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02-16

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Market Selection and Survival of Investment Strategies*

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October 10, 2002

Abstract

The paper analyzes the process of market selection of investment strategies in an incomplete market of short-lived assets. In the model under study, asset payoffs depend on exogenous random factors. Market participants use dynamic investment strategies taking account of available information about current and previous events. It is shown that an investor allocating wealth across the assets according to their conditional expected payoffs eventually accumulates total market wealth, provided the investor's strategy is asymptotically distinct from the portfolio rule suggested by the Capital Asset Pricing Model. This assumption turns out to be essentially necessary for the result.

JEL-Classification: D52, D81, D83, G11.
Keywords: evolutionary finance, portfolio theory, CAPM, investment strategies, market selection, incomplete markets.

*We thank David Easley for helpful comments. Financial support by the national center of competence in research "Financial Valuation and Risk Management" is gratefully acknowledged. The national centers in research are managed by the Swiss National Science Foundation on behalf of the federal authorities.

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1 Introduction

It has long been argued that, in competitive environments, market pressures would eventually select those traders who are better adapted to the prevailing conditions. According to the standard paradigm of economic theory, agents maximize preferences or utilities. From the evolutionary point of view, what matters is not the utility level, but the chances of survival. The evolutionary principle leads to the consideration of the process of economic natural selection among the market participants, or among the strategies of behavior they adopt. This view has been put forward by Alchian (1950), Enke (1951), Penrose (1952) and pursued by many others.

The purpose of the paper is to elaborate on an evolutionary approach to the study of investment strategies in financial markets. This work combines ideas from economic theory and finance. We examine the process of market selection in the framework of incomplete markets with traders using dynamic investment strategies. In the model under study, it is supposed that each trader follows a portfolio rule specifying the allocation of wealth across the available assets at any moment of time and for any history of events. We retain the key feature of economic equilibrium models where a market-clearing mechanism determines prices endogenously in every period. However, we depart from individual utility maximization. Instead, we assume that trading strategies are compared with each other in terms of their abilities to survive under market selection, rather than in the conventional terms of discounted values.

We consider a market with short-lived assets that live only one time period but are identically reborn every next period. The assets are in positive supply, and their payoffs depend on the realization of exogenous states of the world described in terms of a homogeneous finite-state Markov chain. Short sales are ruled out. The focus is on the long-run dynamics of the distribution of wealth across the investors. Following Epstein and Zin (1989, 1991), the model prescribes reinvestment of total wealth and thus precludes consumption. This assumption, in particular, avoids the trade-off between the rate of consumption and the evolutionary fitness of the trading rule in a market.

In the case where only a complete set of Arrow securities is traded, the states of the world are independent and identically distributed, and all the traders use only *simple* portfolio rules (independent of time and observations), our framework reduces to the model considered by Blume and Easley (1992, Section 3). They have demonstrated the remarkable role of the portfolio rule prescribing the investor to distribute wealth between the assets according to the probability of the state in which the asset pays out—this portfolio rule is often referred to as "betting one's beliefs." Blume and Easley (1992) show that if a trader uses this rule, whereas all the others use different (simple) portfolio rules, then the trader will eventually accumulate total market wealth. In other words, the investor will be a *single survivor* in the market selection process.

Apparently, the first who stated the principle underlying the rule of "betting one's beliefs" was Kelly (1956). He showed (in a different context) that this principle leads to the maximization of the expected logarithm of the portfolio growth rate. This idea gave rise to a large body of research—see, e.g., Breiman (1961), Thorp (1971), Algoet and Cover (1988), and Hakansson and Ziemba (1995). A common feature of these papers is that they study single-agent problems with exogenous prices. Blume and Easley (1992) considered an equilibrium model with prices determined endogenously. Nonetheless, due to the completeness of the market (and the special structure of Arrow securities), their result regarding the rule of "betting one's beliefs" can be reduced to the maximization of the expected logarithm of an appropriately defined relative growth rate.

The Blume–Easley approach has been extended by Sandroni (2000) and Blume and Easley (2001) to expected utility maximizing investors with general utility functions. These papers focus, basically, on dynamically complete markets. As regards to incomplete markets, Sandroni (2000) provides a version of the result that traders maximizing the expected logarithm of the growth rate eventually accumulate all wealth—Proposition 1 in Section 2 of the paper cited. This proposition does not apply to the model studied in the present paper (its assumptions do not necessarily hold in our context, and its conclusions do not imply the results we obtain).

Incomplete markets with simple trading strategies and independent identically distributed states of the world have been considered in Evstigneev, Hens, and Schenk-Hoppé (2002) (see also Hens and Schenk-Hoppé (2001) who examine the local dynamics of the market selection process, dealing, however, with more general strategies). There is a significant distinction between models of this type and the previous ones. Owing to market incompleteness and the endogenous mechanism of price formation, the questions studied in this, more general, framework cannot be reduced to the single-agent maximization of growth rates. Usually, the performance of a strategy in a market selection process cannot be determined only by the strategy itself; it depends on the combination of all the strategies employed by the whole group of investors (this combination determines the endogenous asset prices). The main finding in the last two papers is that investors distributing their wealth according to the expected relative payoffs dominate the market.

The present paper continues the study conducted in Evstigneev, Hens, and Schenk-Hoppé (2002). Our goal is to remove two simplifying assumptions that substantially reduce the scope of the models under study. Firstly, we consider general, rather than simple, investment strategies. Secondly, we abandon the assumptions of independence and identical distribution of the random variables describing the states of the world and assume instead that the sequence of these random variables is a homogeneous discrete-time Markov process.

This more general set of assumptions results in a considerably enlarged scope of the theory at hand, with enhanced realism and thus potential applicability of the results. In a financial investment setting, the state of the world is a description of a large and complex set of variables characterizing investors' information, including, among many others, business cycle indicators, central bank policy variables, various firm-level indicators and consumer indices. The complexity underlying the evolution of so many relevant state variables could not possibly be captured by a random process with independent and identically distributed values. Some serial dependence, at least of a Markov nature, must be postulated.

