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Optimal Policy in OG Models

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Abstract

In the present paper general stationary overlapping generations economies with many commodities in every period and many different consumers in every generation are considered. A government maximizes an utilitarian social welfare function, that is the sum of weighted averages of utilities for generations, through fiscal policy, i.e. monetary transfers and taxes. Both situations with and without time discounting are considered. It is shown that if the discount factor is sufficiently close to one then the optimal policy stabilizes the economy, i.e. the equilibrium path has the turnpike property. Moreover the fiscal policy is shown to be time-consistent.

Keywords: Overlapping Generations Economies, Economic Policy, Turnpike Property, Discounting.

JEL-classification: D51, D91, E32, E52, H20.

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1 Introduction

In the heart of any study of normative aspects of economic policy is the discussion of the choice of a welfare criterion. In the early work of Diamond (see [8]) the criterion of Pareto optimality is applied, but this criterion is not very potent from the perspective of economic policy because the set of Pareto optima typically is quite large and contains allocations that treats consumers very differently with respect to welfare. However as noted by Samuelson in [16] the normative aspects of economic policy should rather be analyzed in terms of an utilitarian social welfare function. With utilitarian social welfare functions a unique equilibrium path is typically selected allowing discussions of the dynamic properties as well as implementation of this equilibrium path.

In [16] Samuelson addressed questions with relation to stationarity of equilibrium paths in simple stationary, one-commodity, one-consumer over-lapping-generations economies. He showed that the equilibrium path associated with maximization of an utilitarian social welfare function that is the discounted sum of utilities typically differs from the Pareto optimal equilibrium reached under laissez faire so implementation of the equilibrium path demands an active government. More recently Calvo and Obstfeld (see [7]) considered a continuous time version of Diamond's stationary overlapping generations economies with production where there are two commodities every period (consumption good/capital and labour) and one consumer in every generation and this consumer has time separable preferences. Calvo and Obstfeld considered the same social welfare function as Samuelson did and their results are compatible with earlier findings and in particular they show that equilibrium paths associated with maximization of their social welfare function has the turnpike property.

In the present paper general stationary overlapping generations economies with long-lived consumers and production are considered. Moreover the government is supposed to maximize an utilitarian social welfare function, that is a sum of generations' social welfare functions where these are weighted sums of the utility functions of the consumers in the generations, through fiscal policy, i.e. monetary transfers. However the government has to respect behavior of consumers as well as firms and market clearing, i.e. the government can only influence agents through monetary transfers. Questions concerning dynamic properties such as dynamic consistency, the turnpike property and continuity of fiscal policy with regard to the state of the

economy, and implementation of the equilibrium path are studied.

In the present analysis both situations with and without time discounting by the government are considered. Indeed, an important feature of the overlapping generation model is the existence of equilibria in which future consumption continues to plays a role, i.e. the Samuelson case[1]. Clearly, the only way not to exclude these equilibria from the set of solutions to the government's problem is to consider the undiscounted case. Also no assumptions will be made with regard to time-separability of preferences because time-separability eliminates important sources of intertemporal links. For example, for pure exchange economies time-separability implies that the equilibrium path associated with maximization of the social welfare function is stationary which implies that all consumers consume the same because there are no intertemporal links. Furthermore, no restriction on the lifespan of consumers will be made, so that fluctuations with a period shorter than the lifespan are not excluded a priori[2].

Concerning the turnpike property of equilibrium paths associated with maximization of utilitarian social welfare functions when the government considers feasibility constraints rather than maximization by agents and equilibrium conditions the answer is rather simple: *If the discount factor is sufficiently close to one then the equilibrium has the turnpike property, i.e. it converges to a stationary allocation as time tends to infinity.* This property of the equilibrium path contrasts the fact that equilibria can be quite complex for overlapping generations economies under laissez faire (see [4, 11, 12]). For the implementation part the answer is equally simple: *Fiscal policy in terms of monetary transfers implements equilibria that are associated with maximization of utilitarian social welfare functions.* Moreover dynamic consistency and continuity of the equilibrium path with respect to the state of the economy are usually obtained for free.

It seems to be a common view that economic policy should be directed toward stabilization (see [13, 15]) even though fluctuations may be Pareto optimal as in [12]. The present paper supports this view, but the support is based on intergenerational justice through the utilitarian social welfare function rather than some sort of abstract "social cost" of fluctuations.

The paper is structured as follows: In section 2 a stationary overlapping-

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[1]We thank the Associate Editor for pointing out this important fact.
[2]This remark is due to Karl Shell.

generations economy with an active government is presented; in section 3 the equilibrium associated with maximization of the social welfare function is characterized. Since the undiscounted case is not excluded from the analysis, the meaning of maximal is not straightforward. As in the growth literature, in those circumstances maximality is characterized through the Ramsey-Weizsacker overtaking criterion. Irrespective of the discounting, the characterization is obtained through 3 steps: Transformation of the overlapping generations economy into an optimal growth model; showing that the solution to the planner's problem in the optimal growth model can be implemented as equilibrium in the overlapping generations economy through monetary transfers and; application of a turnpike theorem to the planner's problem for the optimal growth model in order to show that the solution has the turnpike property. The application of the turnpike theorem is not straight forward because the transformation of the overlapping generations economy into an optimal growth model results in production technologies that exhibit decreasing returns to scale as well as constant returns to scale - a case that does not seem to be covered by the various turnpike theorems in the literature. On the one hand, when the future is discounted, the proof by Bewley in [6] can be applied with some modifications. On the other hand, when the government does not discount the future, the proofs of Gale in [10] and McKenzie in [14] are used. Section 4 contains the proofs while some concluding remarks can be found in section 5.

2 The Model

In the present paper stationary overlapping generations economies with production and fiscal policies are considered.

Time extends from zero to infinity with $t \in \mathbf{N} \cup \{0\} = \mathbf{N}_0$. First the consumers and the firms are introduced, second equilibria and steady states are defined and thirdly the government is introduced.

2.1 Commodities, Consumers and Firms

Let L be a set of l commodities in every period. Moreover let $L_c \subset L$ denote a set of l_c consumption commodities, $L_o \subset L$ denote a set of l_o primary commodities and $L_p = L \setminus L_o$ denote a set of l_p producible commodities.

Let I be a set of m consumers in every generation. Consumers live for S periods except for the first $S-1$ generations where the first generation lives for one period, the second generation lives for two periods,.., the $S-1$'th generation lives for $S-1$ periods and all subsequent generations live for S periods. The consumers are described by their consumption sets, endowments, utility functions and shares in firms. Hence the consumers are described by

$$((X_i(t), \omega_i(t), u_i(t), \theta_{i,j}(t))_{t=-S+1}^{-1}, (X_i, \omega_i, u_i)_{s\in\mathbf{N}_0}),$$

where

$$\sum_{t=-S+1}^{-1} \sum_{i\in I} \theta_{i,j}(t) = 1$$

thus only "middle-aged" and "old" consumers are supposed to own shares at the start of the economy.

