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Clubs and the Market*

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Abstract

This paper defines a general equilibrium model with exchange and club formation. Agents trade multiple private goods widely in the market, can belong to several clubs, and care about the characteristics of the other members of their clubs. The space of agents is a continuum, but clubs are finite. It is shown that (i) competitive equilibria exist, and (ii) the core coincides with the set of equilibrium states. The central subtlety is in modeling club memberships and expressing the notion that membership choices are consistent across the population.

JEL Classifications D2, D5, H4

Keywords: Clubs, Continuum Models, Public Goods, Core, Club Equilibrium

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1 Introduction

Consumption is typically a social activity. The company we keep affects our demand for private goods, and our consumption of private goods affects the company we seek. General equilibrium theory focuses on the interactions of consumers with the market, largely ignoring the social aspect of consumption. Club theory focuses on the social activity of consumption, largely ignoring the interactions of individuals with the market. This paper integrates club theory and general equilibrium theory, building a framework in which both markets and relationships matter.

We build here a competitive model of clubs, and thus follow a long tradition initiated by Buchanan (1965). (See Cornes and Sandler (1996) and Kurz (1997) for overviews of the literature.) Marriages, gyms, academic departments, golfing foursomes, and restaurant clienteles are the sorts of clubs we have in mind (but not large organizations like political jurisdictions in the sense of Tiebout (1956)). In addressing competition in club economies, the existing literature has treated economies with a finite number of agents, but such an assumption does not lead to an entirely satisfactory model of clubs or of competition. Club choice is intrinsically indivisible (one joins a club or one does not); as a consequence, the core of a club economy with a finite number of agents may well be empty. Moreover, in a finite economy individuals will generally have market power, so there is no reason to view such economies as perfectly competitive (even when the core is not empty). The clubs literature for the most part has concentrated on special circumstances or approximate notions of core and competition.[1] Our approach is quite different: following Aumann (1964), we build a general equilibrium model with a *continuum* of agents. In this framework, we define a decentralized notion of price-taking equilibrium parallel to the usual notion in exchange economies, show that (exact) equilibrium always exists, that equilibrium allocations are Pareto optimal and belong to the core (so that the core is always non-empty), and that club economies pass a standard test of perfect competition, the coincidence

[1]See Ellickson (1973, 1979), Scotchmer and Wooders (1987), Conley and Wooders (1995), Gilles and Scotchmer (1997) and Scotchmer (1997) for instance.

of the core with the set of equilibrium allocations.[2,3]

Because the theory proposed here is intended as a competitive theory of clubs, we require that clubs be small relative to society as a whole. In our continuum framework, the expression of this requirement is that clubs are finite; clubs are therefore comparable in size to individuals but infinitesimal relative to society. Because our description of a club includes the number of members in the club and their characteristics, however, it is meaningful to ask about the size and composition of clubs that form at equilibrium. Similarly, it is meaningful to ask about the relative numbers of clubs of different types that form at equilibrium (although the absolute number will always be zero or infinite).

We describe a club type by the number and the characteristics of its members and the activity in which the club is engaged. We allow formation of many possible types of clubs and trading of many private goods. The latter is especially important, because agents trade with the market and not just within clubs; if there is a single private good, there will be no trading with the market (only transfers within clubs). We allow each agent to belong to many clubs simultaneously.

Our model is competitive, despite the presence of externalities, because clubs are small and external effects are encapsulated within clubs. We assume that the number of external characteristics of individuals (the characteristics that matter to others) and the number of potential types of club are finite, and that the maximum number of clubs an individual can choose is finite. These assumptions guarantee that our choice space is finite dimensional, greatly simplifying the model and the proofs — but they are not

[2] Cole and Prescott (1997) provide a continuum model in a spirit similar to the present paper, but the objects of choice in that paper are lotteries over bundles of private goods and club memberships. Lotteries overcome the indivisibility problem by making choices divisible. Because we view the indivisibility as fundamental, we prefer to address it head-on. Closer to the present work is the unpublished paper of Makowski (1978), who interprets clubs as organizations formed by entrepreneurs.

[3] Our companion paper, Ellickson, Grodal, Scotchmer and Zame (1997b), treats approximate equilibrium and approximate decentralization in large finite economies.

required for competition. Useful extensions would allow an infinite number of differentiated private goods, an infinite number of differentiated external characteristics and an infinite number of club types, and allow agents to choose an infinite number of club memberships — but such extensions will require an infinite dimensional choice space, and both the model and the proofs will necessarily be much more complicated.[4]

A key to our approach is that we define a club membership as an opening in a specific type of club, available to agents with a specified characteristic, and treat club memberships and private goods in parallel fashion as objects of choice. As in classical general equilibrium theory, where the description of a private good includes all the relevant aspects, so here the description of a club membership includes all the relevant aspects: number of other members, relevant characteristics of the other members, relevant characteristics of the member in question, purpose of the club, resources necessary to carry out that purpose, and institutional arrangements within the club. Just as a small chocolate bar, a large chocolate bar, and a large chocolate bar with almonds are different goods and may have different prices, so membership in a swimming pool club with 20 members, a swimming pool club with 40 members and a tennis club with 40 members are different memberships and may have different prices. Indeed, if gender matters then admission for a female in a coeducational school is different from admission for a male in the same school, and may be priced differently.

Despite the parallel treatment of club memberships and private goods, there are important differences. First, and most importantly, the feasibility condition for trading of club memberships is different from the feasibility condition for trading of private goods. For private goods, feasibility means that demand be equal to supply. For club memberships, feasibility means that choices must be consistent across the population. For example, if a third of the population are women married to men, then a third of the population must be men married to women. This consistency condition must hold simultaneously for all types of clubs. Second, the prices of club memberships

[4]See Mas-Colell (1975) and Ostroy and Zame (1994) for competitive models of exchange economies with infinitely many differentiated commodities.

can be positive, negative or zero (whereas prices of private goods must be positive). This is because club membership prices have two components: a (non-negative) share of the resource to form the club, and a transfer to other members of the club, which may be of any sign. These transfers internalize the externalities of club membership; in equilibrium, the value placed on these externalities will depend on tastes and on the relative supply of various characteristics. Third, club membership is inherently indivisible, with the consequence that consumption sets and preferred sets are not convex. Our continuum framework handles this difference very smoothly.[5]

Our proofs follow lines that are typical of general equilibrium theory, but with many subtleties. The central subtlety is in accommodating the club consistency condition, which has no analog in the general equilibrium literature. Another subtlety arises in the proof of core equivalence, where the separation theorem produces the decentralizing price; this price has both private good and club membership components, and we must be sure that the private good components are not identically zero. A third subtlety arises in the proof of existence of equilibrium, because the nature of club membership prices (which may be positive, negative or zero) means that there is no obvious compact space of prices in which to apply a fixed point argument.

Following this Introduction, Section 2 provides motivating examples. The formal model is described in Section 3. Section 4 discusses welfare theorems and the core. Section 5 addresses the equivalence of the core and the set of equilibrium states and Section 6 addresses the existence of equilibrium. The text outlines the proofs; details are in Section 7.

2 Examples

This Section presents three examples illustrating various aspects of our framework. The first example uses a simple setting similar to Buchanan (1965)

[5]The convexifying effect of the continuum was first noted by Aumann (1964); see Mas-Colell (1977) for a model with indivisible private goods.

to highlight three important features of our approach: 1) clubs are infinitesimal in comparison to society as a whole but, because size is part of the description of a club type, it is meaningful to talk about the optimal size of a club; 2) clubs of any particular type can be replicated with constant returns to scale (i.e., to form two clubs of a particular type requires just twice as many individuals and twice the inputs of commodities as to form a single club); 3) congestion or a finite bound on club size is crucial: without such assumptions, equilibrium may fail to exist.

Example 2.1 Crowding Consider an economy with a continuum of consumers uniformly distributed on $[0, 10]$. There is a single private good; the endowment of consumer k is $e_k = k$. In addition to the private good, consumers have the option of constructing and using a swimming pool, either alone or in a club. Building a pool requires 6 units of the private good. A consumer who consumes x units of the private good derives utility $u(x; 0) = x$ if using no pool and $u(x; n) = 4x/n$ if belonging to a swimming pool club with n members. No one belongs to more than one club.

Normalize the price of the private good to 1, so consumer k has wealth k. Because only the size of clubs (and not their composition) matters to consumers, pool costs will be shared equally within each club, so the price of a membership in a club of size n is an equal share of the cost of production: $q_n = 6/n$. Choosing no pool will yield consumer k the utility k; sharing a pool in a club with n members will yield utility

$$u(k - q_n; n) = \frac{4}{n}\left(k - \frac{6}{n}\right)$$

In equilibrium, consumers stratify by wealth: the wealthiest consumers, those with wealth in $(9, 10]$, have a pool of their own; consumers with wealth in $(6, 9]$ share a pool with one other consumer; the poorest consumers, with wealth in $[0, 6]$, consume the private good but do not enjoy the use of a pool. Clubs of size greater than 2 do not form in equilibrium.

Note how aversion to crowding provides an effective limit on club size; in the absence of any aversion to crowding, larger and larger clubs might be desirable and equilibrium might not exist. Suppose, for instance, that the utility derived from membership in a club of size n were $u(x; n) = 4x$ (rather than $u(x; n) = 4x/n$) and clubs could be of any size. Swimming pools would then be pure public goods, no equilibrium would exist, and a competitive theory of their provision would be inappropriate. ♣

Our second example illustrates the importance of viewing and pricing club memberships for individuals with different characteristics as different commodities. The example is motivated by Arrow's (1972) commentary on Becker's (1957) discussion of market prices and segregation. The example affirms Arrow's insight that, in the absence of price discrimination, profit maximization leads to segregation, but it suggests another possibility as well: integration with price discrimination.

Example 2.2 Segregation Consider an economy with a continuum of consumers uniformly distributed on $[0, 1]$. Consumers in $[0, .3)$ are blue, consumers in $[.3, 1]$ are green. There is a single private good; each consumer has endowment 2. In addition to the private good, consumers have the option of constructing and using a duplex apartment. Building a duplex requires 2 units of the private good. An individual who consumes x units of the private good and no housing derives utility

$$u_B(x; 0) = u_G(x; 0) = x$$

while an individual who consumes x units of the private good and a duplex apartment derives utility according to his own characteristic and that of the occupant of the other half of the duplex:

$$\begin{array}{rclcrcl} u_B(x; BB) & = & 4x & \qquad & u_B(x; BG) & = & 6x \\ u_G(x; GG) & = & 6x & \qquad & u_G(x; BG) & = & 4x \end{array}$$

using the obvious notation. (We omit some profiles because a consumer of type B cannot occupy a GG duplex, and so forth.)

