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Abstract

It is shown that a nominal rigidity may work as a natural mechanism for rational expectations endogenous fluctuations. By a nominal rigidity is meant that some nominal price, here the nominal wage rate, does not adjust within each short period to equate supply and demand (of labor), but adjusts competitively between any two successive periods in response to the excess supply or demand in the first of the periods. Under circumstances that would otherwise exclude endogenous fluctuations, a certain degree of sluggishness in the adjustment of the nominal wage rate implies existence of such fluctuations. The required degree of sluggishness is not unrealistic. The rate of unemployment varies countercyclically along the fluctuations studied, and welfare concerns may motivate stabilization policies attempting at keeping the economy at a steady state, where unemployment is at the natural rate all the time.

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1 Introduction

We demonstrate that a nominal rigidity of traditional Keynesian type may serve as a natural mechanism for indeterminacy and endogenous fluctuations (perfect foresight deterministic cycles and/or rational expectations sunspot equilibria).

We study a monetary dynamic general equilibrium model with no intrinsic uncertainty. By a nominal rigidity we mean that the adjustment of some nominal price is competitive, but not necessarily instantaneous. To be precise we assume that there is a nominal price, in our case the nominal wage rate, that does not adjust *within each short period* to equate the supply and demand of labor in the period, but does adjust competitively *between any two successive periods* in reaction to the excess supply or demand of labor in the first of the periods. The standard, but special, assumption of instantaneous, within the period, competitive wage adjustment is (equivalent to) the particular case where the reaction in the wage rate to an excess demand is infinite.

Our main conclusion is that under some realistic assumptions on economic fundamentals other than the speed of wage adjustment, which rule out the existence of endogenous fluctuations under instantaneous wage adjustment, a lower, but not unrealistically low, speed of adjustment in the nominal wage rate will imply indeterminacy of rational expectations equilibrium and the existence of endogenous fluctuations. In this precise sense a nominal rigidity may serve as a mechanism for endogenous fluctuations, which motivates the term “Keynesian” business cycles.

The analysis in this paper is related to the modern literature on indeterminacy and endogenous fluctuations. This literature has provided several mechanisms for endogenous fluctuations (in various types of models), e.g., strong wealth effects in intertemporal choice, Azariadis (1981), Woodford (1984), Grandmont (1985), Azariadis and Guesnerie (1986), low substitutability between labor and capital in production, Reichlin (1986), Woodford (1986), strong productive externalities, Benhabib and Farmer (1994), Farmer and Guo (1994), countercyclical markups, Gali (1994), to mention just some mechanisms and contributions. Several of these mechanisms require unrealistically large degrees of the factor in question to create indeterminacy, see Schmitt-Grohe (1997), which can, however, often be remedied by introducing additional features. For instance, sector-specific productive externalities in a two-sector model may bring down the degree of (increasing) returns to scale required for indeterminacy to empirically realistic levels, Benhabib and Farmer (1996).

This paper provides a new mechanism for endogenous fluctuations, the kind of nominal rigidity, or sluggish nominal wage adjustment, described above. The mechanism may be of interest for mainly two reasons. First, the degree of sluggishness in wage adjustment required for indeterminacy (under plausible other circumstances) is not empirically unrealistic. Second, the theory we present has some interesting macroeconomic implications of a "Keynesian" flavor:

Since the fluctuations we study are rooted in a sluggish adjustment of wages, output variations will be accompanied by variations in unemployment according to a countercyclical pattern, and substantial fluctuations in output and employment can occur even for low values of the real wage elasticity of labor supply.

Furthermore, our analysis may give a welfare based motivation for *permanent* stabilization policy. The monetary steady state of the model we consider, at which output and unemployment are at their "natural" rates, is better in terms of welfare than endogenous fluctuations according to which output and unemployment fluctuate around the natural rates. This may motivate policies aiming at stabilizing output and unemployment at the natural levels. Such a policy would not (so much) be intended to improve the state of the economy in the single period, but to affect the entire rational expectations dynamics of the economic system by which unemployment keeps on going up and down.[1]

Finally, the present contribution gives a new type of answer to an often heard objection to Keynesian economics: Although the kind of nominal rigidity usually (and realistically) assumed in Keynesian models - the nominal wage rate, say, stays fixed in each short period - does imply that a negative shock causes unemployment, this does not give strong motivation for policy intervention. Unemployment should be expected to disappear by market forces alone, since excess supply of labor should give lower wages, hence lower unemployment etc. So, over a sequence of short periods the competitive reactions of the wage rate should, by themselves, eliminate unemployment. The most often heard, and perfectly sensible, counter objection is that price and wage adjustment may take so long time that policy is needed to

[1] The present paper contains no explicit policy analysis, but points to the need for such analysis, a subject for future research.

improve conditions during the adjustment. The rationale for stabilization policy given here is different. We incorporate explicitly into the analysis the type of wage adjustments between the short periods of fixed wages, that the mentioned objection points to. We show that this may well imply that the full dynamics of the economic system, under rational expectations and without shocks, becomes volatile such that (high) unemployment both disappears *and reappears* all the time by market forces alone, making unemployment a never disappearing economic problem. In other words, even if the time it takes for market forces to eliminate (a high rate of) unemployment is relatively short, unemployment will be an economic problem, since the market forces that make unemployment disappear also make it reappear.

A key feature of the model we study below is the assumed Phillips-curve-like wage adjustment rule. Since this relation is intended to formalize a situation where wages react competitively, but sluggishly, to market signals it will involve the "timing" that a measure of the excess demand for labor in the current period, the current rate of employment, will have an impact on (wage) inflation *from the current period to the next one.* A lot of recent economic theory involves different Phillips-curve-like relations such as the Lucas "surprise" supply function or the so-called new (Keynesian) Phillips curve. These relations formalize preset, and sometimes staggered, wages or prices and therefore they involve the timing that the current rate of unemployment, or the current output gap, has a certain relation to inflation *from the previous period to the current one.* The rational expectations equilibrium dynamics that arise from these two different timings may be very different (as evidenced by the analysis of this paper). Furthermore, as noted by e.g. Gali and Gertler (1999), the timing where unemployment or the output gap leads inflation, seems to be more realistic than the opposite. There seems therefore to be good reasons to investigate the rational expectations dynamics arising from the idea of a sluggish competitive wage adjustment.

In Section 2 we set up the elements of the monetary economy under consideration. Section 3 is concerned with the standard case of instantaneous price adjustment, and hence mainly summarizes some important known results, while Section 4 presents the model with sluggish competitive wage adjustment. Section 5 reports the positive results concerning Keynesian business cycles and gives the intuition. Section 6 offers some concluding remarks.

2 A Simple Monetary Model

To study a nominal rigidity as described a dynamic *monetary* model is required. We will consider a simplification to no capital of the cash-in-advance economy described in Woodford (1986).[2] The exclusion of capital in the production function will imply that the rational expectations dynamic equation we arrive at will be one-dimensional (univariate). This is desirable for two reasons. First, in the one-dimensional case a better intuitive understanding of why and how a nominal rigidity may promote the existence of endogenous rational expectations fluctuations can be obtained. Second, at present the learning stability or instability of sunspot equilibria in the neighborhood of a steady state is better understood for the one-dimensional case, see Evans and Honkapohja (2001).

Time runs discretely in periods indexed by t. In each period the commodities are labor, output, and money. The nominal wage rate and the nominal output price in period t are w_t and p_t respectively. The money stock stays constantly at $M > 0$.

