [4].

Corollary 2 *Any strictly competitive, finite game of perfect information with n outcomes can be solved by $n - 1$ rounds of elimination of weakly dominated strategies.*

5 Application II: Order independence

As another application, we may observe that a corollary of Theorem 1 is the main result on order independence by Marx and Swinkels [7, 8] for pure strategies. For a procedure L, a reduction $L^k(S)$ is *full* if there are no weakly dominated strategies in $L^k(S)$. For a game satisfying TDI, Marx and Swinkels demonstrate that the result of iterative removal by weak dominance does not depend on order (cf. [7], Corollary 1, p. 230).

By Theorem 1, if $L^k(S)$ is a full reduction, then it must be equivalent to a full reduction obtained by removing all weakly dominated strategies in every round. Thus, we have:

Corollary 3 *Let $L^k(S)$ and $\overline{L}^h(S)$ be full reductions, $k, h > 0$. Then $L^k(S) \sim \overline{L}^h(S)$.*

Proof: If $L^k(S)$ is a full reduction then by Theorem 1 it is equivalent to $F^k(S)$ and $F^k(S)$ is a full reduction. Similarly, if $\overline{L}^h(S)$ is a full reduction then it is equivalent to $F^h(S)$ and $F^h(S)$ is a full reduction. Since $F^k(S) = F^h(S)$ and since $\sim$ is transitive we have $L^k(S) \sim \overline{L}^h(S)$. □

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