Write $q_\omega(BB), q_\omega(BG), q_\omega(GG)$ for the prices paid by a consumer of type $\omega = B, G$ for the various kinds of housing. At equilibrium, housing prices for each type of duplex must sum to the production cost of 2; in particular $q_B(BB) = q_G(GG) = 1$. At these prices, blue consumers can obtain utility 2 by choosing no housing, utility 4 by choosing a segregated duplex, and utility $6(2 - q_B(BG))$ by choosing an integrated duplex. Green consumers can obtain utility 2 by choosing no housing, utility 6 by choosing a segregated duplex, and utility $4(2 - q_G(BG))$ by choosing an integrated duplex. If any integrated housing is chosen at equilibrium, optimization by blue consumers entails that $6(2 - q_B(BG)) \geq 4$ and optimization by green consumers entails that $4(2 - q_G(BG)) \geq 6$. Equivalently, $q_B(BG)) \leq 4/3$ and $q_G(BG) \leq 1/2$. Hence $q_B(BG) + q_G(BG) \leq 11/6$. This contradicts the fact that housing prices sum to the cost of production. At the unique equilibrium, therefore, all consumers live in segregated housing. Prices for segregated housing are $q_B(BB) = q_G(GG) = 1$; prices for (undemanded) integrated housing are indeterminate, constrained only by the requirements

$$q_B(BG) \geq \frac{4}{3}\ ,\ q_G(BG) \geq \frac{1}{2}\ ,\ q_B(BG) + q_G(BG) = 2$$

However, segregation is not a necessary conclusion. Suppose that $u_B(x; BG) = 10x$, with all else remaining the same. If no integrated housing is chosen at equilibrium, optimization by blue consumers entails that $10(2 - q_B(BG)) \leq 4$ and optimization by green consumers entails that $4(2 - q_G(BG)) \leq 6$. These inequalities are inconsistent with the fact that housing prices sum to the cost of production, so we conclude some blue consumers and some green consumers live in integrated housing. Because there are more green consumers than blue consumers, some green consumers must live in segregated housing; equating utilities for all green consumers we conclude that housing prices are $q_B(BB) = q_G(GG) = 1, q_B(BG) = 3/2, q_G(BG) = 1/2$. At these prices, green consumers are indifferent between integrated housing and segregated housing, but blue consumers strictly prefer integrated housing. At equilibrium, 3/7 of all green consumers live in integrated housing and the remainder live in segregated housing; blue consumers pay a premium to live in integrated housing. ♣

Our third example shows why allowing for more than one private good matters: Because agents trade with the market, there is an interaction between the demand for club memberships and the demand for private goods; as a result, clubs can be priced out of existence. This phenomenon cannot occur when there is only one private good.

Example 2.3 Marriage and the Market Consider an economy with a continuum of consumers uniformly distributed on $[0, 1]$. Consumers in $[0, \beta)$ are male, and consumers in $[\beta, 1]$ are female, where $0 < \beta < 1$. There are 2 private goods; each consumer has endowment $(10, 10)$. Utility functions are (s means single and m means married):

$$u_M(x_1, x_2; s) = x_1 \qquad u_F(x_1, x_2; s) = x_2$$

$$u_M(x_1, x_2; m) = u_F(x_1, x_2; m) = \frac{5}{2}\sqrt{x_1 x_2}$$

We solve for equilibrium as a function of β, the proportion of males. Consider first the case $\beta < 1/2$. Write q_M, q_F for the gender-specific marriage prices. Because marriage is costless, $q_M + q_F = 0$ so marriage prices represent pure transfers: one sex subsidizes the other. To solve for the equilibrium, we hypothesize that marriage is an equilibrium outcome and work backwards to find prices. Normalize private good prices to sum to 1, so that everyone has wealth 10. Unmarried females (males) buy only good 2 (good 1). Married males and females spend q_M, q_F (respectively) and divide their remaining income between the two private goods. Assume that the "number" of married males is $\bar{m}$; $0 \le \bar{m} \le \beta$. Because utilities within marriage are Cobb-Douglas, market clearing for private goods yields:

$$\begin{aligned} (\beta - \bar{m})\frac{10}{p_1} + \bar{m}\left[\frac{10 - q_M}{2p_1} + \frac{10 - q_F}{2p_1}\right] &= 10 \\ (1 - \beta - \bar{m})\frac{10}{p_2} + \bar{m}\left[\frac{10 - q_M}{2p_2} + \frac{10 - q_F}{2p_2}\right] &= 10 \end{aligned}$$

This uniquely determines private good prices, which are independent of the "number" of marriages: $p_1 = \beta, p_2 = 1 - \beta$. Suppose that some males are married. Because there are more females than males, some females necessarily remain single. Married consumers must receive at least as much utility as if single, and females must be indifferent between the two states (because some females are single at equilibrium). Thus

$$\frac{10}{1-\beta} = \frac{5}{2}\sqrt{\left(\frac{10-q_F}{2\beta}\right)\left(\frac{10-q_F}{2(1-\beta)}\right)}$$

$$\frac{10}{\beta} \leq \frac{5}{2}\sqrt{\left(\frac{10-q_M}{2\beta}\right)\left(\frac{10-q_M}{2(1-\beta)}\right)}$$

Hence we can solve for marriage prices:

$$q_F = 10 - 8\sqrt{\frac{\beta}{1-\beta}} \quad \text{and} \quad q_M \leq 10 - 8\sqrt{\frac{1-\beta}{\beta}}$$

Because $q_M + q_F = 0$ these equations entail that $\beta \geq 1/5$. In the range $1/5 \leq \beta < 1/2$ we can solve uniquely for marriage prices, obtaining $q_F = 10 - 8\sqrt{\frac{\beta}{1-\beta}}$, $q_M = -q_F$. If $\beta = 1/5$ the number of married males is indeterminate; if $1/5 < \beta < 1/2$ all males are married. If $0 < \beta < 1/5$, the hypothesis that some males are married leads to a contradiction, so at equilibrium there are no marriages — but there are equilibria with no marriages.

For $\beta > 1/2$, the analysis is symmetrical, with the roles of men and women reversed. Finally, when $\beta = 1/2$ all males and females are married, but marriage prices are indeterminate.

We summarize the equilibrium correspondence by describing the "number" $\bar{m}$ of married males and the price q_F females pay to enter marriage. Note that $\bar{m}$ is indeterminate for $\beta = 1/5, 4/5$ and q_F is indeterminate for

$0 < \beta < 1/5$, $\beta = 1/2$ and $4/5 < \beta < 1$:

$$\begin{array}{llll}
0 < \beta < 1/5 & : \ \bar{m} = 0 & , & q_F \in \left[+10 - 8\sqrt{\frac{\beta}{1-\beta}}, -10 + 8\sqrt{\frac{1-\beta}{\beta}}\right] \\
\beta = 1/5 & : \ \bar{m} \in [0, \beta] & , & q_F = +10 - 8\sqrt{\frac{\beta}{1-\beta}} \\
1/5 < \beta < 1/2 & : \ \bar{m} = \beta & , & q_F = +10 - 8\sqrt{\frac{\beta}{1-\beta}} \\
\beta = 1/2 & : \ \bar{m} = \beta & , & q_F \in [-2, +2] \\
1/2 < \beta < 4/5 & : \ \bar{m} = 1 - \beta & , & q_F = -10 + 8\sqrt{\frac{1-\beta}{\beta}} \\
\beta = 4/5 & : \ \bar{m} \in [0, 1 - \beta] & , & q_F = -10 + 8\sqrt{\frac{1-\beta}{\beta}} \\
4/5 < \beta < 1 & : \ \bar{m} = 0 & , & q_F \in \left[+10 - 8\sqrt{\frac{\beta}{1-\beta}}, -10 + 8\sqrt{\frac{1-\beta}{\beta}}\right]
\end{array}$$

When the sex ratio is not extreme, the sex that is in short supply is subsidized to enter marriage. When the sex ratio is extreme, the subsidy that would be required is so large that the more populous sex prefers to remain single; marriage is priced out of existence. ♣

For an example illustrating how easily our approach handles the possibility that individuals choose to belong to more than one club, see our working paper Ellickson, Grodal, Scotchmer and Zame (1997a).

3 Club Economies

3.1 Private Goods

Throughout, there are $N \geq 1$ perfectly divisible, publicly traded private goods; thus the space of private goods is $\mathbf{R}^N$. For $x, x' \in \mathbf{R}^N$, we write $x \geq x'$ to mean $x_i \geq x'_i$ for each i, $x > x'$ to mean that $x \geq x'$ but $x \neq x'$, and $x >> x'$ to mean that $x_i > x'_i$ for each i. We write $|x| = \sum_{n=1}^{N} |x_n|$.

3.2 Clubs

We will describe a *club type* by the number and characteristics of its members and the activity in which the club is engaged.

Let Ω be a finite set of *external characteristics* of potential members of a club. An element $\omega \in \Omega$ is a complete description of the characteristics of an individual that are relevant for the *other* members of a club. We call these characteristics "external" because they are the aspects of agents that create "externalities" within clubs, and because they are observable. Such characteristics might include sex, intelligence, appearance — even tastes and endowments, to the extent that such characteristics can be observed.

A *profile* (of a club) is a function $\pi : \Omega \to \mathbf{Z}_+ = \{0, 1, \ldots\}$ describing the external characteristics in a club. For $\omega \in \Omega$, $\pi(\omega)$ represents the number of members of the club having external characteristic ω. For π a profile, write $|\pi| = \sum_{\omega \in \Omega} \pi(\omega)$ for the total number of members.

There is a finite set Γ of *activities* available to a profile of agents. (Γ is simply an abstract set.) We interpret activities as public projects in the sense of Ellickson (1979) and Mas-Colell (1980). An activity may incorporate a shared facility, a code of behavior or a publicly professed ideology. Agents may rank activities differently, and an individual's ranking may depend on his/her consumption of private goods. Activities are not traded.

A *club type* is a pair $c = (\pi, \gamma)$ consisting of a profile and an activity $\gamma \in \Gamma$. We take as given a finite set of possible club types $\mathbf{Clubs} = \{(\pi, \gamma)\}$. (In particular, there is a bound on club size.) Formation of the club (π, γ) requires a total input of private goods equal to $\mathbf{inp}(\pi, \gamma) \in \mathbf{R}^N_+$.