Furthermore, restricting consideration to simple investment strategies amounts to asking each trader to commit to one and the same constant strategy for the entire duration of the process, as if the agent had no access to any relevant information throughout. This is hardly compatible with the real-life behavior of investors, who typically react with considerable frequency to a whole array of economic indices. Consequently, it seems imperative to model investors' behavior as reflecting unfolding events and disclosed information.[1]

To outline our main result, let us denote by λ^* the portfolio rule that requires a trader to allocate wealth across the assets in accordance with their relative conditional expected payoffs. (In the Markov rather than i.i.d. setting, we have to deal with conditional, rather than unconditional, expectations.) Our main result is that, in any—complete or incomplete—market for short-lived assets, *a trader following the rule λ^* eventually accumulates total market wealth, provided the trader's strategy λ^* is asymptotically distinct from the CAPM rule.* The latter prescribes investing into the *market portfolio.*[2] A trader using the CAPM rule keeps a constant fraction of market

[1] While our setting is formally not game-theoretic since agents are not payoff-maximizing, an instructive partial analogy exists between the two concepts of strategy described here and those in standard dynamic games. Simple and general strategies in our evolutionary setting correspond, respectively, to the notions of open-loop and closed-loop (with or without history-dependence) strategies in dynamic games.

[2] This investment rule is suggested by the Capital Asset Pricing Model (CAPM) and Tobin's mutual fund theorem—see, e.g., Magill and Quinzii (1996, Theorem 17.3 and Proposition 16.15).

wealth. Thus the trader can neither accumulate total market wealth nor be driven out of the market. Investing into the market portfolio means mimicking the "average" portfolio (therefore the CAPM strategy does not, typically, belong to the class of simple strategies).

We prove that the λ^*-trader accumulates total market wealth at exponential rate if λ^* is bounded away from the CAPM rule for "sufficiently many" time periods. More precisely, we impose the following condition: there exists a random number $\kappa > 0$ such that, almost surely, the distance between λ^* and the CAPM rule is greater than κ in n_t periods during every time-horizon of length t, where $\liminf_{t\to\infty} n_t/t > 0$. Remarkably, this requirement turns out to be not only sufficient but also necessary for the λ^*-trader to be a single survivor in the market selection process, accumulating wealth at exponential rate (see Theorem 2 in Section 3). The need for such a requirement arises here due to the added complexity of strategic behavior. To the best of our knowledge, the above result has no counterparts in the related literature.

Although our analysis is complete within the present framework, there are certainly many desirable extensions of the model. One can mention, for instance, long-lived assets, changes in the market structure, endogenous asset supply, and variations of the investment-consumption ratio. These generalizations are left to future research.

For recent studies on evolutionary finance dealing with different (albeit related) models and questions see Brock and Hommes (1998), Brock, Hommes, and Wagener (2001), Sandroni (2000), Blume and Easley (2001) and references therein.

The paper is organized as follows. Section 2 introduces the model. The main results are presented in Section 3. All the proofs are relegated to the Appendix.

2 Model

Let S be a finite set and s_t, $t = 0, 1, 2, ...$, a homogeneous Markov chain with transition function $p(\sigma|s)$, specifying the conditional probabilities $P\{s_{t+1} = \sigma|\ s_t = s\}$. The random variable s_t describes the "state of the world" at time t. We consider a market with K assets. Their life cycle is one time period, but they are identically reborn every next period. The total amount of each asset k in the market is a positive constant $V_k > 0$, exogenously given in the model. One unit of asset k issued at time t yields a payoff $A_k(s_{t+1}, s_t) \geq 0$

at time $t+1$. We assume

$$\sum_{k=1}^{K} A_k(\sigma, s) > 0 \tag{1}$$

for all $\sigma, s \in S$.

There are I investors (traders) $i = 1, ..., I$ acting in the market. Every investor i at each time $t = 0, 1, 2, ...$ selects a *portfolio*

$$h_t^i = (h_{1,t}^i, ..., h_{K,t}^i),$$

where $h_{k,t}^i$ is the number of units of asset k in the portfolio h_t^i. Generally, h_t^i depends on the history $s^t = (s_0, ..., s_t)$ of the process s_t up to time t:

$$h_t^i = h_t^i(s^t).$$

We will often omit the argument s^t when this does not lead to ambiguity. As is explained below, this model allows portfolios to depend on past and current market prices. We will assume that, at each time $t \geq 1$ and in every random situation s^t, the asset market clears:

$$\sum_{i=1}^{I} h_{k,t}^i(s^t) = V_k. \tag{2}$$

According to this equality, demand for each asset k (the sum on the left-hand side of (2)) is equal to its supply, V_k.

If investor i possesses a portfolio $h_t^i = (h_{k,t}^i)$ at time $t \geq 0$, then her wealth w_{t+1}^i at time $t+1$ can be expressed as

$$w_{t+1}^i = \sum_{k=1}^{K} A_k(s_{t+1}, s_t)\, h_{k,t}^i.$$

For every i, a strictly positive number w_0^i is given—the *initial wealth* of investor i. In view of (2), we have

$$\sum_{i=1}^{I} w_{t+1}^i = \sum_{k=1}^{K} A_k(s_{t+1}, s_t)\, V_k, \quad t \geq 0. \tag{3}$$

The variable

$$w_t = \sum_{i=1}^{I} w_t^i,$$

specifies *aggregate market wealth* at time $t \geq 0$.

