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Monotonicity of Social Optima With Respect to Participation Constraints

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Abstract

In this paper we consider solutions which select from the core. For games with side payments with at least four players, it is well-known that no core-selection satisfies monotonicity for all coalitions; for the particular class of core-selections found by maximizing a social welfare function over the core, we investigate whether such solutions are monotone for a given coalition. It is shown that if this is the case then the solution actually maximizes aggregate coalition payoff on the core. Furthermore, the social welfare function to be maximized exhibits larger marginal social welfare with respect to the payoff of any member of the coalition. The results may be used to show that there are no monotonic core selection rules of this type in the context of games *without* side payments.

JEL classification: I10

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1 Introduction

Much of recent literature in game theory deals with the tension between social values and the self-seeking actions of individual agents. If we consider allocation rules of a given society this tension implies that allocation rules based on social values must take into account the incentive constraints of individual subgroups of society.

A particular version of this social allocation problem is that of finding a suitable solution to a coalitional game. Here, the tension between social values and individual actions becomes a question of defining solution concepts that upholds social values but does so under various participation constraints, for example, in the form of coalitional stability (or core-constraints). For instance, social value may consist of maximizing the smallest excess of any coalition as in the nucleolus by Schmeidler [12] (which agrees with the core-constraints), or finding Lorenz dominating allocation(s) in the core as in Dutta and Ray [2] and Hougaard, Peleg and Thorlund-Petersen [6].

In the present paper social values are represented by a social welfare function. Hence, we consider allocations that maximizes social welfare with respect to participation constraints in the form of coalitional stability. In other words, we focus on the broad class of

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solutions that maximizes a (strictly concave) social welfare function over the set of core allocations. As such, the relevant coalitional games may be thought of as games where the value of the characteristic function is given in monetary terms - for example, surplus sharing games or (dually) cost sharing games.

Our particular interest relates to the monotonicity properties of such solutions. Now, assuming that the participation constraint of a single coalition is increased, all others coalitional constraints kept fixed (and without leaving the core empty), then the constrained maximization of social welfare ensures that the coalition as a whole is either unaffected or better off than status quo. But we are interested in more than that; the constrained maximization of social welfare should (weakly) increase the payoff of all individuals in the coalition so that individual incentives to change (that is to affect the participation constraints) are in agreement with social values. In other words, social values shall not prevent social progress.

As such, a suitable allocation rule ought to be monotonic in payoff's for all possible coalitions. However, a well known result by Young [15] states that no solution that selects allocations in the core satisfies monotonicity for all coalitions. The result demonstrates that there is a trade-off between coalitional stability (that is, selecting from the core) and coalitional monotonicity.

As both coalitional stability and coalitional monotonicity are central properties many papers have examined the extent to which these two conficting goals may be reconciled. For example, on the subclass of convex games the Shapley value is indeed coalitionally monotonic (Shapley [13], Sprumont [14], Rosenthal [11]), and so is the Dutta-Ray solution (Hokari, [5], Hougaard, Peleg and Østerdal [7]), whereas the nucleolus is not (Hokari [4]).

Keeping in mind Young's general impossibility result, we focus on monotonicity with respect to the worth of any particular coalition (in the following called S-monotonicity) - for example, in the present context it is obvious that all core selection rules are i-monotonic. In this paper, we show that S-monotonic allocation rules that maximizes a social welfare function over the set of core allocations can be characterized in terms of the "social desirability" of giving payoff to the members of coalition S in the sense that their total payoff is maximal given core constraints. Moreover, "social desirability" can be expressed in terms of marginal social welfare of each individual's payoff. We also show that no solution maximizing a social welfare function is monotonic with respect to the grand coalition.

While there are S-monotonic core selections on the domain of all coalitional games with nonempty core, this turns out to be a feature that is closely connected with the presence of side payments. Indeed, if coalitional games *without* side payments are considered, then the previously derived results (characterizing S-monotonic core selection rules) may be exploited to yield an impossibility result: Even for a fixed coalition S there cannot be core selection rules that are monotonic. This shows that the non-monotonicity in coalitional worth is a feature which is inherent in any solution satisfying the coalitional stability (core) conditions.

The paper is structured as follows: In Section 2, we give the necessary definitions (in the context of coalitional games with side payments); the basic characterization result is given in Section 3; it exploits the characterization of games with nonempty core by balancedness and exploits some details concerning suitable balanced families of coalitions, which have been collected in an appendix. In Section 4, we establish the relationship between S-monotonicity and the partial derivatives of the social welfare function and in Section 5 we consider special coalitions. In Section 6 we extend the investigation to coalitional games without side payments and give the impossibility result in this context, and finally, Section 7 has the function of an appendix containing the statement and proof of the intermediary result needed in Section 3.