A *club membership* is an opening in a particular club type for an agent of a particular external characteristic; i.e., a triple $m = (\omega, \pi, \gamma)$ such that $(\pi, \gamma) \in \mathbf{Clubs}$ and $\pi(\omega) \geq 1$. (An agent can belong to a club only if the description of that club type includes one or more members with his/her external characteristics.) Write $\mathcal{M}$ for the set of club memberships.

Each agent may choose to belong to many clubs or to none. A *list* is a function $\ell : \mathcal{M} \to \{0, 1, \ldots\}$, where $\ell(\omega, \pi, \gamma)$ specifies the number of memberships of type (ω, π, γ). Write:

$$\mathbf{Lists} = \{\ell : \ell \text{ is a list }\}$$

for the set of lists. **Lists** is a set of functions from $\mathcal{M}$ to $\{0, 1, \ldots\}$, but we frequently view it as a subset of $\mathbf{R}^{\mathcal{M}}$, which is the set of functions from $\mathcal{M}$ to $\mathbf{R}$. We also assume throughout that there is an exogenously given upper bound M on the number of memberships an individual may choose.

3.3 Agents

The set of agents is a nonatomic finite measure space $(A, \mathcal{F}, \lambda)$. That is, A is a set, $\mathcal{F}$ is a σ-algebra of subsets of A and λ is a non-atomic measure on $\mathcal{F}$ with $\lambda(A) < \infty$.

A complete description of an agent $a \in A$ consists of his/her external characteristics, choice set, endowment of private goods and utility function.[6] An external characteristic is an element $\omega_a \in \Omega$. The choice set X_a specifies the feasible bundles of private goods and club memberships, so $X_a \subset \mathbf{R}^N \times \mathbf{Lists}$. For simplicity, we assume the only restriction on private good consumption is that it be non-negative, so $X_a = \mathbf{R}^N_+ \times \mathbf{Lists}(a)$ for some subset $\mathbf{Lists}(a) \subset \mathbf{Lists}$.[7] We assume that an individual can only belong to a club type offering memberships with his/her external characteristic; formally, $\ell(\omega, \pi, \gamma) = 0$ if $l \in \mathbf{Lists}(a)$, $(\omega, \pi, \gamma) \in \mathcal{M}$ and $\omega \neq \omega_a$. By assumption, the number of memberships an individual may choose is bounded by M, so $|\ell| \leq M$ for each $\ell \in \mathbf{Lists}(a)$. The endowment is $e_a \in \mathbf{R}^N_+$. The utility function is defined over private goods consumptions and club memberships and is thus a mapping $u_a : X_a \to \mathbf{R}$. We assume throughout that utility functions $u_a(\cdot, \ell)$ are continuous and strictly monotone in private goods.

3.4 Club Economies

A *club economy* $\mathcal{E}$ is a mapping $a \mapsto (\omega_a, X_a, e_a, u_a)$ for which:

- the external characteristic mapping $a \mapsto \omega_a$ is a measurable function

[6] We find it convenient to use utility functions rather than preferences.

[7] Thus we incorporate into consumption sets various kinds of restrictions on club memberships. For instance, we may forbid some consumers to enter some club types.

- the consumption set correspondence $a \mapsto X_a$ is a measurable correspondence
- the endowment mapping $a \mapsto e_a$ is an integrable function
- the utility mapping $(a, x, \ell) \mapsto u_a(x, \ell)$ is a (jointly) measurable function (of all its arguments)[8]

We assume that the *aggregate endowment* $\bar{e} = \int_A e_a \, d\lambda(a)$ is strictly positive, so all private goods are represented in the aggregate.

3.5 States

A *state* of a club economy is a measurable mapping

$$(x, \mu) : A \to \mathbf{R}^N \times \mathbf{R}^{\mathcal{M}}$$

A state describes choices for each individual agent, ignoring feasibility at the level of the individual and at the level of society. *Individual feasibility* means $(x_a, \mu_a) \in X_a$. *Social feasibility* entails market clearing for private goods and consistent matching of agents.

We say that a membership vector $\bar{\mu} \in \mathbf{R}^{\mathcal{M}}$ is *consistent* if for every club type $(\pi, \gamma) \in$ **Clubs**, there is a real number $\alpha(\pi, \gamma)$ such that

$$\bar{\mu}(\omega, \pi, \gamma) = \alpha(\pi, \gamma)\pi(\omega)$$

for each $\omega \in \Omega$. (The coefficient $\alpha(\pi, \gamma)$ may be interpreted as the "number" of clubs of type (π, γ) accounted for in $\bar{\mu}$.) A choice function $\mu : B \to$ **Lists** is *consistent for* B if the corresponding aggregate membership vector $\bar{\mu} = \int_B \mu_a d\lambda(a) \in \mathbf{R}^{\mathcal{M}}$ is consistent. (The aggregate membership vector $\bar{\mu}$ counts the "number" of memberships in each club type chosen by individuals in B

[8]This measurability requirement is equivalent to the usual requirement on measurability of preferences.

of each characteristic. Consistency is the requirement that these numbers are in the same proportion as in the club types themselves.) Write

$$\mathbf{Cons} = \{\bar{\mu} \in \mathbf{R}^{\mathcal{M}} : \bar{\mu} \text{ is consistent } \}$$

Cons is a subspace of $\mathbf{R}^{\mathcal{M}}$. Because individuals choose non-negative numbers of club memberships, feasible club choices induce aggregate membership vectors in the positive part $\mathbf{Cons}_+ \subset \mathbf{Cons}$.

The state (x, μ) is *feasible for* the measurable subset $B \subset A$ if it satisfies the following requirements:

(i) **Individual Feasibility** $(x_a, \mu_a) \in X_a$ for each $a \in B$

(ii) **Material Balance**

$$\int_B x_a \, d\lambda(a) \; + \; \int_B \sum_{(\omega,\pi,\gamma)\in\mathcal{M}} \frac{1}{|\pi|}\mathbf{inp}(\pi,\gamma)\, \mu_a(\omega,\pi,\gamma)\, d\lambda(a) = \int_B e_a \, d\lambda(a)$$

(iii) **Consistency** $\int_B \mu_a \, d\lambda(a)$ is consistent.

The state (x, μ) is *feasible* if it is feasible for the set A itself.

A state of the economy will generally have "many" clubs of each club type. Because members of a club care only about the external characteristics of other members, and not about their identities, it is not necessary to distinguish different clubs of the same club type.

3.6 Pareto Optimality and the Core

We distinguish "weak" and "strong" notions of Pareto optimality and the core, and give a condition under which they coincide.

We say a feasible state (x, μ) is *weakly Pareto optimal* if there is no feasible state (x', μ') such that $u_a(x'_a, \mu_a) > u_a(x_a, \mu_a)$ for almost all $a \in A$; we say (x, μ) is *strongly Pareto optimal* if there is no feasible state (x', μ')

such that $u_a(x'_a, \mu'_a) \geq u_a(x_a, \mu_a)$ for almost all $a \in A$ and $u_{a'}(x'_{a'}, \mu'_{a'}) > u_{a'}(x_{a'}, \mu_a)$ for all a' in some subset $A' \subset A$ having positive measure. We say (x, μ) is in the *weak core* if there is no subset $B \subset A$ of positive measure and state (x', μ') that is feasible for B such that $u_b(x'_b, \mu'_b) > u_b(x_b, \mu_b)$ for almost every $b \in B$; we say (x, μ) is in the *strong core* if there is no subset $B \subset A$ of positive measure and state (x', μ') that is feasible for B such that $u_b(x'_b, \mu'_b) \geq u_b(x_b, \mu_b)$ for every $b \in B$ and $u_{b'}(x'_{b'}, \mu'_{b'}) > u_{b'}(x_{b'}, \mu_{b'})$ for all b' in some subset $B' \subset B$ having positive measure. The strong Pareto set is a subset of the weak Pareto set, and the strong core is a subset of the weak core. The following assumption, adapted from Gilles and Scotchmer (1997), guarantees that the strong and weak Pareto sets coincide and that the strong and weak cores coincide.

We say that *endowments are desirable* if for every agent a and every list $\ell \in \mathbf{Lists}(a)$, $u_a(e_a, 0) > u_a(0, \ell)$. That is, each agent would prefer to remain single and consume his endowment rather than to belong to any feasible set of clubs and consume no private goods.[9]

Proposition 3.1 *If endowments are desirable, then the weak and strong Pareto sets coincide and the weak and strong cores coincide.*

For the proof, see our working paper: Ellickson, Grodal, Scotchmer and Zame (1997a). When endowments are desirable, we omit modifiers and refer unambiguously to Pareto optimality and the core.

3.7 Equilibrium

Competitive prices will be $(p, q) \in \mathbf{R}^N_+ \times \mathbf{R}^{\mathcal{M}}$; p is a vector of prices for private goods and q is a vector of prices for club memberships. Because utility functions are assumed monotone in private goods, the prices of private

[9] Desirability of endowments is weaker than the *indispensability* assumption of Mas-Colell (1980), which in our framework would be $u_a(0, \ell) = \min_{(x^*, \ell^*) \in X_a} u_a(x^*, \ell^*)$ for every $\ell \in \mathbf{Lists}(a)$.

goods will be non-negative, but prices of club memberships may be positive, negative or zero.

A *club equilibrium* consists of a feasible state (x, μ) and prices $(p, q) \in \mathbf{R}^N_+ \times \mathbf{R}^{\mathcal{M}}, p \neq 0$ such that

(1) **Budget Feasibility for Individuals** For almost all $a \in A$,

$$(p, q) \cdot (x_a, \mu_a) = p \cdot x_a + q \cdot \mu_a \leq p \cdot e_a$$

(2) **Optimization** For almost all $a \in A$:

$$(x'_a, \mu'_a) \in X_a \text{ and } u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a) \Rightarrow p \cdot x'_a + q \cdot \mu'_a > p \cdot e_a$$

(3) **Budget Balance for Club Types** For each $(\pi, \gamma) \in$ **Clubs**:

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot \mathbf{inp}(\pi, \gamma)$$

Thus, at an equilibrium, individuals optimize subject to their budget constraints and the sum of membership prices in a given club type is just enough to pay for the inputs to clubs of that type.

A *club quasi-equilibrium* satisfies (1), (3) and (2′) instead of (2):

(2′) **Quasi-Optimization** For almost all $a \in A$:

$$(x'_a, \mu'_a) \in X_a \text{ and } u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a) \Rightarrow (p, q) \cdot (x'_a, \mu'_a) \geq p \cdot e_a$$

That is, nothing that is feasible and strictly preferred can cost strictly less than agent a's wealth. An equilibrium is necessarily a quasi-equilibrium.