For $t \in \mathbf{N}_0$ consumers are supposed to satisfy the following assumptions

A.1 Consumption sets, X_i, are the nonnegative orthants of $\mathbf{R}^{Sl_c}$, i.e.

$$X_i = \mathbf{R}_+^{Sl_c}.$$

A.2 Endowments, ω_i, are primary commodities, i.e.

$$\omega_i \in \mathbf{R}_+^{Sl_o}.$$

A.3 Endowments of consumption commodities are positive, i.e.

$$\omega_i^{s,k} \in \mathbf{R}_{++}$$

for some $s \in \{1, \ldots, S\}$ and some $k \in l_c \cap l_o$.

A.4 Utility functions, u_i, are twice differentiable with positive first-order derivatives and negative definite Hessian matrix.

For $t \in \{-S+1, \ldots, -1\}$ consumers are defined as follows

A.1' $X_i(t) = \mathbf{R}_+^{(S+t)l_c}$

A.2' $\omega_i(t) = (\omega_i^{1-t}, \ldots, \omega_i^S)$

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A.3' There exists $x_i(t) \in \mathbf{R}_+^{-tl_c}$ such that $u_i(t)$ is defined by

$$u_i(t)(x) = u_i(x_i(t), x).$$

Hence the first $S - 1$ generations are truncations of subsequent generations. All assumptions are more or less standard within the differentiable framework. With some abuse of notation endowments and consumption plans are from time to time taken to be vectors in $\mathbf{R}^{Sl}$ or $\mathbf{R}^l$ rather than $\mathbf{R}^{Sl_c}$ or $\mathbf{R}^{l_c}$.

Let J be a set of n firms. Firms are infinitely lived and they are described by their production sets and stocks of output in period 0. Firms are supposed to satisfy the following assumptions

A.5 Production sets, Y_j, allow transformations of some commodities in one period into producible commodities in the subsequent period, i.e.

$$Y_j \subset \mathbf{R}_-^{l_j^1} \times \mathbf{R}_+^{l_j^2}$$

where $L_j^1 \subset L$ is a set of l_j^1 commodities and $L_j^2 \subset L_p$ is a set of l_j^2 commodities.

A.6 Zero production is possible, i.e.

$$0 \in Y_j.$$

A.7 Primary commodities are necessary for production, i.e. suppose that $(y^1, y^2) \in Y_j$ then

$$y^{1,k} = 0 \text{ for all } k \in L_o \Rightarrow y^2 = 0.$$

A.8 There exist twice differentiable functions, $f_j : \mathbf{R}_-^{l_j^1} \times \mathbf{R}_+^{l_j^2}$, with positive first-order derivatives and positive definite Hessian matrix on the orthogonal complement to Df_j, such that

$$y \in Y_j \Leftrightarrow f_j(y) \leq 0$$

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As for the consumers all assumptions are more or less standard within the differentiable framework. With some abuse of notation production plans are from time to time taken to be vectors in $\mathbf{R}^{2l}$ or $\mathbf{R}^l$ rather than $\mathbf{R}^{l_j^1} \times \mathbf{R}^{l_j^2}$, $\mathbf{R}^{l_j^1}$ or $\mathbf{R}^{l_j^2}$.

Total resources and the aggregate production set, $\sum_{j \in J} Y_j$, are supposed to satisfy the following assumptions

A.9 There is a positive amount of every primary commodity, i.e.

$$\omega = \sum_{i \in I} \sum_{s=1}^{S} \omega_i^s \in \mathbf{R}_{++}^{l_o}$$

A.10 There exist production plans, $(y_j^1, y_j^2)_{j \in J}$, such that all commodities are available, i.e.

$$\sum_{i \in I} \sum_{s=1}^{S} \omega_i^s + \sum_{j \in J} (y_j^1 + y_j^2) \in \mathbf{R}_{++}^l.$$

Let $p_t \in \mathbf{R}_+^l$ be commodity prices, $q_t \in \mathbf{R}_+^n$ asset prices and $p_{m,t}$ money price in period t. The government controls the fiscal policy, i.e. it makes lump-sum transfers and taxes to the consumers as well as to the firms. Let $(((\tau_{i,t})_{i \in I})_{t=-S+1}^{\infty}, ((\sigma_{j,t})_{j \in J})_{t=0}^{\infty})$ be a fiscal policy, where $\tau_{i,t} = (\tau_{i,t}^s)_{s=1}^{S}$ is the transfer to consumer i in generation t and $\sigma_{j,t}$ the transfer to firm j in period t.

Consumers maximize their utilities subject to their budget constraints. For $t \in \mathbf{N}_0$ the consumers solve the following problems

$$\max \quad u_i(x^1, \ldots, x^S)$$

$$\text{s.t.} \quad \sum_{s=1}^{S} (p_{s+t-1} \cdot (x^s - \omega_i^s) - p_{m,s+t-1} \tau_{i,s+t-1}^s)$$
$$+ \sum_{s=1}^{S-1} \theta^s \cdot (\pi_{s+t} + q_{s+t} - q_{s+t-1}) = 0.$$

where $\theta^s \in \mathbf{R}^n$ is the portfolio and $\pi_{s+t} \in \mathbf{R}^n$ is the net profit of the firms.

For $t \in \{-S+1, \ldots, -1\}$ the consumers solve the following problems

$$\max \quad u_i(x_i(s), x^{1-t}, \ldots, x^S)$$

$$\text{s.t.} \quad \sum_{s=1-t}^{S} (p_{s+t-1} \cdot (x^s - \omega_i^s) - p_{m,s+t-1}\tau_{i,s+t-1}^s)$$
$$+ \sum_{s=1-t}^{S-1} \theta^s \cdot (\pi_{s+t} + q_{s+t} - q_{s+t-1}) \quad = \quad \theta_i(t) \cdot (\pi_0 + q_0).$$

Only fiscal policies that are compatible with the price of money being positive are considered therefore the price of money is normalized to one. Moreover only prices on commodities and assets that exclude arbitrage are considered, i.e.

$$q_t \quad = \quad \pi_{t+1} + q_{t+1} \quad (NA)$$

for all t, because otherwise there is no solution to the consumers' maximization problems. Hence the consumers' maximization problems reduce to

$$\max \quad u_i(x^0, \ldots, x^{S-1})$$

$$(CP)$$

$$\text{s.t.} \quad \sum_{s=1}^{S} (p_{s+t-1} \cdot (x^s - \omega_i^s) - \tau_{i,s+t-1}^s) \quad = \quad 0$$

for consumer i in generation t with $t \in \mathbf{N}_0$, while they reduce to

$$\max \quad u_i(x_i(s), x^{1-t}, \ldots, x^S)$$

$$(CP')$$

$$\text{s.t.} \quad \sum_{s=1-t}^{S} (p_{s+t-1} \cdot (x^s - \omega_i^s) - \tau_{i,s+t-1}^s) \quad = \quad \theta_i(t) \cdot (\pi_0 + q_0)$$

for consumer i in generation t with $t \in \{-S+1, \ldots, -1\}$. Let $x_{i,t} = (x_{i,s+t-1}^s)_{s=1}^S$ denote the consumption plan of consumer i in generation t and let $x_t = (x_{i,t})_{i \in I}$ denote the consumption plans of consumers of generation t.