2 Preliminaries

A coalitional game with side payments, is a pair (N, v) where $N = \{1, \ldots, n\}$ is a nonempty finite set of players, and v is a function (called the characteristic function of the game) that associates a positive real number $v(S)$ with each subset S of N, such that $v(\emptyset) = 0$. The set of all such games is denoted $\mathcal{G}$. For fixed N, the set of all games in $\mathcal{G}$ with player set N can be identified with a closed subset of $\mathbf{R}^{2^N}$, where 2^N is the power set of N. As N is fixed in the following we shall write v instead of (N, v) for notational simplicity.

Let $v \in \mathcal{G}$. A payoff vector is a member x of $\mathbf{R}^N_+$; each payoff vector gives rise to an additive set function such that $x(S) = \sum_{i \in S} x_i$. We use the notation x_S for the projection of the payoff vector x on $\mathbf{R}^S_+$, where $S \subset N$.

For $v \in \mathcal{G}$, the *core* of v is the subset $C(v)$ of $\mathbf{R}^N_+$ defined by

$$C(v) = \{x \in \mathbf{R}^N_+ \mid x(S) \geq v(S) \text{ for all } S \subset N \text{ and } x(N) = v(N)\}.$$

In the following we restrict our attention to the class of games for which the core is non-empty, i.e. the class $\mathcal{B} \subset \mathcal{G}$ of balanced games, cf. Bondareva [1]. For later reference, we give the definition of balancedness and introduce some notation.

Let e_i denote the ith unit vector in Euclidean space, and let $\triangle_N = \{x \in \mathbf{R}^N_+ \mid \sum_{h=1}^n x_h = 1\} = \text{conv}(e_1, \ldots, e_n)$. We denote by $\triangle_S$ the face of $\triangle_N$ determined by $S \subset N$, that is $\triangle_S = \{x \in \triangle_N \mid x_i = 0 \text{ for } i \notin S\}$, and we let $\hat{e}_S = |S|^{-1} \sum_{i \in S} e_i$ be the barycenter of $\triangle_S$, for $S \subset N$. Clearly $\hat{e}_{\{i\}} = e_i$ for each $i \in N$. A family $\mathcal{D}$ of coalitions is balanced if there are non-negative weights λ_T, for $T \in \mathcal{D}$, such that $\sum_{T \in \mathcal{D}, i \in T} \lambda_T = 1$ for each $i \in N$, or, equivalently, if $e_N \in \text{conv}(\{\hat{e}_T \mid T \in \mathcal{D}\})$. A game v is balanced if $\sum_{T \in \mathcal{D}} \lambda_T v(T) \leq v(N)$ for every balanced family of coalitions.

A *solution* on $\mathcal{B}$ is a function $\phi : \mathcal{B} \to \mathbf{R}^N_+$, for which each game $v \in \mathcal{B}$ allocates $v(N)$ among the players, that is $\phi(v) \in \{x \in \mathbf{R}^N_+ \mid x(N) = v(N)\}$. A solution is a *core selection* if $\phi(v) \in C(v)$ for each $v \in \mathcal{B}$.

We shall be interested in a particular class of core selections: Suppose that $u : \mathbf{R}^N_+ \to \mathbf{R}$ is strictly concave and differentiable, and define the *u-selection* ϕ^u by

$$\phi^u(v) = \text{argmax}\{u(x) \mid x \in C(v)\}$$

(note that since $C(v)$ is a closed and convex subset, there is a unique maximizer of u on $C(v)$).

Since a u-selection only depends on the core constraints, well-known solution concepts such as the nucleolus - or the Shapley value of convex games[1] - cannot be represented as a u-selection. Contrary to selections such as the nucleolus or the Shapley value, a u-selection

[1] A game v is convex if $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$ for all $S, T \subseteq N$.

satisfies a version of independence of irrelevant alternatives in the sense that if a point $x \in C(v)$ is a u-selection for some game v then this point is also the u-selection for any game v' for which the core is a subset of $C(v)$ that contains x.

