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Abstract

This paper studies the evolution of market shares of portfolio rules in incomplete markets with short-lived assets. Prices are determined endogenously. The performance of a portfolio rule in the process of continuous reinvestment of wealth is determined by the market share eventually conquered in competition with other portfolio rules. Using random dynamical systems theory, we derive necessary and sufficient conditions for the evolutionary stability of portfolio rules. In the case of Markov (in particular i.i.d.) payoffs these local stability conditions lead to a simple portfolio rule that is the unique evolutionary stable strategy. This rule possesses an explicit representation. Moreover, it is demonstrated that mean-variance optimization is not evolutionary stable while the CAPM-rule always imitates the best portfolio rule and survives.

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1 Introduction

We consider an incomplete asset market where a finite number of portfolio rules manage capital by iteratively reinvesting in a fixed set of assets. Assets are short lived but identically "re-born" in every period. Their payoffs depend on a stationary process in discrete time. Portfolio rules are nonnegative vectors of expenditure shares for assets which may depend on the past observations. The set of portfolio rules we consider is not restricted to those generated by expected utility maximization. It may as well include investment rules favored by behavioral finance models. Consumption is exogenously determined. In every period in time the available market capital is given by the total payoff of the assets. Portfolio rules compete for this market capital—the endogenous price process provides a *market selection mechanism* along which some strategies gain market capital while others lose. The particular case of Arrow securities and simple portfolio rules has been introduced in the seminal paper of Blume and Easley (1992).

The framework studied here is well suited to analyze the performance of large institutional investors like pension plans, insurance companies or mutual funds. Those institutions are investing on asset markets that do not provide complete insurance against all possible risks. This seems to be the natural case. Savings and withdrawals are exogenously determined because they are the pensions and the indemnities to be payed or they are the liquidity demanded by the clients of the institutions. Institutional investors do have a considerable impact on asset market prices, face relatively small transaction costs, and their investment horizon is potentially infinite. Finally it is not clear at all that institutional investors maximize some infinite horizon expected utility function.

In this paper we derive a description of the market selection process from a random dynamical systems perspective. In each period in time, the evolution of the distribution of market capital (wealth shares) is determined by a map that depends on the exogenous process determining the asset payoffs. An equilibrium in this model is provided by a distribution of wealth shares across portfolio rules that is invariant under the market selection process. It turns out that (provided there are no redundant assets) every invariant distribution of market shares is generated by a *monomorphic* population, i.e. all traders with strictly positive wealth use the same portfolio rule. A criterion for evolutionary stability as well as evolutionary instability is derived for such monomorphic populations. Roughly speaking a portfolio rule is evolutionary stable if it has the highest exponential growth rate in any population where itself determines market prices. This implies that an evolutionary stable investment strategy is robustness against the entry of new portfolio rules. In

a sense an evolutionary stable population plays the "best response against itself."

The stability criterium for the robustness of invariant distributions with respect to the entry of new portfolio rules singles out one portfolio rule, say λ^*, that is the unique *evolutionary stable strategy*, i.e. it drives out any mutations. Moreover, this investment strategy can successfully invade any other invariant distribution. According to this rule one should divide wealth proportionally to the expected relative payoffs of the assets. If asset payoffs are determined by a Markov process this rule is given by an explicit formula which is quite easy to apply in actual markets. In the i.i.d. case it even does not require the knowledge of the actual probabilities driving the payoff process. The sample mean of the payoffs is an unbiased estimator for the future expected payoffs. The effect of this rule on asset prices is that it equalizes the expected relative returns of all assets—in particular asset pricing is *risk neutral*.

Global stability results for a particular version of the evolutionary finance model presented here are given in Evstigneev, Hens, and Schenk-Hoppé (2002).

In the case of Arrow securities and i.i.d. payoffs the investment strategy λ^* becomes the *Kelly rule* (Kelly 1956). This portfolio rule maximizes the expected logarithm of relative returns. Since with Arrow securities "prices do not matter" (in the sense explained in Section 4), this rule coincides with the well known rule "betting your beliefs" (Breiman 1961) according to which income should be divided proportionally to the probability of the states. Blume and Easley (1992) have obtained the result that the investor whose beliefs are closest to the Kelly rule eventually gathers total market wealth.

In the incomplete market case, market prices matter in the evolution of wealth shares. In particular an absolute fitness criterion for investment strategies does not exist and the method used in Blume and Easley (1992) cannot be applied. To overcome this difficulty we exploit the idea of evolutionary stability which so far has not been used in any portfolio theory. The assumption of short-lived assets retained here is not satisfactory with regards to possible applications. It will have to be generalized in future research. For the complete market case Blume and Easley (2000) and Sandroni (2000) have recently investigated the case of long-lived assets in an economy with rational investors who maximize expected discounted utility over an infinite time-horizon. However, as Blume and Easley (2000) made perfectly clear, this result is ultimately linked to Pareto-efficiency. Complete markets are therefore essential in their approach.

We also apply the stability criterium obtained here to demonstrate that mean-variance optimization can be invaded by any completely diversified

portfolio rule while the CAPM-rule, which prescribes buying the market portfolio, is able to always imitate the best portfolio rule and thus survives.

In passing it is worthwhile to mention the relation between the evolutionary portfolio theory literature and the classical finance approach to maximize the expected growth rate of wealth for some exogenously given return process. In a series of papers, Hakansson (1970), Thorp (1971), Algoet and Cover (1988), and Karatzas and Shreve (1998), among others, have explored this maximum growth perspective. Computing the maximum growth portfolio is a non-trivial problem. Even if one restricts attention to i.i.d. returns, when markets are incomplete, there is no explicit solution to this investment problem in general. Numerical algorithms to compute the maximum growth portfolio have been provided by Algoet and Cover (1988) and Cover (1984, 1991), but so far practical decisions are rarely based on these ideas. Our result is interesting also in this respect because the simple portfolio rule that we obtain shows that considering the equilibrium consequences of expected growth rate maximization does not make matters more complicated but indeed much easier.

In the next section we present the economic model which has the mathematical structure of a random dynamical system. Then we define the equilibrium concepts and the stability notions, Section 3. In Section 4 we discuss and generalize Blume and Easley's result to general diagonal markets. Section 5 presents our main results which will be proved using a series of propositions that are also of independent interest. In Section 6 we analyze the evolutionary fitness of portfolio rules based on mean-variance optimization. We study the issue of under-diversified portfolios, and discuss the implication of the CAPM investment strategy. All proofs are relegated to the Appendix.

2   The Model

Time is discrete and indexed by t. The possible states of nature are determined in each period in time by the realization of a stationary stochastic process with values in some measurable space $(S, \mathcal{S})$. Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ denote its canonical realization as a metric dynamical system on the path space, i.e. Ω is the sample path space with representative element $\omega = (..., \omega_{-1}, \omega_0, \omega_1, ...)$, $\mathcal{F} = \mathcal{S}^{\mathbb{Z}}$ is the corresponding σ-algebra, $\mathbb{P}$ is the associated probability measure, and θ is the shift map (defined by $\theta\omega(\cdot) = \omega(1 + \cdot)$). The family θ^t, $t \in \mathbb{Z}$, (where θ^t denotes the t-times iterate of θ) defines a measurable flow on Ω, i.e. $\theta^{t+u} = \theta^t \circ \theta^u$ for all $u, t \in \mathbb{Z}$, $\theta^0 = \mathrm{id}_\Omega$, and θ is measurable and measurably invertible. Stationarity implies $\theta\mathbb{P} = \theta^{-1}\mathbb{P} = \mathbb{P}$. ω_t denotes the state of nature at time t, and the sequence of observations up to the end of

period t is referred to as ω^t. $\mathcal{F}^t = \sigma\{\omega_u \mid u \leq t\}$ denotes the corresponding information set. A sequence of random variables $(\xi_t)_{t \in \mathbb{Z}}$ such that ξ_t is measurable with respect to $\mathcal{F}^t$ is called *adapted* to the filtration $(\mathcal{F}^t)_{t \in \mathbb{Z}}$. By definition, an $\mathcal{F}^t$-measurable random variable can only depend on ω^t.

There are finitely many investors $i = 1, ..., I$ endowed with wealth $w_0^i > 0$ at time 0. Assets $k = 1, ..., K$ with $K \geq 2$, live for one period only but are identically re-born in every period. Their payoffs $A_t^k(\omega)$ are assumed to be adapted. We make the following assumption.