A representative firm produces output y_t from labor input l_t in period t according to the production function $y_t = l_t^{1/\beta}$, where $\beta \geq 1$, covering constant returns, $\beta = 1$, as a special case.

An infinitely lived representative worker obtains utility in period t according to the von Neumann-Morgenstern utility function $\sum_{s=t}^{\infty} \delta^{s-t} \left[u(c_s)/\delta - v(n_s)\right]$, where δ is the discount factor, $0 < \delta < 1$, and c_s and n_s are consumption and labor supply in period s respectively. On u and v we impose differentiability assumptions and: $u' > 0$, $u'' < 0$, $u'(0) = \infty$, $v' > 0$, $v'' > 0$, $v'(0) = 0$, and $v'(1) = \infty$ (maximal labor supply in a period is one). In case the firm earns positive profits ($\beta > 1$) these go to a separate representative capitalist who does not work, but obtain utility from consumption according to the utility function $\sum_{s=t}^{\infty} \delta^{s-t} u(c_s)$.

Both the worker and the capitalist are assumed to be subject to cash-in-advance constraints. In any period t, the value of a consumer's consumption cannot exceed the amount of money m_t held at the beginning of the period, $p_t c_t \leq m_t$. This gives

[2] The equilibrium conditions derived below are equivalent to the equilibrium conditions of an appropriately defined overlapping generations model, as explained by Woodford (1986). For the present analysis the period length should be thought of as the amount of time during which a nominal price (the wage rate) is insensitive to current excess demand (for labor), which should be relatively short. The model interpretation with an infinitely lived agent and cash-in-advance constraints is therefore to be preferred.

rise to a "transactions demand" for money. Two remarks on the role of the capitalist in the present analysis may be clarifying. First, we make the assumption that profits go to a separate capitalist class because it has been analyzed elsewhere how it affects the existence of endogenous fluctuations if the labor supply decision is influenced by profit incomes, Tvede (1991) and Jacobsen (2000). Here we want to focus on the nominal rigidity as a (potential) source of endogenous fluctuations. Second, the assumption that also the capitalist is subject to a cash-in-advance requirement is made for simplicity. It will give us the simple demand curve M/p_t for output in any period, which is identical to the demand curve one would have if there were no profits. In the particular case $\beta = 1$, profits are zero and the capitalist class plays no role.

The consumers are also subject to traditional budget constraints stating that in any period the increase in money holding equals what is earned minus what is spent, and money holdings must always be positive. However, if the cash-in-advance constraints are binding in all periods the optimal behaviors will be as follows. The capitalist will in each period spend on consumption exactly the profits received in the preceding period, while the worker will over the succession of any two periods $(t, t+1)$, decide on n_t, m_{t+1}, and c_{t+1} to maximize $E_t u(c_{t+1}) - v(n_t)$, subject to $m_{t+1} = w_t n_t$, and $c_{t+1} = m_{t+1}/p_{t+1}$. Hence, labor supply n_t in period t is the solution to maximizing $E_t u([w_t/p_{t+1}]\, n_t) - v(n_t)$ with respect to n_t, where the expectation E_t is taken with respect to p_{t+1} at time t. We only consider rational expectations equilibria, so the subjectively expected distribution for p_{t+1} at time t is equal to the equilibrium distribution for p_{t+1} conditional on current realizations p_t, y_t etc. The above problem has, for any $w_t > 0$ and distribution over $p_t > 0$, a unique solution n_t strictly between zero and one given by the first order condition,

$$v'(n_t) = E_t u'(\frac{w_t}{p_{t+1}} n_t) \frac{w_t}{p_{t+1}}. \tag{1}$$

Let $n(\omega_t)$ be the labor supply curve under deterministic expectations, that is, the solution in n_t to $v'(n_t) = u'(\omega_t n_t)\omega_t$, where $\omega_t := w_t/p_{t+1}$. Log-differentiation gives for the elasticity $\varepsilon(\omega_t) := n'(\omega_t)\omega_t/n(\omega_t)$,

$$\varepsilon(\omega_t) = \frac{1 - R(\omega_t n_t)}{R(\omega_t n_t) + N(n_t)} > -1, \tag{2}$$

where $n_t = n(\omega_t)$, $R(c) := -u''(c)c/u'(c)$, and $N(n) := v''(n)n/v'(n)$.

The behaviors derived above are conditional on the cash-in-advance constraints being binding in all periods. Along an equilibrium that stays sufficiently close to a steady state the cash-in-advance constraints will indeed be binding in all periods simply because $u'(c_t) > \delta\,(p_t/p_{t+1})\,u'(c_{t+1})$ for all t whenever c_t and c_{t+1} are always sufficiently close to a common c, and nominal prices are close to being constant (which they must be close to a steady state). In the continuation it will therefore be of importance to identify dynamic equilibria which are (arbitrarily) close to a steady state. We will do so, but also briefly point to some dynamic "equilibria" which are not necessarily close to a steady state, but could well be.

It will everywhere be assumed that the output and money markets clear instantaneously and competitively. The labor market possibly has a sluggish competitive price (wage) adjustment implying that supply and demand of labor are not necessarily equal in all periods. If labor supply is larger than or equal to labor demand, employment will be given by demand. In case labor demand should exceed labor supply in a period we assume that the firm will, nevertheless, be able to buy the amount of labor it demands through forced overwork etc., but this state of labor shortage will have an impact on the wage increase from the current to the next period. Employment is hence assumed to be given by labor demand always and the firm will never be rationed in demand for labor, which simplifies the analysis. However, the assumption that employment can be larger than labor supply is not necessary for anything that follows. Under some plausible assumptions on the fundamentals of the considered economy labor supply will exceed labor demand in all periods along the equilibria studied, so there will be no forced overwork in equilibrium.

The worker may be rationed in supply of labor and experience that not all of n_t, but only the smaller demanded quantity l_t, can be sold. This will have an implication for how much the worker can actually consume next period, but should not affect n_t, the amount of labor the worker *would like* to sell in period t.

The fact that the cash-in-advance constraints are binding in all periods implies that the money holdings of the worker and the capitalist at the beginning of period t are last period's incomes, $w_{t-1}l_{t-1}$ and $(p_{t-1}y_{t-1} - w_{t-1}l_{t-1})$, respectively, so total money demand is $p_{t-1}y_{t-1}$, while the output demands in period t are the output values of these money holdings, $w_{t-1}l_{t-1}/p_t$ and $(p_{t-1}y_{t-1} - w_{t-1}l_{t-1})/p_t$ respectively, giving a total demand of $p_{t-1}y_{t-1}/p_t$. Since equilibrium in the money market implies $M = p_{t-1}y_{t-1}$, the AD-curve is simply,

$$y_t = \frac{M}{p_t}. \tag{3}$$

Profit maximization by the firm (the marginal product $l_t^{(1-\beta)/\beta}$ equal to w_t/p_t) implies a supply of output in period t, or an AS-curve,

$$y_t = \left(\frac{p_t}{\beta w_t}\right)^{\frac{1}{\beta-1}}. \tag{4}$$

Instantaneous clearing of the output market, AD = AS, then implies that in any period t, for any given w_t, the nominal output price adjusts to ensure,

$$w_t = \frac{1}{\beta}\left(\frac{1}{M}\right)^{\beta-1} p_t^{\beta}, \tag{5}$$

implying, by also using $p_t = M/y_t$, that,

$$\omega_t := \frac{w_t}{p_{t+1}} = \frac{y_{t+1}}{\beta y_t^{\beta}}. \tag{6}$$

3 Instantaneous Adjustment of the Nominal Wage

We first consider briefly the standard case of overall instantaneous price adjustment where in each period, in addition to the adjustment of p_t just described, w_t adjusts to ensure $n_t = l_t$. Inserting $n_t = l_t = y_t^{\beta}$ and (6) into the first order condition (1) gives,[3]

$$\beta y_t^{\beta} v'(y_t^{\beta}) = E_t y_{t+1} u'(\frac{y_{t+1}}{\beta}). \tag{7}$$

The temporary equilibrium condition (7), just usually with $\beta = 1$, and the rational expectations and/or learning dynamics associated to it, have been studied in numerous contributions, e.g. Azariadis (1981), Woodford (1984), Grandmont (1985), Azariadis and Guesnerie (1986), Woodford (1990), Evans and Honkapohja (2001). The following is a brief account of known results meant for comparison.