In the exchange case, the possibility that quasi-equilibrium is not an equilibrium is frequently viewed as a mere technical problem, and can be ruled out by various simple assumptions (such as strictly monotone preferences and strictly positive aggregate endowments). In the presence of indivisibilities (such as club memberships), however, the issue is more subtle. The following example illustrates the problems that may arise when private goods are used as inputs to club activities. (See Gilles and Scotchmer (1997) for an example illustrating the problems that may arise when endowments are not desirable.)

Example 3.2 There are two private goods, a single external characteristic ω and a single club type $c = (2, \gamma)$ consisting of two people, and requiring inputs $\mathbf{inp}(c) = (2, 0)$. Agents can choose at most one club membership. All agents are identical, with endowment $(1, 1)$, and utility function:

$$\begin{aligned} u(x, 0) &= 1 - e^{-x_1 - x_2} \\ u(x, \delta_{(\omega,c)}) &= \sqrt{x_1} + \sqrt{x_2} \end{aligned}$$

where $\delta_{(\omega,c)}$ is the list with unique membership (ω, c). Because endowments are desirable, the weak and strong cores coincide. In a core state, all agents belong to clubs and consume $x = (0, 1)$, so the entire supply of good 1 is used as input to the club activity. However, this state is not an equilibrium because the marginal rate of substitution of good 1 for good 2 is infinite, so the price ratio p_1/p_2 would have to be infinite also. On the other hand, the state is a quasi-equilibrium with prices $p = (1, 0)$, $q(\omega, c) = 1$. (This is not an equilibrium, because good 2 is free and every agent desires more of it.) ♣

In the familiar exchange setting, a quasi-equilibrium may fail to be an equilibrium if some agents are in the "minimum expenditure situation:" consumptions require expenditures exactly equal to wealth, and slightly smaller expenditures are not possible. As the example above illustrates, this situation can arise easily in club economies because private goods are used as inputs to club activities, and club choices are indivisible. The following assumption (cf. Mas-Colell (1985)) is one of several that will guarantee that a quasi-equilibrium is an equilibrium.

Let $\mathcal{E}$ be a club economy and let (x, μ) be a feasible state. Write δ_j for the consumption bundle consisting of one unit of good j and nothing else. Say that (x, μ) is *club linked* if whenever $I \cup J = \{1, \ldots, N\}$ is a partition of the set of private goods and $x_{ai} = 0$ for all $i \in I$ and almost all $a \in A$, then for almost all $a \in A$ there exist $r \in \mathbf{R}_+$, $j \in J$ such that

$$u_a(e_a + r\delta_j, 0) > u_a(x_a, \mu_a)$$

That is, if, as in Example 3.2, the entire social endowment of the private goods in I is used to produce club activities, then for almost all agents a, there is some good $j \notin I$ and some sufficiently large level of consumption of good j such that agent a would prefer consuming his endowment together with this large level of good j, and belong to no clubs, rather than consume the bundle x_a in the club memberships μ_a. Say that $\mathcal{E}$ is *club irreducible* if every feasible allocation is club linked.

Proposition 3.3 *Let $\mathcal{E}$ be a club economy for which endowments are desirable. If $(x, \mu), (p, q)$ is a club quasi-equilibrium and (x, μ) is club linked, then $p >> 0$ and $(x, \mu), (p, q)$ is a club equilibrium.*

Proof We show first that all private good prices are strictly positive. If not, let I be the set of indices for which $p_i > 0$, and let $J \neq \emptyset$ be the complementary set of indices. Fix $i \in I$. If $x_{ai} \neq 0$ for some set of consumers having positive measure, then some of these consumers could sell a small amount of their consumption of x_i and buy an unlimited quantity of x_j (for any $j \in J$) and be strictly better off with a lower expenditure. This would contradict the quasi-equilibrium conditions. We conclude that, for each $i \in I$, $x_{ai} = 0$ for almost all $a \in A$. Club linkedness guarantees that all consumers would prefer to consume their endowments plus a large quantity of some commodity j rather than their quasi-equilibrium consumption. Since aggregate endowments of private goods are strictly positive, the endowments of some consumers have a strictly positive value and, by continuity of preferences, those consumers would prefer to consume a very large fraction of their endowment plus a large quantity of commodity x_j rather than their quasi-equilibrium consumption. Again, this would contradict the quasi-equilibrium conditions, so we conclude that all private good prices are strictly positive.

If $(x,\mu),(p,q)$ is not an equilibrium, then there is an agent a who is quasi-optimizing, but not optimizing. Hence there is a choice $(x',\mu') \in X_a$ which is strictly preferred to agent a's quasi-equilibrium choice and costs no more than his endowment. Desirability of endowments entails that $x' \neq 0$, so $p \cdot x' > 0$. Continuity of preferences entails that there is a bundle x'' such that $p \cdot x'' < p \cdot x'$, $(x'',\mu') \in X_a$ and (x'',μ') is strictly preferred to agent a's quasi-equilibrium choice, but costs strictly less than his endowment. This is a contradiction, so the proof is complete. ■

3.8 Pure Transfers

Equilibrium requires that the sum of membership prices in each club type exactly pays for the required inputs. An equivalent notion will be more convenient in proofs. Say that $q \in \mathbf{R}^{\mathcal{M}}$ is a *pure transfer* if $q \in \mathbf{Trans}$, defined as:

$$\mathbf{Trans} = \{q \in \mathbf{R}^{\mathcal{M}} : q \cdot \mu = 0 \text{ for each } \mu \in \mathbf{Cons}\}$$

Thus for each club type (π,γ) and $q \in \mathbf{Trans}$, $\sum_{\omega\in\Omega} \pi(\omega) q(\omega,\pi,\gamma) = 0$.

A *pure transfer equilibrium* is a feasible state (x,μ) and prices $(p,q) \in \mathbf{R}^N_+ \setminus \{0\} \times \mathbf{R}^{\mathcal{M}}$ such that

(1) **Budget Feasibility** For almost all $a \in A$:

$$p \cdot x_a + q \cdot \mu_a + \sum_{(\omega,\pi,\gamma)} p \cdot \frac{1}{|\pi|} \mathbf{inp}(\pi,\gamma)\mu_a(\omega,\pi,\gamma) \;\leq\; p \cdot e_a$$

(2) **Optimization** For almost all $a \in A$:

$$\begin{array}{ll} \text{if} & (x'_a,\mu'_a) \in X_a \text{ and } u_a(x'_a,\mu'_a) > u_a(x_a,\mu_a) \\ \text{then} & p \cdot x'_a + q \cdot \mu'_a + \sum_{(\omega,\pi,\gamma)} p \cdot \frac{1}{|\pi|} \mathbf{inp}(\pi,\gamma)\mu'_a(\omega,\pi,\gamma) \;>\; p \cdot e_a \end{array}$$

(3) **Pure Transfers** $q \in \mathbf{Trans}$

We define a pure transfer quasi-equilibrium in the obvious way.

The following lemma tells us that equilibrium (respectively quasi-equilibrium) and pure transfer equilibrium (respectively pure transfer quasi-equilibrium) are equivalent notions; we leave the simple proof to the reader.

Lemma 3.4 *Let $\mathcal{E}$ be a club economy and let $q, q^* \in \mathbf{R}^{\mathcal{M}}$ be such that*

$$q^*(\omega, \pi, \gamma) = q(\omega, \pi, \gamma) + p \cdot \frac{1}{|\pi|}\mathbf{inp}(\pi, \gamma)$$

Then: $(x, \mu), (p, q)$ is a pure transfer equilibrium (respectively, pure transfer quasi-equilibrium) if and only if $(x, \mu), (p, q^*)$ is a club equilibrium (respectively, club quasi-equilibrium).

4 The Welfare Theorems and the Core

For exchange economies, the first welfare theorem asserts that equilibrium states are Pareto optimal. The corresponding result is easily established for club economies.

Theorem 4.1 *Every club equilibrium state of a club economy belongs to the weak core and, in particular, is weakly Pareto optimal. If endowments are desirable, every club equilibrium state belongs to the strong core and, in particular, is strongly Pareto optimal.*

Proof Let $\mathcal{E}$ be a club economy and let (x, μ) be an equilibrium state, supported by the prices $p \in \mathbf{R}^N_+ \setminus \{0\}, q \in \mathbf{R}^{\mathcal{M}}$. If (x, μ) is not in the weak core, there is a subset $B \subset A$ of positive measure and a state (y, ν) that is feasible for B and preferred to (x, μ) by every member of B. Feasibility of (y, ν) for the coalition B entails the material balance condition:

$$\int_B y_a \, d\lambda(a) \; + \; \int_B \sum_{(\omega,\pi,\gamma)\in\mathcal{M}} \frac{1}{|\pi|}\mathbf{inp}(\pi, \gamma) \, \nu_a(\omega, \pi, \gamma) \, d\lambda(a) = \int_B e_a \, d\lambda(a)$$

and the budget balance condition for each club type (π, γ):

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot \mathbf{inp}(\pi, \gamma)$$

Combining these with the consistency condition, we conclude that

$$\int_B (p, q) \cdot (y_a, \nu_a) \, d\lambda(a) = \int_B p \cdot e_a \, d\lambda(a)$$

Hence there is a set $B' \subset B$ having positive measure for which

$$(p, q) \cdot (y_b, \nu_b) \le p \cdot e_b$$

for every $b \in B'$. Since (y, ν) is unanimously preferred to (x, μ) by members of B, this contradicts the equilibrium nature of (x, μ). We conclude that (x, μ) is in the weak core, as desired. That (x, μ) is weakly Pareto optimal follows immediately by taking $B = A$ in the argument above. If endowments are desirable, the weak and strong cores coincide and the sets of weak and strong Pareto sets coincide, so the proof is complete. ■

For exchange economies, the second welfare theorem asserts that every Pareto optimal allocation can be supported as an equilibrium conditional on a suitable reallocation of endowments. Surprisingly, the second welfare theorem fails in club economies; for examples we refer the reader to our working paper.[10]

5 Core/Equilibrium Equivalence

In this section we establish that non-atomic club economies pass a familiar test of perfect competition: The core coincides with equilibrium states.

[10]For exchange economies with a finite number of agents, the second welfare theorem depends on convexity of consumption sets and preferences — requirements that fail in club economies because club memberships are indivisible. However, for exchange economies with a continuum of agents, the second welfare theorem does not require convexity of consumption sets or preferences.