Firms are infinitely lived, their net profits are as usual given by

$$p^t \cdot y_{j,0}^2 + \sigma_{j,0} + \sum_{t \in \mathbf{N}_0} (p_t \cdot y_{j,t}^1 + p_{t+1} \cdot y_{j,t+1}^2 + \sigma_{j,t})$$

where $((y^1_{j,t}, y^2_{j,t+1})_{j \in J})_{t \in \mathbf{N}_0}$ satisfy $\quad f_j(y^1_{j,t}, y^2_{j,t+1}) \leq 0$ for all t and j. According to this definition profits may be infinite. An example is provided by a stationary plan with constant prices, a situation that is likely to occur in the undiscounted case. We cure this problem by adopting the usual Ramsey-Weizsacker overtaking criterion.

Definition 1 *A sequence* $(a_t)_{t \in \mathbf{N}_0} \in \mathbf{R}$ **overtakes** *a sequence* $(b_t)_{t \in \mathbf{N}_0} \in \mathbf{R}$ *if there exists* T_0 *such that*

$$\sum_{t=0}^{T}(a_t - b_t) \geq 0 \text{ for all } T \geq T_0$$

A sequence that cannot be overtaken is called **strongly optimal**.

Note that if $\sum_{t=0}^{\infty} a_t < \infty$ then the overtaking criterion is equivalent to the usual criterion: $\sum_{t=0}^{\infty} a_t \geq \sum_{t=0}^{\infty} b_t$ for all sequences (b_t).

Firms are assumed to maximize profits in the above sense, with $a_t = p_t \cdot y^1_{j,t} + p_{t+1} \cdot y^2_{j,t+1} + \sigma_{j,t}$. This is equivalent to that firms maximize their instant profit because technologies display intertemporal separability. Hence the firms' maximization problem reduce to

$$\begin{aligned} \max \quad & p_t \cdot y^1_{j,t} + p_{t+1} \cdot y^2_{j,t+1} + \sigma_{j,t} \\ & \text{s.t.} \quad f_j(y^1_{j,t}, y^2_{j,t+1}) \leq 0 \end{aligned} \qquad (FP)$$

provided that there exists a solution. Let $y_{j,t} = (y^s_{j,s+t-1})^2_{s=1}$ denote the production plan of firm j in period t and let $y_t = (y_{j,t})_{j \in J}$ denote the production plans of firms in period t.

2.2 Equilibria and Steady States

Equilibria and steady states have to be defined in order to study how the government uses its instruments. Equilibria are prices and transfers such that markets clear provided that consumers maximize their utilities and producers maximize their profits.

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Definition 2 *An equilibrium is a sequence of prices, monetary transfers and profit taxes $(p_t, q_t, \tau_t, \sigma_t)_{t \in \mathbf{N}_0} \in \mathbf{R}_+^l \times \mathbf{R}_+^n \times \mathbf{R}^{Sm} \times \mathbf{R}^n$ such that markets clear:*

$$\sum_{s=1}^{S} \sum_{i \in I} (x_{i,t}^s - \omega_i^s) = \sum_{j \in J} \sum_{s=1}^{2} y_{j,t}^s$$

where

- *$(x_{i,s+t-1}^s)_{s=1}^{S}$ solves (CP) for $((p_{s+t-1}, \tau_{i,s+t-1}^s)_{s=1}^{S}, \sigma_{j,t})$ for all $i \in I$, $j \in J$ and all $t \in \mathbf{N}_0$,*

- *$(x_{i,s+t-1}^s)_{s=1-t}^{S}$ solves (CP') for $(p_{s+t-1}, \tau_{i,s+t-1}^s)_{s=1-t}^{S}$ and π_0, where*

  $$\pi_{j,0} = p_0 \cdot y_{j,0}^2 + \sigma_{j,0}$$

  for all $j \in J$, for all $i \in I$ and all $t \in \{-S+1, \ldots, -1\}$,

- *$(y_{j,s+t-1}^s)_{s=1}^{2}$ solves (FP) for $((p_{s+t-1})_{s=1}^{2}, \sigma_{j,t})$ for all $j \in J$ and all $t \in \mathbf{N}_0$.*

Let

$$K_t = \sum_{j \in J} y_{j,t}^2$$

be the stock of output in period t then for an equilibrium, $(p_t, q_t, \tau_t, \sigma_t)_{t \in \mathbf{N}_0}$, let $((x_t)_{t=-S+1}^{\infty}, (y_t)_{t \in \mathbf{N}_0})$ denote the associated equilibrium allocation and let

$$E(K_0) \subset \prod_{t=-S+1}^{-1} \prod_{i \in I} X_i(t) \times \prod_{t \in \mathbf{N}_0} \prod_{i \in I} X_i \times \prod_{t \in \mathbf{N}_0} \prod_{j \in J} Y_j$$

denote the set of equilibrium allocations, i.e. $((x_t)_{t=-S+1}^{\infty}, (y_t)_{t \in \mathbf{N}_0}) \in E(K_0)$ if and only if there exists an equilibrium such that $((x_t)_{t=-S+1}^{\infty}, (y_t)_{t \in \mathbf{N}_0})$ is the associated equilibrium allocation. Steady states are equilibria where prices, taxes and transfers are constant up to multiplication by a scalar.

Definition 3 *A steady state is an equilibrium, $(p_t, q_t, \tau_t, \sigma_t)_{t \in \mathbf{N}_0}$, for which there exist prices, transfers and taxes, $(p, q, \tau, \sigma) \in \mathbf{R}_+^l \times \mathbf{R}_+^n \times \mathbf{R}^{Sm} \times \mathbf{R}^n$ and a scalar $\beta \in \mathbf{R}_{++}$ such that $(p_t, q_t, \tau_t, \sigma_t) = \beta^t (p, q, \tau, \sigma)$ for all $t \in \mathbf{N}_0$*

Since all consumers face the same relative prices and all consumers of the same type receive the same transfer at steady states, the associated allocation is stationary.