The perfect foresight dynamics associated to (7) is $\beta y_t^{\beta} v'(y_t^{\beta}) = y_{t+1}u'(y_{t+1}/\beta)$. For any $y_{t+1} > 0$, this equation has a unique solution $y_t < 1$ defining (globally) a

[3] Throughout we express equilibrium conditions in terms of output. Of course, (7) can be changed into an equilibrium condition in the price level by substituting M/p_t for y_t.

function $y_t = f(y_{t+1})$. A monetary steady state is a $y > 0$, such that $y = f(y)$, or $\beta y^{\beta-1} v'(y^\beta) = u'(y/\beta)$. Under our assumptions there is a unique such $y < 1$. At this steady state the intertemporal real wage is constantly $\omega = 1/(\beta y^{\beta-1})$ from (6). As a particular case of the later equation (23), the slope of f at steady state is $f'(y) = \varepsilon(\omega)/[\beta(1+\varepsilon(\omega))]$. If $f'(y) < -1$ (> 1 is not possible since $\varepsilon > -1$), which is equivalent to $\varepsilon(\omega) < -\beta/(1+\beta)$ specializing to $\varepsilon(1) < -\frac{1}{2}$ for $\beta = 1$, perfect foresight equilibrium is indeterminate. Indeterminacy implies the existence of perfect foresight deterministic cycles (not necessarily close to the monetary steady state), truly stochastic stationary sunspot equilibria close to the cycles, and truly stochastic stationary sunspot equilibria arbitrarily close to the steady state. (We provide some definitions later in connection with the more general model). Some stationary sunspot equilibria are learning stable under an appropriately defined adaptive learning scheme, and this holds true also for some sunspot equilibria close to the steady state, the latter result being an application of the recent result in Evans and Honkapohja (2001).

The condition $\varepsilon(\omega) < -\beta/(1+\beta)$ ($< -\frac{1}{2}$) is not empirically plausible, and not strictly necessary for rational expectations endogenous fluctuations. However, an assumption of $\varepsilon(\omega') \geq 0$ for all ω' implies determinacy, it makes y globally stable according to f,[4] which obviously rules out deterministic cycles and, due to results in Azariadis and Guesnerie (1986) or Grandmont (1986), also stationary sunspot equilibria.

4 Sluggish Adjustment of the Nominal Wage

We now generalize the model with instantaneous competitive wage adjustment to a situation where the nominal wage rate possibly reacts sluggishly, but still competitively, to an excess supply/demand.

In any short period t, in which the wage rate is w_t, a Keynesian type of short run equilibrium given by AS = AD is established, exactly as described in Section 2 above. Hence, given w_t, the nominal price p_t adjusts in accordance with (5), implying a level of production $y_t = M/p_t$, and a certain level of employment $l_t = y_t^\beta$. In period t there will also be a labor supply n_t depending, as given by (1), on w_t and

[4] From (23) of the Appendix, f will be everywhere positively sloped and since f has only one (positive) intersection with the 45°-line this has to be "from above".

the expectations concerning p_{t+1}. To distinguish logically between individual and aggregate magnitudes we will let total employment in period t be denoted by L_t, and total labor supply by N_t. In equilibrium these will equal l_t and n_t respectively, but the individual worker takes L_t and N_t, and hence the rate of employment, $e_t := L_t/N_t$, as given.

Between the short periods t and $t+1$, market forces (the invisible hand) imply that the nominal wage rate reacts to the excess supply/demand of labor, or the degree of labor shortage, as measured by e_t, in a way that obeys,

$$\frac{E_t w_{t+1}}{w_t} = P\left(e_t\right)\left(\frac{E_t p_{t+1}}{p_t}\right)^{\alpha}, \; P'(e) > 0 \text{ and } 0 \leq \alpha \leq 1. \tag{8}$$

This Phillips-curve type relation states that nominal wage inflation from t to $t+1$ in expected terms depends positively on the excess supply/demand of labor, or the degree of labor shortage, in period t and on expected inflation from t to $t+1$, where α measures the degree to which expected inflation builds itself into wage inflation with (8) allowing anything between "fully" ($\alpha = 1$) and "not at all" ($\alpha = 0$). Alternatively, α measures the degree to which it is the real, as opposed to the nominal, wage rate that reacts to a given demand pressure on the labor market. The relationship (8) is an essential element of the analysis and we have some comments about it.

First, (8) is not derived from optimization like, e.g., the new (Keynesian) Phillips curve based on monopolistic competition and preset, staggered wages or prices, see for instance Clarida, Gali, and Gertler (1999). In this sense (8) is ad hoc. It is, however, no more ad hoc than the standard assumption of immediate wage adjustment (a very specific assumption about wage adjustment); in fact, the relation (8) *generalizes* the model with instantaneous market clearing. One particular case for P is that it is vertical at $e_t = 1$, as indicated by the thick lines in Figure 1, a) and b), illustrating different possible shapes of P. (Formally this particular case is included below by proceeding in terms of the inverse of P). With a vertical P the economy's equilibrium condition, as derived below, will specialize to exactly (7) corresponding to instantaneous wage adjustment. However, in the wide range of functions P allowed by (8) the vertical one seems special.

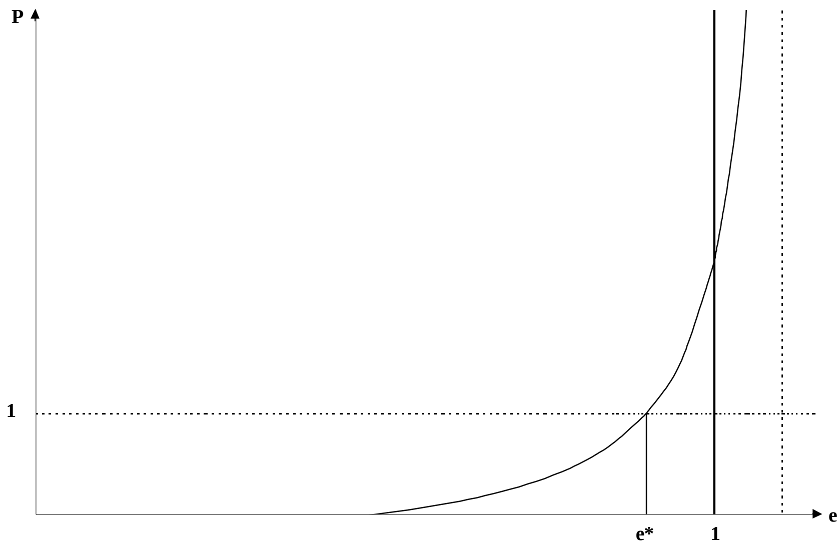

Figure 1, a)