Assumption 1 *For all t, (1) $A_t^k(\omega) \geq 0$ for all k and all ω; (2) for each k there exists a set $\Omega_k \in \mathcal{F}$ with $\mathbb{P}(\Omega_k) > 0$ such that $A_t^k(\omega) > 0$ for all $\omega \in \Omega_k$; and (3) $\sum_{k=1}^{K} A_t^k(\omega) > 0$ for all ω.*

This assumption ensures that all assets yield non-negative payoffs in all states of nature, each asset has a strictly positive payoff for a set of states of non-zero measure, and total payoff of all assets is strictly positive in every state.

In each period in time t every investor selects a portfolio $a_t^i = (a_{1,t}^i, ..., a_{K,t}^i)$ with values in $\mathbb{R}_+^K$. $a_t^i : \Omega \to \mathbb{R}_+^K$ is assumed to be adapted. Given the portfolio a_t^i at time t, the investor's wealth in period $t + 1$ is given by,

$$w_{t+1}^i = \sum_{k=1}^{K} A_{t+1}^k(\omega)\, a_{k,t}^i \tag{1}$$

Letting $\rho_{k,t}$ denote the price of asset k in period t, then—provided that the agent's wealth is positive—his *budget shares* are given by,

$$\lambda_{k,t}^i := \frac{\rho_{k,t}\, a_{k,t}^i}{w_t^i}$$

We define the *trading strategy* of investor i as a sequence of budget shares $\lambda_t^i = (\lambda_{1,t}^i, ..., \lambda_{K,t}^i)_{t \geq 0}$. Since a_t^i and $\rho_{k,t}$ are adapted for all t so is each budget share λ_t^i.

Assuming that every investor exhausts his budget in all periods in time, i.e. the portfolio is chosen such that $\sum_{k=1}^{K} \rho_{k,t}\, a_{k,t}^i = w_t^i$ for all $t \geq 0$, every trading strategy λ_t^i takes values in the unit simplex $\Delta^K := \{x \in \mathbb{R}_+^K | \sum_{k=1}^{K} x_k = 1\}$.

The market-clearing prices are given by,

$$\rho_{k,t} = \frac{1}{S_t^k} \sum_{i=1}^{I} \lambda_{k,t}^i\, w_t^i \tag{2}$$

where $S_t^k > 0$, assumed to be adapted, is the total exogenous supply of asset k at time t. There are no transaction costs. For the market selection process

to be well defined, we need to guarantee that equilibrium prices ρ_t are always positive. A sufficient condition for this is that some trading strategy with positive initial wealth is *completely mixed*, i.e. it has only strictly positive budget shares in every period in time and in every state of nature. We make the following assumption.

Assumption 2 *In every market there is some trading strategy λ_t^i with initial wealth $w_0^i > 0$ that is completely mixed, i.e. $\lambda_t^i(\omega) \in \text{int}\Delta^K$ for all $\omega \in \Omega$.*

It is clear from Assumption 1 and equation (1) that any completely mixed trading strategy with strictly positive wealth in a period of time maintains strictly positive wealth in all future periods. This ensures that prices are always well-defined.

Taking into account how equilibrium prices are determined, we obtain a recursive formula for the total wealth of each consumer. Consumer i's wealth in period $t + 1$ is given by,

$$w_{t+1}^i = \sum_{k=1}^K A_{t+1}^k(\omega)\, S_t^k\, \frac{\lambda_{k,t}^i\, w_t^i}{\sum_{j=1}^I \lambda_{k,t}^j\, w_t^j} \tag{3}$$

and the total market wealth in period $t+1$, $W_{t+1} := \sum_i w_{t+1}^i$, can be equated as,

$$W_{t+1} = \sum_{i=1}^I w_{t+1}^i = \sum_{k=1}^K A_{t+1}^k(\omega)\, S_t^k \tag{4}$$

Note that in the definition of next period wealth we have assumed that no investor saves or withdraws any wealth. Our results carry over to the case of identical saving rates.

The prices of the assets, normalized by the market wealth, are given by,

$$q_{k,t} := \frac{\rho_{k,t}}{W_t} = \frac{1}{S_t^k} \sum_{i=1}^I \lambda_{k,t}^i\, \frac{w_t^i}{W_t} \tag{5}$$

i.e. the normalized price is a convex combination of the trading strategies for asset k over the wealth shares w_t^i/W_t of investors. $q_{k,t}$ is adapted.

From (3) and (4) we obtain a recursive formula for the evolution of the wealth (or market) shares $r_t^i := w_t^i/W_t$,

$$r_{t+1}^i = \sum_{k=1}^K \frac{A_{t+1}^k(\omega)\, S_t^k}{\sum_{l=1}^K A_{t+1}^l(\omega)\, S_t^l} \frac{\lambda_{k,t}^i\, r_t^i}{\sum_{j=1}^I \lambda_{k,t}^j\, r_t^j} \tag{6}$$

Finally we define the relative payoff of asset k as,

$$R^k_{t+1}(\omega) := \frac{A^k_{t+1}(\omega) \, S^k_t}{\sum_{l=1}^K A^l_{t+1}(\omega) \, S^l_t}$$

Assumption 3 *(1) The relative payoff of each asset is a stationary random variable, i.e. $R^k_t(\omega) = R^k(\omega_t) = R^k(\theta^t\omega)$.*
(2) All strategies are stationary, i.e. $\lambda^i_t(\omega) = \lambda^i(\omega^t) = \lambda^i(\theta^t\omega)$.

For any given set of stationary adapted strategies $(\lambda^i)_{i=1,\dots,I}$ the evolution of wealth shares (6) can be written as,

$$r_{t+1} = f(\theta^{t+1}\omega, r_t) \tag{7}$$

where

$$f_i(\theta^{t+1}\omega, r) = \sum_{k=1}^K R^k(\theta^{t+1}\omega) \, \frac{\lambda^i_k(\theta^t\omega) \, r^i}{\sum_{j=1}^I \lambda^j_k(\theta^t\omega) \, r^j} \tag{8}$$

We refer to equation (7) as the *market selection process* in the following.

The dynamical description of the market selection process employs the framework of *random dynamical systems* (Arnold 1998). The market selection process (7) generates a random dynamical system in the following sense. Let $f(\omega) := f(\omega, \cdot) : \Delta^I \to \Delta^I$. Define $\varphi(t, \omega, r) = f(\theta^t\omega) \circ \dots \circ f(\theta\omega)r$ for all $t \geq 1$, and $\varphi(0, \omega, r) = r$. In words, $\varphi(t, \omega, r)$ is the vector of wealth shares of all investors at time t when the initial distribution of market shares is r and the sequence of realizations of states ω prevails.

The family of maps $\varphi(t, \omega, r)$ form a *random dynamical system* on the unit simplex Δ^I. That is, $\varphi : \mathbb{N} \times \Omega \times \Delta^I \to \Delta^I$, $(t, \omega, r) \mapsto \varphi(t, \omega, r)$ is a $\mathcal{B}(\mathbb{N}) \otimes \mathcal{F} \otimes \mathcal{B}(\Delta^I), \mathcal{B}(\Delta^I)$ measurable[1] mapping such that $\varphi(0, \omega) = \mathrm{id}_{\Delta^I}$ and $\varphi(s + t, \omega) = \varphi(t, \theta^s\omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{N}$ and all $\omega \in \Omega$. We refer the reader to the monograph by Arnold (1998) for any additional information.

It is important to emphasize that the random dynamical system generated by (7) depends on the trading strategies pursued by the investors. That is, for any set of strategies (λ^i) there is a unique random dynamical system generated by (7).

It is straightforward to see that there is no need to consider the evolution of market shares on the level of individual investors. For example, suppose two investors with strictly positive wealth follow the same portfolio rule, say $\lambda^1 = \lambda^2$. Then the ratio of both investors' wealth does not change over time because, in every period in time, they hold portfolios that are identical

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[1]$\mathcal{B}$ denotes the Borel σ-algebra, and $\mathcal{B}(\Delta^I) := \mathcal{B}(\mathbb{R}^I \cap \Delta^I)$ is the trace σ-algebra.

up to a multiple. The dynamics of the market selection process does not change qualitatively if we identify investors with the same portfolio rule. The economic justification is that these investors can set up a fund which is owned according to the proportions of the initial contribution.

3   Evolutionary Stability

In this section we introduce the stability concepts needed to analyze the long term behavior of the wealth shares under the market selection process.

Given a random dynamical system for a set of stationary and adapted trading strategies (λ^i), one is particularly interested in those wealth shares that evolve in a stationary fashion over time. Here we restrict ourselves to deterministic distributions of market shares that are fixed under the market selection process (7).[2] To specify this notion, we recall the definition of a deterministic fixed point in the framework of random dynamical systems. Let a set of strategies (λ^i) be given, and denote by φ the associated random dynamical system.