EXAMPLE 2.1 In *convex games,* the core contains a unique Lorenz-maximal element (see e.g. the Egalitarian solution of Dutta and Ray [2]) that can be found by maximizing the following strictly concave function $u^L(x) = -\sum_{i=1}^n x_i^2$ for $x \in C(v)$. Thus, the selection of the Lorenz-maximal point in the core is in fact a u-selection as defined above. In fact, as long as we consider convex games (or other classes of games for which the Lorenz-maximum is unique) we could have chosen any separable function $u(x) = \sum_{i=1}^n f(x_i)$ where f is strictly concave by Theorem 108 of Hardy et al. [3]. For balanced games in general there may by many Lorenz-maxima in the core and consequently the specific form of u becomes important.[2]

Let $S \subset N$ be a nonempty coalition. A solution ϕ on $\mathcal{B}$ is called S-monotonic if for all $v, w \in \mathcal{B}$ and $i \in S$,

$$[v(S) < w(S), v(T) = w(T) \text{ if } T \neq S] \;\Rightarrow\; \phi_i(v) \leq \phi_i(w).$$

The solution ϕ is said to be *coalitionally monotonic* if ϕ is S-monotonic for all $S \subseteq N$. It is well-known (cf. Young [15] and Housman and Clark [8]) that on the class of balanced games no core selection satisfies coalitional monotonicity (for $|N| \geq 4$). In other words, all core selections, including those of the form ϕ^u considered here, must violate S-monotonicity for some coalition S. We shall return to this result (in the context of core selections of the type ϕ^u considered here) at a later stage, where it will emerge as a simple corollary of our characterization to follow.

Since S-monotonicity (for $S \neq N$) is a way of formulating that society wants to give all agents in coalition S incentive to increase the total worth of S, it is also to be expected that the choice within the core results in a favourable position of coalition S. A straight forward way to favour S would be to use a *utilitarian* social welfare function in the sense that $u(x) = \sum_{i \in S} x(i)$; however, this social welfare function does not satisfy our general condition of strict concavity. Thus, we say that ϕ^u is *S-utilitarian* if ϕ^u maximizes $x(S)$ over all $x \in C(v)$, for each $v \in \mathcal{B}$. In the next section, we show that the intuitive relationship between S-monotonicity and S-utilitarianism holds, at least when games and coalitions of very small size are disregarded.

3 A characterization of S-monotonic core selections

The main result of this section demonstrates that if the u-selection ϕ^u is S-monotonic, then it is also S-utilitarian and vice versa. In this sense it looks as if the social welfare function singles out the total payoff of the coalition S as the overall criterion to be maximized given the core constraints. As we shall see in Section 4, the result that social welfare maximization works "as if" the total payoff of the members of S is maximized over the core, is indeed sustained by the properties of u (the social welfare criterion) which will

[2]By Theorem 2 in Hougaard et al. [6] any Lorenz-maximum in the core of a balanced game is the maximizer of some additive and concave social welfare function.

exhibit higher marginal social welfare of the payoffs of members in S than of the payoffs of individuals not in S.

We record the first main result:

THEOREM 3.1 *For $|N| \geq 4$ and $|S| \geq 3$, the u-selection ϕ^u is S-monotonic if and only if it is S-utilitarian.*

PROOF: "If": Assume that ϕ^u is S-utilitarian, and consider two games $v, w \in \mathcal{B}$ such that

$$w(S) > v(S),\ w(T) = v(T) \text{ for all } T \neq S.$$

Since ϕ^u is S-utilitarian, we have that $x = \phi^u(v)$ satisfies $x(S) > v(S)$ (otherwise $C(w)$ would be empty). But then x belongs also to $C(w)$ (only the worth of S has changed), and since $C(w) \subset C(v)$, we conclude that x maximizes u on $C(w)$, that is $\phi^u(w) = x$, so that trivially $\phi^u_i(w) \geq \phi^u_i(v)$ for all $i \in S$.

"Only if": Assume the core selection is not S-utilitarian; then there is a game v such that the u-maximizing core-allocation x satisfies

$$x(S) < \max_{y \in C(v)} y(S).$$

Without loss of generality we may assume that $v(N) = 1$ and that $x(S) = v(S)$, since otherwise we consider the game v' with $v'(T) = v(T)$ for $T \neq S$ and $v'(S) = x(S)$.

By Lemma 7.1 (in the Appendix) we may assume that v is such that for some $i \in S$, the family $\mathcal{C}$ of coalitions T with $x(T) = v(T)$ either do not contain i, or they are equal to S. Moreover, the family $\mathcal{C}$ cannot be balanced: Suppose to the contrary that there were λ_T, $T \in \mathcal{C}$ such that $\sum_{T \in \mathcal{C}:\, i \in T} \lambda_T = 1$ for each $i \in N$. Then we have

$$\sum_{T \in \mathcal{C}} \lambda_T x(T) = \sum_{T \in \mathcal{C}} \lambda_T v(T) = v(N),$$

since $C(v) \neq \emptyset$, and since $x(S) < \max_{y \in C(v)} y(S)$, there is $\varepsilon > 0$ such that the game v' with

$$v'(T) = v(T) \text{ for } T \neq S,\ v'(S) = v(S) + \varepsilon$$

satisfies $C(v') \neq \emptyset$. But then $\sum_{T \in \mathcal{C}} \lambda_T v'(T) > v(N)$, a contradiction, showing that $\mathcal{C}$ cannot be balanced.