2.3   The Government

The government maximizes a social welfare function and it utilizes fiscal policy, i.e. monetary transfers, in order to do so. Obviously any reasonable social welfare function for dynamic economies should satisfy Pareto optimality as well as time consistency. However these criteria are not very potent from the perspective of economic policy because any path of transfers that is compatible with an optimal allocation satisfy them. Therefore the social welfare function of the government is supposed to be a utilitarian social welfare function, i.e. the social welfare function is the possibly discounted sum of generations' social welfare functions and generations' social welfare functions are weighted sums of the utility functions of the consumers in the generations. Furthermore the discount factors as well as the weights on the consumers are chosen to be independent of time. This social welfare function is very simple and perhaps even natural, moreover it satisfies time consistency.

As usual the value of the discounting applied by the government plays a major role in the analysis. Furthermore, an important characteristic of the overlapping generation model is the existence of equilibria in which future consumption still plays a role, i.e. the Samuelson case. Therefore, both situations in which the government discounts and does not discount the future need to be considered.

The aim of the government is to maximize its social welfare function subject to the equilibrium conditions, i.e. the government can make transfers but it cannot choose consumption plans or production plans. However, since the case with no discounting is included in the analysis, maximality is characterized through the Ramsey-Weizsacker overtaking criterion as formalized in Definition 1. Let $\delta \in \,]0,1]$ be the time discount factor, then the aim of the government is to find a sequence that cannot be overtaken by any other sequence subject to the equilibrium conditions, i.e.

$$((\delta^t \sum_{i \in I} \lambda_i u_i(x_i(t), x_{i,t}))^{-1}_{t=-S+1}, (\delta^t \sum_{i \in I} \lambda_i u_i(x_{i,t}))_{t \in \mathbf{N}_0})$$
$$\text{is strongly optimal subject to} \qquad (GP)$$
$$\exists (y_t)_{t \in \mathbf{N}_0} : ((x_t)^{\infty}_{t=-S+1}, (y_t)_{t \in \mathbf{N}_0}) \in E(K_0)$$

In the case with $\delta < 1$ this boils down to the more usual maximization problem

$$\max \quad \sum_{t=-S+1}^{-1} \delta^t \sum_{i \in I} \lambda_i u_i(x_i(t), x_{i,t}) + \sum_{t \in \mathbf{N}_0} \delta^t \sum_{i \in I} \lambda_i u_i(x_{i,t})$$

$$\text{s.t.} \quad \exists (y_t)_{t \in \mathbf{N}_0} : ((x_t)_{t=-S+1}^{\infty}, (y_t)_{t \in \mathbf{N}_0}) \in E(K_0)$$

The government selects its monetary transfers in order to maximize its social welfare function, but some monetary transfers may be compatible with several equilibria. In this case monetary policies a la Grandmont (see [13]) may enable the government to select the most favorable equilibrium. In the present paper it is simply assumed that the government selects the most favorable equilibrium in case of several equilibria.

3   Optimal Policy and Turnpike

The problem of the government as stated in the previous section seems to be quite intractable because the government has to take maximizing behavior of consumers and firms and market clearing into account, i.e. only allocations in $E(K_0)$ are admissible for the government. However it is possible to transform the problem of the government into the planner's problem of an optimal growth model and in that model analyse the problem of the government. Then, known results for the growth models can be used to prove that the equilibrium chosen by the government exists and that its path has the turnpike property. The next theorem provides the existence result.

Theorem 1 *For all* $\delta \in]0, 1]$ *there exists a solution to the government's problem,* (GP).

This result is pretty trivial provided that (GP) is transformed into an optimal growth model with one consumer. However, it is a necessary step toward the turnpike result. Indeed, as is clearly shown by its proof, the transformation of the stationary overlapping generation economy into an optimal growth model results in an optimal growth model that is stationary - except for the first periods where utility functions differ. Then, if the first periods are disregarded a usual **stationary** optimal growth model is obtained. The standard assumptions that ensure that an equilibrium allocation exists in

the stationary optimal growth model are adopted. Let $((x_t)_{t=-S+1}^{\infty}, (y_t)_{t \in \mathbf{N}_0})$ be the equilibrium allocation for the initial overlapping generations economy associated with the solution to the government's problem as found in Theorem 1. Similarly, for a suitable initial capital the **stationary** optimal growth model admits a **stationary** equilibrium allocation. Let (x^*, y^*) be the stationary equilibrium allocation for the overlapping generations economy associated with the stationary allocation of the stationary growth model with one consumer. Then consider

A.11 There exists $\delta^* \in [0,1[$ such that if $\delta \in]\delta^*, 1]$ then $x_{i,t}^{s,k} > 0$ for all $i \in I$, all $k \in L_c$, all $s \in \{1, ..., S\}$ and all $t \in \{0, \ldots, S-1\}$ and $\sum_{j \in J} y_{j,S-1}^{2,k} > 0$ for all $k \in L_p$.

A.12 There exist $\alpha \in \mathbf{R}_{++}$ and $\delta^* \in [0,1[$ such that if $\delta \in]\delta^*, 1]$ then $x_i^{*,k} \geq \alpha$ for every i and all k and $\sum_{j \in J} y^{*,2,k} \geq \alpha$ for all j and all k.

A.11 implies that the initial stock of output for the planner's stationary problem in the optimal growth model is strictly positive in all coordinates. A.12 implies that in the stationary solution all producible commodities are produced.

Under these further assumptions the following turnpike result is obtained.

Theorem 2 *There exist $\delta^* \in]0,1[$ and $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}) \in \mathbf{R}_+^{Slm} \times \mathbf{R}^{2ln}$ such that if $\delta \in]\delta^*, 1]$ and $((x_t)_{t=-S+1}^{\infty}, (y_t)_{t \in \mathbf{N}_0})$ is the allocation associated with the solution to the government's problem then*

$$\lim_{t \to \infty} |(x_t, y_t) - (x^*, y^*)| = 0$$

Theorem 2 implies that if the discount factor is sufficiently close to one then the equilibrium associated with the maximization of the utilitarian social welfare function converges to a steady state as time tends to infinity. Furthermore, the associated steady state price is of the form $p_t = \delta^t p$. Hence the government stabilizes the economy in the sense that it removes fluctuations through its fiscal policy.

The planner's stationary problem generates a well defined value function, at least for $\delta < 1$. From the properties of utility functions it can then be deduced that the value function is strictly concave so that there is a unique solution to the planner's problem according to [3]. Therefore the policy function that is the solution to the planner's problem is continuous and the fiscal policy varies continuously with the evolution of the economy.