Definition 1 $\bar{r} \in \Delta^I$ *is called a (deterministic)* fixed point *of φ if, for all $\omega \in \Omega$ and all t,*

$$\bar{r} = \varphi(t, \omega, \bar{r}). \tag{9}$$

The distribution of market shares $\bar{r}$ is said to be invariant *under the market selection process (7).*

By the definition of $\varphi(t, \omega, r)$ the condition (9) is equivalent to $\bar{r} = \varphi(1, \omega, \bar{r})$ for all ω, i.e. a deterministic state is fixed under the one-step map if and only if it is fixed under all t-step maps. A technical remark is in order. The assumption of stationarity discharges us from using the common "almost surely," because any condition or result with this additional restriction can be transferred into a "for all ω" statement by restricting the space Ω to an invariant subset of full $\mathbb{P}$-measure. This claim holds here because time is discrete.

If $r^i = 0$, then $\varphi^i(t, \omega, r) = 0$ by (7). Therefore, in any set of trading strategies each unit vector in Δ^I (i.e. each vertex) is a fixed point. In words, the state in which one investor possesses the entire market does not change over time.

We are particularly interested in those invariant distributions of market shares which are stable under the market selection process. Loosely speaking,

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[2]See e.g. Schenk-Hoppé (2001) for an application of stochastic invariant distributions—commonly referred to as random fixed points.

stability means that small perturbations of the initial distribution of wealth shares do not have a long-run effect. If an invariant distribution of wealth shares is stable, every sample path starting in a neighborhood of this invariant distribution at time zero is asymptotically identical to the sample path of the invariant distribution of the wealth shares. We will need different notions of stability; they are defined as follows.

Definition 2 *An invariant distribution of market shares $\bar{r} \in \Delta^I$ is called* stable, *if $\lim_{t \to \infty} \|\varphi(t, \omega, r) - \bar{r}\| = 0$ for all r in a neighborhood of $\bar{r}$ for all ω. The neighborhood may depend on ω.*

Given a locally stable invariant distribution of wealth shares $\bar{r}$, then any initial distribution of market shares in a small neighborhood of $\bar{r}$ is asymptotically identical to $\bar{r}$ as time tends to infinity. It is crucial to allow dependence of the neighborhood on ω in the above definition. In general there is no proper *deterministic* neighborhood of a fixed point $\bar{r}$ that is attracted to $\bar{r}$, i.e. the intersection of all (maximal) neighborhoods can only consist of $\bar{r}$. This point is illustrated in Schenk-Hoppé (2002, Section 4.2).

As discussed above, a distribution of market shares r is a distribution over populations of investors pursuing the same strategy within each group. Thus the market share r^i denotes that fraction of the total market wealth belonging to the players of strategy λ^i. Under this assumption it is clear that all strategies λ^i are different from each other, i.e. $\lambda^i \neq \lambda^{i'}$ for all i, i', $i \neq i'$.

The above definition refers to the stability of a distribution of wealth in a population with given strategies. However, one would also like to have a notion of stability in the case that new strategies occur on the market. We first note that the structure of the market selection process (7) implies the following extension property. Let $(\lambda^i)_{i \in \mathcal{I}}$, $\mathcal{I} = \{1, ..., I\}$, be any set of investment strategies. Suppose $\bar{r}$ is an invariant distribution of wealth shares for the corresponding random dynamical system on Δ^I. Then for any set $(\lambda^j)_{j \in \mathcal{J}}$, $\mathcal{J} = \{1, ..., J\}$ with $J \geq 0$ ($\mathcal{J} = \emptyset$, if $J = 0$), of strategies, $(\bar{r}, 0, ..., 0) \in \Delta^{I+J}$ (J-times zero) is an invariant distribution of wealth shares for the random dynamical system on Δ^{I+J} associated to the set of strategies $((\lambda^i)_{i \in \mathcal{I}}, (\lambda^j)_{j \in \mathcal{J}})$.

In the following definition we assume that for a given set of strategies $(\lambda^i)_{i \in \mathcal{I}}$, any set of strategies $(\lambda^j)_{j \in \mathcal{J}}$ is distinct in the sense that with strictly positive probability (1) $\lambda^j \neq \lambda^i$ for all $j \in \mathcal{J}$, $i \in \mathcal{I}$ and (2) $\lambda^j \neq \lambda^{j'}$ for all $j, j' \in \mathcal{J}$, $j \neq j'$, i.e. a set of new and distinct strategies is added.

Definition 3 *An invariant distribution of market shares $\bar{r} \in \Delta^I$ (for given trading strategies $(\lambda^i)_{i \in \mathcal{I}}$) is called* evolutionary stable, *if for all $J \geq 0$ and*

all $(\lambda^j)_{j\in\mathcal{J}}$, $(\bar{r}, 0, ..., 0) \in \Delta^{I+J}$ is stable for the random dynamical system with trading strategies $((\lambda^i)_{i\in\mathcal{I}}, (\lambda^j)_{j\in\mathcal{J}})$.

A trading strategy is called evolutionary stable, *if the invariant distribution of market shares* $1 \in \Delta^1$ *is evolutionary stable.*

For each evolutionary stable distribution of market shares there exits an entry barrier (a random variable here) below which an arbitrary number of new portfolio rules do not drive out the incumbent players. Any perturbation, if sufficiently small, does not change the long-run behavior of the distribution of market shares. The market selection process asymptotically leaves the mutants with no market share while the market is shared between the incumbents as unchanged. Note that stability refers to the distribution of market shares and not to the set of strategies. It may well be the case that for a given set of strategies there are two different stable invariant distributions of wealth shares one of which being evolutionary stable and the other not.

Finally, a corresponding local stability criterion is introduced.

Definition 4 *An invariant distribution of market shares* $\bar{r} \in \Delta^I$ *is called* locally evolutionary stable, *if for all* $J \geq 0$ *there exists a random variable* $\delta(\omega) > 0$ *such that* $(\bar{r}, 0, ..., 0) \in \Delta^{I+J}$ *is locally stable for all sets of portfolio rules* $((\lambda^i)_{i\in\mathcal{I}}, (\lambda^j)_{j\in\mathcal{J}})$ *with* $\min_{i\in\mathcal{I}} \max_{j\in\mathcal{J}} \|\lambda^i(\omega) - \lambda^j(\omega)\| < \delta(\omega)$ *for all* ω.

A portfolio rule is called locally evolutionary stable, *if the invariant distribution of market shares* $1 \in \Delta^1$ *is locally evolutionary stable.*

A locally evolutionary stable distribution of market shares is evolutionary stable with respect to local mutations. That is, the strategies that can be pursued by all mutants are limited to small deviations from existing strategies.

4 Diagonal Payoff Markets

This section extends the selection results for simple strategies in a complete market of Arrow securities and finitely many states given by Blume and Easley (1992). We rephrase their model and results in the framework of random dynamical systems theory—allowing for stationary portfolio rules and an arbitrary ergodic payoff process with a general set of states. However we retain the assumption that asset payoffs are diagonal.

Let $(S_k)_{k=1,...,K}$ be a measurable partition of S into sets with strictly positive measure, i.e. $p_k := \mathbb{P}(... \times S \times S_k \times S \times ...) > 0$ for all k. Then $R^k(\omega) \in \{0, 1\}$ and $R^k(\omega) = 1$ if and only if $\omega_0 \in S_k$. A trading strategy of

investor i is a stationary random variable $\lambda^i : \Omega \to \Delta^K$, $\lambda_k^i(\theta^t \omega)$ being her wealth share invested in asset k at time t. By the diagonal payoff structure we may unambiguously denote by $\lambda_{\omega_{t+1}}^i$ the wealth share invested in that asset k with $\omega_{t+1} \in S_k$, i.e. $\lambda_{\omega_{t+1}}^i = \sum_k R^k(\omega_{t+1}) \lambda_k^i(\omega^t)$.

Due to the assumption of diagonal payoffs (7) simplifies to,

$$ r_{t+1}^i = \frac{\lambda_{\omega_{t+1}}^i \, r_t^i}{\sum_{j=1}^I \lambda_{\omega_{t+1}}^j \, r_t^j} $$

The evolution of the ratio of the market shares of any two investors, say i and j, using completely mixed trading strategies can then be written as,

$$ \frac{r_{t+1}^i}{r_{t+1}^j} = \frac{\lambda_{\omega_{t+1}}^i}{\lambda_{\omega_{t+1}}^j} \, \frac{r_t^i}{r_t^j} $$

because the normalized asset price, $\sum_{j=1}^I \lambda_{\omega_{t+1}}^j \, r_t^j$, cancels out for diagonal securities.