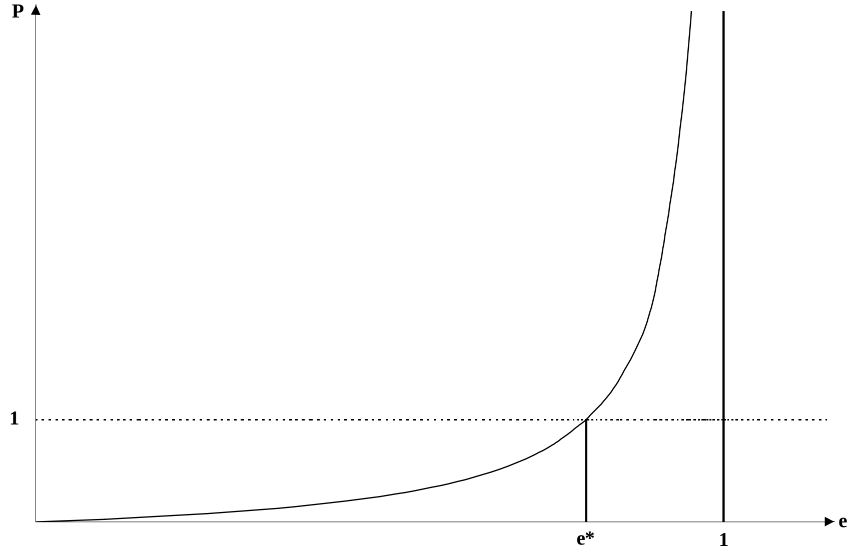

Figure 1, b)

Second, as discussed in the Introduction, the fact that (8) formalizes competitive market forces possibly operating over (real) time justifies the "timing" assumed in (8), that the rate of employment in period t has an impact on the nominal wage increase from t to $t+1$, rather than a certain connection to the wage/price increase from $t-1$ to t as in a model with preset wages. This implies that (8) may have more realistic properties than optimization based new Phillips-curves. According to (8), excess demand (here measured by the employment rate, but alternatively measured by the output gap) leads inflation (in time), whereas according to the new Phillips curve inflation leads the output gap. As noted by Fuhrer and Moore (1995), and Gali and Gertler (1999), the first feature is the more realistic one.

Third, given current realizations and expectations, (8) imposes a restriction on the *expected value* $E_t w_{t+1}$ of next period's nominal wage, not on the wage rate w_{t+1} itself. In other words, market forces are assumed to work such that the degree of labor shortage determines nominal wage increases (over what is caused by expected price increases) "up to" the expected value $E_t w_{t+1}/w_t$ of the wage increase, and hence with a precision that allows w_{t+1} to be a stochastic variable given current realizations and expectations. The actual process for w_{t+1} will be determined in a full equilibrium as defined below. This means that w_{t+1} is not completely fixed, or predetermined, from period t, but w_{t+1} is subject to a (nominal) rigidity since the mean of w_{t+1} is predetermined from period t, and insensitive to the supply and demand of labor in period $t+1$.

If it were w_{t+1} rather than $E_t w_{t+1}$ that entered (8), then w_{t+1} would be determined deterministically given the current variables and the equilibrium distribution over p_{t+1}, but then, due to (5), p_{t+1} would also be determined deterministically ruling out a stochastic process for p_{t+1} in rational expectations equilibrium. This would rule out the possibility of stochastic sunspot equilibria altogether! (The only exception would be the case where (8) did *not* determine w_{t+1}, because P were vertical, that is, the standard case of immediate wage adjustment). With the (8) as written we assume a degree of precision in the market determination of wage increases that does not rule out the existence of sunspot equilibria a priori under a sluggish wage adjustment.[5]

The natural rate of employment is defined by $P(e^*) = 1$, that is, as a degree of employment that does not imply any increase or decrease in wage inflation over what is caused by expected price inflation. Such an e^* is assumed to exist, see also Figure 1, and it is natural to assume that $e^* < 1$ except in the case where P is

[5] Insisting that it should be w_{t+1} that entered (8) would, interestingly, make the kind of stochastic sunspot equilibria often studied in standard models of immediate price adjustment a "shaky" one. In the space of all possible P-functions, the phenomenon would only pertain to the particular case of a vertical P (perhaps just locally, but still a knife edge case). The slightest finiteness in the slope of P, that is, the slightest departure from price adjustment being literally instantaneous, would rule out the phenomenon. Expressed differently, if one insists on continuity in the equilibrium correspondence as the slope of P goes to infinity it cannot be w_{t+1} that should appear in (8), but it can be $E_t w_{t+1}$.

vertical at one, where $e^* = 1$. All equilibria studied will be "on the P-function" in all periods. A P-function as the one shown in Figure 1, a) has a piece where $e_t > 1$, so an equilibrium may involve forced overwork, while a P-function as in Figure 1, b) will imply that $e_t < 1$ in all periods. In any case, an assumption of $e^* < 1$ will imply that all equilibria sufficiently close to steady state will have $e_t < 1$ in all periods since in steady state the rate of employment is exactly e^*. This verifies the earlier claim that the possibility of $e_t > 1$ is not important.

From (5), $E_t w_{t+1}/w_t = E_t p_{t+1}^{\beta}/p_t^{\beta}$. Using the approximation $E_t p_{t+1}^{\beta} \cong (E_t p_{t+1})^{\beta}$, (more precise the closer β is to one), one gets $E_t w_{t+1}/w_t = (E_t p_{t+1}/p_t)^{\beta}$, so (8) becomes,

$$\left(\frac{E_t p_{t+1}}{p_t}\right)^{\beta-\alpha} = P\left(e_t\right). \tag{9}$$

Note that $\beta = \alpha = 1$ is a highly special case where the rate of employment must be at the natural rate in all periods in any equilibrium, not only in steady state.

To include formally as a special case the vertical P, we proceed in terms of the inverse of P, that is, the function $H := P^{-1}$ from $(0, \infty)$ into $(0, \infty)$, that gives the rate of employment in period t needed to create a certain (excess) wage inflation from t to $t + 1$, and we allow H to be flat at one. Taking the inverse on both sides in (9) gives,

$$e_t = H\left(\left(\frac{E_t p_{t+1}}{p_t}\right)^{\beta-\alpha}\right), \; H'(z) \geq 0 \text{ for all } z \geq 0. \tag{10}$$

The vertical P corresponds to $H(z) = 1$ for all z. The natural rate of employment is $e^* = H(1)$. The relation between the elasticity of P, $\pi(e) := P'(e)e/P(e) > 0$, and that of H, $\eta(z) := H'(z)z/H(z) > 0$, is $\eta(z) := 1/\pi(H^{-1}(z)) = 1/\pi(e)$.

To derive the equilibrium condition under sluggish wage adjustment first note that from (10),

$$N_t = L_t/H\left(\left(\frac{E_t p_{t+1}}{p_t}\right)^{\beta-\alpha}\right). \tag{11}$$

Inserting that in equilibrium $N_t = n_t$, $L_t = l_t = y_t^{\beta}$, and $p_t = M/y_t$ gives,

$$n_t = y_t^{\beta}/H\left(\left(y_t E_t \frac{1}{y_{t+1}}\right)^{\beta-\alpha}\right). \tag{12}$$

Finally, inserting this expression for n_t, as well the expression for w_t/p_{t+1} stated in

(6), into the first order condition (1) gives,

$$\beta y_t^\beta v'\left(\frac{y_t^\beta}{H\left(\left(y_t E_t \frac{1}{y_{t+1}}\right)^{\beta-\alpha}\right)}\right) = E_t y_{t+1} u'\left(\frac{y_{t+1}}{\beta H\left(\left(y_t E_t \frac{1}{y_{t+1}}\right)^{\beta-\alpha}\right)}\right). \qquad (13)$$

This is the temporary equilibrium condition under a possibly sluggish wage adjustment. It states for any given distribution over y_{t+1}, what y_t must fulfill in order to be an equilibrium output of period t. A rational expectations (in particular a perfect foresight) dynamic equilibrium is a stochastic (deterministic) process for y_t, such that (7) is fulfilled for all t. If $H(z) = 1$ for all z, then (13) reduces to (7), verifying the claim, that the rational expectations equilibrium condition for overall instantaneous price adjustment is a special case.