Theorem 5.1 *Let $\mathcal{E}$ be a non-atomic club economy in which endowments are desirable and uniformly bounded above. Then every core state can be supported as a club quasi-equilibrium and every core state that is club linked can be supported as a club equilibrium. In particular, if $\mathcal{E}$ is club irreducible, then the core coincides with the set of club equilibrium states.*

The proof in Section 7 parallels the Vind (1964) and Schmeidler (1969) proofs of Aumann's core equivalence theorem for exchange economies: Construct a preferred net trade correspondence and an aggregate net preferred set, and apply the Lyapunov convexity theorem to show that the aggregate net preferred set is convex. Use the core property to show that the aggregate net preferred set is disjoint from a cone that represents feasible net trades for all coalitions. Find a price that separates the aggregate net preferred set from this cone and show that this is a quasi-equilibrium price. Use club linkedness to conclude that the quasi-equilibrium is an equilibrium.

The argument contains two surprises that are not present in the exchange case. The first is that we require endowments to be bounded. As the following example demonstrates, this is not merely an artifact of the proof. If endowments are unbounded, the core may not coincide with the set of equilibrium states and equilibrium may not exist.

Example 5.2 There is a single consumption good, two external characteristics M, F (male, female), and a single club type c consisting of one agent of each characteristic (i.e., monogamous marriage) and requiring no inputs. Agents can choose at most one membership. Agents in the intervals $[0, 1/2)$ and $[1/2, 1]$ are respectively males and females. Males love marriage and females hate it:

$$\begin{aligned} u_a(x, 0) &= x && \text{for all } a \in [0, 1] \\ u_a(x, \delta_{(M,c)}) &= 2x && \text{for all } a \in [0, 1/2) \text{ (males)} \\ u_a(x, \delta_{(F,c)}) &= 1 - e^{-x} && \text{for all } a \in [1/2, 1] \text{ (females)} \end{aligned}$$

where $\delta_{(\omega,c)}$ is a list with a single membership (ω, c). Endowments are $\frac{1}{\sqrt{a}}$. It is easily checked that the initial state is the unique element of the core but is not an equilibrium: There is no upper bound on the amount men would pay to enter a marriage, because males are willing to give up half their endowment to enter a marriage, and male endowments are unbounded. But no female is willing to enter a marriage at any price. ♣

The other surprise is that it will not be good enough to find prices (p, q) that separate the aggregate net preferred set from the cone representing feasible net trades; we must also be sure that $p \neq 0$. To achieve this we will separate the aggregate net preferred set from a cone that is larger than the cone representing feasible net trades. To show that the aggregate net preferred set is disjoint from this cone, we will need to show that if (y, ν) is a state, $B \subset A$ is a coalition, and ν is "nearly" consistent for B, then there is a large subset $B' \subset B$ such that ν is (exactly) consistent for B'. This idea is formalized in Lemma 7.1.

6 Existence of Equilibrium

Theorem 6.1 *Let $\mathcal{E}$ be a non-atomic club economy. If endowments are desirable and uniformly bounded above, then a club quasi-equilibrium exists. If in addition $\mathcal{E}$ is club irreducible, then a club equilibrium exists.*

The structure of the argument will be familiar: construct an excess demand correspondence, use a fixed point theorem to find a zero, and show that this zero is an equilibrium. However, there are many subtleties:

- The balance condition for private goods is that the excess demand for private goods is 0, but the balance condition for club memberships is that the demand for club memberships is in **Cons**.
- In equilibrium, prices for private goods must be positive, but prices for club membership prices may be positive, negative or 0. Hence the relevant space of all prices is not a proper cone, and the usual forms of the excess demand lemma will not apply.

- Private good prices can be normalized to sum to 1, but club membership prices admit no obvious normalization or bound.
- We assume that all private goods are present in the aggregate, but not that all external characteristics are present. Further, some club types might not be chosen at equilibrium. In effect we must construct reservation prices for club memberships which are not available or are not chosen.

As the following example illustrates, club membership prices may be indeterminate and unbounded.

Example 6.2 There is a single consumption good, and two external characteristics, $\Omega = \{M, F\}$. There are two club types c_1, c_2, each consisting of one male (M) and one female (F), and requiring no inputs. Agents can choose at most 2 memberships. Agents in $[0, 1/2)$ are male; agents in $[1/2, 1]$ are female. Each agent's endowment is 1. Utility functions are:

$$\begin{array}{lclcl} u_a(x, 0) & = & x & & \\ u_a(x, \delta_{(\omega,c_1)}) & = & u_a(x, \delta_{(\omega,c_2)}) & = & 1 - e^{-x} \\ u_a(x, 2\delta_{(\omega,c_1)}) & = & u_a(x, 2\delta_{(\omega,c_2)}) & = & 1 - e^{-x} \\ u_a(x, \delta_{(\omega,c_1)} + \delta_{(\omega,c_2)}) & = & 2x & & \end{array}$$

where $\omega = M, F$ according to whether agent a is male or female, $2\delta_{(\omega,c)}$ is the list consisting of 2 memberships of type (ω, c) and $\delta_{(\omega,c_1)} + \delta_{(\omega,c_2)}$ is the list consisting of one membership of type (ω, c_1) and one membership of type (ω, c_2). Thus, both males and females hate belonging to a single club or two clubs of the same type, but love belonging to two clubs of different types. The core consists of a single point: all agents choose one club of each type and consume their endowments. This state is supported as an equilibrium by any private good prices and club membership prices such that $p > 0$ and:

$$\begin{array}{rcl} q(M, c_1) + q(F, c_1) & = & 0 \\ q(M, c_2) + q(F, c_2) & = & 0 \\ q(M, c_1) + q(M, c_2) & = & 0 \\ q(F, c_1) + q(F, c_2) & = & 0 \end{array} \qquad \clubsuit$$

The proof circumvents the indeterminacy and unboundedness of club membership prices by focusing on list prices, which can be bounded in the following way. Normalize private good prices to sum to 1. By assumption, individual endowments are uniformly bounded, so individual incomes are uniformly bounded. Hence if $q \in \mathbf{R}^{\mathcal{M}}$ is a vector of membership prices, $\ell \in$ **Lists** is a list, and if the list price $q \cdot \ell$ exceeds the bound on individual incomes, then the demand for ℓ will be 0. Thus the upper bound on individual incomes provides an upper bound for list prices. To construct a lower bound for list prices (keeping in mind that list prices might be negative), we show that, for lists chosen at equilibrium, if some individuals are paying large negative list prices, then others are paying large positive list prices. The construction we use is formalized in Lemma 7.2.

The existence proof in Section 7 has 8 steps:

1 For each positive integer k, construct a perturbed economy $\mathcal{E}^k$ by adjoining a few agents of each external characteristic, with utility functions linear in private good consumption.

2 For the perturbed economy $\mathcal{E}^k$, use the adjoined agents to identify a compact set of prices in which to find an equilibrium.

3-6 Construct an excess demand correspondence for $\mathcal{E}^k$, and find a fixed point that maximizes the value of excess demand. Show that this is an equilibrium for the perturbed economy: excess demand for private goods equals 0 and demand for club memberships is consistent.

7-8 The previous steps give an equilibrium for each $\mathcal{E}^k$, with associated prices (p^k, q^k). Now show that there is a sequence of such equilibrium prices that satisfy a uniform bound independent of k. Take limits of these uniformly bounded equilibrium prices as $k \to \infty$ and apply Fatou's lemma to construct an equilibrium for $\mathcal{E}$.

7 Proofs

We begin by formalizing the idea that if a choice function ν is "nearly" consistent for a coalition B then it is exactly consistent for some large subset of B. We need some notation. For $L \subset \mathbf{R}^{\mathcal{M}}$ write $\mathrm{conv}\,(L)$ for its convex hull. Let

$$\begin{aligned} \mathbf{Lists}_M &= \{\ell \in \mathbf{Lists} : |\ell| \leq M\} \\ \mathcal{D} &= \{L \subset \mathbf{Lists}_M : \mathrm{conv}\,(L) \cap \mathbf{Cons} = \emptyset\} \\ D &= \inf\{\mathrm{dist}\,(\mathrm{conv}\,(L), \mathbf{Cons}) : L \in \mathcal{D}\} \end{aligned}$$

Lemma 7.1 *Let $B \subset A$ be a measurable set of positive measure and let $\nu : B \to \mathbf{Lists}_M$ be a measurable function. Then there is a measurable subset $B' \subset B$ such that*

$$\int_{B'} \nu_b \, d\lambda(b) \;\in\; \mathbf{Cons}$$

$$\lambda(B') \geq \lambda(B) - \frac{1}{D}\mathrm{dist}\,(\int_B \nu_b \, d\lambda(b)\;, \mathbf{Cons}) \qquad (1)$$

Proof If $\mathbf{Lists}_M \subset \mathbf{Cons}$, then $\mathrm{dist}\,(\int_B \nu_b \, d\lambda(b)\;, \mathbf{Cons}) = 0$, $\mathcal{D} = \emptyset$ and $D = \infty$, so we may take $B' = B$. Assume therefore that $\mathbf{Lists}_M \not\subset \mathbf{Cons}$. For $\ell \in \mathbf{Lists}_M$, write $B_\ell = \{b \in B : \nu_b = \ell\}$. Let $L = \{\ell \in \mathbf{Lists}_M : \lambda(B_\ell) > 0\}$. Note that $\sum_{\ell \in L} \lambda(B_\ell) = \lambda(B)$ and

$$\int_B \nu_b \, d\lambda(b) = \sum_{\ell \in L} \lambda(B_\ell)\, \ell = \lambda(B) \sum_{\ell \in L} \frac{\lambda(B_\ell)}{\lambda(B)}\, \ell$$

In particular

$$\int_B \nu_b \, d\lambda(b) \in \lambda(B)\, \mathrm{conv}\,(L)$$

If $\mathrm{conv}\,(L) \cap \mathbf{Cons} = \emptyset$ then the right hand side of (1) is non-positive, so we may take $B'_\ell = \emptyset$ for each ℓ and $B' = \bigcup_{\ell \in L} B'_\ell = \emptyset$. We therefore assume $\mathrm{conv}\,(L) \cap \mathbf{Cons} \neq \emptyset$.

Consider the linear programming problem:

$$\text{maximize} \qquad \sum_{\ell \in L} \beta_\ell$$

$$\text{subject to} \quad 0 \le \beta_\ell \le \lambda(B_\ell)$$
$$\sum_{\ell \in L} \beta_\ell \ell \in \mathbf{Cons}$$

The feasible set for this problem is non-empty (it contains the origin), so this problem has a solution. Let $\{\beta_\ell : \ell \in L\}$ be any such solution.