Consider now the family

$$\mathcal{D} = \mathcal{C} \cup \{N \backslash \{i, j\}, (N \backslash S) \cup \{j\}\},$$

where $j \in S$, $j \neq i$, is chosen such that $x'_j > x_j$ for some $x' \in C(v)$ by Lemma 7.1. We claim that $\mathcal{D}$ is not balanced. Indeed, if $\mathcal{D}$ were balanced, then the weight λ_S would be equal to 1 since S is the only coalition in $\mathcal{D}$ containing i. But then the weights of any coalition intersecting S, including the coalitions $N \backslash \{i, j\}$ and $(N \backslash S) \cup \{j\}$, must be 0, so that the family $\mathcal{C}$ is balanced, contrary to what has been shown above.

We claim that the game w with

$$w(T) = \begin{cases} x(T) & \text{if } T \in \mathcal{D},\ T \neq S, \\ v(S) + \varepsilon & \text{for } T = S, \\ v(S) & \text{otherwise} \end{cases}$$

is balanced for $\varepsilon > 0$ small enough: Indeed, if $\mathcal{H}$ is a balanced family, then it must contain some coalition T such that $x(T) > v(T)$, and choosing ε smaller than $x(T) - v(T)$ for all such T we get the claim.

Let $z = \text{argmax}_{y \in C(w)} u(y)$; we now show that $z(\{i\}) < x(\{i\})$. Indeed, we must have $z(\{i,j\}) \leq x(\{i,j\})$ since $w(N\backslash\{i,j\}) = x(N\backslash\{i,j\})$. Also $z(N\backslash S \cup \{j\}) \geq x(N\backslash S \cup \{j\})$. Since $z(N\backslash S) \leq x(N\backslash S) - \varepsilon$, we must have that $z(\{j\}) > x(\{j\})$. But then it follows that $z(\{i\}) < x(\{i\})$. □

4 Applications of the characterization

With the result of Theorem 3.1, we may obtain further properties of the selected core elements, for example that if a u-selection is S monotonic then all members of S have a strictly higher marginal utility than all non-members:

THEOREM 4.1 *For $|N| \geq 4$ and $|S| \geq 3$, if ϕ^u is S-monotonic, then $u'_i(x) > u'_j(x)$, for all $x \in int\mathbf{R}^N_+$, $i \in S$ and $j \in N\backslash S$.*

PROOF: Let $x \in \text{int}\,\mathbf{R}^N_+$ be arbitrary, and choose a game $v \in \mathcal{B}$ such that $\phi^u(v) = x$ (such a game exists, e.g. the additive game defined by x). By Theorem 3.1, we may assume that $x(N\backslash S) = v(N\backslash S)$. Let w be the game such that $w(T) = x(T)$ if either (i) $T \subseteq N\backslash S$ or (ii) $N\backslash S \subset T$ and $T = T' \cup (N\backslash S)$ for some T' such that $x(T') = v(T)$; for all other coalitions, $w(T) = 0$.

By our construction, $x \in C(w)$, so $w \in \mathcal{B}$. For each coalition S' such that the inequality $x(S') \geq v(S')$ is binding (i.e. an equality) at x in the game v, we have either $S' \subset N\backslash S$, so that $x(S') = w(S')$ by condition (i) above, or $S' \cap S \neq \emptyset$. Let $D = (N\backslash S)\backslash S'$; by (ii), $w(S' \cup (N\backslash S)) = x(S' \cup (N\backslash S))$, and from this together with $x(S' \cup (N\backslash S)) = x(S') + x(D)$ we get that

$$x(S') = w(S' \cup (N\backslash S)) - w(D) = v(S')$$

so that the same binding equality must hold in the game x. Consequently there is a neighborhood U if x such that $C(w) \cap U \subset C(v) \cap U$, and therefore $\phi^u(w) = x$.

We conclude that there is a game w such that $x \in \phi^u(w)$ and for all coalitions T with $w(T) = x(T)$, if some $i \in S$ belongs to T, then so does each $j \in N\backslash S$. Now x solves the constrained maximization problem

$$\begin{aligned}
&\max u(x') \\
&s.t. \\
&x'(T) \geq w(T),\, T \subset N, \\
&x'(N) = w(N),
\end{aligned}$$

with Kuhn-Tucker conditions

$$u'_i(x) = -\sum_{T \in \mathcal{T}, i \in T} \lambda_T - \mu,\; i = 1, \ldots, n,$$

where for each T, $\lambda_T > 0$ only if $x(T) = w(T)$ (the complementary slackness condition). Since each $j \in N\backslash S$ belongs to all the coalitions T having some member from S, we get that $u'_i(x) \geq u'_j(x)$ for all $i \in S$, $j \in N\backslash S$.