5 Keynesian Business Cycles

5.1 Perfect Foresight Dynamics

The perfect foresight dynamics associated to (13) appears by deleting the expectations operators. We write this as,

$$G(y_t, y_{t+1}) := \beta y_t^\beta v'\left(\frac{y_t^\beta}{H\left((y_t/y_{t+1})^{\beta-\alpha}\right)}\right) / u'\left(\frac{y_{t+1}}{\beta H\left((y_t/y_{t+1})^{\beta-\alpha}\right)}\right) = y_{t+1}. \qquad (14)$$

A monetary steady state is a constant solution $y_t = y > 0$,

$$\beta y^{\beta-1} v'\left(\frac{y^\beta}{e^*}\right) = u'\left(\frac{y}{\beta e^*}\right), \qquad (15)$$

where it is used that $H(1) = e^*$. It follows from our assumptions on u and v that there is a unique monetary steady state. At this steady state, since output is constantly y, it follows from $y_t = M/p_t$ that also the nominal price level is constant, and then from (5) that the nominal wage rate is constant. The rate of employment must then be at the natural rate all the time, $e_t = e^*$ for all t. (Formally this follows from (10) with $E_t p_{t+1}/p_t = p_{t+1}/p_t = 1$). Also the intertemporal real wage is constant and given by (6), $\omega_t = \omega = 1/(\beta y^{\beta-1})$.

It is a direct consequence of the Implicit Function Theorem that if the derivative of the function G with respect to y_t measured at the steady state ($y_t = y_{t+1} = y$) is not zero, then (14) defines a differentiable function $y_t = f(y_{t+1})$ in a neighborhood around the steady state. To prove Lemma 1 below is a matter of using this fact and conducting the appropriate differentiation (proofs are given in Appendix).

Lemma 1 *If* $\beta[1+\varepsilon(\omega)] - (\beta-\alpha)\eta(1) \neq 0$*, then there are two neighborhoods around* y*, such that for* y_{t+1} *chosen in the first of these neighborhoods there is a unique* y_t *in the second neighborhood solving (14), and the function* $y_t = f(y_{t+1})$ *thus (locally) defined is continuously differentiable.*

We will everywhere assume that the condition in Lemma 1 is fulfilled. This amounts to eliminating a knife edge case. Hence the perfect foresight dynamic $y_t = f(y_{t+1})$ will be locally well-defined around the steady state y. For some results we will need that the f is globally well-defined which motivates the following lemma.

Lemma 2 *Assume that there are* $\underline{e}$ *and* $\overline{e}$ *such that* $0 < \underline{e} < H(z) < \overline{e}$*, and* $\varepsilon(\omega') \geq 0$ *for all* ω'*, and* $\eta(z) < \frac{\beta}{\beta-\alpha}$ *for all* z *(or* $\beta = \alpha$*). Then for any* $y_{t+1} > 0$*, there is a unique* y_t *solving (14), and the function* $y_t = f(y_{t+1})$ *thus (globally) defined is continuously differentiable and fulfils* $0 \leq f < \overline{e}$*.*

The assumption in Lemma 2 can be expressed that the actual P-function is not too far from the vertical one - P is squeezed in between $\underline{e}$ and $\overline{e}$ as illustrated in Figure 1, a) and has a relatively high elasticity $\pi(e) > (\beta-\alpha)/\beta$ - and labor supply is increasing in the real wage. Finally we will need the slope of f at steady state.

Lemma 3 *The derivative of* f *measured at the monetary steady state, i.e. at* $y_t = y_{t+1} = y$*, is*

$$f'(y) = \frac{\varepsilon(\omega) - (\beta-\alpha)\eta(1)}{\beta[1+\varepsilon(\omega)] - (\beta-\alpha)\eta(1)}. \tag{16}$$

5.2 Endogenous Fluctuations

Write the temporary equilibrium condition (13) in the general form $\widetilde{Z}(y_t, \mu_{t+1}) = 0$, as studied by Chiappori, Geoffard, and Guesnerie (1992), where $\widetilde{Z}$ is a function defined on pairs of a current output level y_t and a probability distribution μ_{t+1} over future output y_{t+1}. Hence, in the more general formulation, the perfect foresight dynamic is $Z(y_t, y_{t+1}) := \widetilde{Z}(y_t, \delta_{y_{t+1}}) = 0$, where $\delta_{y_{t+1}}$ is the distribution assigning probability one to y_{t+1}, and a (monetary) steady state is a $y > 0$, such that $Z(y, y) =$

0. If $Z_1'(y,y) \neq 0$, then $Z(y_t, y_{t+1}) = 0$ implicitly defines a differentiable function $y_t = f(y_{t+1})$ around the steady state y, and $f'(y) = -Z_2'(y,y)/Z_1'(y,y)$.

A perfect foresight k-period cycle consists of k different output levels y_i, $i = 1, ..., k$, such that $Z(y_1, y_2) = 0, Z(y_2, y_3) = 0, ..., Z(y_k, y_1) = 0$, or equivalently, if f is everywhere well-defined, $y_1 = f(y_2), y_2 = f(y_3), ..., y_k = f(y_1)$. A rational expectations stationary sunspot equilibrium of order k consists of k different output levels y_i and distributions $\mu(y_i)$, $i = 1, ..., k$, where each distribution only assigns positive probability to $y_1, ..., y_k$, such that $\widetilde{Z}(y_i, \mu(y_i)) = 0$ for $i = 1, ...k$, and the Markov matrix of transition probabilities $M := (\mu(y_i))_{i=1}^k$ is irreducible. (If people believe that a Markov chain over states $s_1, ..., s_k$, with transition probabilities M, although this is extrinsic to the economic system, governs this system such that state s_i will imply output y_i, then this theory will be correct/self-fulfilling). If one distribution $\mu(y_i)$ assigns strictly positive probability to at least two different output levels, the sunspot equilibrium is truly stochastic.

5.3 Rational Expectations Stationary Sunspot Equilibria

Since our economic model falls within the generality of Chiappori, Geoffard, and Guesnerie (1992), it follows (from their Theorem 3), that if perfect foresight dynamics is indeterminate in the sense $f'(y) > 1$ or $f'(y) < -1$, then there are truly stochastic stationary sunspot equilibria of any order arbitrarily close to the monetary steady state.

The denominator in (16) may be negative. In that case $f'(y) > 1$. It may be positive, in which case $f'(y) < 1$, and $f'(y) < -1$ if and only if,

$$\varepsilon(\omega) < 2\frac{\beta - \alpha}{1+\beta}\eta(1) - \frac{\beta}{1+\beta}. \tag{17}$$

Hence, whenever this condition is fulfilled, either $f'(y) > 1$ or $f'(y) < -1$. This suffices for,

Theorem 1 *Assume (17). Then for any neighborhood around y, and for any $k \geq 2$, there is a truly stochastic stationary sunspot equilibrium of order k, $(y_i, \mu(y_i))_{i=1}^k$, for which all output levels y_i are in the neighborhood.*

Note that in any type of equilibrium the processes for p_t and w_t associated to the equilibrium process for y_t are given by $p_t = M/y_t$ and (5).