4 Proofs

4.1 Proof of Theorem 1

Consider first the planner's problem where the objective of the government is to maximize its social welfare function subject to feasibility conditions rather than equilibrium conditions. Then the planner's problem, denoted (PP), is to find a sequence that cannot be overtaken by any other sequence subject to the feasibility conditions, i.e.

$$((\delta^t \sum_{i \in I} \lambda_i u_i(x_i(t), x_{i,t}))_{t=-S+1}^{-1}, (\delta^t \sum_{i \in I} \lambda_i u_i(x_{i,t}))_{t \in \mathbf{N}_0}) \text{ is strongly optimal}$$

$$\text{s.t.} \begin{cases} \sum_{s=1}^{S} \sum_{i \in I}(x_{i,t}^s - \omega_i^s) &=& \sum_{j \in J} \sum_{s=1}^{2} y_{j,t}^s \\ \\ f_j(y_{j,t}^1, y_{j,t+1}^2) &\leq& 0 \text{ for all } j \\ \\ K_0 \text{ is fixed.} \end{cases} \qquad (PP)$$

Clearly the maximal value of the planner's problem is at least as high as the maximal value of the government's problem because the planner has more direct instruments at hand than the government.

The proof then relies on the transformation of the overlapping generations economy into an optimal growth model. This involves three steps: Transformation of the overlapping generations economy with S periods of life into an economy with 2 periods of life (this transformation was introduced in [2]); transformation of the overlapping generations economy with 2 periods of life into an optimal growth model and; introduction of artificial firms[3].

For the transformation of the economy with S periods of life into an economy with 2 periods of life let the periods $(S-1)t, ..., (S-1)(t+1)-1$ be identified with period t for all $t \in \mathbf{N}_0$. Then there are $(S-1)l$ commodities in every period, $(S-1)m$ consumers in every generation and $(S-1)n$ firms.

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[3]David Cass has informed the authors that unified models of overlapping generations economies and models of capital accumulation were used by L. Benveniste in the early eighties in an unpublished paper.

For $i \in \{1, \ldots, m\}$ let

$$\overline{\omega}_i^1 = \begin{pmatrix} \omega_i^1 \\ \vdots \\ \omega_i^{S-1} \end{pmatrix}, \qquad \overline{x}_{i,t}^1 = \begin{pmatrix} x_{i,(S-1)t}^1 \\ \vdots \\ x_{i,(S-1)(t+1)-1}^{S-1} \end{pmatrix},$$

$$\overline{\omega}_i^2 = \begin{pmatrix} \omega_i^S \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \overline{x}_{i,t+1}^2 = \begin{pmatrix} x_{i,(S-1)(t+1)}^S \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and $\overline{u}_i = \lambda_i u_i$. The variables related to all other consumers are defined in a similar way. Then for $i \in \{(S-2)m+1, \ldots, (S-1)m\}$ let

$$\overline{\omega}_i^1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \omega_{i-(S-2)m}^1 \end{pmatrix}, \qquad \overline{x}_{i,t}^1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{i-(S-2)m,(S-1)(t+1)-1}^1 \end{pmatrix},$$

$$\overline{\omega}_i^2 = \begin{pmatrix} \omega_{i-(S-2)m}^2 \\ \vdots \\ \omega_{i-(S-2)m}^S \end{pmatrix}, \qquad \overline{x}_{i,t+1}^2 = \begin{pmatrix} x_{i-(S-2)m,(S-1)(t+1)}^2 \\ \vdots \\ x_{i-(S-2)m,(S-1)(t+2)-1}^S \end{pmatrix},$$

and $\overline{u}_i = \delta^{S-2} \lambda_{i-(S-2)m} u_{i-(S-2)m}$. For the first $S-1$ generations $\overline{\omega}_i^1 = \overline{x}_{i,t}^1 = 0$ and $\overline{\omega}_i^2$ and $\overline{x}_{i,t}^2$ is defined as for all other generations and utility functions are defined by

$$\overline{v}_i = \delta^{s-1} \lambda_{i-(s-1)m} u_{i-(s-1)m}\big(x_{i-(s-1)m}(S-s), \cdot\big)$$

for $i \in \{(s-1)m+1, \ldots, sm\}$ and $s \in \{1, \ldots, S-1\}$.

For $j \in \{1, \ldots, n\}$ let

$$\overline{y}_{j,t}^1 = \begin{pmatrix} y_{j,(S-1)t}^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \overline{y}_{j,t}^2 = \begin{pmatrix} 0 \\ y_{j,(S-1)t+1}^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

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with
$$(\overline{y}^1_{j,t}, \overline{y}^2_{j,t}) \ \in \ Y_j \ \Leftrightarrow \ f_j(\overline{y}^1_{j,t}, \overline{y}^2_{j,t}) \ \leq \ 0$$

so production takes place within periods. The variables related to all other firms up to $i = (S-2)n$ are defined in a similar while for $j \in \{(S-2)n + 1, \ldots, (S-1)n\}$ let

$$\overline{y}^1_{j,t} \ = \ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y^1_{j-(S-1)n+1,(S-1)(t+1)-1} \end{pmatrix}, \quad \overline{y}^2_{j,t} \ = \ \begin{pmatrix} y^2_{j-(S-1)n+1,(S-1)t} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

with
$$(\overline{y}^1_{j,t}, \overline{y}^2_{j,t+1}) \ \in \ Y_j \ \Leftrightarrow \ f_j(\overline{y}^1_{j,t}, \overline{y}^2_{j,t+1}) \ \leq \ 0$$

so that for these firms production takes place between periods. Finally

$$\overline{K}_t \ = \ \sum_{j=(S-2)n+1}^{(S-1)n} \overline{y}^2_{j,t}$$

is the stock of output in period t.

For the transformation of the economy with 2 periods of life into an optimal growth model let the consumption of consumers in generation t be identified with consumption of consumers in period t, but consumption in the first period of life are considered to be different commodities. So there are $(S-1)^2 lm + (S-1)l$ commodities in every period, $(S-1)m$ consumers and $(S-1)n$ firms.

For $i = 1$ let

$$\overline{\overline{\omega}}_i \ = \ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \overline{\omega}^1_i + \overline{\omega}^2_i \end{pmatrix}, \quad \overline{\overline{x}}_{i,t} \ = \ \begin{pmatrix} \overline{x}^1_{i,t} \\ 0 \\ \vdots \\ 0 \\ \overline{x}^2_{i,t+1} \end{pmatrix},$$

and $\overline{\overline{u}}_i = \overline{u}_i$. The variables related to all other consumers are defined in a

similar way. Then for $i = (S-1)m$ let

$$\overline{\overline{\omega}}_i \;=\; \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \overline{\omega}_i^1 + \overline{\omega}_i^2 \end{pmatrix}, \quad \overline{\overline{x}}_{i,t} \;=\; \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \overline{x}_{i,t}^1 \\ \overline{x}_{i,t+1}^2 \end{pmatrix},$$

and $\overline{\overline{u}}_i = \overline{u}_i$ and $\overline{\overline{v}}_i = \overline{v}_i$.