In order to complete the proof of the theorem, assume that there is $u'_i(x) = u'_j(x)$ for some $i \in S$, $j \in N \backslash S$. Since the map $f : \mathbf{R} \to \mathbf{R}$ with

$$f(t) = u(x_1, \dots, x_i + t, \dots, x_j - t, \dots, x_n)$$

is differentiable and strictly concave in a neighborhood of $t = 0$, with $f'(0) = u'_i(x) - u'_j(x) = 0$, we have that $f'(t) > 0$ for $t < 0$ and $f'(t) < 0$ for $t > 0$, from which we conclude that $u'_i(x') < u'_j(x')$ for some x' in a neighborhood of x, contradicting what was shown above. We conclude that $u'_i(x) > u'_j(x)$ for all $i \in S$, $j \in N \backslash S$. □

As mentioned in the previous section, Young [15] showed that no core selection rule can be coalitionally monotonic on games with five or more players, a result that was strengthened by Housman and Clark [8], showing that this result holds also for games with four players. In the particular case of u-selections, this follows from Theorem 4.1.

COROLLARY 4.2 *Let* $|N| \geq 4$. *No core allocation method maximizing a social welfare function is coalitionally monotonic.*

PROOF: Theorem 4.1 cannot be satisfied for all $S \subset N$ for any u-selection. □

We may remark that on the particular class of convex games it was shown in Hougaard, Peleg and Østerdal [7], that if the social welfare function is separable, i.e. $u(x) = \sum_{i=1}^{n} u_i(x_i)$ then ϕ^u (called a Generalized Lorenz-solution) satisfies S-monotonicity for all $S \subseteq N$. In other words, for convex games, some u-selections may satisfy coalitional monotonicity.

5 Special coalitions

Clearly, the grand coalition has its own interest since all agents of society ought to have incentive to increase the total worth to be shared. It is well known that there are core allocation methods that are *aggregate monotonic* (N-monotonic) as for example the *per capita nucleolus* (cf. e.g. Young [15]). However, unfortunately it can be shown that u-selections cannot be aggregate monotonic. In this sense there will always be members of society who have incentives to block actions that increase the total worth of society independent of the choice of social welfare function.

THEOREM 5.1 *Let* $|N| \geq 3$. *Then there exists no* N*-monotonic* u*-selection* ϕ^u.

PROOF: Since u is strictly convex, the set of gradients $u'(x)$ for $x \in \operatorname{int} \mathbf{R}^N_+$ is an open set, and in particular, there is $x \in \operatorname{int} \mathbf{R}^N_+$ such that the coordinates $u'_i(x)$, for $i \in N$, possibly after a renumbering, satisfy

$$\{u'_1(x), ..., u'_{n-3}(x)\} < u'_{n-2}(x) < \{u'_{n-1}(x), u'_n(x)\}$$

(where $A < r$ and $r < B$ for $r \in \mathbf{R}$, $A, B \subset \mathbf{R}$ means that $a < r$ and $r < b$ for each $a \in A$, $b \in B$). Now, define the game v as follows: $v(\{i\}) = x_i$ for all $i \neq \{n-2\}$, and $v(\{n-2\}) = x_{n-2} - \lambda$, for some $\lambda > 0$. Moreover, let $v(\{n-2\}, \{n-1\}) = x_{n-2} + x_{n-1}$, $v(\{n-2\}, \{n\}) = x_{n-2} + x_n$, $v(N) = x_1 + ... + x_n$ and $v(T) = 0$ otherwise. Then $C(v) = \{x\} = \phi^u(v)$ (indeed, only the payoff of player $n-2$ can be reduced without violating the constraints of the singleton coalitions, but any such reduction must be matched by an equal increase in the payoffs of both $n-1$ and n, which is impossible).

Now, define a new game w where $w(N) = v(N) + \varepsilon$ for some small enough $\varepsilon > 0$, and $w(T) = v(T)$ for $T \neq N$. In $C(w)$, the payoffs of all individuals may be increased, but by our assumptions on $u'(x)$, the value of u grows more if the payoff is increased for players $n-1$ and n than if any other player gets larger payoff. Indeed, increasing x_n by ε will produce tghe highest increase in u; but then x_{n-2} may be reduced by this same ε amount without violating the constraint defined for the coalition $\{n-2, n\}$, and this amount ε may be shifted from $n-2$ to $n-1$, so that the constraint defined for $\{n-2, n-1\}$ is satisfied, once again producing an increase in u. It follows that

$$\phi^u(w) = (x_1, ..., x_{n-3}, x_{n-2} - \varepsilon, x_{n-1} + \varepsilon, x_n + \varepsilon),$$

contradicting N-monotonicity. □

Finally, Theorems 3.1 and 4.1 do not consider cases where $|S| < 3$. Clearly, it follows directly from the core conditions that all u-selections are monotonic with respect to single player coalitions. However, if $|S| = 2$ problems with monotonicity may occur:

Example 5.2 Let $N = \{1, \ldots, 7\}$, $S = \{1, 2\}$, and $u(x) = \sum_{i=1}^{7}(\alpha_i x_i + \sqrt{x_i + 1})$, where $\alpha = (10, 1, 100, 100, 100, 100, 100)$. Moreover, let the game v be defined as follows; $v(N) = 3, v(1,2) = v(1,3) = v(1,4) = v(2,5) = v(2,6) = v(2,7) = 1$, and $v(T) = 0$ otherwise. We observe that $\phi^u(v) = (0, 1, 1, 1, 0, 0, 0)$. Now, define a new game w as $w(1,2) = 2$ and $w(T) = v(T)$ otherwise. The new solution is $\phi^u(w) = (1.33, 0.66, 0, 0, 0.33, 0.33, 0.33)$ where player 2 is worse off, i.e., the solution ϕ^u is not $\{1, 2\}$-monotonic.

6 Core selections for games without side payments

In this section, we extend some of the results of the previous sections to games without side payments. For a given set of players $N = \{1, \ldots, n\}$, a game without side payments V assigns to each coalition S a subset $V(S)$ of $\mathbf{R}^S$, assumed to be nonempty, closed and comprehensive (meaning that $V(S) - \mathbf{R}^S_+ \subset V(S)$).

A payoff vector in V is an element of $V(N)$. The *core* of V is the set of payoff vectors x which are Pareto optimal in the sense that if $y \in V(N)$ and $y_i \geq x_i$ for all $i \in N$, then $y = x$ and *undominated:* there is no coalition S such that $x_S \in \operatorname{int} V(S)$. The core of the game V is denoted $C(V)$.

We shall make use of a particular class of games without side payments in the sequel: Let v be a game with side payments and let $\lambda \in \triangle_N$. Then the λ-weighted game with side payments $V_{\lambda,v}$ is the game without side payments defined by

$$V_{\lambda,v}(T) = \{x \in \mathbf{R}^T_+ \mid \sum_{i \in T} \lambda_i x_i \leq v(T)\}.$$

Let $\mathcal{V}$ be a family of games without side payments with non-empty cores; a core selection on $\mathcal{V}$ is a map ψ which to each game $V \in \mathcal{V}$ assigns an element $\psi(V)$ of $C(V)$. If $u : \mathbf{R}^N_+ \to \mathbf{R}$ is a function which is differentiable and strictly concave, we define a core selection ψ^u to be a u-selection if

$$\psi^u(V) \in \{x \in C(V) \mid u(x) \geq u(x'), \text{ all } x' \in C(V)\}.$$

The notion of S-monotonicity introduced previously for games with side payments cannot be transferred immediately to games without side payments, but a reasonable way of capturing the notion that increased power of the coalition makes all its members better off might be the following definition: A core selection ψ is S-monotonic if $\psi_i(V') \geq \psi_i(V)$, all $i \in S$, for all games V, V' such that $V(T) = V'(T)$ for all $T \neq S$ and $V(S) \subset V'(S)$.

As it will turn out presently, the notion of S-monotonicity is of limited relevance in the context of games without side payments, at least when attention is restricted to selections which are obtained by maximizing a social welfare function of the core of a game. Indeed, we prove the following impossibility theorem:

THEOREM 6.1 *Let $|N| \geq 4$ and let $\mathcal{V}$ be a family of games without side payments containing all the games $V_{\lambda,v}$ for $\lambda \in \triangle_N$, $v \in \mathcal{B}$, and let $u : \mathbf{R}_+^N \to \mathbf{R}$ be differentiable and concave. If ψ^u is a core selection on $\mathcal{V}$ and $S \subset N$ is a coalition with $|S| \geq 3$, then ψ^u is not S-monotonic.*

PROOF: Choose $\lambda \in \operatorname{int} \triangle_N$ arbitrarily and consider the set of all λ-weighted games with side payments V_λ. Trivially, the game V_λ is transformed to a game with side payments by the coordinate transformation

$$x \mapsto h_\lambda(x) = (\lambda_1 x_1, \ldots, \lambda_n x_n), \tag{1}$$

which has the inverse transformation

$$y \mapsto h_\lambda^{-1}(y) = \left(\frac{y_1}{\lambda_1}, \ldots, \frac{y_n}{\lambda_n}\right).$$