Fix any initial market shares $r_0^i > 0$ and $r_0^j > 0$. Then the asymptotic behavior of the ratio of the two market shares is given by,

$$ \lim_{T \to \infty} \frac{1}{T} \, \ln \frac{r_T^i}{r_T^j} = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^T \ln \frac{\lambda_{\omega_t}^i}{\lambda_{\omega_t}^j} = \mathbb{E} \ln \frac{\lambda_{\omega_0}^i}{\lambda_{\omega_0}^j} $$

The equality on the far right-hand side of the last equation holds by the Birkhoff ergodic theorem. The expected value is finite if the strategies are completely mixed. Consequently we obtain,

$$ \lim_{T \to \infty} \frac{1}{T} \, \ln \frac{r_T^i}{r_T^j} > 0 \ \text{ if and only if } \ \mathbb{E} \ln \lambda_{\omega_0}^i > \mathbb{E} \ln \lambda_{\omega_0}^j \tag{10} $$

The equation on the left-hand side implies that for all small enough $\varepsilon > 0$, $\ln r_T^i(\omega) \geq T\varepsilon + \ln r_T^j(\omega)$ for large T. Since $\ln r_T^i(\omega) \leq 0$ for all T and all ω, $\ln r_T^j(\omega) \to -\infty$ as $T \to \infty$. Thus, we find that $r_T^i(\omega) \to 1$.

This result implies the following asymptotic behavior of the market selection process. Those investors who are closest to maximizing the expected logarithm of the wealth shares will eventually dominate the market—regardless of the initial distribution of wealth shares in the population. Equation (10) is an absolute fitness criterion in the sense that it assigns a value to any investment strategy which is independent of the population under consideration. However, note that even in the case discussed here the surviving population depends on the strategies present in the population.

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The best choice an investor can make in a period t, were strategies not necessarily adapted, is to set $\lambda_k^i = 1$ (and $\lambda_n^i = 0$ for all $n \neq k$) if and only if $\omega_{t+1} \in S_k$. However this requires information from period $t+1$.

Among all adapted strategies the optimal portfolio rule is given by the random variable $\lambda(\omega) = \lambda(\omega^0) \in \Delta^K$ that maximizes

$$\mathbb{E}\Big[\ln \Big(\sum_{k=1}^K R^k(\theta\omega)\, \lambda_k(\omega) \Big) \mid \mathcal{F}^0 \Big] = \mathbb{E}[\ln(\lambda_{\omega_1}) \mid \mathcal{F}^0] \tag{11}$$

where $\lambda_{\omega_1} = \lambda_k(\omega^0)$ if and only if $\omega_1 \in S_k$.

Two particular cases are considered in detail. Let the state of nature follow an i.i.d. process, then the payoffs $R^k(\theta\omega)$ are independent of the past $\mathcal{F}^0$. Thus (11) becomes $\sum_{k=1}^K p_k \ln \lambda_k(\omega)$. Maximizing this expression over all adapted strategies, one finds $\lambda_k^\star = p_k$. The right-hand side of equation (10) becomes

$$\sum_{k=1}^K p_k \ln \lambda_k^i > \sum_{k=1}^K p_k \ln \lambda_k^j$$

When the number of states and assets is the same, i.e. $K = S$, Blume and Easley (1992) case is obtained. Consequently—as in Blume and Easley (1992)—we obtain that those investors who are closest to maximizing the expected logarithm of the budget shares λ_k will eventually dominate the market. The strategy maximizing the expected logarithm of the budget shares λ_k is "betting your beliefs," i.e. $\lambda_k = p_k$ for all k. Notice that it is not important to have a correct assessment of the probability of the states of nature but only of the event S_k in which the corresponding asset pays off.

Finally suppose, more generally, that the state of nature is determined by a Markov process with time-homogenous transition probability $P(\omega_{t+1}, \omega_t)$. Then (11) becomes

$$\sum_{k=1}^K P(S_k, \omega_0) \ln \lambda_k(\omega)$$

Thus $\lambda_k^\star(\omega) = P(S_k, \omega_0)$, i.e. the portfolio rule depends on the last observation ω_0. Stationarity gives $\lambda_k^\star(\theta^t \omega) = P(S_k, \omega_t)$. Any further information from the history is not helpful in the Markov case. It is obvious that $\lambda_k^\star(\theta^t \omega) = P(S_k, \omega_t)$ is the unique long-run outcome of the market-selection process in competition with any other portfolio rules.

5 The Main Results

We now turn to the study of the long-run outcome of the market selection process in the general incomplete market case. In this section it is shown that there exists a unique evolutionary stable strategy in any incomplete market. This portfolio rule has an explicit representation in the Markov (in particular i.i.d.) case. Moreover, we can show that any other investment strategy is not even locally evolutionary stable.

As pointed out above, we restrict our analysis to invariant deterministic distributions of wealth shares. It is first proved that these invariant distributions correspond to monomorphic populations if there are no redundant assets. We then derive necessary conditions for the (local) stability and instability of such invariant wealth distributions in the general case in which portfolio rules are adapted. The central mathematical tool is the multiplicative ergodic (or Oseledets's) theorem for random dynamical systems.

These conditions are used to single out one particular portfolio rule $\lambda^\star$ as being evolutionary stable if all investors use simple portfolio rules, i.e. strategies that are independent of past observations. In the i.i.d. case we prove that $\lambda^\star$ is the only evolutionary stable portfolio rule in the set of all adapted strategies. We further show that any other portfolio rule can be driven out even by portfolio rules arbitrary close to it, i.e. it is not even locally evolutionary stable.

With a general payoff matrix we can no longer benefit from the cancellation of prices in the evolution of relative wealth shares (an essential property that was used in Section 4) and there are some important conceptual differences to the case of diagonal securities. In contrast to that case there is no longer an absolute fitness criterion for the survival of trading strategies. The growth rate of any trading strategy now depends essentially on the population in which it lives. Restricting attention to the question of local stability of deterministic invariant distributions circumvents these problems and is still sufficient to single out a unique evolutionary stable trading strategy. Before presenting the main results of the paper, we derive two auxiliary results that are also of independent interest.

We make the following assumption throughout the remainder of the paper.

Assumption 4 *There are no redundant assets in the sense that different portfolios do not yield the same payoff almost surely, i.e. for any two portfolios $a^1, a^2 : \Omega \to \Delta^K$ with $a^1(\omega) \neq a^2(\omega)$ on a set of strictly positive measure, $\sum_k R^k(\theta\omega)(a_k^1(\omega) - a_k^2(\omega)) \neq 0$ on a set of strictly positive measure.*

Let us address two specific cases. Suppose there are only finitely many

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states of nature $s \in S$ and the payoff matrix $(R^k(s))_k^s$ has full rank. If the state of nature is determined by an i.i.d. process, then Assumption 4 is satisfied if each state s occurs with strictly positive probability, i.e. $p_s > 0$ for all $s \in S$. If the state of nature follows a Markov process, Assumption 4 is satisfied in particular if all transition probabilities $p_{s,u} > 0$, $s, u \in S$. In the Markov case, Assumption 4 corresponds to the notion of the absence of conditionally redundant assets, see Evstigneev, Hens, and Schenk-Hoppé (2002).

We have already noted that every distribution of market shares in which the players of only one portfolio rule possess the entire market wealth is invariant under the market selection process (and is a deterministic fixed point). Moreover, if there are no redundant assets there is also a converse to this observation as the following result shows.

Proposition 1 *Only one portfolio rule can have strictly positive wealth in every population of strategies with a (deterministic) invariant distribution of wealth shares.*

In Proposition 1 all deterministic invariant distributions of wealth shares are characterized. We next derive a sufficient condition for the stability of such fixed points under Assumptions 1–4. The following result is central to the proof of the main results.

Proposition 2 *Let the state of nature be determined by an ergodic process. Given any set of adapted portfolio rules (λ^i). The invariant distribution of market shares $\bar{r} = e_n$ being concentrated on the players of the completely mixed n-th strategy is*
(i) stable, if

$$\mathbb{E} \ln \left(\sum_{k=1}^{K} R^k(\theta\omega) \frac{\lambda_k^i(\omega)}{\lambda_k^n(\omega)} \right) < 0 \quad \text{for all } i \neq n; \tag{12}$$

(ii) unstable, if

$$\mathbb{E} \ln \left(\sum_{k=1}^{K} R^k(\theta\omega) \frac{\lambda_k^i(\omega)}{\lambda_k^n(\omega)} \right) > 0 \quad \text{for some } i \neq n. \tag{13}$$

The conditions in Proposition 2 have the following interpretation. In a situation in which the prices of all assets are determined by the stationary portfolio rule λ^n we can measure the exponential growth rate of other, competing portfolio rules. If the invariant distribution of wealth shares $\bar{r} = e_n$

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is stable then the strategy λ^n, which completely determines the prices, has a higher growth rate in a neighborhood of this distribution of market shares than all other portfolio rules in the population. However, if there is at least one strategy that has a higher growth rate for these prices, $\bar{r} = e_n$ is unstable and the λ^n-player does not reobtain total market wealth after a slight deviation from the possessing-everything situation. From the condition (13) any monomorphic population playing $\lambda^n(\omega)$ can be successfully invaded by a trader maximizing the expected logarithm given the prices $\lambda^n(\omega)$. In particular any incumbent who maximizes expected utility according to incorrect beliefs $\tilde{p}$ will be driven out by the trader maximizing expected logarithmic utility according to the correct beliefs p.