For the firms let

$$\overline{\overline{y}}_{j,t}^1 \;=\; \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \overline{y}_{j,t+1}^1 \end{pmatrix}, \quad \overline{\overline{y}}_{j,t}^2 \;=\; \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \overline{y}_{j,t+1}^2 \end{pmatrix},$$

for all $j \in \{1, \ldots, (S-1)n\}$. So for $j \in \{1, \ldots, (S-2)n\}$ production takes place within periods and for $j \in \{(S-2)n+1, \ldots, (S-1)n\}$ production takes place between periods. Note that production only involves the last $(S-1)l$ commodities. Finally let

$$\overline{\overline{K}}_{-1} \;=\; \sum_{j=(S-2)n+1}^{(S-1)n} \overline{\overline{y}}_{j,-1}^2 \;\text{ and }\; \overline{\overline{K}}_t \;=\; \sum_{j=(S-2)n+1}^{(S-1)n} \overline{\overline{y}}_{j,t}^2 + \sum_{k=1}^{(S-1)^2 lm} \overline{\overline{z}}_{k,t}^2$$

be the stock of output in period t.

Hence in order to produce the first commodities $(S-1)l$ artificial firms are introduced for every consumer so $(S-1)^2 lm$ artificial firms are introduced. Actually many of the artificial firms are not needed because consumers only consume Sl_c commodities rather than $2(S-1)l$ commodities but in order to keep the notation at a reasonable level all $(S-1)l$ artificial firms are introduced for every consumer. They produce the first commodities by use of the last $(S-1)l$ commodities with one-to-one technologies. Their production sets are $Z_k \subset \mathbf{R}_-^{(S-1)^2 lm+(S-1)l} \times \mathbf{R}_+^{(S-1)^2 lm+(S-1)l}$, their inputs are $\overline{\overline{z}}_{k,t}^1$ and their outputs are $\overline{\overline{z}}_{k,t}^2$ for all $k \in \{1, \ldots, (S-1)^2 lm\}$ with

$$\overline{\overline{z}}_{k,t}^{1,k'} \;=\; 0 \quad \text{for} \quad k' \neq k - (S-1)l(i-1) + (S-1)^2 lm$$

$$\overline{\overline{z}}_{k,t}^{2,k'} \;=\; 0 \quad \text{for} \quad k' \neq k$$

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for all $k \in \{(S-1)l(i-1)+1, \ldots, (S-1)li\}$ and all $i \in \{1, \ldots, (S-1)m\}$. Moreover

$$(\overline{\overline{z}}^1_{k,t}, \overline{\overline{z}}^2_{k,t+1}) \in Z_k \Leftrightarrow g_k(\overline{\overline{z}}^1_{k,t}, \overline{\overline{z}}^2_{k,t+1}) \leq 0$$

where

$$g_k(\overline{\overline{z}}^1_{k,t}, \overline{\overline{z}}^2_{k,t+1}) = \overline{\overline{z}}^{1,k-(S-1)l(i-1)+(S-1)^2 lm}_{k,t} + \overline{\overline{z}}^{2,k}_{k,t+1}$$

for all $k \in \{(S-1)l(i-1)+1, \ldots, (S-1)li\}$ and all $i \in \{1, \ldots, (S-1)m\}$.

For all artificial firms production takes place between periods. It may seem a little strange to distinguish between consumption in the first period of life for consumers in the same generation but in the proof of the turnpike property it turns out to be very useful as will become clear - hopefully. A difficulty is that some of the production technologies exhibit decreasing returns to scale while other production technologies exhibit constant returns to scale. Indeed, in the literature on turnpike theory in optimal growth models all production technologies are assumed to exhibit either decreasing returns to scale as in [6] or constant returns to scale as in [17, 18] (see also [19]).

A useful property of the model that is not altered by the inclusion of artificial firms is that for any given initial capital feasible consumptions are bounded, i.e. there exists an upper bound, $a \in \mathbf{R}$ such that if $(\overline{x}_t, \overline{\overline{y}}_t, \overline{\overline{z}}_t)_{t \in \mathbf{N}_0}$ is an equilibrium allocation $\sup_{t \in \mathbf{N}_0} |\overline{x}_t| \leq a$. Indeed, unbounded consumptions may only be caused by the real firms (with decreasing returns to scale) and not by the artificial firms so that Lemma 7.1 in [6] applies (it is also possible to use Lemma 1 and Lemma 3 in [17]).

In the following, the artificial technologies are transformed into decreasing returns to scale technologies. First, modify the artificial firms such that

$$g_k(\overline{\overline{z}}^1, \overline{\overline{z}}^2) = \overline{\overline{z}}^{1,k-(S-1)l(i-1)+(S-1)^2 lm} - h_\beta(-\overline{\overline{z}}^{2,k})$$

for all $k \in \{(S-1)l(i-1)+1, \ldots, (S-1)li\}$ and all $i \in \{1, \ldots, (S-1)m\}$ where

$$h_\beta(z) = (1-\beta)z + \beta \ln(1+z)$$

with $\beta \in [0,1]$. Second, modify the utility functions such that

$$\overline{\overline{u}}_i(\overline{\overline{x}}) = \overline{\overline{u}}_i((h_\beta^{-1}(\overline{\overline{x}}^k))_{k=1}^{(S-1)^2 lm}, (\overline{\overline{x}}^k)_{k=(S-1)^2 lm+1}^{(S-1)^2 lm+(S-1)l}).$$

Hence as β increases output of the artificial firms decreases and the utilities of output from artificial firms increases. Clearly, consumptions of the last

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$(S-1)$ commodities, productions, inputs of artificial production and the maximal value of the planner's stationary problem are independent of β. Then there exists β_0 such that if $\beta \in]0, \beta_0[$ then the modified utility functions have negative definite Hessian matrix for all $\overline{\overline{x}}$ with $|\overline{\overline{x}}| \leq a$, where a is the upper bound on consumption found in the previous paragraph.