The mapping $u : \mathbf{R}_+^N \to \mathbf{R}$ gives rise to a map u_λ defined on the transformed coordinates by

$$u_\lambda(y_1, \ldots, y_n) = u\left(\frac{y_1}{\lambda_1}, \ldots, \frac{y_n}{\lambda_n}\right).$$

Assume now that the core selection ψ^u defined on the family $\mathcal{V}$ of games without side payments is S-monotonic. For each $\lambda \in \operatorname{int} \triangle_N$, the restriction of ψ^u to the set of λ-weighted games $V_{\lambda,v}$ for $v \in \mathcal{B}$ gives rise to a core selection ψ^{u_λ} on the set of all balanced games (with side payments) $\mathcal{B}$; indeed, the transformation in (1) takes $C(V_\lambda)$ to the core of the transformed game with side payments, and if $x \in C(V_\lambda)$ maximizes u, then $h_\lambda(x)$ maximizes u_λ on the core of the transformed game. Also, u_λ is differentiable and strictly concave.

Finally, we see that if ψ^u is S-monotonic on the set of λ-weighted games with side payments V_λ, then ψ^{u_λ} is S-monotonic on $\mathcal{G}$. Applying now Theorem 4.1, we get that for any $y \in \operatorname{int} \mathbf{R}_+^N$, we have $(u_\lambda)'_i(y) > (u_\lambda)'_j(y)$ for all $i \in S$, $j \in N \backslash S$, and using the definition of u_λ, we get that

$$\frac{u'_i(x)}{\lambda_i} > \frac{u'_j(x)}{\lambda_j}, \text{ all } i \in S,\ j \in N \backslash S$$

at any $x \in \operatorname{int} \mathbf{R}_+^N$. But since λ was chosen arbitrarily in $\operatorname{int} \triangle_N$, we have a contradiction. □

7 Appendix: Proof of main lemma

In this section we state and prove the main lemma which was used in the proof of Theorem 3.1.

Lemma 7.1 *Let $|N| \geq 3$, let $S \subset N$ be a coalition with $|S| \geq 2$. Suppose that $v \in \mathcal{B}$ is a such that $x = \phi^u(v)$ belongs to $\operatorname{int} \mathbf{R}_+^N$ and satisfies $x(S) = v(S)$ and $x'(S) > v(S)$ for some $x' \in C(v)$. Then there is a game v' and $i \in S$ such that*

(i) $\phi^u(v') = \phi^u(v) = x$,

(ii) if $v'(T) = x(T)$ for some $T \subset N$, then either $i \notin T$, or $T = S$.

(iii) $C(v')$ contains points x' with $x'_j > x_j$, for some $j \in S$, $j \neq i$.

Proof: Normalizing if necessary we may assume that $v(N) = 1$. Since x maximizes u on $C(v)$ it solves the maximization problem

$$\begin{aligned}&\max\, u(x)\\ &s.t.\\ &x(T) \geq v(T),\, T \subset N,\, T \neq N,\\ &x(N) = 1.\end{aligned}$$

In this maximization problem, if T is such that $x(T) > v(T)$, then we may remove the constraint $x(T) \geq v(T)$ and still have the solution x; indeed, we may focus only on the set of zero-excess coalitions $\mathcal{C}$ satisfying $x(T) = v(T)$ (including S).

Next, note that we search for solutions y over vectors in $H = \operatorname{aff}\triangle_N = \{x \mid \sum_{i=1}^n x_i = 1\}$, and therefore we may as well replace the objective function u by $\hat{u} = u_{|H}$, its restriction to H,

$$\hat{u}'_i = \frac{\partial \hat{u}}{\partial x_h}(x),\ i = 1, \ldots, n.$$

Then the family of halfspaces (in H) consisting of

$$\{x' \in H \mid x'(T) \geq v(T)\} = \left\{x' \in H \,\middle|\, \sum_{i\in T} x'_i \geq \sum_{i \in T} x_i\right\}$$

for T belonging to the family $\mathcal{C} = \{S' \subset N \mid x(S') = v(S')\}$ and the halfspace $\{x' \in H \mid \hat{u}' \cdot x' > \hat{u}' \cdot x\}$, have empty intersection (since otherwise there would be a point x' close to x satisfying all constraints and giving a larger value of the objective function).