It is assumed that every trader i selects a portfolio by using the following procedure. The trader chooses an *investment strategy*—a sequence of functions

$$\lambda_t^i = (\lambda_{1,t}^i, ..., \lambda_{K,t}^i),\ \lambda_t^i = \lambda_t^i(s^t), \quad t \geq 0, \tag{4}$$

such that

$$\lambda_{k,t}^i > 0,\ \sum_{k=1}^{K} \lambda_{k,t}^i = 1, \tag{5}$$

and assigns the share $\lambda_{k,t}^i$ of the budget w_t^i for purchasing asset k at time t. If every investor i has chosen a strategy $(\lambda_{k,t}^i)$, the equation

$$\rho_{k,t} = \frac{1}{V_k} \sum_{i=1}^{I} \lambda_{k,t}^i \, w_t^i \tag{6}$$

determines the market clearing price $\rho_{k,t} = \rho_{k,t}(s^t)$ of asset k at any time $t \geq 0$. The portfolio h_t^i of investor i can be expressed by the formula

$$h_{k,t}^i = \frac{\lambda_{k,t}^i \, w_t^i}{\rho_{k,t}}, \quad k = 1, 2, ..., K, \quad t \geq 0.$$

From the last and the previous equations, we find

$$h_{k,t}^i = V_k \frac{\lambda_{k,t}^i \, w_t^i}{\sum_{j=1}^{I} \lambda_{k,t}^j \, w_t^j}. \tag{7}$$

This leads to the following formula expressing the wealth w_{t+1}^i of investors $i = 1, 2, ..., I$ at time $t + 1$ through their wealth at time t:

$$w_{t+1}^i = \sum_{k=1}^{K} A_k(s_{t+1}, s_t) \, V_k \frac{\lambda_{k,t}^i \, w_t^i}{\sum_{j=1}^{I} \lambda_{k,t}^j \, w_t^j}. \tag{8}$$

Since $w_0^i > 0$, we obtain by way of induction that $w_t^i > 0$ for each t (see (1) and (5)). From this we conclude that the evolution of the *relative market shares* of the investors,

$$r_t^i = \frac{w_t^i}{w_t},$$

is governed by the equations

$$r_{t+1}^i = \sum_{k=1}^{K} R_k(s_{t+1}, s_t) \frac{\lambda_{k,t}^i \, r_t^i}{\sum_{j=1}^{I} \lambda_{k,t}^j \, r_t^j}, \quad i = 1, ..., I, \tag{9}$$

where

$$R_k(s_{t+1}, s_t) = \frac{A_k(s_{t+1}, s_t)\, V_k}{\sum_{m=1}^{K} A_m(s_{t+1}, s_t)\, V_m}.$$

The numbers $R_k(s_{t+1}, s_t)$ characterize the *relative* (*normalized*) *payoffs* of the assets $k = 1, 2, ..., K$. We have $R_k(s_{t+1}, s_t) \geq 0$ and

$$\sum_{k=1}^{K} R_k(s_{t+1}, s_t) = 1. \tag{10}$$

The above notion of an investment strategy includes, in particular, portfolio rules described in terms of traders' demand functions. Suppose investor i chooses her budget share $\lambda^i_{k,t}$ as a function $\Lambda^i_{k,t}(s^t, \rho_t, w^i_t)$ of the current and past observations s^t, the vector $\rho_t = (\rho_{k,t})$ of the prevailing market prices and wealth w^i_t. In other words, suppose $\Lambda^i_{k,t}(s^t, \rho_t, w^i_t)$ is the demand function of investor i. Then the equation determining the equilibrium (market clearing) price vector ρ_t takes on the form

$$\rho_{k,t} = \frac{1}{V_k} \sum_{i=1}^{I} \Lambda^i_{k,t}(s^t, \rho_t, w^i_t) w^i_t.$$

Assuming an equilibrium is realized and solving the above equation, we express ρ_t as a function of s^t and $w^1_t, ..., w^I_t$. Since $w^i_t = w^i_t(s^t)$, we obtain that ρ_t and hence $\lambda^i_{k,t}$ are functions of s^t, which agrees with our notion of a strategy.

The main focus of this work is on the analysis of the evolution of the relative market shares r^i_t depending on the choice of the strategies λ^i_t, $i = 1, 2, ..., I$. We are interested primarily in those strategies that allow an investor to *survive*, i.e., to keep a positive relative market share in the limit, and, moreover, that allow the investor to *dominate the market*, i.e., to accumulate in the limit all market wealth. A central role is played by the following notion. We say that an investor i (or the strategy $\lambda^i = (\lambda^i_{k,t})$) is a *single survivor* in the selection process (9) if

$$\lim r^i_t = 1 \tag{11}$$

almost surely (a.s.). Condition (11) implies $\lim r^j_t = 0$ a.s. for all $j \neq i$, which means that, in the limit, investor i accumulates all market wealth. If the sequence r^i_t involved in (11) converges to 1 at an exponential rate, we shall say that the strategy λ^i *dominates* the others *exponentially*.

It is an important problem to identify those strategies which enable an investor using them to become a single survivor. Hens and Schenk-Hoppé

(2001) and Evstigneev, Hens, and Schenk-Hoppé (2002) have analyzed this problem within two different settings: local and global, respectively. The latter paper deals with a special case of the model in which

(i) the random variables s_t are independent and identically distributed;

(ii) the functions A_k (and hence R_k) depend only on s_{t+1};

(iii) $V_k = 1$;

(iv) the expected values $ER_k(s_t)$ are strictly positive;

(v) the functions $R_1(s), ..., R_K(s)$ are linearly independent (the absence of redundant assets).