For $\delta < 1$, the planner's problem for the overlapping generations economy with S periods of life, (PP), translates into the following objective for the optimal growth model

$$\max \quad \gamma^{-1} \sum_{i=1}^{(S-1)m} \overline{\overline{v}}_i(\overline{\overline{x}}_{i,-1}) + \sum_{t \in \mathbf{N}_0} \gamma^t \sum_{i=1}^{(S-1)m} \overline{\overline{u}}_i(\overline{\overline{x}}_{i,t})$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{(S-1)m} (\overline{\overline{x}}_{i,t} - \overline{\overline{\omega}}_i) = \sum_{j=1}^{(S-1)n} (\overline{\overline{y}}^1_{j,t} + \overline{\overline{y}}^2_{j,t}) + \sum_{k=1}^{(S-1)^2 lm} (\overline{\overline{z}}^1_{k,t} + \overline{\overline{z}}^2_{k,t}) \\[2ex] f_j(\overline{\overline{y}}^1_{j,t}, \overline{\overline{y}}^2_{j,t}) \leq 0 \text{ for all } j \in \{1, \ldots, (S-2)n\} \\[2ex] f_j(\overline{\overline{y}}^1_{j,t}, \overline{\overline{y}}^2_{j,t+1}) \leq 0 \text{ for all } j \in \{(S-2)n+1,, \ldots, (S-1)n\} \\[2ex] g_k(\overline{\overline{z}}^1_{k,t}, \overline{\overline{z}}^2_{k,t+1}) \leq 0 \text{ for all } k \\[2ex] \overline{\overline{K}}_{-1} \text{ is fixed,} \end{cases}$$

where $\gamma = \delta^{S-1}$. The case $\delta = 1$ can be stated easily with the help of the overtaking criterion.

In order to transform the optimal growth model with $(S-1)m$ consumers into an optimal growth model with one consumer, let the maximal aggregate utility of the consumers be the utility of the aggregate consumer, i.e. let

$$\overline{\overline{\omega}} = \sum_{i=1}^{(S-1)m} \overline{\overline{\omega}}_i,$$

$$\overline{\overline{u}}(x) = \max \sum_{i=1}^{(S-1)m} \lambda_i \overline{\overline{u}}_i(x_i), \text{ s.t. } \sum_{i=1}^{(S-1)m} x_i = x,$$

and let $\overline{\overline{v}}$ be defined as $\overline{\overline{u}}$. Then the optimal growth model with $(S-1)m$ consumers is transformed into an optimal growth model with one consumer.

The sequel of the proof of Theorem 1 depends on whether the government discounts the future or not.

In the case $\delta < 1$, according to [5, 6] the optimal growth model with 1 consumer has an equilibrium, $(\overline{\overline{p}}_t)_{t=-1}^{\infty}$, and the equilibrium allocation, $(\overline{\overline{x}}_t, \overline{\overline{y}}_t, \overline{\overline{z}}_t)_{t=-1}^{\infty}$, is Pareto optimal. Clearly the equilibrium allocation solves the planner's problem for the optimal growth model with one consumer.

When there is no discounting, [5, 6] cannot be applied. In this case it is useful to restate the model in its reduced form and use the fact that strict concavity of the reduced utility function directly follows from the strict concavity of production as well as of the original utility function. The other usual assumptions of the turnpike literature with no discounting follow directly from our assumptions and from the fact that feasible consumptions are bounded (see above). Once the existence of an optimal stationary path is proven, all other paths can be compared to it by means of a well defined "value loss" function. This function, which represents the loss of not being at the stationary state, is strictly concave. This fact implies that all paths converge to the stationary path and therefore that a strongly optimal path always exists. Theorem 6.1 in [14] or Theorem 9 in [10] formally prove this result which, in our setting ensure the existence of the strongly optimal path for $t \geq 0$. Finally, the value function being continuous it is then straightforward to complete this path to obtain a strongly optimal path for the problem with $t \geq -1$.

Implementation in the overlapping generations economy of the equilibrium for the optimal growth model has to be considered. Let $((x_t)_{t=-S+1}^{\infty}$, $(y_t)_{t \in \mathbf{N}_0})$ be the associated allocation for the overlapping generations economy - this allocation is found by going through the steps used in transforming the overlapping generations economy into an optimal growth economy in reverse order. The associated commodity prices $(p_t)_{t \in \mathbf{N}_0}$ are obtained by considering gradients of utility functions and of productions functions: 1. all gradients are normalized such that if two agents use or produce a strictly positive amount of some commodity then their normalized gradients are identical; 2. for all commodities which are used in strictly positive amounts by at least one consumer or firm the normalized gradients are used as prices and; 3. for all commodities which are not used in strictly positive amount by any consumer or firm the maximal gradients of the agents who have this commodity in their consumption or production sets are used as prices. Concerning 1. it is possible to normalize gradients as described because the allocation in

consideration is a solution to the planner's problem and this implies that different agents have identical marginal rates of substitution. Concerning 3. taking the price as the maximal normalized gradient implies that the price is not smaller than agents' normalized gradients for a commodity that they do not use. Note that all prices are different from zero and infinity because they are either normalized gradients or max over a finite set of normalized gradients.

The allocation $((x_t)_{t=-S+1}^{\infty}, (y_t)_{t\in\mathbf{N}_0})$ solves the planner's problem, (PP), because otherwise the equilibrium allocation, $(\overline{\overline{x}}_t, \overline{\overline{y}}_t, \overline{\overline{z}}_t)_{t=-1}^{\infty}$, would not be a solution to the planner's problem in the optimal growth model with one consumer. Profits can be taxed away by a suitable negative transfer to the firms[4]. In this case, the non-arbitrage condition, (NA), implies that the prices of assets are constant. Therefore, assets behave exactly like (bubble) money. Since their prices can be chosen arbitrarily, let $q_{j,t} = 0, j \in J$, and all $t \in \mathbf{N}_0$. The budget constraint is then

$$\sum_{s=1-t}^{S} p_{s+t-1} \cdot (x_{i,s+t-1}^s - \omega_i^s) = \sum_{s=1-t}^{S} \tau_{i,s+t-1}^s$$

for all $i \in I$ and all $t \in \{-S+1, \dots, -1\}$ and

$$\sum_{s=1}^{S} p_{s+t-1} \cdot (x_{i,s+t-1}^s - \omega_i^s) = \sum_{s=1}^{S} \tau_{i,s+t-1}^s$$

for all $i \in I$ and all $t \in \mathbf{N}_0$. Then $(p_t, q_t, \tau_t, \sigma_t)_{t\in\mathbf{N}_0}$ is an equilibrium with $((x_t)_{t=-S+1}^{\infty}, (y_t)_{t\in\mathbf{N}_0})$ as the associated equilibrium allocation and this allocation solves the problem of the government because it is Pareto optimal for the associated general equilibrium growth model.   Q.E.D.

Remark In theorem 1 the sum of the transfers to every consumer is determined while the profile of transfers over periods is undetermined. This fact is related to the irrelevance of government budget deficits when lump-sum taxes and transfers are available.

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[4]When $\delta = 1$, zero profits is a necessary condition for implementation of the strongly optimal path, but not for existence of the solution to the government's problem. On the other hand, these taxes are superfluous when there is discounting.