For each ℓ, write $\alpha_\ell = \lambda(B_\ell) - \beta_\ell \ge 0$. Write $L' = \{\ell : \alpha_\ell > 0\}$. If $L' = \emptyset$ we are done, so assume not. Write $\alpha = \min\{\alpha_\ell : \ell \in L'\}$. If $\mathrm{conv}\,(L') \cap \mathbf{Cons} \neq \emptyset$ there are non-negative real numbers ϵ_ℓ summing to 1 with $\sum_{L'} \epsilon_\ell \ell \in \mathbf{Cons}$. Set $\beta^*_\ell = \beta_\ell + \epsilon_\ell \alpha$ for $\ell \in L'$ and $\beta^*_\ell = \beta_\ell$ for $\ell \notin L'$. Then $\{\beta^*_\ell : \ell \in L\}$ satisfies the constraints in the linear programming problem and yields a larger value of the objective, contradicting the choice of $\{\beta_\ell\}$ as the solution. We conclude that $\mathrm{conv}\,(L') \cap \mathbf{Cons} = \emptyset$.

For each $\ell \in \mathbf{Lists}_M$, non-atomicity of λ guarantees that we can choose $B'_\ell \subset B_\ell$ such that $\lambda(B'_\ell) = \beta_\ell$. Set

$$B' = \bigcup_{\ell \in L} B'_\ell$$

By construction,

$$\int_{B'} \nu_b \, d\lambda(b) = \sum_{\ell \in L} \lambda(B'_\ell)\ell \;\in\; \mathbf{Cons}$$

We need only estimate $\lambda(B')$. Because $\mathbf{Cons}$ is a linear subspace, for every $x \in \mathbf{R}^{\mathcal{M}}, y \in \mathbf{Cons}, r \in \mathbf{R}_+$, we have $\mathrm{dist}\,(x-y, \mathbf{Cons}) = \mathrm{dist}\,(x, \mathbf{Cons})$ and $\mathrm{dist}\,(rx, \mathbf{Cons}) = r\mathrm{dist}\,(x, \mathbf{Cons})$. Hence,

$$\begin{aligned}
\mathrm{dist}\,(\sum_{\ell \in L} \lambda(B_\ell)\ell, \mathbf{Cons}) &= \mathrm{dist}\,((\sum_{\ell \in L} \lambda(B_\ell)\ell - \sum_{\ell \in L} \beta_\ell \ell)\,, \mathbf{Cons}) \\
&= \mathrm{dist}\,(\sum_{\ell \in L'} (\lambda(B_\ell) - \beta_\ell)\ell, \mathbf{Cons}) \\
&= \mathrm{dist}\,(\sum_{\ell \in L'} \alpha_\ell \ell, \mathbf{Cons}) \\
&= \mathrm{dist}\,\left(\sum_{\ell \in L'} \alpha_\ell \sum_{\ell \in L'} \left[\frac{\alpha_\ell}{\sum_{\ell \in L'} \alpha_\ell}\right] \ell\,, \mathbf{Cons}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (\sum_{\ell \in L'} \alpha_\ell) \operatorname{dist} \left(\left[\sum_{\ell \in L'} \frac{\alpha_\ell}{\sum_{\ell \in L'} \alpha_\ell} \right] \ell \,, \mathbf{Cons} \right) \\
&\geq (\sum_{\ell \in L'} \alpha_\ell) \operatorname{dist} (\operatorname{conv}(L'), \mathbf{Cons}) \\
&= (\sum_{\ell \in L'} [\lambda(B_\ell) - \beta_\ell]) \operatorname{dist} (\operatorname{conv}(L'), \mathbf{Cons}) \\
&\geq D \sum_{\ell \in L'} [\lambda(B_\ell) - \beta_\ell] \\
&= D \sum_{\ell \in L} [\lambda(B_\ell) - \beta_\ell] \\
&= D[\lambda(B) - \lambda(B')]
\end{aligned}
$$

Rearranging terms yields the desired inequality (1). ■

With this result in hand, we turn to the proof of core equivalence.

Proof of Theorem 5.1 Let (x, μ) be a core state. We show that (x, μ) can be supported as a pure transfer quasi-equilibrium.

Step 1 For each agent a, consider the preferred set

$$\Phi(a) = \{(x, \ell) \in X_a : u_a(x, \ell) > u_a(x_a, \mu_a)\}$$

For each club type $(\pi, \gamma) \in \mathbf{Clubs}$, $\frac{1}{|\pi|}\mathbf{inp}(\pi, \gamma)$ is the bundle of goods each member of (π, γ) would be required to contribute to the club (π, γ) if inputs were imputed equally to all members. For $\ell \in \mathbf{Lists}$, define

$$\tau(\ell) = \sum_{(\omega,\pi,\gamma)\in\mathcal{M}} \ell(\omega, \pi, \gamma) \frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma)$$

This is the total bundle of goods that an individual choosing list ℓ would have to contribute if inputs were imputed equally to all members of all clubs.

For each agent a, write

$$\psi(a) = \{(x, \ell) \in \mathbf{R}^N \times \mathbf{R}^{\mathcal{M}} : (x + e_a - \tau(\ell), \ell) \in \Phi(a)\}$$

and $\Psi(a) = \psi(a) \cup \{0\}$. It is easily checked that Ψ is a measurable correspondence. Let

$$Z = \int_A \Psi(a) \, d\lambda(a)$$

Step 2 In view of the Lyapunov convexity theorem, Z is a non-empty convex subset of $\mathbf{R}^N \times \mathbf{R}^{\mathcal{M}}$. (See Hildenbrand (1974).)

Step 3 Write $\mathbf{1} = (1, \ldots, 1) \in \mathbf{R}^N_+$. By assumption, endowments are uniformly bounded; say $e_a \leq W\mathbf{1}$ for each $a \in A$. Set

$$C = \{(\bar{x}, \bar{\mu}) \in \mathbf{R}^N \times \mathbf{R}^{\mathcal{M}} : \bar{x} < 0\,,\ \bar{\mu} \in \mathbf{Cons}\}$$

C is a convex cone in $\mathbf{R}^N \times \mathbf{R}^{\mathcal{M}}$. The core property of (x, μ) implies that $Z \cap C = \emptyset$ and hence that Z can be separated from C by prices (p, q). Unfortunately, it might happen that $p = 0$. (See Example 5.2.) To guarantee $p \neq 0$, we separate Z from a "fatter" cone. Define

$$C^* = \{(\bar{x}, \bar{\mu}) \in \mathbf{R}^L \times \mathbf{R}^{\mathcal{M}} : \bar{x} < -\frac{W}{D}\mathrm{dist}\,(\bar{\mu}, \mathbf{Cons})\mathbf{1}\}$$

We claim that $Z \cap C^* = \emptyset$.

Suppose that $Z \cap C^* \neq \emptyset$; we construct a blocking coalition. Choose $(x^*, \mu^*) \in Z \cap C^*$. By definition, there is a measurable selection $a \mapsto (y_a, \nu_a)$ from the correspondence Ψ such that

$$(x^*, \mu^*) = \int_A (y_a, \nu_a)\, d\lambda(a)$$

Let $B = \{a \in A : (y_a, \nu_a) \in \psi(a)\}$ be the set of agents for whom (y_a, ν_a) is in their net preferred set. Note that $\lambda(B) > 0$ and

$$(x^*, \mu^*) = \int_B (y_a, \nu_a)\, d\lambda(a) \tag{2}$$

We now apply Lemma 7.1 to choose $B' \subset B$ such that

$$\int_{B'} \nu_a\, d\lambda(a) \quad \in \quad \mathbf{Cons} \tag{3}$$

$$\lambda(B') \quad \geq \quad \lambda(B) - \frac{1}{D}\mathrm{dist}\,(\int_B \nu_a\, d\lambda(a)\ , \mathbf{Cons}) \tag{4}$$

We assert that B' is a blocking coalition. To see this, note first that, because endowments are bounded above by $W\mathbf{1}$, net preferred sets are bounded below by $-W\mathbf{1}$. Hence

$$\int_B y_a d\lambda(a) \geq -\lambda(B)W\mathbf{1}$$

Equation (2) and the fact that $(x^*, \mu^*) \in C^*$ entail that

$$\int_B y_a d\lambda(a) < -\frac{W}{D}\text{dist}\,(\int_B \nu_a d\lambda(a)\,, \mathbf{Cons})\mathbf{1}$$

and hence,

$$\text{dist}\,(\int_B \nu_a d\lambda(a)\,, \mathbf{Cons}) < \lambda(B)D$$

Together with (4), this implies that $\lambda(B') > 0$.

Define $(\hat{x}, \hat{\mu})$ by $(\hat{x}_a, \hat{\mu}_a) = (y_a + e_a - \tau(\nu_a), \nu_a)$. To see that $(\hat{x}, \hat{\mu})$ is feasible for B' note first that equation (2) and the definition of C^* entail that

$$\begin{aligned} x^* = \int_B y_a \, d\lambda \;&\leq\; -\frac{W\mathbf{1}}{D}\text{dist}\,(\mu^*, \mathbf{Cons}) \\ &=\; -\frac{W\mathbf{1}}{D}\text{dist}\,(\int_B \nu_a d\lambda(a)\,, \mathbf{Cons}) \end{aligned} \tag{5}$$

Because integration is additive,

$$\int_B y_a \, d\lambda = \int_{B'} y_a \, d\lambda + \int_{B \setminus B'} y_a \, d\lambda \tag{6}$$

By the bound on endowments and the definition of excess demand,

$$\int_{B \setminus B'} y_a \, d\lambda \geq -\lambda(B \setminus B')W\mathbf{1} \tag{7}$$

Combining equations (4), (5), (6) and (7) yields $\int_{B'} y_a \, d\lambda(a) \leq 0$, so

$$\int_{B'} [\hat{x}_a + \tau(\nu_a)] \, d\lambda(a) \leq \int_{B'} e_a \, d\lambda$$

which is the material balance condition. Equation (3) is consistency, so we conclude that $(\hat{x}, \hat{\mu})$ is feasible for B'. By construction, $(\hat{x}, \hat{\mu})$ is preferred to (x, μ) by every member of B', so this contradicts the assumption that (x, μ) is a core state. We conclude that $Z \cap C^* = \emptyset$, as asserted.