The analysis in the paper is restricted to the consideration of only *simple* strategies $\lambda^i = (\lambda^i_{k,t})$, i.e., those for which the budget shares $\lambda^i_{k,t}(s^t)$ do not depend on t and s^t. For that model, the following result has been obtained (Evstigneev, Hens, and Schenk-Hoppé 2002, Theorem 3.1). *If one of the investors $i = 1, ..., I$, say $i = 1$, uses the simple strategy $\lambda^* = (\lambda^*_k)$ defined by*

$$\lambda^*_k = ER_k(s_t) \tag{12}$$

while all the other investors $i \neq 1$ use different simple strategies $\lambda^i \neq \lambda^*$, then investor 1 is a single survivor in the market selection process (9). This result generalizes that of Blume and Easley (1992), dealing with the case of Arrow securities ($S = \{1, 2, ..., K\}$, $A_k(s) = 0$ if $s \neq k$ and $A_k(s) = 1$ if $s = k$). Furthermore, the strategy (12) defined in terms of the expected payoffs may be regarded as a development of the Kelly rule of "betting one's beliefs" (Kelly 1956). Originally designed in connection with gambling problems, this rule has been successfully employed in portfolio theory (Thorp 1971, Aurell, Baviera, Hammarlid, Serva, and Vulpiani 2000).

In this work, we intend to obtain versions of the above result applicable to the more general model we have described in the present section. What is most essential in this generalization is that we are going to leave the framework of simple strategies and allow the investors to employ strategies using information about the history of the process s_t—see the definition in (4) and (5). In this context, we can define a direct analog of the strategy λ^* given by (12). As it turns out, we cannot, generally, guarantee λ^* to be a single survivor. Nevertheless, we show that this conclusion does obtain under a natural sufficient condition, having a clear economic meaning. We also provide a necessary and sufficient condition for an investor using the strategy λ^* to be a single survivor dominating the others exponentially. Precise statements of the results are given in the next section.

3 Results

Consider the random dynamical system (9) describing the evolution of the relative market shares $r_t^i(s^t)$ of the investors $i = 1, 2, ..., I$. Note that if $r_t = (r_t^i)$ is a strictly positive vector, then, as is easily seen from (9), (10) and (5), r_{t+1} is a strictly positive vector as well. Thus $r_t = r_t(s^t)$ is a random process with values in the relative interior Δ_+^I of the unit simplex

$$\Delta^I = \{x = (x^1, ..., x^I) \in \mathbb{R}^I : \; x^i \geq 0, \; \sum x^i = 1\}.$$

The initial state $r_0 = (r_0^1, ..., r_0^I) \in \Delta_+^I$, from which this process starts, is fixed ($r_0^i = w_0^i / \sum w_0^j$).

We will analyze the above random dynamical system under the following assumptions.

(A.1) The functions

$$R_k^*(s) := \sum_{\sigma \in S} p(\sigma|s)\, R_k(\sigma, s), \quad k = 1, 2, ..., K, \tag{13}$$

take on strictly positive values for each $s \in S$.

(A.2) For every $s \in S$, the functions $R_1(\cdot, s), ..., R_K(\cdot, s)$ restricted to the set

$$\Pi(s) = \{\sigma \in S : \; p(\sigma|s) > 0\}$$

are linearly independent.

According to (A.1), the conditional expectation

$$R_k^*(s) = E[R_k(s_{t+1}, s_t) \mid s_t = s] \tag{14}$$

of the relative payoff $R_k(s_{t+1}, s_t)$ of every asset k given $s_t = s$ is strictly positive at each state s. Assumption (A.2) means the absence of *conditionally redundant* assets. The term "conditionally" refers to the fact that the functions $R_k(\cdot, s)$, $k = 1, ..., K$, are linearly independent on the set $\Pi(s)$—the support of the conditional distribution $p(\sigma|s)$.

In what follows, we will restrict attention to those investment strategies $\lambda = (\lambda_{k,t})$ that satisfy the following additional assumption.

(B) The coordinates $\lambda_{k,t}(s^t)$ of the vectors $\lambda_t(s^t)$ are bounded away from zero by a strictly positive non-random constant ρ (that might depend on the strategy λ, but not on k, t and s^t).

In (5), we included in the definition of a strategy the condition $\lambda_{k,t} > 0$ (such strategies are sometimes termed *completely mixed*). Assumption (B) contains the additional requirement of uniform strict positivity of $\lambda_{k,t}$.

A key role in our analysis will be played by the strategy $\lambda^* = (\lambda^*_{k,t}(s_t))$ defined according to the formula

$$\lambda^*_{k,t}(s_t) = R^*_k(s_t), \tag{15}$$

where $R^*_k(s)$ is the conditional expectation of $R_k(s_{t+1}, s_t)$ given $s_t = s$ (see (13) and (14)). This is the direct analog of the strategy of "betting one's beliefs", which takes on, in the case of independent identically distributed variables s_t, the form (12). Note that $\lambda^*_k(s_t) = \lambda^*_{k,t}(s_t)$ does not explicitly depend on t, and, furthermore, $\lambda^*_k(s_t)$ is a function of only the current state s_t of the process (s_t), rather than the whole history s^t of it. This implies, by virtue of (A.1) and in view of the finiteness of S, that the strategy λ^* satisfies condition (B).

To proceed further, we need to describe a recursive method of constructing strategies based on (Markovian) decision rules. Suppose one of the traders, say 1, has a privilege of making her investment decision at time t with full information about the current market structure r_t and the actions $\lambda^2_t(s^t)$, $\lambda^3_t(s^t), ..., \lambda^I_t(s^t)$ that have just been undertaken by all the other traders $2, 3, ..., I$. Formally, the decision of investor 1 is specified by a function

$$f_t(r, l^2, ..., l^I),\ r \in \Delta^I_+,\ l^j \in \Delta^K_+ \quad (j = 2, 3, ..., K)$$

taking values in Δ^K_+. Suppose such functions—*decision rules*—are given for all $t = 0, 1, 2, ...$ Furthermore, suppose investors $2, ..., I$ have chosen some strategies $\lambda^2_t, ..., \lambda^I_t$ $(t = 0, 1, 2, ...)$. Then we can construct a strategy $\lambda^1_t(s^t)$, $t = 0, 1, ...,$ of investor 1 by using the formula

$$\lambda^1_t(s^t) = f_t(r_t, \lambda^2_t, ..., \lambda^I_t), \tag{16}$$

where $r_t = r_t(s^t)$ and $\lambda^j_t = \lambda^j_t(s^t)$, $j = 2, ..., I$.