4.2 Proof of Theorem 2

The transformation of the stationary overlapping generation economy into an optimal growth model results in an optimal growth model that is stationary - except for the first period where utility functions differ. If the first period is disregarded then the planner's **stationary** problem is obtained, and in the case $\delta < 1$ it reads

$$\max \quad \sum_{t \in \mathbf{N}_0} \gamma^t \sum_{i=1}^{(S-1)m} \bar{\bar{u}}_i(\bar{\bar{x}}_{i,t})$$

$$\text{s.t.} \quad \begin{cases} \displaystyle\sum_{i=1}^{(S-1)m} (\bar{\bar{x}}_{i,t} - \bar{\bar{\omega}}_i) &=& \displaystyle\sum_{j=1}^{(S-1)n} (\bar{\bar{y}}_{j,t}^1 + \bar{\bar{y}}_{j,t}^2) + \sum_{k=1}^{(S-1)^2lm} (\bar{\bar{z}}_{k,t}^1 + \bar{\bar{z}}_{k,t}^2) \\[2ex] f_j(\bar{\bar{y}}_{j,t}^1, \bar{\bar{y}}_{j,t}^2) &\leq& 0 \text{ for all } j \in \{1, \ldots, (S-2)n\} \\[2ex] f_j(\bar{\bar{y}}_{j,t}^1, \bar{\bar{y}}_{j,t+1}^2) &\leq& 0 \text{ for all } j \in \{(S-2)n+1, , \ldots, (S-1)n\} \\[2ex] g_k(\bar{\bar{z}}_{k,t}^1, \bar{\bar{z}}_{k,t+1}^2) &\leq& 0 \text{ for all } k \\[2ex] \bar{\bar{K}}_0 \text{ is fixed.} \end{cases}$$

When $\delta = 1$, maximality is stated with the help of the overtaking criterion (see Definition 1). The planner's problem is studied in the literature on turnpikes where it is proven that a **stationary** solution exists.

Lemma 1 *Consider the planner's stationary problem then there exist $\delta^* \in [0, 1[$, $\bar{\bar{K}}^* \in \mathbf{R}_+^{(S-1)^2lm+(S-1)l}$ and $(\bar{\bar{x}}^*, \bar{\bar{y}}^*, \bar{\bar{z}}^*)$ such that if $\delta \in]\delta^*, 1]$ and*

$$\sum_{j=(S-2)n+1}^{(S-1)n} \bar{\bar{y}}_{j,0}^2 + \sum_{k=1}^{(S-1)^2lm} \bar{\bar{z}}_{k,0}^2 = \bar{\bar{K}}^*$$

and $(\bar{\bar{x}}_t, \bar{\bar{y}}_t, \bar{\bar{z}}_t)_{t \in \mathbf{N}_0}$ solves the planner's stationary problem then

$$(\bar{\bar{x}}_t, \bar{\bar{y}}_t, \bar{\bar{z}}_t) = (\bar{\bar{x}}^*, \bar{\bar{y}}^*, \bar{\bar{z}}^*)$$

for all $t \in \mathbf{N}_0$.

Proof For $\delta < 1$ it follows directly from [6]. When $\delta = 1$, the result follows from Theorem 6.1 in [14]. Q.E.D.

Under A.11 and A.12 a second lemma is also obtained.

Lemma 2 *Consider the planner's stationary problem and let $(\overline{\overline{x}}^*, \overline{\overline{y}}^*, \overline{\overline{z}}^*)$ be the associated stationary allocation. Then for every compact set,*

$$C \subset \mathbf{R}_{++}^{(S-1)^2 lm + (S-1)l},$$

there exists $\delta_C^* \in [0, 1[$ such that if $(\overline{\overline{x}}_t, \overline{\overline{y}}_t, \overline{\overline{z}}_t)_{t \in \mathbf{N}_0}$ solves the planner's stationary problem then

$$\lim_{t \to \infty} |(\overline{\overline{x}}_t, \overline{\overline{y}}_t, \overline{\overline{z}}_t) - (\overline{\overline{x}}^*, \overline{\overline{y}}^*, \overline{\overline{z}}^*)| = 0$$

for all $\overline{\overline{K}} \in C$.

Proof When $\delta < 1$, the proof is an application of theorem (4.5) in [6]. Indeed, since C is compact and $\overline{\overline{K}} \in C$ there exists an upper bound, $a \in \mathbf{R}$, such that if $(\overline{\overline{x}}_t, \overline{\overline{y}}_t, \overline{\overline{z}}_t)_{t \in \mathbf{N}_0}$ is an equilibrium allocation associated with $\overline{\overline{K}} \in C$ then

$$\sup_{t \in \mathbf{N}_0} |\overline{\overline{x}}_t| \leq a.$$

Therefore there exists β_C such that if $\beta \in]0, \beta_C[$ then the modified utility functions have negative definite Hessian matrix for all $\overline{\overline{x}}$ with $|\overline{\overline{x}}| \leq a$. Hence for $\beta \in]0, \beta_C[$ theorem (4.5) in [6] can be applied to the modified optimal growth model in order to obtain lemma 2.

The case with no discounting, $\delta = 1$, can be treated similarly. The result then follows from Theorem 8 in [10]. Q.E.D.

Proof of Theorem 2 Assumptions A.11 and A.12 ensures that the initial stock of output for the stationary problem is strictly positive in all coordinates and that all producible commodities are produced at the stationary solution. Then, for the optimal growth model the planner's stationary problem has the turnpike property according to lemma 2. Clearly, this implies that the planner's problem has the turnpike property. Then the overlapping generations economy has the turnpike property by construction. Q.E.D.

5   Concluding Remarks

In the present paper the optimal policy of a government with a utilitarian social welfare function, that is a possibly discounted sum of generations' social welfare functions where these are weighted sums of the utility functions of the consumers in the generations, is considered. Both situations with and without time discounting by the government are considered in order to include equilibria in which future consumption matters. Through a transformation of the government's problem into an optimal growth model it is shown that if the discount factor is sufficiently close to one then the government stabilizes the economy, i.e. the equilibrium has the turnpike property.

It should be emphasized that the limit of optimal policies when the discount rate goes to zero coincides with the solution found for zero discounting (see [9] for a proof). This fact provides a way to obtain the optimal policy when there is no discounting which is sometimes easier than looking directly at the overtaking criterion.

The focus of the paper is on the characterization of the global optimum of the planner. Of course whether or not it is implementable depends on the available information and instruments, so that the optimum may be in this sense infeasible. Indeed, if the government has incomplete information about the types of individual consumers, then fiscal policies of the form considered in the present paper become infeasible. However, with a suitable notion of distance at hand, a similar analysis allows to select the best policy given the constraints.

The present model can also be interpreted as an overlapping generation economy with bequests. In this case the welfare weights are endogenously determined by the market and the turnpike property concerns the competitive equilibria reached under laissez faire. The condition for convergence is then that consumers are sufficiently concerned by future generations.

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