Step 4 We now use the separation theorem to find prices $(p, q) \in \mathbf{R}^N \times \mathbf{R}^{\mathcal{M}}$, $(p, q) \neq (0, 0)$ such that

$$\begin{aligned} (p, q) \cdot (\bar{x}, \bar{\mu}) \leq 0 &\qquad \text{for each } (\bar{x}, \bar{\mu}) \in C^* \\ (p, q) \cdot z \geq 0 &\qquad \text{for each } z \in Z \end{aligned}$$

Because C^* contains the cone $-\mathbf{R}^N_{++} \times \{0\}$, it follows that $p \geq 0$. Because C^* contains the subspace $\{0\} \times \mathbf{Cons}$, q vanishes on $\mathbf{Cons}$ and hence $q \in \mathbf{Trans}$.

We claim that $p \neq 0$. For suppose $p = 0$. By construction, $(p, q) \neq (0, 0)$ so $q \neq 0$. Hence there is a $\bar{\mu} \in \mathbf{R}^{\mathcal{M}}$ such that $q \cdot \bar{\mu} > 0$. For $\varepsilon > 0$ sufficiently small, $(-\mathbf{1}, \varepsilon\bar{\mu}) \in C^*$, so $(p, q) \cdot (-\mathbf{1}, \varepsilon\bar{\mu}) \leq 0$. However

$$(p, q) \cdot (-\mathbf{1}, \varepsilon\bar{\mu}) = (0, q) \cdot (-\mathbf{1}, \varepsilon\bar{\mu}) = \varepsilon q \cdot \bar{\mu}$$

which, by our choice of $\bar{\mu}$, is positive. This is a contradiction, so we conclude that $p \neq 0$, as desired.

We claim that $(x, \mu), (p, q)$ is a pure transfer quasi-equilibrium. Feasibility is guaranteed by assumption. To verify budget feasibility, write

$$\begin{aligned} E_1 &= \{a \in A : p \cdot [x_a + \tau(\mu_a)] + q \cdot \mu_a > p \cdot e_a\} \\ E_2 &= \{a \in A : p \cdot [x_a + \tau(\mu_a)] + q \cdot \mu_a < p \cdot e_a\} \end{aligned}$$

E_1 is the set of agents for whom expenditure exceeds income, and E_2 is the set of agents for whom income exceeds expenditure. We must show $\lambda(E_1) = 0$ (so that almost all agents choose in their budget sets). Measurability of the endowment mapping e implies that E_1, E_2 are measurable sets. Feasibility of (x, μ) implies that if $\lambda(E_1) > 0$, then $\lambda(E_2) > 0$. Strict monotonicity of preferences in private goods means that, for each $a \in A$ and each $\varepsilon > 0$, the choice vector $(x_a + \varepsilon\bar{e}, \mu_a)$ is strictly preferred to (x_a, μ_a). Hence if $a \in E_2$ then there is an $\varepsilon_a > 0$ such that $(x_a + \varepsilon_a\bar{e}, \mu_a)$ costs strictly less than e_a and is strictly preferred to (x_a, μ_a). We may choose ε_a to be a measurable function of a. Define $(\hat{x}, \hat{\mu}) : A \to \mathbf{R}^N_+ \times \mathbf{R}^{\mathcal{M}}$ by

$$(\hat{x}, \hat{\mu}) = \begin{cases} (x_a + \varepsilon_a\bar{e} + \tau(\mu_a) - e_a, \mu_a) & \text{if } a \in E_2 \\ (0, 0) & \text{otherwise} \end{cases}$$

By construction, $(\hat{x}, \hat{\mu})$ is a measurable selection from the correspondence Ψ, so $\int_A (\hat{x}_a, \hat{\mu}_a)\, d\lambda(a) \in Z$. However, our construction guarantees that

$$(p, q) \cdot \int_A (\hat{x}_a, \hat{\mu}_a)\, d\lambda(a) = \int_A (p, q) \cdot (\hat{x}_a, \hat{\mu}_a)\, d\lambda(a) < 0$$

which contradicts the fact that (p, q) separates Z from C^*. We conclude that $\lambda(E_1) = 0$; almost all agents choose in their budget sets.

To check the quasi-optimization conditions, let E_3 be the set of agents who are not quasi-optimizing in their budget sets. Let $H \subset \mathbf{R}^n_+$ be any countable dense subset. By definition, $a \in E_3$ if and only if there is a choice vector $(y_a, \nu_a) \in X_a = \mathbf{R}^n_+ \times \mathbf{Lists}_a$ which is strictly preferred to (x_a, μ_a) and costs strictly less than a's endowment; equivalently (because H is dense), $a \in E_3$ if and only if there is a choice vector $(y_a, \nu_a) \in X_a = H \times \mathbf{Lists}_a$ which is strictly preferred to (x_a, μ_a) and costs strictly less than a's endowment. It follows that E_3 is a measurable set, and that the choices (y_a, ν_a) can be chosen to depend measurably on $a \in E_3$. Suppose $\lambda(E_3) > 0$. Define $(\bar{x}, \bar{\mu}) : A \to \mathbf{R}^N_+ \times \mathbf{R}^{\mathcal{M}}$ by

$$(\bar{x}_a, \bar{\mu}_a) = \begin{cases} (y_a + \tau(\nu_a) - e_a, \nu_a) & \text{if } a \in E_3 \\ (0, 0) & \text{otherwise} \end{cases}$$

By construction, $(\bar{x}, \bar{\mu})$ is a measurable selection from the correspondence Ψ so $\int_A (\bar{x}_a, \bar{\mu}_a)\, d\lambda(a) \in Z$. However, our construction guarantees that

$$(p, q) \cdot \int_A (\hat{x}_a, \hat{\mu}_a)\, d\lambda(a) = \int_A (p, q) \cdot (\hat{x}_a, \hat{\mu}_a)\, d\lambda(a) < 0$$

which contradicts the fact that (p, q) separates Z from C^*. We conclude that $\lambda(E_3) = 0$: almost all agents are quasi-optimizing.

It follows that $(x, \mu), (p, q)$ is a pure transfer quasi-equilibrium. Setting

$$q^*_m = q_m + \frac{1}{|\pi|}\, p \cdot \mathbf{inp}(\pi, \gamma)$$

for each $m \in \mathcal{M}$ yields a quasi-equilibrium $(x, \mu), (p, q^*)$. If (x, μ) is club linked, it follows from Proposition 3.3 that $(x, \mu), (p, q^*)$ is an equilibrium.

Finally, if $\mathcal{E}$ is club irreducible, then every feasible state is club linked and hence every core state can be supported as an equilibrium. By Theorem 4.1, every equilibrium state belongs to the core. Hence the core coincides with the set of equilibrium states. ■

We now turn to the existence of equilibrium. We begin with a lemma which will allow us to construct upper and lower bounds for list prices. By analogy with a notion from cooperative game theory, we say that a subset $L \subset \mathbf{Lists}_M$ is *strictly balanced* if there are strictly positive real numbers $\{\epsilon_L(\ell) : \ell \in L\}$ (called *balancing weights*) such that $\sum_{\ell \in L} \epsilon_L(\ell)\ell \in \mathbf{Cons}$.

Lemma 7.2 *There is a constant $R^* > 0$ such that: If $L \subset \mathbf{Lists}_M$ is a strictly balanced collection and $q \in \mathbf{Trans}$ is a pure transfer then*

$$\max_{\ell \in L} q \cdot \ell \ \geq \ -R^* \min_{\ell \in L} q \cdot \ell$$

Proof For each strictly balanced collection of lists L, fix balancing weights $\{\epsilon'_L(\cdot)\}$. By definition, balancing weights are strictly positive so the sum $\beta_L = \sum_{\ell \in L} \epsilon'_L(\ell)$ is strictly positive. Define $\epsilon_L(\cdot) = \frac{1}{\beta}\epsilon_L(\cdot)$. Because **Cons** is a subspace, $\{\epsilon_L(\cdot)\}$ is also a family of balancing weights for L, and of course this family sums to 1. Set

$$R^* = \min\{\epsilon_L(\ell) : L \text{ is a strictly balanced collection }, \ell \in L\}$$

R^* is well-defined and $R^* > 0$ because balancing weights are strictly positive and the set of strictly balanced collection of lists is finite.

To see that R^* has the desired property, consider a strictly balanced collection L with balancing weights $\epsilon_L(\cdot)$ constructed above. Observe that

$$\sum_{\ell \in L} \epsilon_L(\ell) q \cdot \ell = q \cdot \sum_{\ell \in L} \epsilon_L(\ell)\ell = 0$$

Set $L_+ = \{\ell \in L : q \cdot \ell \geq 0\}$ and $L_- = L \setminus L_+$. Collect L_+ terms on the left hand side and L_- terms on the righthand side to obtain:

$$\sum_{\ell \in L_+} \epsilon_L(\ell) q \cdot \ell = -\sum_{q \in L_-} \epsilon_L(\ell) q \cdot \ell \tag{8}$$

Because the coefficients $\epsilon_L(\ell)$ are positive and sum to 1, we have:

$$\max_{\ell \in L} q \cdot \ell = \left[\sum_{\ell \in L} \epsilon_L(\ell)\right] \max_{\ell \in L} q \cdot \ell \geq \left[\sum_{\ell \in L_+} \epsilon_L(\ell)\right] \max_{\ell \in L_+} q \cdot \ell \geq \sum_{\ell \in L_+} \epsilon_L(\ell) q \cdot \ell \tag{9}$$

Because $q \cdot \ell < 0$ for each $\ell \in L_-$, we have:

$$- \sum_{q \in L_-} \epsilon_L(\ell) q \cdot \ell \geq - \min_{\ell \in L_-} \epsilon_L(\ell) \min_{\ell \in L_-} q \cdot \ell \geq - \min_{\ell \in L} \epsilon_L(\ell) \min_{\ell \in L} q \cdot \ell \qquad (10)$$

Combining (8), (9) and (10) and recalling the definition of R^* yields the desired inequality. ■

Proof of Theorem 6.1 Assume without loss that $\lambda(A) = 1$. By assumption, aggregate endowment $\bar{e}$ is strictly positive and individual endowments are uniformly bounded above; say that $\bar{e} \geq w\mathbf{1} >> 0$ and that $e_a \leq W_0 \mathbf{1}$ for all $a \in A$. Write $W = \max\{W_0, 1\}$. As in the proof of Theorem 5.1, write

$$\tau(\ell) = \sum_{(\omega,\pi,\gamma) \in \mathcal{M}} \ell(\omega, \pi, \gamma) \frac{1}{|\pi|} \mathbf{inp}(\pi, \gamma)$$