The quantities on the left-hand side of (12) resp. (13) are the Lyapunov exponents of the linearization of the random dynamical system generated by (7) (the market selection mechanism) at the vertex of the simplex Δ^I, i.e. the fixed point in which only one investor holds total market wealth. The eigenspaces correspond to the vertices of the simplex.

Note that stability resp. instability of a status quo strategy λ^n is determined by pair-wise comparisons with all other strategies in the population. This is due to the fact that the linearization at each vertex of the simplex Δ^I is a diagonal matrix. Evolutionary stability of a strategy λ^n thus means that condition (i) in Proposition 2 is satisfied for all adaptive strategies $\lambda^i \neq \lambda^n$. There is no need to check for stability of λ^n in any possible pool of strategies.

The main result of our paper is based on the observation that, allowing for all possible mutations, only one particular strategy satisfies the necessary condition for stability derived in Proposition 2.

Denote by

$$g(\lambda^i, \lambda^n) := \mathbb{E} \ln \left(\sum_{k=1}^{K} R^k(\theta\omega) \frac{\lambda_k^i(\omega)}{\lambda_k^n(\omega)} \right) \tag{14}$$

the exponential growth rate appearing on the left-hand side of (12) and (13).

Suppose there is some portfolio rule $\lambda^\star$ such that $g(\lambda, \lambda^\star) < 0$ for all $\lambda \neq \lambda^\star$. Then we obtain

$$g(\lambda^\star, \lambda) = \mathbb{E} \ln \left(\sum_{k=1}^{K} R^k(\theta\omega) \frac{\lambda_k^\star(\omega)}{\lambda_k(\omega)} \right) \geq \mathbb{E} \ln \left(\sum_{k=1}^{K} R^k(\theta\omega) \frac{\lambda_k(\omega)}{\lambda_k^\star(\omega)} \right)^{-1} > 0$$

by the Jensen inequality because, for every ω, $R^k(\theta\omega)$ is a probability measure on $\{1, ..., K\}$. Summarizing this result, we can state the following corollary.

Corollary 1 *Suppose all investors employ completely mixed portfolio rules. Then existence of a portfolio rule that is evolutionary stable implies that all other portfolio rules cannot be evolutionary stable.*

The task is to show that there is an evolutionary stable portfolio rule and to give an explicit formula for this rule.

Assume momentarily that the state of nature is determined by an i.i.d. process. Then the relative payoff of asset k, $R^k(\theta^{t+1}\omega) = R^k(\omega_{t+1})$, is independent of the past realizations of the state of nature. The probability measure on the sample path space $\Omega = S^{\mathbb{Z}}$ is given by the product measure $\mathbb{P} = \mu^{\mathbb{Z}}$, where μ is the distribution of the state in each period in time. In this case,

$$g(\lambda^i, \lambda^n) = \int_{S^{\mathbb{N}}} \int_S \ln\left(\sum_{k=1}^K R^k(s) \frac{\lambda_k^i(\omega^0)}{\lambda_k^n(\omega^0)}\right) \mu(ds)\, \mu^{\mathbb{N}}(d\omega^0) \qquad (15)$$

where $\omega^0 = (..., \omega_{-1}, \omega_0)$ is the observed history at time zero.

If investors do not base their portfolio decision on any past observations (for instance because they know that the state is determined by an i.i.d. process), their portfolio rule is determined by a *simple trading strategy*, i.e. a deterministic vector of budget shares $(\lambda(\omega) \equiv \lambda \in \Delta^K)$. Under this assumption the above equation simplifies to

$$g(\lambda^i, \lambda^n) = \int_S \ln\left(\sum_{k=1}^K R^k(s) \frac{\lambda_k^i}{\lambda_k^n}\right) \mu(ds) \qquad (16)$$

If we want to check for evolutionary stability of a simple strategy λ^n in the set of all simple strategies, we need to show that condition (i) in Proposition 2 holds for all $\lambda^i \in \Delta^K$ with $\lambda^i \neq \lambda^n$, i.e. the term in (16) is strictly less than zero. Note that for fixed $\lambda^n \in \text{int}\Delta^K$ (or for fixed $\lambda^i \in \text{int}\Delta^K$) (16) defines a map from Δ^K to $\mathbb{R}$. This observation is also true for both conditions in Proposition 2 if the state process is ergodic and investors only employ simple strategies.

The situation is more complicated if, for instance, the status quo strategy λ^n depends on the observation ω^0. Then the quantity (15) also depends on the past—albeit the payoffs of all assets are independent of the past. The stability condition in Proposition 2 depends on the past only through the investor's belief that past observations matter in choosing a portfolio.

If we want to check for evolutionary stability of a strategy λ^n in this general case it suffices prove the following. For every possible past ω^0 and all deterministic vectors $\lambda^i \in \Delta^K$ it holds

$$\int_S \ln\left(\sum_{k=1}^K R^k(s) \frac{\lambda_k^i}{\lambda_k^n(\omega^0)}\right) \mu(ds) \leq 0$$

and the inequality is strict on a set of strictly positive $\mathbb{P}$-measure. Thus we are back to check stability for simple strategies. And the above equation also defines a map from Δ^K to $\mathbb{R}$.

To check for the absence of evolutionary stability of a strategy $\lambda^n(\omega^0)$ it is sufficient to show that for every possible past ω^0 there exists a deterministic vector $\lambda^i \in \Delta^K$ such that

$$\int_S \ln \left(\sum_{k=1}^K R^k(s) \frac{\lambda_k^i}{\lambda_k^n(\omega^0)} \right) \mu(ds) \geq 0$$

and the inequality is strict on a set of strictly positive $\mathbb{P}$-measure. This defines a map $\omega^0 \mapsto \lambda^i$. The above inequality implies that the (instability) condition (ii) in Proposition 2 is fulfilled for $\lambda^i(\omega^0)$. However, it is important to point out that one has to make sure that the resulting map is measurable. This is true without any further assumptions, for instance, if $\{s\} \in \mathcal{S}$ for all $s \in S$. Then every past $\omega^0 \in \mathcal{F}^0$.

Let us finally discuss the Markovian case in which the state of nature is determined by a Markov process with transition probability P. We find

$$g(\lambda^i, \lambda^n) = \int_{S^\mathbb{N}} \int_S \ln \left(\sum_{k=1}^K R^k(s) \frac{\lambda_k^i(\omega^0)}{\lambda_k^n(\omega^0)} \right) P(ds, \omega_0) \, \mathbb{P}^0(d\omega^0) \qquad (17)$$

where $\mathbb{P}^0$ is the distribution of the sample path $\omega^0 \in S^\mathbb{N}$.

If investors only employ Markovian strategies, i.e. $\lambda(\omega^0) = \lambda(\omega_0)$ depends only on the last state of nature, then (17) simplifies to

$$g(\lambda^i, \lambda^n) = \int_S \int_S \ln \left(\sum_{k=1}^K R^k(s) \frac{\lambda_k^i(u)}{\lambda_k^n(u)} \right) P(ds, u) \, \mu(du) \qquad (18)$$

where μ is the stationary distribution of the state of nature in each period. The (absence of) evolutionary stability of a strategy can be checked analogously to the procedure explained for the i.i.d. case.

We have the following result.

Theorem 1 *Let the state of nature be determined by an ergodic process. Suppose investors only employ simple strategies, i.e. $\lambda(\omega) \equiv \lambda \in \Delta^K$. Then the simple strategy $\lambda^\star$ defined by,*

$$\lambda_k^\star = \mathbb{E} R^k(\omega) \qquad (19)$$

for $k = 1, ..., K$ is evolutionary stable, and no other strategy is locally evolutionary stable.

The portfolio rule $\lambda^\star$ divides wealth according to the expected relative payoffs of the assets. For a given asset market of the structure discussed in this paper, the strategy is very simple to compute; it requires a minimum of easily accessible information.