Let us consider a particular decision rule $f = (f_1, ..., f_K)$ (which does not explicitly depend on t) defined by

$$f(r, l^2, ..., l^I) = \sum_{j=2}^{I} \frac{r^j}{1 - r^1}\, l^j. \tag{17}$$

Here $r = (r^1, ..., r^I) \in \Delta^I_+$, $l^j = (l^j_1, ..., l^j_K) \in \Delta^K_+$, and so the vector $f = (f_1, ..., f_K)$ belongs to Δ^K_+. Note that the vector f is a convex combination of the vectors $l^2, ..., l^I$ with weights $r^j(1-r^1)^{-1}$. This implies, in particular, the following: if the coordinates l^j_k of the vectors l^j are bounded away from 0 by a constant $\rho > 0$, then the coordinates f_k of f are bounded away from 0 by the same constant. Consequently, if the strategies λ^2_t,...,λ^I_t satisfy condition

(B), the strategy (16) satisfies condition (B) as well. *In what follows, we will use the notation $f = (f_k)$ for the specific decision rule described in (17).*

The decision rule (17) has a number of remarkable properties. First of all, observe the following. Suppose investor 1 employs the strategy $\lambda_t^1(s^t)$ defined by (16) in terms of the decision rule (17). Then we have

$$\lambda_{k,t}^1 = \sum_{j=1}^{I} \lambda_{k,t}^j \, r_t^j, \tag{18}$$

which, in view of (9), yields

$$r_{t+1}^1 = r_t^1.$$

Thus, if investor 1 uses the strategy generated by the decision rule (17), then, regardless of what strategies are used by the others, the relative market share of this investor remains constant over time. This observation leads to the following conclusion. If one of the traders $2, ..., I$ uses the strategy λ^*, she cannot be a single survivor, as long as trader 1 uses the strategy (16), (17) and, consequently, keeps a constant positive market share $r_t^1 = r_0^1$ for all t.

Further, we can see that the portfolio of investor 1, who uses the strategy λ_t^1 defined in terms of the decision rule (17), is given by

$$h_{k,t}^1 = V_k \frac{\lambda_{k,t}^1 w_t^1}{\sum_{j=1}^{I} \lambda_{k,t}^j \, w_t^j} = V_k \frac{\lambda_{k,t}^1 \, r_t^1}{\sum_{j=1}^{I} \lambda_{k,t}^j \, r_t^j} = V_k r_t^1,$$

for all $k = 1, 2, ..., K$ (see (7) and (18)). Thus the vector $h_t^1 = (h_{1,t}^1, ..., h_{K,t}^1)$ turns out to be proportional to the *market portfolio*, i.e., the vector

$$(V_1, ..., V_k),$$

whose components indicate the amounts of assets $k = 1, 2, ..., K$ traded at the market. According to the well-known Tobin mutual fund theorem (Magill and Quinzii 1996, Proposition 16.15), portfolios having this structure result from the mean-variance optimization in the Capital Asset Pricing Model (CAPM). Therefore it is natural to term the decision rule (17) the *CAPM decision rule* and the strategy generated by it the *CAPM strategy.* The CAPM decision rule plays a key role in the formulation of the main results below.

The notions we have just described pertain to investor 1. We can introduce analogous notions for any $m \in \{1, 2, ..., I\}$. To this end, consider the vector function

$$f^m(r, l^1, ..., l^{m-1}, l^m, ..., l^I) = \sum_{j \neq m} \frac{r^j}{1 - r^m} \, l^j$$

of $r = (r^1, ..., r^I) \in \Delta_+^I$ and $l^j = (l_1^j, ..., l_K^j) \in \Delta_+^K$. This function specifies the CAPM decision rule for trader m. Given strategies $\lambda_t^j(s^t)$ of all the other traders $j \in \{1, 2, ..., I\} \backslash \{m\}$, the CAPM strategy of m is defined by $\lambda_t^m = f^m(r_t, \lambda_t^1, ..., \lambda_t^{m-1}, \lambda_t^{m+1}, ..., \lambda_t^I)$. Those properties we discussed for $m = 1$, extend to an arbitrary m.

In Theorem 1 below, we describe a condition sufficient for the strategy (15) to be a single survivor. We consider the dynamical system (9), assuming that the investors $i \in \{1, 2, ..., I\}$ use some strategies $\lambda^i = (\lambda_t^i)$ satisfying requirement (B). We define

$$\zeta_t = (\zeta_{1,t}, ..., \zeta_{K,t}) = f(r_t, \lambda_t^2, ..., \lambda_t^I),$$

where f is the CAPM decision rule (17). The symbol $|\cdot|$ denotes the sum of the absolute values of the coordinates of a finite-dimensional vector.

Theorem 1 *Let investor 1 use the strategy $\lambda^1 = \lambda^*$ defined by (15). Let the following condition be fulfilled:*
(C) With probability 1, we have

$$\liminf_{t \to \infty} |\lambda^*(s_t) - \zeta_t| > 0. \tag{19}$$

Then investor 1 is a single survivor, and, moreover,

$$\liminf_{t \to \infty} \frac{1}{t} \ln \frac{r_t^1}{1 - r_t^1} > 0 \tag{20}$$

almost surely.

Property (20) means that the relative market share of investor 1 tends to one at an exponential rate, whereas the relative market shares of all the other investors vanish at such rates, and so the strategy λ^* dominates the others exponentially.