The condition for the denominator in (16) being positive is,

$$\varepsilon(\omega) > \frac{\beta - \alpha}{\beta}\eta(1) - 1. \tag{18}$$

If both (17) and (18) are fulfilled, then $f'(y) < -1$. The recent analysis of Evans and Honkapohja (2001) suggests that exactly in this case ($f'(y) < -1$, as here implied by (17) and (18)), there will be stationary sunspot equilibria arbitrarily close to the steady state which are learning stable under a natural adaptive learning rule.[6]

5.4 Perfect Foresight Deterministic Cycles

If both (17) and (18) are fulfilled, implying $f'(y) < -1$, and f is globally well-defined and bounded above, as under the assumptions of Lemma 2, it follows from the standard "mirror image argument" that deterministic cycles exist: Just draw the function f in a diagram with y_{t+1} along the horizontal axis and $y_t = f(y_{t+1})$ along the vertical axis, and in the same diagram draw f with y_t along the horizontal and y_{t+1} along the vertical axis. The latter is the mirror image of f around the 45°-line. If, at the monetary steady state, $f'(y) < -1$, and f stays below the "ceiling" $\overline{e}$, then there is an intersection between the two drawn f-functions at a (y_1, y_2), where $0 < y_1 < y < y_2$. Then $y_1 = f(y_2)$ from the first drawn function, and $y_2 = f(y_1)$ from the second, and (y_1, y_2) is a two-period cycle. It follows from a results in Azariadis and Guesnerie (1986) that there are also truly stochastic stationary sunspot equilibria close to a cycle.

Theorem 2 *Under the assumptions of Lemma 2 and (17) and (18), deterministic cycles and truly stochastic stationary sunspot equilibria close to each cycle exist.*

It is not known about the equilibria pointed to in Theorem 2 that they can be found arbitrarily close to the steady state y. It is hence not known either, that the cash-in-advance constraints are binding in all periods. Therefore Theorem 1 is our main result on the existence of endogenous fluctuations. We report Theorem 2 for two reasons. First, the equilibria pointed to *may well be* close to the steady state. Second, if they are, then the *deterministic* cycles pointed to will be equilibria also if there were no expectations operator on w_{t+1} in (8).

[6] Our equilibrium condition (13) does not fall completely within the generality of the one studied by Evans and Honkapohja (2001), $y_t = E_t F(y_{t+1})$, where F may be non-linear. However, the full analysis done in Evans and Honkapohja (2001) bears over to a generality including (13).

5.5 The Plausibility of, and the Intuition for, Endogenous Keynesian Business Cycles

For $\eta = 0$ (or $\pi = \infty$), the case of instantaneous wage adjustment, the condition (17) of Theorem 1 specializes to the standard one, $\varepsilon(\omega) < -\beta/(1+\beta)$. Since sluggish wage adjustment means $\eta > 0$, whenever $\beta > \alpha$ the condition in Theorem 1 imposes less restrictions on ε under non-instantaneous wage adjustment. In particular, there are realistic restrictions on the remaining parameters, such that with $\eta = 0$ the condition (17) cannot be fulfilled, and indeed neither deterministic cycles nor sunspot equilibria exist, while for η sufficiently large, or π sufficiently small, (17) is fulfilled. One such restriction is $\varepsilon(\omega') \geq 0$ for all ω'. In this precise sense the considered type of nominal rigidity can serve as a mechanism for endogenous fluctuations.

We now argue that not only is it a logical possibility that a nominal rigidity may give rise to endogenous fluctuations, but for *plausible* calibrations of ε, α, and β, the requirement on η, or π, for the existence and learning stability of endogenous fluctuations is not unrealistic.

It is natural to calibrate the parameters such that $1/\beta$ is approximated by labor's share, where $\beta \cong 3/2$ is reasonable, and to consider for α the most usual assumption of expected price inflation building itself perfectly into wage inflation, that is, $\alpha = 1$. Note that lower values for α only tend to make the basic condition (17) for fluctuations more easily fulfilled. Given these α and β, Figure 2 shows the combinations of $\varepsilon(\omega)$ and $\eta(1)$ which are compatible with the conditions of Theorem 1 and 2, with the labels [17] and [18] referring to the conditions (17) and (18) respectively.

Everywhere below the line [17] endogenous fluctuations exist arbitrarily close to the monetary steady state. One should compare with the indicated little piece where $-1 < \varepsilon < -\beta/(1+\beta)$ and $\eta = 0$, giving fluctuations under instantaneous market clearing. Realistically, ε is not far away from zero, so a plausible sufficient condition on the speed of wage adjustment for endogenous fluctuations is $\eta(1) > 3/2$, or $\pi(e^*) < 2/3$. Everywhere below [17] and above [18] indeterminacy takes the form $f'(y) < -1$, so this is the area in which there are learning stable stationary sunspot equilibria close to the steady state. Under the additional requirements of Lemma 2, among them $\eta < \beta/(\beta - \alpha) = 3$ as indicated by the dashed vertical line, deterministic cycles exist. For ε close to zero, the condition for being in this area is approximately $3/2 < \eta(1) < 3$, or $1/3 < \pi(e^*) < 3/2$.

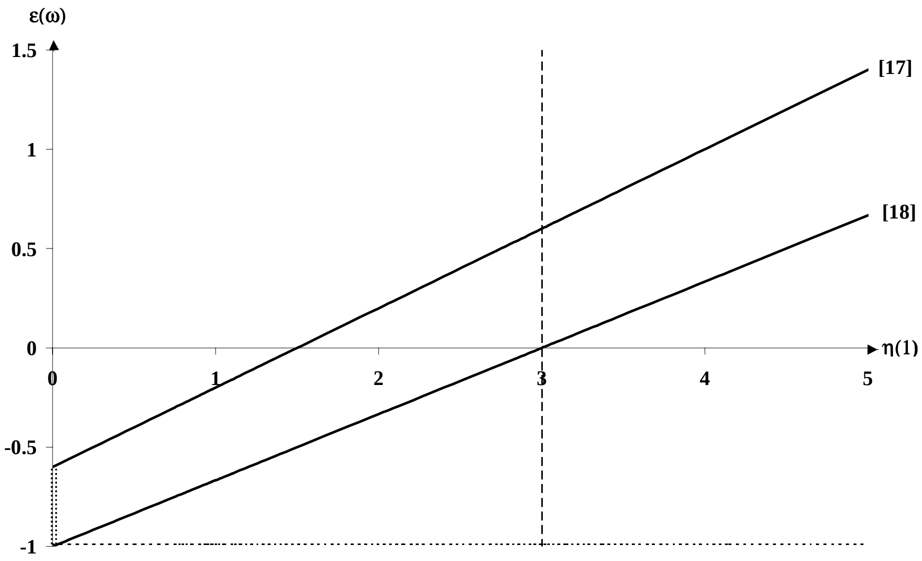

Figure 2

As said, the period length in the present analysis should be thought of as the amount of time during which the nominal wage rate stays fixed. From the evidence reported in Gali and Gertler (1999), such a period could perhaps be thought of as up to 3/4 of a year for nominal output prices. If wages can be considered somewhat more sticky than prices, a period length of approximately one year seems reasonable. A rule of thumb in connection with Phillips-curves is that each percentage point that the rate of unemployment is below the natural rate implies inflation to be one half of a percentage point above "core inflation".[7] If this can be taken to point to a Phillips relation like, $\log E_t w_{t+1} - \log w_t = \frac{1}{2}(e_t - e^*) + (\log E_t p_{t+1} - \log p_t)$, or $E_t w_{t+1}/w_t = \exp(\frac{1}{2}(e_t - e^*))E_t p_{t+1}/p_t)$, then in connection with the model studied in this paper one would have $P(e) = \exp(\frac{1}{2}(e - e^*))$, and hence $\pi(e) = \frac{1}{2}e$ and $\pi(e^*) \cong \frac{1}{2}$, since e^* should be thought of as close to one. This places $\pi(e^*)$ in the interval yielding both existence and learning stability of stationary sunspot equilibria close to the steady state and existence of deterministic cycles. For what these empirical exercises are worth, at least they suggest that the nominal rigidity mechanism for endogenous fluctuations is not in radical contradiction to empirical regularities.