Let us consider the case in which the state of nature can only take finitely many values $s = 1, ..., S$ in detail. Under the assumption of stationarity $\mathbb{P}\{\omega \mid \omega_t = s\} \equiv p_s$ with $p_s > 0$ by assumption. Therefore, the portfolio rule $\lambda^\star$ in Theorem 1 becomes $\lambda_k^\star = \sum_{s=1}^{S} p_s R^k(s)$. It is straightforward to see that, in the case of diagonal securities, we reobtain the result by Blume and Easley (1992, Section 3). In this case $\lambda^\star$ corresponds to the Kelly rule of "betting one's beliefs."

The assumption in Theorem 1 that all investors employ simple strategies even though the state is determined by a general ergodic process can be criticized. However, recall that even if the state is i.i.d. the past observations can enter into the market selection mechanism (and thus in the condition for local (in)stability) through investors' beliefs. That is, if a trader adopts a proper adapted portfolio rule then the past matters. Note that the stability criteria in Proposition 2 require an integration over the entire history of the process because the strategies may depend on the past observations.

We have the following general result in the i.i.d. case.

Theorem 2 *Let the state of nature be determined by an i.i.d. process. Then $\lambda_k^\star = \mathbb{E}R^k$, $k = 1, ..., K$, is the only evolutionary stable portfolio rule.*

Moreover, if $\mathcal{S}$ is the power set of the set of states S, then we find that all other completely mixed adapted strategies are not even locally evolutionary stable.

The technical assumption that $\mathcal{S}$ is the power set of the set of states S is fulfilled, for instance, if S is countable (or finite) and $\mathcal{S}$ is the Borel σ-field. We need this condition to ensure measurability of a strategy that is constructed in the proofs of the next two results to ensure that no portfolio rule different to $\lambda^\star$ can be locally evolutionary stable.

Theorem 3 generalizes this result to the Markovian case. The proof of Theorem 3 is a straightforward extension of the proofs of Theorems 1 and 2 and is therefore omitted.

Theorem 3 *Let the state of nature be determined by a Markov process (with transition probability P). Then the adaptive strategy $\lambda^\star$ defined by,*

$$\lambda_k^\star(\omega_0) = \mathbb{E}(R^k(\omega_1) \mid \omega_0) = \int_S R^k(s) \, P(ds, \omega_0) \tag{20}$$

for $k = 1, ..., K$ is the only evolutionary stable portfolio rule.

Moreover, if $\mathcal{S}$ is the power set of the set of states S, then we find that all other completely mixed adapted strategies are not even locally evolutionary stable.

The strategy $\lambda^\star$ can be interpreted as a Nash equilibrium in the following way. For simplicity we restrict the discussion to the i.i.d. case. Recall the definition of the function g in (16). $g(\alpha, \beta)$ measures the asymptotic exponential growth rate of a strategy α in a population in which all asset prices are determined by strategy β. Using Proposition 2, the assertion of Theorem 1 can be stated as:

For all $\alpha \neq \lambda^\star$,

$$
\begin{aligned}
g(\lambda^\star, \lambda^\star) &> g(\alpha, \lambda^\star) \quad \text{and;} \\
g(\alpha, \alpha) &< g(\beta, \alpha) \quad \text{for some } \beta \text{ in every neighborhood of } \alpha.
\end{aligned}
$$

That is to say $\lambda^\star$ is the unique symmetric Nash equilibrium in a game with payoff function g. Moreover, $\lambda^\star$ is also a *strict* Nash equilibrium. Therefore $\lambda^\star$ is the unique evolutionary stable strategy in the sense of Maynard Smith and Price (1973).

6 Mean-Variance Optimization

In this section we analyze the evolutionary fitness of portfolio rules based on mean-variance optimization. For clarity of presentation we restrict the analysis to simple strategies.

The mutual fund theorem states that given all investors build portfolios according to the mean-variance-criterion, then every investor will hold a combination of the riskless asset and the market portfolio in any capital market equilibrium. Even though it is very questionable whether indeed all investors use mean-variance-optimization, investing a big share of wealth in the market portfolio is a very common behavior.

We extend our previous model by incorporating a strategy that enables an investor to buy the market portfolio. This extension relates our model to the classical CAPM[3] results.

It is well known that in practice mean-variance portfolios are often under-diversified, i.e. they typically put positive weight on very few assets only. To cure this defect it is then usually suggested to modify the mean-variance portfolio by devoting some positive but small share of the budget on every asset in

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[3]See also Sciubba (1999) for an analysis of CAPM-trading rules in the original Blume and Easley (1992) setup with diagonal securities.

the portfolio, ensuring that the portfolio is completely mixed. We show in the next section that this commonly used "quick fix" of the under-diversification problem is indeed an improvement of the mean-variance portfolio.

6.1 The CAPM strategy

Consider an investor, say γ, who wants to buy a fraction of the market portfolio which is the vector of total stock of the assets. In our model each asset is supplied in one unit. Phrased in terms of budget shares, to buy the market portfolio, an investor has to divide his wealth proportional to the asset prices, i.e.,

$$\lambda_{k,t}^{\gamma} = \frac{\rho_{k,t}}{\sum_{l=1}^{K} \rho_{l,t}} \equiv q_{k,t}$$

where $k = 1, ..., K$. This trading strategy depends on the equilibrium prices in the current period. An investor who buys the market portfolio has therefore to give a demand function to the auctioneer. This calls for an extension of our previous analysis.

Suppose all other investors pursue simple trading strategies $\lambda^i \in \Delta^K$, $i = 1, ..., I, i \neq \gamma$. Then the market-clearing condition becomes,

$$q_{k,t} = \sum_{i \neq \gamma} \lambda_k^i\, r_t^i + q_{k,t}\, r_t^{\gamma}$$

and thus

$$q_{k,t} = \frac{1}{1 - r_t^{\gamma}} \sum_{i \neq \gamma} \lambda_k^i\, r_t^i$$

The evolution of the market wealth of the CAPM investor can be equated as

$$r_{t+1}^{\gamma} = \sum_{k=1}^{K} R^k(\theta^{t+1}\omega)\, \frac{\lambda_{k,t}^{\gamma}\, r_t^{\gamma}}{q_{k,t}} = \sum_{k=1}^{K} R^k(\theta^{t+1}\omega)\, r_t^{\gamma} = r_t^{\gamma}$$

Summarizing our findings we can state the following result.

Proposition 3 *The market share of a CAPM investor is constant in any population in which all other players pursue simple strategies. In particular, a CAPM investor will never vanish nor dominate the market.*

The intuition behind this result is given by the representation of the normalized market-clearing price in the model with only simple strategies (5). The normalized equilibrium price equals the relative market wealth invested in that asset. If one player dominates the market in the long-run and

asymptotically own the entire market wealth, the asset price will reflect the trading strategy of this investor. The CAPM investor mimics this strategy because he distributes his wealth according to the relative value of the assets.

From an evolutionary point of view it can be concluded that investing in the market portfolio is a strategy with strong resistance against the market selection mechanism. Hence even though buying the market portfolio may not be in accordance with mean-variance optimization (because not everybody uses it) it is a convenient rule which automatically imitates the most successful trading strategy!

6.2 Diversification

We assume that the number of states of nature is finite, $s = 1, ..., S$.

Corollary 2 *Suppose $\hat{\lambda}$ is an under-diversified simple strategy, i.e. $\hat{\lambda}_k = 0$ for at least one k. Denote by $\hat{\lambda}_k^\varepsilon := (1 - \varepsilon)\hat{\lambda}_k + \varepsilon/S$, $0 < \varepsilon \leq 1$, the corresponding ε-completed strategy. Then $\hat{\lambda}^\varepsilon$ is robust against $\hat{\lambda}$-mutants for all sufficiently small $\varepsilon > 0$, i.e. the distribution of wealth shares that assigns total wealth to the $\hat{\lambda}^\varepsilon$-player is stable in the population $(\hat{\lambda}^\varepsilon, \hat{\lambda})$.*

Even though using the "quick fix" to prevent under-diversification is better than investing according to the under-diversified portfolio rule, it is clear from the main result Theorem 1, that ε-completed under-diversified simple strategies are not locally stable (if they do not coincide with $\lambda^\star$). However, we next show that the situation for ε-completed portfolio rules $\hat{\lambda}^\varepsilon$ is even worse. Any completely mixed simple strategy drives out $\hat{\lambda}^\varepsilon$ for all small enough $\varepsilon > 0$.