Condition (C) can be restated as follows: there exists a strictly positive random variable κ such that, almost surely,

$$|\lambda^*(s_t) - \zeta_t(s^t)| \geq \kappa \tag{21}$$

for all t large enough. The last inequality requires that the actions $\lambda^*(s_t)$ prescribed by the strategy λ^* should differ by not less than $\kappa > 0$ from the actions

$$\zeta_t(s^t) = (\zeta_{1,t}(s^t), ..., \zeta_{K,t}(s^t)), \;\; \zeta_{k,t}(s^t) = \sum_{j=2}^{I} \frac{r_t^j(s^t)}{1 - r_t^1(s^t)} \lambda_{k,t}^j(s^t),$$

prescribed by the CAPM decision rule. Here, we do not assume that there is at least one investor who indeed employs the CAPM rule; we need it only as an indicator, a proper deviation of which from λ^* guarantees λ^* to be a single survivor.

In concrete instances, it might not be easy to verify condition (C) directly. Therefore we provide another hypothesis, (C.1), which is stronger than (C) but can conveniently be checked in various examples.

(C.1) There exists a strictly positive random variable κ such that, with probability 1, *the distance between the vector $\lambda^*(s_t) \in \mathbb{R}^K$ and the convex hull of the vectors $\lambda_t^2(s^t), ..., \lambda_t^I(s^t) \in \mathbb{R}^K$ is not less than κ for all t large enough.*

Clearly (C.1) implies (C) because $\zeta_t = f(r_t, \lambda_t^2, ..., \lambda_t^I)$ is a convex combination of $\lambda_t^2, ..., \lambda_t^I$.

Condition (C), which is sufficient for investor 1 to be a single survivor, turns out to be close to a necessary one. The theorem below provides a version of hypothesis (C) that is necessary and sufficient for the conclusion of Theorem 1 to hold.

Theorem 2 *Investor 1 using the strategy (15) is a single survivor in the market selection process, and, moreover, dominates the others exponentially, if and only if the following condition is fulfilled:*
(C.2) There exists a random variable $\kappa > 0$ such that

$$\liminf_{T\to\infty} \frac{1}{T} \#\big\{t \in \{0, ..., T\} : |\lambda^*(s_t) - \zeta_t(s^t)| \geq \kappa\big\} > 0 \tag{22}$$

with probability 1.

The symbol # in the above formula stands for the number of elements in a finite set.

Observe that (C.2) follows from (C). Indeed, (C) is equivalent to the existence of a random variable κ for which, almost surely, inequality (21) is fulfilled for all t large enough. In this case, the limit in (22) is equal to 1. The limit in (22) may be thought of as a *density* (in the set of natural numbers) of those natural numbers t for which inequality (21) holds. Hypothesis (C.2) only requires this density to be strictly positive, whereas (C) says that (21) should hold from some t on.

Let us return to Theorem 1. From this theorem, it follows immediately that if the relation

$$\liminf_{t\to\infty} \frac{1}{t} \ln \frac{r_t^1}{1 - r_t^1} \leq 0 \tag{23}$$

holds with positive probability, then, with positive probability, there exists a (random) sequence t_k such that

$$|\lambda^*(s_{t_k}) - \zeta_{t_k}(s^{t_k})| \to 0. \tag{24}$$

Can we make a stronger statement about convergence in (24) if we strengthen (23) appropriately? A result along these lines is provided by the next theorem.

Theorem 3 *Let the following condition be satisfied:*
(D.1) There exists a random variable $0 < \gamma < 1$ *such that* $E \ln \gamma > -\infty$ *and*

$$r_t^1 < 1 - \gamma$$

a.s. for all t.

Then we have

$$|\lambda^*(s_t) - \zeta_t| \to 0 \quad a.s.$$

We will actually prove Theorem 3 under a weaker assumption:
(D.2) The expectations

$$E[\ln(1 - r_t^1)]$$

do not converge to $-\infty$.

Clearly (D.1) is stronger than both (D.2) and (23), but (D.2) does not necessarily imply (23). Condition (D.1) holds, for example, if one of the investors $i = 2, ..., I$ uses the CAPM strategy (and so her relative market share remains constant). Then, as Theorem 3 asserts, the difference between the budget shares of investor 1 prescribed by the strategy λ^* and the budget shares prescribed by the CAPM decision rule converges a.s. to zero.

Appendix

A.1 Proofs of the Main Results

Theorem 1 is a direct consequence of Theorem 2.

Proof of Theorem 2. By using (9), we write

$$\begin{aligned} \frac{1 - r_{t+1}^1}{1 - r_t^1} &= \frac{\sum_{i=2}^I r_{t+1}^i}{1 - r_t^1} \\ &= \sum_{k=1}^K R_k(s_{t+1}, s_t) \frac{(1 - r_t^1)^{-1} \sum_{i=2}^I \lambda_k^i r_t^i}{q_{k,t}} = \sum_{k=1}^K R_k(s_{t+1}, s_t) \frac{\zeta_{k,t}}{q_{k,t}}, \end{aligned}$$

where

$$q_{k,t} = \sum_{m=1}^I \lambda_k^m r_t^m = \lambda_k^1 r_t^1 + (1 - r_t^1) \frac{\sum_{i=2}^I \lambda_k^i r_t^i}{1 - r_t^1} = \lambda_k^1 r_t^1 + \zeta_{k,t}(1 - r_t^1).$$

Consequently,

$$1 - r^1_{t+1} = \sum_{k=1}^{K} R_k(s_{t+1}, s_t) \frac{\zeta_{k,t}(1 - r^1_t)}{\lambda^1_k r^1_t + \zeta_{k,t}(1 - r^1_t)}, \tag{25}$$

and

$$r^1_{t+1} = \sum_{k=1}^{K} R_k(s_{t+1}, s_t) \frac{\lambda^1_k r^1_t}{\lambda^1_k r^1_t + \zeta_{k,t}(1 - r^1_t)}. \tag{26}$$

For each $t = 1, 2, ...,$ consider the random variable

$$D_t = \ln \frac{r^1_t (r^1_{t-1})^{-1}}{(1 - r^1_t)(1 - r^1_{t-1})^{-1}}.$$

We have

$$D_1 + ... + D_T = \ln \frac{r^1_T}{(1 - r^1_T)} - \ln \frac{r^1_0}{(1 - r^1_0)}, \tag{27}$$