[7] See for instance footnote 20 in Chapter 5 of Romer (2001).

To provide an intuition for how the assumed nominal rigidity promotes endogenous fluctuations it is clarifying to focus on a two-period deterministic cycle. Consider a sequence of outputs where y_t goes from high to low back to high etc. over the periods. How can this be sustained as an equilibrium outcome under rational expectations (perfect foresight)? Due to the decreasing AD-curve (M/p_t), the price level p_t must be low when output is high and vise versa. Due to the AS-curve also w_t must be low when output is high, so the intertemporal real wage under correct expectations, $\omega_t = w_t/p_{t+1}$, must be low exactly when output is high. With instantaneous wage adjustment, output is in each period given directly by labor supply, so labor supply needs to be high exactly when the intertemporal real wage is low. In other words, the labor supply must be (sufficiently) decreasing in the real wage. This explains the standard condition $\varepsilon(\omega) < -\beta/(1+\beta)$. Sluggish wage adjustment, or the short run non-clearing of the labor market associated to it, implies a separation between labor supply on the one side and employment and output on the other. For the considered volatility in output it is still required that y_t is high in periods where ω_t is low, but this is possible without the low ω_t implying a high labor supply. It can suffice for having a high level of employment when output is high that the rate of unemployment is low, and a low rate of unemployment in periods where output is high will exactly tend to create high wages and prices in the (next) periods where output is low. This is as required and expected along the considered path. Thus sluggish wage adjustment can substitute for a perversely sloped labor supply curve as a mechanism for perfect foresight equilibrium cycles. This also explains why a certain degree of sluggishness may be required. From Figure 1, a), e.g., it is clear that if the P-function is close to the vertical thick one ($\underline{e}$ and $\overline{e}$ are both close to one), then only little separation between employment and labor supply is possible. This separation is, however, essential in the above argument. To have a sufficient amount of separation the P-curve must be sufficiently flat, that is, π must be sufficiently low.

6 Concluding Remarks

Shortly stated our conclusion is that sluggishness in the competitive adjustment of nominal wages may make the phenomenon of rational expectations endogenous fluctuations (more) plausible: Under realistic assumptions on other fundamentals, which exclude the existence of rational expectations fluctuations under instantaneous wage adjustment, a certain *and not unrealistic* degree of sluggishness in the adjustment of nominal wages will imply the existence and learning stability of stationary sunspot equilibria close to a monetary steady state. The fact that such fluctuations are rooted in a sluggish nominal wage adjustment has several implications that may be of macroeconomic interest:

1. Since in any period the rate of employment relates to the growth factor of output as $(y_t E_t(1/y_{t+1}))^{\beta-\alpha} = P(e_t)$, there will (for $\beta > \alpha$) be a tendency that unemployment is high in periods where output is low. Furthermore, a volatile equilibrium as studied here typically involves that output and employment fluctuate around the output and employment levels of the monetary steady state. So, over the Keynesian cycle, the rate of unemployment tends to vary countercyclically and around the natural rate.

2. This also means that the rigidity of the nominal wage rate makes unemployment a *permanent* problem in the sense that it makes unemployment go back to higher levels all the time. A two-period cycle, according to which unemployment will be above the natural rate every second period, and below every other second period, gives a sharp illustration. As discussed above, the period length can perhaps be thought of as around a year. So, according to a two-period cycle, unemployment above the natural rate disappears by market forces in one year, which is fast, but also reappears in one year. In the long run unemployment will be above the natural rate half of the time. Along other fluctuations something similar, but less sharp, happens.

3. From the concavity of utility functions it follows directly that variations in activity and unemployment around the steady state are bad for welfare in the sense that the households would be better off if the economy were permanently at the monetary steady state with unemployment at the natural rate all the time. Nominal rigidities may thus motivate permanent stabilization policies attempting at keeping the rate of unemployment at the natural rate all the time, since they may well, in the absence of such policies, imply permanent fluctuations in output and unemployment (around the natural rates) that are bad for welfare.

4. Because of the separation between labor supply and employment possible

with a sluggish wage adjustment, there can be substantial fluctuations in output and employment even for (numerically) low values of the elasticity of labor supply. It is well known that in models where labor supply and employment are tied together it is difficult to create realistic fluctuations in output for realistically low values of the elasticity of labor supply.

The present contribution is meant as a first investigation into the nominal rigidity mechanism for endogenous fluctuations. At least two directions for future research suggest themselves. One is explicit policy analysis in a model as the one studied here, but with a government. In particular it seems of interest to investigate if stabilization policies typically advocated by Keynesians and rooted in nominal rigidities, e.g. high government demand when output is low, tends to be stabilizing (eliminate indeterminacy) or destabilizing (creating indeterminacy). Another subject for future research is an extension of the model framework to include also capital, mainly in order to investigate in more depth to what extent the properties of rational expectations endogenous fluctuations rooted in nominal rigidities are in accordance with observed empirical regularities.

A Proofs

Proof of Lemma 1. As noted in the text this is just a matter of showing that the condition expressing that the derivative of the function G defined in (14) with respect to y_t measured at $y_t = y_{t+1} = y$ is not zero. For convenience we restate (14) here,

$$G(y_t, y_{t+1}) := \beta y_t^\beta v' \left(\frac{y_t^\beta}{H\left((y_t/y_{t+1})^{\beta-\alpha}\right)} \right) / u' \left(\frac{y_{t+1}}{\beta H\left((y_t/y_{t+1})^{\beta-\alpha}\right)} \right) = y_{t+1}. \quad (19)$$

We first find the derivative $G_1'(y_t, y_{t+1})$ at an arbitrary point and then measure at steady state. For later use we will express the derivative in terms of the elasticity $G'(y_t, y_{t+1})y_t/G(y_t, y_{t+1})$. By log-differentiation of G one finds, not writing everywhere what is inside $v'(\cdot)$, $u(\cdot)$, $H(\cdot)$ etc.,

$$\begin{aligned}
\frac{G_1'(y_t, y_{t+1})y_t}{G(y_t, y_{t+1})} &= \beta + \frac{v''}{v'} \frac{H\beta y_t^{\beta-1} - y_t^\beta H'(\beta-\alpha)\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha-1}\frac{1}{y_{t+1}}}{H^2} y_t \\
&\quad + \frac{u''}{u'} \frac{y_{t+1}}{\beta} \frac{H'(\beta-\alpha)\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha-1}\frac{1}{y_{t+1}}}{H^2} y_t \\
&= \beta + N\left(\frac{y_t^\beta}{H\left((y_t/y_{t+1})^{\beta-\alpha}\right)} \right) \left[\beta - (\beta-\alpha)\eta\left(\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha} \right) \right] \\
&\quad - R\left(\frac{y_{t+1}}{\beta H\left((y_t/y_{t+1})^{\beta-\alpha}\right)} \right) (\beta-\alpha)\eta\left(\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha} \right).
\end{aligned}$$