Corollary 3 *Given any completely mixed simple strategy λ^c and any under-diversified simple strategy $\hat{\lambda}$. Then $\hat{\lambda}^\varepsilon$, defined in Corollary 2, is not robust against λ^c-mutants for all sufficiently small $\varepsilon > 0$, i.e. the distribution of wealth shares that assigns total wealth to the $\hat{\lambda}^\varepsilon$-player is not stable in the population $(\hat{\lambda}^\varepsilon, \lambda^c)$.*

Appendix

Proof of Proposition 1. We prove the statement by contraposition. Let λ^i, $i \in \mathcal{I}$, be a family of adapted trading strategies such that $\lambda^i(\omega) \neq \lambda^j(\omega)$ for some $i, j \in \mathcal{I}$, $i \neq j$ on a set $\bar{\Omega}$ of strictly positive $\mathbb{P}$-measure. Let $r \in \Delta^I$ with $r^i r^j > 0$. We will show that r cannot be invariant.

Since $\lambda^i(\omega), \lambda^j(\omega) \in \text{int}\Delta^K$ for all ω, and $r^i r^j > 0$, $\lambda^i(\omega)\, r^i \neq \lambda^j(\omega)\, r^j$ for all $\omega \in \bar{\Omega}$ and there is no $c > 0$ such that $\lambda^i(\omega)\, r^i = c\, \lambda^j(\omega)\, r^j$. This further implies that the portfolios $a^i := (\lambda_k^i\, r^i / \sum_{l=1}^I \lambda_k^j\, r^l)_k \neq (\lambda_k^j\, r^j / \sum_{l=1}^I \lambda_k^j\, r^l)_k =: a^j$ for all $\omega \in \bar{\Omega}$ and their payoff-vectors are not multiples. Due to the non-redundancy Assumption 4, either $f_i(\omega, r)$ or $f_j(\omega, r)$ in equation (7) is not a constant, i.e. takes on different values with strictly positive probability. Hence r is not invariant in the sense of Definition 1. $\qquad\square$

Proof of Proposition 2. The proof is mainly an application of the multiplicative ergodic theorem for random dynamical systems on manifolds, see Arnold (1998, Chapter 4).

The random dynamical system describing the evolution of wealth shares is defined on the simplex Δ^I, an $I-1$-dimensional manifold with boundary. We therefore transform the system and consider a conjugate random dynamical system on a subset of the Euclidean space.

Define the projection of the unit simplex,

$$D^{I-1} := \left\{ y \in \mathbb{R}^{I-1} \mid y_i \geq 0, \sum_{i=1}^{I-1} y_i \leq 1 \right\} \subset \mathbb{R}_+^{I-1}.$$

Further, for each $n \in \mathcal{I}$ define the map

$$h_n : D^{I-1} \to \Delta^I, \; h_n(y_1, ..., y_{I-1}) := \left(y_1, ..., y_{n-1}, 1 - \sum_{i=1}^{I-1} y_i, y_n, ..., y_{I-1} \right).$$

with inverse
$$h_n^{-1}(x_1, ..., x_I) := (x_1, ..., x_{n-1}, x_{n+1}, ..., x_I).$$

h_n is a C^∞-diffeomorphism. We obtain the conjugate random dynamical system on D^{I-1},
$$\psi_n(t, \omega) := h_n^{-1} \circ \phi(t, \omega) \circ h_n.$$

Due to the definition of the space D^{I-1}, we can take directional derivatives in the direction of all unit vectors at all points in the interior of D^{I-1} relative to $\mathbb{R}_+^{I-1}$. That is we can determine the Jacobian of the conjugate system at all points in $\{y \in \mathbb{R}_+^{I-1} \mid \sum_{i=1}^{I-1} y_i < 1\}$ (which is an invariant set for the random dynamical system ψ_n).

Note that the origin of $\mathbb{R}^{I-1}$ corresponds to the nth vertex of the unit simplex. The stability properties of these two fixed points are identical because of the C^∞-equivalence of both random dynamical systems.

For notational simplicity we assume without loss of generality that $n = I$. Then the partial derivatives of $\psi_I(1, \omega, y) = h_I^{-1} \circ \phi(1, \omega) \circ h_I(y)$ are given

by,

$$\frac{\partial \psi_I^i(1,\omega,y)}{\partial y_m} = -\sum_{k=1}^{K} \frac{R^k(\theta\omega)\,(\lambda_k^m(\omega) - \lambda_k^I(\omega))\,\lambda_k^i(\omega)\,y_i}{(\sum_{j=1}^{I-1}\lambda_k^j(\omega)\,y_j + \lambda_k^I(\omega)\,(1 - \sum_{j=1}^{I-1}y_j))^2}$$

for all $i \neq m$, and by

$$\begin{aligned}
\frac{\partial \psi_I^i(1,\omega,y)}{\partial y_m} &= -\sum_{k=1}^{K} \frac{R^k(\theta\omega)\,(\lambda_k^m(\omega) - \lambda_k^I(\omega))\,\lambda_k^m(\omega)\,y_m}{(\sum_{j=1}^{I-1}\lambda_k^j(\omega)\,y_j + \lambda_k^I(\omega)\,(1 - \sum_{j=1}^{I-1}y_j))^2} \\
&\quad + \sum_{k=1}^{K} \frac{R^k(\theta\omega)\,\lambda_k^m(\omega)}{\sum_{j=1}^{I-1}\lambda_k^j(\omega)\,y_j + \lambda_k^I(\omega)\,(1 - \sum_{j=1}^{I-1}y_j)}
\end{aligned}$$

for all $i = m$.

The stability properties of $e_I \in \Delta^I$ can be determined by evaluating the Jacobian of ψ_I at the origin and applying the multiplicative ergodic theorem of Oseledets. It will be shown that condition (12), resp. (13), ensures that the top Lyapunov exponent of this linear system is strictly negative, resp. positive. Results by Wanner (1995), see Arnold (1998, Theorem 7.5.6), ensure that the dynamic behavior of the linearized system carries over (locally) to the nonlinear stochastic system.

From the above expressions, we obtain the Jacobian of ψ_I at $y = (0,...,0)$. It is a diagonal matrix with entry,

$$A_{m,m}(\omega) := \sum_{k=1}^{K} R^k(\theta\omega)\,\frac{\lambda_k^m(\omega)}{\lambda_k^I(\omega)}$$

The multiplicative ergodic theorem, Arnold (1998, Theorem 4.2.6), implies that the Lyapunov exponents of the fixed point $y = 0$ of ψ_I are given by $\lim_{T\to\infty}\frac{1}{T}\ln|\prod_{t=0}^{T} A_{m,m}(\theta^t\omega)|$, $m = 1,...,I-1$. The integrability condition of the multiplicative ergodic theorem is satisfied if $\mathbb{E}\ln|A_{m,m}(\omega)| < \infty$ for all $m = 1,...,I-1$. By the Birkhoff ergodic theorem, we find that this limit is equal to

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\ln\left|\sum_{k=1}^{K} R^k(\theta^{t+1}\omega)\,\frac{\lambda_k^m(\theta^t\omega)}{\lambda_k^I(\theta^t\omega)}\right| = \mathbb{E}\ln\left|\sum_{k=1}^{K} R^k(\theta\omega)\,\frac{\lambda_k^m(\omega)}{\lambda_k^I(\omega)}\right| \quad (21)$$

Thus the integrability condition of the multiplicative ergodic theorem is satisfied if the right-hand side of (12) (which is identical to the right-hand side of (13)) in Proposition 2 is well-defined, i.e. the expected value is finite.

Zero is a stable fixed point of ψ_I if the term in (21) is strictly negative for all $m = 1,...,I-1$. If (21) is strictly positive for some m, then zero is locally

unstable. Due to the diagonal structure of the Jacobian, the eigenspaces correspond to the linear spaces spanned by the unit vectors (restricted to the positive orthant $\mathbb{R}_+^{I-1}$).

The stability of the original system on Δ^I at the fixed point $e_I \in \Delta^I$ is determined by the Lyapunov exponents (21). The corresponding eigenspaces are given by the vertices. All summands in (21) are positive and we thus have obtained conditions (12) and (13) of the proposition. $\qquad\square$

Proof of Theorem 1. Obviously, $\lambda^\star$ is a completely mixed strategy, i.e. $\sum_{k=1}^K \lambda_k^\star = 1$ and $\lambda_k^\star > 0$ for all k. Next we define the auxiliary function,

$$g_\beta(\alpha) := \mathbb{E} \ln \left(\sum_{k=1}^K R^k(\omega) \frac{\alpha_k}{\beta_k} \right) \tag{22}$$

in accordance with Proposition 2. For each fixed strategy $\beta \in \text{int}\Delta^K \subset \mathbb{R}^K$, $g_\beta : \text{int}\Delta^K \to \mathbb{R}$. $g_\beta(\alpha)$ is the Lyapunov exponent of the distribution of wealth that assigns total wealth to the 'status quo' population that plays strategy β in a market in which α is the only the alternative strategy.