By duality, we have that 0 can be written as a nonnegative sum of the normals of these halfspaces (see e.g. Rockafellar [10]). Clearly, $\hat{u}'$ is a normal of $\{x' \in H \mid \hat{u}' \cdot x' > \hat{u}' \cdot x\}$, whereas for any halfspace $\{x' \mid x'(T) = v(T)\}$, the vector $\hat{e}_N - \hat{e}_T$ is a normal; thus, we have that

$$\lambda_0 u' + \sum_{T \in \mathcal{C}} \lambda_T[\hat{e}_N - \hat{e}_T] = 0,\, \lambda_T \geq 0,\, T \in \mathcal{C}. \tag{2}$$

By our construction, $S \in \mathcal{C}$; also, $\lambda_T > 0$ for each T by the minimality of $\mathcal{C}$. Multiplying all λ_i by some constant if necessary, we may assume that $\sum_{T\in\mathcal{C}} \lambda_T = 1$; finally, we may replace u by the function $\lambda_0 u$ without changing the core selection ϕ^u, so that we may assume that $\lambda_0 = 1$. Consequently, (2) can be written as

$$u' + \hat{e}_N = \sum_{T\in\mathcal{C}} \lambda_T \hat{e}_T,\ \lambda_T \geq 0,\ T \in \mathcal{C}. \tag{3}$$

Conversely, if

$$u' + \hat{e}_N = \sum_{T\in\mathcal{C}'} \mu_T \hat{e}_T,\ \mu_T \geq 0,\ T \in \mathcal{C}',\ \sum_{T\in\mathcal{C}'} \mu_T = 1, \tag{4}$$

for some family of coalitions $\mathcal{C}'$, then x maximizes u on the set of vectors x' in $\triangle_N$ satisfying $\sum_{i\in T} x'_i \geq \sum_{i\in T} x_i$, for each $T \in \mathcal{C}'$

We claim that the family $\mathcal{C}'$ in (4) can be chosen such that $S \in \mathcal{C}'$ and some $i \in S$ is in no other coalition in $\mathcal{C}'$, so that we have proved part (i) and (ii) for the game v' with $v'(T) = \sum_{i\in T} x_i$ for $T \in \mathcal{C}'$ and $v'(T) = 0$ otherwise, if we can find a family of coalitions containing S such that $\hat{u}'(x) + \hat{e}_N$ belongs to the convex hull of the associated barycenters.

Let $z \in \triangle_N$ be arbitrary, then $z = \sum_{i=1}^n z_i \hat{e}_i$, and put $\hat{S} = \{i \in S \mid z_i \neq 0\}$. If $\hat{S}$ has less than 2 members, then there are $i, j \in S$ with $z_i = z_j = 0$, and the family $\mathcal{C}' = \{\{i\} \mid z_i > 0\} \cup S$ satisfies our claim; also (iii) is satisfied since $C(v')$ contains points x' with $x'_j > x_j$.

Thus, we may assume that $|\hat{S}| \geq 2$. Let $i^0 \in \hat{S}$ be such that $z_{i^0} = \min_{i\in\hat{S}} z_i$, let

$$\lambda_{\hat{S}} = z_{i^0}|\hat{S}|,\ \lambda_{\{i\}} = z_i - z_{i^0},\ i \in \hat{S},\ \lambda_{\{j\}} = z_j,\ j \notin \hat{S} \text{ and } \lambda_S = 0 \text{ if } S \neq \hat{S},$$

and let $\mathcal{C}'$ be the family consisting of $\{\{i\} \mid z_i > 0,\ i \neq i^0\}$, $\hat{S}$ and S. Then

$$\sum_{T\in\mathcal{C}'} \lambda_T = \sum_{i\in\hat{S},\, i\neq i^0} (z_i - z_{i^0}) + \sum_{i\notin S} z_i + z_{i^0}|\hat{S}| = \sum_{i=1}^n z_i = 1,$$

and

$$\begin{aligned}
\sum_{T\in\mathcal{C}'} \lambda_T \hat{e}_T &= \sum_{i\in\hat{S}, i\neq i^0} (z_i - z_{i^0})\hat{e}_i + \sum_{i\notin S} z_i \hat{e}_i + z_{i^0}|\hat{S}|\hat{e}_{\hat{S}} \\
&= \sum_{i\neq i^0} z_i \hat{e}_i + \left[z_{i^0}|\hat{S}|\hat{e}_{\hat{S}} - z_{i^0}|\hat{S}| \sum_{i\in\hat{S}, i\neq i^0} \frac{1}{|\hat{S}|}\hat{e}_i \right] \\
&= \sum_{i\neq i^0} z_i \hat{e}_i + z_{i^0}\hat{e}_{i^0} = z,
\end{aligned}$$

showing that our claim is satisfied with $i \in S \backslash \hat{S}$ if this set is nonempty, $i = i^0$ otherwise. Finally, by our construction we have that for any $j \in S \backslash \{i^0\}$ we have that $C(v')$ contains elements x' with $x'_j > x_j$, proving (iii). □

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