Therefore, (20) holds if and only if

$$\liminf_{T \to \infty} \frac{1}{T} (D_1 + ... + D_T) > 0 \quad \text{a.s.}$$

By virtue of assumption (B), for every set of strategies $(\lambda^i_{k,t})$, $i = 1, ..., I$, we consider, there exists a constant H such that $(\min_{i,k} \lambda^i_k)^{-1} \leq H$. For this H, we have

$$H^{-1} \leq \frac{r^i_{t+1}}{r^i_t} \leq H, \quad i = 1, ..., I.$$

This implies

$$H^{-1} \leq \frac{1 - r^1_{t+1}}{1 - r^1_t} \leq H$$

because $1 - r^1_t = \sum_{m=2}^{I} r^m_t$. Consequently, the random variables D_t are uniformly bounded.

We have the following identity

$$\frac{1}{T} \sum_{t=1}^{T} D_t = \frac{1}{T} \sum_{t=1}^{T} E(D_t | s^{t-1}) + \frac{1}{T} \sum_{t=1}^{T} [D_t - E(D_t | s^{t-1})].$$

Since the random variables D_t are uniformly bounded, we can apply to the process of martingale differences $B_t := D_t - E(D_t | s^{t-1})$ the strong law of large numbers (Hall and Heyde 1980, Theorem 2.19), which yields $T^{-1}(B_1 + ... + B_T) \to 0$ with probability 1. Thus, we have

$$\liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} D_t = \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} E(D_t|s^{t-1}), \tag{28}$$

and so (20) is equivalent to

$$\liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} E(D_t|s^{t-1}) > 0 \quad \text{a.s.} \tag{29}$$

By using (25), (26), we write

$$E[D_t|s^{t-1}] = E[\ln \frac{r_t^1 (r_{t-1}^1)^{-1}}{(1-r_t^1)(1-r_{t-1}^1)^{-1}} |s^{t-1}]$$

$$= \sum_{\sigma\in S} p(\sigma|s_{t-1}) \ln \frac{\sum_k R_k(\sigma, s_{t-1}) \frac{\lambda_{k,t-1}^1}{\lambda_{k,t-1}^1 r_{t-1}^1 + \zeta_{k,t-1}(1-r_{t-1}^1)}}{\sum_k R_k(\sigma, s_{t-1}) \frac{\zeta_{k,t-1}}{\lambda_{k,t-1}^1 r_{t-1}^1 + \zeta_{k,t-1}(1-r_{t-1}^1)}}, \tag{30}$$

where

$$\zeta_{k,t-1} = \zeta_{k,t-1}(s^{t-1}) = \frac{\sum_{i=2}^{I} \lambda_{k,t-1}^i r_{t-1}^i}{1-r_{t-1}^1}, \tag{31}$$

$$\lambda_{k,t-1}^i = \lambda_{k,t-1}^i(s^{t-1}),\ r_{t-1}^i = r_{t-1}^i(s^{t-1}), \tag{32}$$

$$\lambda_{k,t-1}^1 = \lambda_{k,t-1}^1(s_{t-1}) = R_k^*(s_{t-1}).$$

Let us use Lemma 1 (see Section A.2 below) to estimate the expression in (30). In view of this lemma, we have

$$E(D_t|s^{t-1}) \geq \delta_\rho(|R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})|), \tag{33}$$

where ρ is the strictly positive constant bounding away from zero the coordinates of λ_t^i. Denote by $N(T) = N(T, s^T)$ the set of those $t \in [0, T]$ for which $|R^*(s_t) - \zeta_t(s^t)| \geq \kappa$. We have

$$\liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} E(D_t|s^{t-1}) \geq \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \delta_\rho(|R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})|)$$

$$\geq \liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \delta_\rho(|R^*(s_t) - \zeta_t(s^t)|) \geq \liminf_{T\to\infty} \frac{1}{T} \sum_{t\in N(T-1)} \delta_\rho(|R^*(s_t) - \zeta_t(s^t)|)$$

$$\geq \delta_\rho(\kappa) \cdot \liminf_{T\to\infty} \frac{1}{T} \#\{N(T-1)\} > 0,$$

where the last inequality follows from (C.2). Thus we have established (29), which is equivalent to (20).

Now, suppose that (20), and hence (29), hold. By virtue of Lemma 1, we find

$$E(D_t|s^{t-1}) \leq L_\rho \cdot |R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})|,$$

and so (29) yields

$$\liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} d_t > 0 \quad \text{a.s.,} \tag{34}$$

where $d_t = |R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})|$.

Denote by $\bar{\kappa}$ the strictly positive random variable which is equal a.s. to the lim inf in (34) and set $\kappa = \bar{\kappa}/2$. We claim that

$$\liminf \frac{1}{T} \#\big\{t \in \{1, ..., T\} : d_t \geq \kappa\big\} > 0, \tag{35}$$

which is equivalent to (C.2). Indeed, suppose the contrary. Then there is a sequence T_k such that

$$\frac{1}{T_k} \#\big\{t \in \{1, ..., T_k\} : d_t \geq \kappa\big\} \to 0. \tag{36}$$