What is inside $N(\cdot)$ must be n_t (formally, $y_t^\beta/H = l_t/(l_t/n_t) = n_t$), and what is inside $R(\cdot)$ must be $\omega_t n_t$ (formally, $y_{t+1}/(\beta H) = \left[y_{t+1}/(\beta y_t^\beta\right] \left[y_t^\beta/H\right] = \omega_t n_t$ where we used (6) on the way). Hence, not writing everywhere what is inside $\eta(\cdot)$,

$$\begin{aligned}
\frac{G_1'(y_t, y_{t+1})y_t}{G(y_t, y_{t+1})} &= \beta + N(n_t)\left[\beta - (\beta-\alpha)\eta\right] - R(\omega_t n_t)(\beta-\alpha)\eta \\
&= \beta\left[1 + N(n_t)\right] - (\beta-\alpha)\eta\left[R(\omega_t n_t) + N(n_t)\right] \\
&= \left[R(\omega_t n_t) + N(n_t)\right]\left(\beta\frac{1+N(n_t)}{R(\omega_t n_t)+N(n_t)} - (\beta-\alpha)\eta \right) \quad (20) \\
&= \left[R(\omega_t n_t) + N(n_t)\right]\left(\beta\frac{1+N(n_t)}{R(\omega_t n_t)+N(n_t)} - \beta + \beta - (\beta-\alpha)\eta \right) \\
&= \left[R(\omega_t n_t) + N(n_t)\right]\left[\beta\frac{1-R(\omega_t n_t)}{R(\omega_t n_t)+N(n_t)} + \beta - (\beta-\alpha)\eta) \right]
\end{aligned}$$

$$= \quad [R(\omega_t n_t) + N(n_t)] \left[\beta(1+\varepsilon(\omega_t)) - (\beta-\alpha)\eta\left(\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha}\right)\right],$$

where (2) was used in the last line. Measuring where $y_t = y_{t+1} = y$ and $\omega_t = \omega$ gives that $G_1'(y,y) \neq 0 \Leftrightarrow \beta\,[1+\varepsilon(\omega)] - (\beta-\alpha)\eta(1) \neq 0$. ■

Proof of Lemma 2. First note from the last line in (20), that $\varepsilon \geq 0$ and $\eta < \beta/(\beta-\alpha)$ or $\beta = \alpha$ everywhere, as assumed in the lemma, implies that $G_1' > 0$ everywhere, so under these conditions the function G is strictly increasing in y_t everywhere.

Consider now a given $y_{t+1} > 0$. We want to show that there is a unique positive y_t that solves (19), and that $y_t < \overline{e}$. As y_t goes to zero, $H\left((y_t/y_{t+1})^{\beta-\alpha}\right)$ goes to a certain value greater than or equal to $\underline{e} > 0$. It follows that what is inside v' goes to zero, and what is inside u' does not go to infinity, so u' does not go to zero. Hence $G(y_t, y_{t+1})$ goes to zero as y_t goes to zero. Since $H\left((y_t/y_{t+1})^{\beta-\alpha}\right)$ stays below $\overline{e}$, as y_t increases from zero, at some point $y_t/H\left((y_t/y_{t+1})^{\beta-\alpha}\right)$ will go up to one, (y_t must go up to $H(y_t/y_{t+1})$) and then v' will go to infinity. Then $G(y_t, y_{t+1})$ will also go to infinity since what is inside u' does not go to zero. Therefore, given $y_{t+1} > 0$, there are solutions in y_t to (19), and all such solutions fulfill that $0 < y_t < H(y_t/y_{t+1}) \leq \overline{e}$. Uniqueness of the solution follows from the fact that $G(y_t, y_{t+1})$, the left hand side in (19) is strictly increasing in y_t. Differentiability of f follows again from the Implicit Function Theorem. ■

Proof of Lemma 3. From total log-differentiation of the equation implicitly defining f, namely $G(y_t, y_{t+1}) = y_{t+1}$, one gets,

$$\frac{dy_t}{dy_{t+1}} \frac{y_{t+1}}{y_t} = \frac{1 - \frac{G_2'(y_t,y_{t+1})y_{t+1}}{G_2'(y_t,y_{t+1})}}{\frac{G_1'(y_t,y_{t+1})y_t}{G(y_t,y_{t+1})}}. \tag{21}$$

Here the last line of (20) gives an expression for the denominator. A similar expression for the numerator is found by log-differentiation of G with respect to y_{t+1},

$$\begin{aligned} \frac{G_2'(y_t, y_{t+1})y_{t+1}}{G(y_t, y_{t+1})} &= \frac{v''}{v'} y_t^{\beta} \frac{H'(\beta-\alpha)\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha-1} \frac{y_t}{y_{t+1}^2}}{H^2} y_{t+1} \\ &\quad - \frac{u'}{u''} \frac{1}{\beta} \frac{H + y_{t+1}H'(\beta-\alpha)\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha-1} \frac{y_t}{y_{t+1}^2}}{H^2} y_{t+1} \end{aligned}$$

$$
\begin{aligned}
&= N\left(\frac{y_t^{\beta}}{H\left((y_t/y_{t+1})^{\beta-\alpha}\right)}\right)(\beta-\alpha)\eta\left(\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha}\right) \\
&\quad +R\left(\frac{y_{t+1}}{\beta H\left((y_t/y_{t+1})^{\beta-\alpha}\right)}\right)\left[1+(\beta-\alpha)\eta\left(\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha}\right)\right].
\end{aligned}
$$

Here again, what is inside $N(\cdot)$ must be n_t, and what is inside $R(\cdot)$ must be $\omega_t n_t$. Then,

$$
\begin{aligned}
1-\frac{G_2'(y_t,y_{t+1})y_{t+1}}{G(y_t,y_{t+1})} &= 1-N(n_t)(\beta-\alpha)\eta-R(w_t n_t)\left[1+(\beta-\alpha)\eta\right] \\
&= 1-R(w_t n_t)-(\beta-\alpha)\eta\left[R(w_t n_t)+N(n_t)\right] \qquad (22) \\
&= \left[R(w_t n_t)+N(n_t)\right]\left[\frac{1-R(w_t n_t)}{R(w_t n_t)+N(n_t)}-(\beta-\alpha)\eta\right] \\
&= \left[R(w_t n_t)+N(n_t)\right]\left[\varepsilon(\omega_t)-(\beta-\alpha)\eta\left(\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha}\right)\right],
\end{aligned}
$$

where (2) was used again in the last line. Inserting the last lines of (20) and (22) into (21) gives the slope of f at an arbitrary point where f is well-defined,

$$
f'(y_{t+1}) = \frac{y_t}{y_{t+1}}\frac{\varepsilon(\omega_t)-(\beta-\alpha)\eta\left(\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha}\right)}{\beta(1+\varepsilon(\omega_t))-(\beta-\alpha)\eta\left(\left(\frac{y_t}{y_{t+1}}\right)^{\beta-\alpha}\right)}. \qquad (23)
$$

Measuring at the monetary steady state, $y_t = y_{t+1} = y$ and $\omega_t = \omega$, gives for $f'(y)$ what is stated in Lemma 3. ■

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