By Proposition 2 the first assertion of the theorem follows if we can show that $g_{\lambda^\star}(\alpha) < 0$ for all $\alpha \in \text{int}\Delta^K$ with $\alpha \neq \lambda^\star$.

We prove that $g_\beta(\alpha)$ is strictly concave for all $\beta \in \text{int}\Delta^K$ and that $g_{\lambda^\star}(\alpha)$ takes on its maximum value at $\alpha = \lambda^\star$.

To ensure strict concavity it suffices to show that $\alpha \mapsto g_\beta(\alpha)$ is strictly concave on the space $\mathbb{R}_{++}^K$, because restriction of the domain to the linear subspace $\text{int}\Delta^K$ preserves strict concavity. The function $\ln \sum_{k=1}^K (R^k(\omega)\,\alpha_k/\beta_k)$ is concave for all ω and—due to the no-redundancy Assumption 4—strictly concave on a set of positive measure. Therefore $g_\beta(\alpha)$ is strictly concave for each fixed $\beta \in \text{int}\Delta^K$.

We can now employ that $\lambda^\star$ is the unique maximum of $g_{\lambda^\star}(\alpha)$ on $\text{int}\Delta^K$ if all directional derivatives at this point are zero.

The partial derivative of $g_\beta(\alpha)$ with respect to the i-th component α_i is given by

$$\frac{\partial g_\beta(\alpha)}{\partial \alpha_i} = \mathbb{E} \frac{R^i(\omega)/\beta_i}{\sum_{k=1}^K R^k(\omega)\frac{\alpha_k}{\beta_k}}$$

Observe that interchanging integration and differentiation is allowed because $\ln(\sum_{k=1}^K R^k(\omega)\,\alpha_k/\beta_k)$ is integrable for each fixed α (follows from $\mathbb{E} R^k(\omega) \leq 1 < \infty$ for all k) and $\mathbb{E}(R^i(\omega)/\sum_{k=1}^K R^k(\omega)) \leq 1 < \infty$ (this follows from the fact that $R^k(\omega) \geq 0$ for all k and all ω by assumption). The last equation implies

$$\frac{\partial g_{\lambda^\star}(\lambda^\star)}{\partial \alpha_i} = \mathbb{E} \frac{R^i(\omega)}{\lambda_i^\star} = \mathbb{E} \frac{R^i(\omega)}{\mathbb{E} R^i} \equiv 1$$

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for all $i = 1, ..., K$, since $\sum_{k=1}^{K} R^k(\omega) \equiv 1$ for all ω.

The directional derivative of $g_{\lambda^\star}$ in the direction $(d\alpha_1, ..., d\alpha_K)$ with the restriction $\sum_{k=1}^{K} d\alpha_k = 0$ (which is a vector in the simplex) is equated as

$$\sum_{i=1}^{K} \frac{\partial g_{\lambda^\star}(\lambda^\star)}{\partial \alpha_i} \, d\alpha_i = 0.$$

Corollary 1 ensures that any portfolio rule different to $\lambda^\star$ is not evolutionary stable.

Let us next prove that any strategy $\beta \neq \lambda^\star$ with $\beta \in \text{int}\Delta^{\text{K}}$ is not locally evolutionary stable. Since—due to the i.i.d. assumption—the dependence of the strategy β on the past ω^{-1} does not affect the expected value in Proposition 2, cf. equation (16), we can restrict our analysis to simple strategies β. A strategy $\beta \neq \lambda^\star$ is not locally evolutionary stable, if for any neighborhood of β there exists an α such that $g_\beta(\alpha) > 0$. It suffices to show that the directional derivative of g_β at β is strictly positive in one direction.

Since $\beta \neq \lambda^\star$ and both are points in the simplex there exists $i \neq j$ with $\beta_i > \lambda_i^\star$ and $\beta_j < \lambda_j^\star$. Note that we have assumed a minimum of two assets.

The directional derivative of g_β at β in the direction $d\alpha$ given by $d\alpha_i = -1/2$, $d\alpha_j = 1/2$, and zero otherwise, is given by,

$$\sum_{k=1}^{K} \frac{\partial g_\beta(\beta)}{\partial \alpha_k} \, d\alpha_k = \sum_{k=1}^{K} \frac{\mathbb{E}R^k}{\beta_k} \, d\alpha_k = \frac{1}{2} \left(\frac{\lambda_j^\star}{\beta_j} - \frac{\lambda_i^\star}{\beta_i} \right) > 0.$$

$\square$

Proof of Theorem 2. This result follows mainly from an application of the proof of Theorem 1. For any adapted strategy $\lambda^i(\omega) = \lambda^i(\omega^0)$ with $\lambda^i(\omega) \neq \lambda^\star$ on a set $\bar{\Omega}$ of strictly positive $\mathbb{P}$-measure we have the following. Fix any $\omega^{-1} \in S^{\mathbb{N}}$. Then $g_{\lambda^\star}(\lambda^i(\omega^0)) \leq 0$ for all ω and the inequality is strict for all $\omega \in \bar{\Omega}$, cf. proof of Theorem 1. This implies that condition (i) in Proposition 2 holds. Therefore $\lambda^\star$ is evolutionary stable.

To prove that no other adapted strategy can be evolutionary stable, fix any adapted strategy $\lambda^i(\omega) = \lambda^i(\omega^0)$ with $\lambda^i(\omega) \neq \lambda^\star$ on a set $\bar{\Omega}$ with $\mathbb{P}(\bar{\Omega}) > 0$. For every fixed $\omega^0 \in S^{\mathbb{N}}$, we employ the procedure that has been applied in proof of Theorem 1 to find a strategy such that for a strategy, say λ^j, $g_{\lambda^i(\omega^0)}(\lambda^j) \geq 0$ where the inequality is strict for all $\omega \in \bar{\Omega}$.

If $\mathcal{S}$ is the power set of S, then $\mathcal{F}$ is the power set of $S^{\mathbb{Z}}$. Therefore the strategy $\lambda^j(\omega) := \lambda^j(\omega^0)$ is measurable. By construction it is also adapted and in any prescribed neighborhood of λ^i. We have defined λ^j in the way

that condition (ii) in Proposition 2 holds for the pair of strategies λ^i, $\lambda^n = \lambda^j$. As a result the adapted strategy λ^i is not evolutionary stable. $\square$

Proof of Corollary 2. According to Proposition 2(i) it suffices to show that

$$\mathbb{E} \ln \left(\sum_{k:\hat{\lambda}_k > 0}^{K} R^k(\omega) \frac{\hat{\lambda}_k}{(1-\varepsilon)\hat{\lambda}_k + \varepsilon/S} \right) < 0$$

for all small $\varepsilon > 0$. The left-hand side of this equation is strictly increased by omitting ε/S in the denominator. We thus obtain the sufficient condition,

$$\mathbb{E} \ln \left(\sum_{k:\hat{\lambda}_k > 0} R^k(\omega) \right) \leq \ln(1-\varepsilon) \tag{23}$$

Since there is at least one k such that $\hat{\lambda}_k = 0$, we find that $\sum_{k:\hat{\lambda}_k > 0} R^k(\omega) < 1$ on a set of positive measure (the term is bounded by 1 for all ω), the left-hand side of (23) $\mathbb{E} \ln \left(\sum_{k:\hat{\lambda}_k > 0} R^k(\omega) \right) < 0$. Therefore (23) is satisfied for all small enough ε. $\square$

Proof of Corollary 3. Again we employ Proposition 2. The local instability result says that the assertion of the Corollary is true, if

$$\mathbb{E} \ln \left(\sum_{k=1}^{K} R^k(\omega) \frac{\lambda_k^c}{(1-\varepsilon)\hat{\lambda}_k + \varepsilon/S} \right) > 0 \tag{24}$$

for all small $\varepsilon > 0$.

Noting that

$$\sum_{k=1}^{K} \frac{R^k(\omega) \lambda_k^c}{(1-\varepsilon)\hat{\lambda}_k + \varepsilon/S} = \sum_{k:\hat{\lambda}_k > 0} \frac{R^k(\omega) \lambda_k^c}{(1-\varepsilon)\hat{\lambda}_k + \varepsilon/S} + \sum_{k:\hat{\lambda}_k = 0} R^k(\omega) \frac{S \lambda_k^c}{\varepsilon}$$

$\lambda_k^c > 0$ for all k, and $R^k(\omega) > 0$ on a set of positive measure for all k with $\hat{\lambda}_k = 0$, we find that the left-hand side of (24) tends to infinity as $\varepsilon \to 0$. $\square$

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