For each k denote by M_k (resp. N_k) the set of those $t \in \{1, T_k\}$ for which $d_t \geq \kappa$ (resp. $d_t < \kappa$). Then we have

$$\frac{1}{T_k} \sum_{t=1}^{T_k} d_t = \frac{1}{T_k} \sum_{t\in M_k} d_t + \frac{1}{T_k} \sum_{t\in N_k} d_t \leq 2 \cdot \frac{1}{T_k} \#(M_k) + \kappa \tag{37}$$

because $d_t \leq 2$. According to (36), $(T_k)^{-1} \cdot \#(M_k) \to 0$. Consequently,

$$\liminf \frac{1}{T_k} \sum_{t=1}^{T_k} d_t \leq \kappa < \bar{\kappa},$$

which contradicts the definition of $\bar{\kappa}$. □

Proof of Theorem 3. Consider the nonnegative random variables $v_t = \delta_\rho(|R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})|)$. By using (33), we write $Ev_t \leq E[E(D_t|s^{t-1})] = ED_t$, which yields, in view of (27),

$$\sum_{t=1}^{T} Ev_t \leq E \ln \frac{r^1}{1 - r_T^1} + C \leq -E \ln(1 - r_T^1) + C,$$

where C is some constant. According to (D.1), the expectations $-E\ln(1-r_T^1)$ do not converge to $+\infty$. Therefore the non-negative sums $Ev_1 + ... + Ev_T$ are bounded by a constant C_1. Consequently,

$$E \lim_{T\to\infty} \sum_{t=0}^{T} v_t = E \lim_{T\to\infty} \inf \sum_{t=0}^{T} v_t \le \liminf \sum_{t=0}^{T} Ev_t \le C_1$$

by virtue of the Fatou lemma. Thus, we obtain $\sum_{t=0}^{\infty} v_t < \infty$ a.s., hence $v_t \to 0$ a.s., and so $|R^*(s_{t-1}) - \zeta_{t-1}(s^{t-1})| \to 0$ a.s. □

A.2 An Auxiliary Result

Let S be a finite set, and, for each $s \in S$, let $p(\sigma|s)$ $(\sigma \in S)$ be a probability distribution on S:

$$p(\sigma|s) \ge 0, \ \sum_{\sigma} p(\sigma|s) = 1.$$

For every $\sigma \in S$, let $R(\sigma, s) = (R_1(\sigma, s), ..., R_k(\sigma, s))$ be a vector in the simplex Δ^K satisfying (A.1) and (A.2) for all $s \in S$.

Let $\rho > 0$ be a number such that $R_k^*(s) > \rho$, $s \in S$ (see (13)). Denote by $\Delta^K(\rho)$ the set of those vectors $(b_1, ..., b_K)$ in Δ^K that satisfy $b_k \ge \rho$, $k = 1, ..., K$. Consider the function

$$\begin{aligned}\Phi(s,\kappa,\mu) \;&=\; \sum_{\sigma\in S} p(\sigma|s) \ln \sum_{k=1}^{K} R_k(\sigma,s) \frac{R_k^*(s)}{R_k^*(s)\kappa + (1-\kappa)\mu_k} \\ &\quad - \sum_{\sigma\in S} p(\sigma|s) \ln \sum_{k=1}^{K} R_k(\sigma,s) \frac{\mu_k}{R_k^*(s)\kappa + (1-\kappa)\mu_k}\end{aligned}$$

of $s \in S$, $\kappa \in [0,1]$ and $\mu = (\mu_k) \in \Delta^K(\rho)$.

Lemma 1 *There exists a constant L_ρ and a function $\delta_\rho(\gamma) \ge 0$ of $\gamma \in [0, \infty)$ satisfying the following conditions:*
(a) The function $\delta(\cdot)$ is non-decreasing, and $\delta_\rho(\gamma) > 0$ for all $\gamma > 0$.
(b) For any $s \in S$, $\kappa \in [0,1]$ and $\mu = (\mu_k) \in \Delta^K(\rho)$, we have

$$L_\rho |R^*(s) - \mu| \ge \Phi(s,\kappa,\mu) \ge \delta_\rho(|R^*(s) - \mu|). \qquad (38)$$

Proof. It follows from (Evstigneev, Hens, and Schenk-Hoppé 2002, Lemma 3.1) that, for all $s \in S$, $\kappa \in [0,1]$ and any $\mu \in \Delta_+^K$, $\mu \ne R^*(s)$, the value of $\Phi(s,\kappa,\mu)$ is strictly positive. Fix some $\gamma_0 > 0$ for which the set $W(s,\gamma) = \{\mu \in \Delta_\rho^K : |R^*(s) - \mu| \ge \gamma\}$ is non-empty for all $s \in S$, $\gamma \in [0, \gamma_0]$ and define

$$\delta_\rho(s,\gamma) = \inf\{\Phi(s,\kappa,\mu) : \ \kappa \in [0,1], \ \mu \in W(s,\gamma)\}$$

if $\gamma \in [0, \gamma_0]$ and $\delta_\rho(s, \gamma) = \delta_\rho(s, \gamma_0)$ if $\gamma > \gamma_0$. Since $\Phi(s, \kappa, \mu)$ is continuous and strictly positive on the compact set $[0, 1] \times W(s, \gamma)$ $(\gamma > 0)$, the function $\delta_\rho(s, \gamma)$ takes on strictly positive values for $\gamma > 0$. Clearly this function is non-decreasing in γ. Fix some s, consider any $\mu \in \Delta_\rho^K$ and define $\gamma = |R^*(s) - \mu|$. Then we have $\mu \in W(s, \gamma)$, and so

$$\Phi(s, \kappa, \mu) \geq \delta_\rho(s, \gamma) = \delta_\rho(s, |R^*(s) - \mu|).$$

From this we can see that the sought-for function $\delta_\rho(\gamma)$ can be defined as

$$\delta_\rho(\gamma) = \min_{s \in S} \delta_\rho(s, \gamma).$$

We can write $\Phi(s, \kappa, \mu) = \Phi(s, \kappa, \mu) - \Phi(s, \kappa, R_k^*(s))$ since the latter term is zero. The function $\Phi(s, \kappa, \mu)$ is differentiable in $\mu \in \Delta_+^K$ and its gradient $\Phi'_\mu(s, \cdot, \cdot)$ is continuous, and hence bounded, on the compact set $[0, 1] \times \Delta_\rho^K$. This implies the existence of the Lipschitz constant L_ρ in (38). □

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