Step 1 Fix an integer $k > 0$. Choose a family $\{A^k_\omega : \omega \in \Omega\}$ of pairwise disjoint intervals in $\mathbf{R}$, each of length $1/k$. Set

$$A^k = A \cup \bigcup_{\omega \in \Omega} A^k_\omega$$

The agent space for the perturbed economy $\mathcal{E}^k$ is $(A^k, \mathcal{F}^k, \lambda)$, where $\mathcal{F}^k$ is the σ-algebra generated by $\mathcal{F}$ and the Lebesgue measurable subsets of $\cup_{\omega \in \Omega} A^k_\omega$, λ^k is λ on A and Lebesgue measure on $\cup_{\omega \in \Omega} A^k_\omega$. Note that $\lambda^k(A^k) = 1 + \frac{|\Omega|}{k}$. External characteristics, consumption sets, endowments and utility functions of agents in A are just as in the original club economy $\mathcal{E}$. For agents $a \in A^k_\omega$, we define:

$$\begin{aligned} \omega_a &= \omega \\ X_a &= \mathbf{R}^N_+ \times \{\ell \in \mathbf{Lists}_M : \ell(\omega', \pi, \gamma) = 0 \text{ if } \omega' \neq \omega\} \\ e_a &= W\mathbf{1} \\ u_a(x, \ell) &= |x| \text{ for all } (x, \ell) \in X_a \end{aligned}$$

Step 2 The demand functions of the added agents are such that, for commodity prices near the boundary of the simplex and for membership prices that are large in absolute value, aggregate excess demand for commodities will be impossibly large. As a consequence, we can write down compact price sets that contain an equilibrium price for $\mathcal{E}^k$. To define these sets, write $M^* = \max\{|\pi| : (\pi,\gamma) \in \mathbf{Clubs}\}$. Choose a real number $\varepsilon > 0$ so small that

$$[1-(N-1)\varepsilon]\left[\frac{W}{kN\varepsilon} - W(1+\frac{|\Omega|}{k})\right] - \varepsilon(N-1)W(1+\frac{|\Omega|}{k}) > 0$$

Having chosen ε, choose a real number $R > 0$ so big that $R > 2|\tau(\ell)|$ for all $\ell \in \mathbf{Lists}_M$ and

$$[1-(N-1)\varepsilon]\left[\frac{R}{2kNM^*} - W(1+\frac{|\Omega|}{k})\right] - \varepsilon(N-1)W(1+\frac{|\Omega|}{k}) > 0$$

Of course ε, R depend on k. Define a price simplex for private goods and a bounded price set for club memberships:

$$\begin{aligned} \Delta_\varepsilon &= \{p \in \mathbf{R}^N_+ : \sum_{n\in N} p_n = 1 \text{ and } p_n \geq \varepsilon \text{ for each } n\} \\ Q_R &= \{q \in \mathbf{Trans} : |q_m| \leq R \text{ for all } m \in \mathcal{M}\} \end{aligned}$$

Step 3 We define an excess demand correspondence. Let $p \in \Delta_\varepsilon, q \in Q_R$. For each agent $a \in A$, write

$$B(a,p,q) = \{(x,\ell) \in X_a : p\cdot x + q\cdot \ell + p\cdot\tau(\ell) \leq p\cdot e_a\}$$

This is agent a's budget set, assuming that he must pay an equal share of the inputs to club activities. Let

$$\begin{aligned} d(a,p,q) &= \text{argmax } \{u_a(x,\ell) : (x,\ell) \in B(a,p,q)\} \\ \zeta(a,p,q) &= \{(x+\tau(\ell),\ell) - (e_a,0) : (x,\ell) \in d(a,p,q)\} \end{aligned}$$

be agent a's demand set and excess demand set. Excess demand sets are uniformly bounded because endowments are bounded, private good prices are bounded away from 0 and club membership prices are bounded above and below. The correspondence $(a,p,q) \to \zeta(a,p,q)$ is measurable; for each a, the correspondence $(p,q) \to \zeta(a,p,q)$ is upper hemi-continuous since endowments are assumed to be desirable. Define the aggregate excess demand correspondence $Z : \Delta_\varepsilon \times Q_R \to \mathbf{R}^N_+ \times \mathbf{R}^{\mathcal{M}}$ as the integral of individual excess demand correspondences:

$$Z(p,q) = \int_{A^k} \zeta(a,p,q)\, d\lambda(a)$$

Because it is the integral of an upper hemi-continuous correspondence with respect to a non-atomic measure, Z is upper hemi-continuous, with compact, convex, non-empty values.

Step 4 We find a fixed point of the excess demand correspondence. Individual income comes from selling endowments and (perhaps) receiving subsidies for club memberships. The value of the each individual's endowment is bounded by W. Because club membership prices lie in the interval $[-R, +R]$ and individuals can choose no more than M club memberships, subsidies for club memberships are bounded by MR. Because private good prices are bounded below by ε, individual demand for each private good is bounded above by $\frac{1}{\varepsilon}(W + RM)$, and individual excess demand for each private goods lie between $-W$ and $\frac{1}{\varepsilon}(W+RM)$. Hence aggregate excess demand for private goods lies in the compact set

$$X = \{x \in \mathbf{R}^N : -\lambda(A^k)W \le x_n \le \lambda(A^k)\frac{1}{\varepsilon}(W + RM) \text{ for each } n\}$$

Because individuals are constrained to demand at most M club memberships, aggregate demands for club memberships lie in the set

$$C = \{\bar{\mu} \in \mathbf{R}^{\mathcal{M}}_+ : \sum_{m \in \mathcal{M}} \bar{\mu}(m) \le \lambda(A^k)M\}$$

Define a correspondence $\Phi : \Delta_\varepsilon \times Q_R \times X \times C \to \Delta_\varepsilon \times Q_R \times X \times C$ by

$$\Phi(p,q,x,\bar{\mu}) = \Big[\text{argmax } \{(p^*,q^*) \cdot (x,\bar{\mu}) : (p^*,q^*) \in \Delta_\varepsilon \times Q_R\}\Big] \times Z(p,q)$$

It is easily checked that Φ is upper hemi-continuous with compact convex values. Hence Kakutani's fixed point theorem guarantees that Φ has a fixed point. Thus there is a price pair $(p^k, q^k) \in \Delta_\varepsilon \times Q_R$ and a consumption/club membership pair $(z^k, \bar{\mu}^k) \in Z(p^k, q^k)$ such that

$$(p^k, q^k) \cdot (z^k, \bar{\mu}^k) = \max\Big\{(p^*,q^*) \cdot (z^k, \bar{\mu}^k) \ : \ (p^*,q^*) \in \Delta_\varepsilon \times Q_R\Big\}$$

Walras's law implies that $(p^k, q^k) \cdot (z^k, \bar{\mu}^k) = 0$.

Step 5 We show in several steps that $z^k = 0$ and $\bar{\mu}^k \in \mathbf{Cons}$.

Step 5.1 We show first that $q^k \cdot \bar{\mu}^k = 0$. Suppose not. We obtain a contradiction by looking at excess demands at prices p^k, q^k of agents in $A^k \backslash A$. Because $0 \cdot \bar{\mu}^k = 0$, maximality and the definition of Φ imply $q^k \cdot \bar{\mu}^k > 0$. Maximality entails that $q^k \in \mathrm{bdy}\, Q_R$ so that $|q^k_m| = R$ for some $m \in \mathcal{M}$. Budget balance for club types means that if some price has large magnitude and is positive then some other price must have large magnitude and be negative. Thus there is a membership m^* such that $q^k_{m^*} \leq -R/M^*$. An agent $b \in A^k_{\omega^*}$ could obtain a subsidy of R/M^* by choosing the membership m^* and no other. Such an agent, finding all private goods to be perfect substitutes and deriving no utility from club memberships, will consume only the least expensive private good(s) and club memberships with non-positive prices. Because $R > 2|\tau(\ell)|$ for each ℓ, the wealth used on inputs to clubs is less than $\frac{R}{2}$. Thus, b's demand for the least expensive private good — which we may as well suppose is good 1 — is at least

$$d_1(b, p^k, q^k) \geq \frac{R}{2NM^*}$$

Because $\lambda(A^k_\omega) = 1/k$ and individual excess demands are bounded below by $-W\mathbf{1}$, aggregate excess commodity demand z^k satisfies

$$\begin{aligned} z^k_1 &\geq \frac{1}{k}\frac{R}{2NM^*} - W(1 + \frac{|\Omega|}{k}) \\ z^k_n &\geq -W(1 + \frac{|\Omega|}{k}) \qquad \text{if } n > 1 \end{aligned}$$

Define $p \in \Delta_\varepsilon$ by:

$$\begin{aligned} p_1 &= 1 - (N-1)\varepsilon \\ p_n &= \varepsilon \qquad \text{if } n > 1 \end{aligned} \tag{11}$$

A little algebra shows that

$$p \cdot z^k \geq [1 - (N-1)\varepsilon]\left[\frac{R}{2kNM^*} - W(1 + \frac{|\Omega|}{k})\right] - \varepsilon(N-1)W(1 + \frac{|\Omega|}{k})$$

Our choices of R, ε guarantee that the right side is strictly positive, so

$$(p, 0) \cdot (z^k, \bar{\mu}^k) > 0 = (p^k, q^k) \cdot (z^k, \bar{\mu}^k)$$

which contradicts maximality. We conclude that $q^k \cdot \bar{\mu}^k = 0$, as desired.

Step 5.2 We show next that $\bar{\mu}^k \in$ **Cons**. If not, we could find a pure transfer $q^* \in$ **Trans** such that $q^* \cdot \bar{\mu}^k > 0$ and hence could find $q^{**} \in Q_R$ such that $q^{**} \cdot \bar{\mu}^k > 0$, contradicting maximality.

Step 5.3 We claim that $p_n^k > \varepsilon$ for each n. Suppose not. We once again obtain a contradiction by considering the excess demand of agents $b \in A^k \setminus A$. As before, we note that each such agent will consume only the least expensive private good(s) and club memberships with non-positive prices. It follows that b's demand for the least expensive private good — which we may as well suppose is good 1 — is at least

$$d_1(b, p^k, q^k) \geq \frac{W}{N\varepsilon}$$

As before, this means that aggregate excess commodity demand z^k satisfies

$$\begin{aligned} z_1^k &\geq \frac{1}{k}\frac{W}{N\varepsilon} - W(1 + \frac{|\Omega|}{k}) \\ z_n^k &\geq -W(1 + \frac{|\Omega|}{k}) \qquad \text{if } n > 1 \end{aligned}$$

Defining p as in (11), a little algebra yields

$$p \cdot z^k \geq [1 - (N-1)\varepsilon]\left[\frac{W}{kN\varepsilon} - W(1 + \frac{|\Omega|}{k})\right] - \varepsilon(N-1)W(1 + \frac{|\Omega|}{k})$$

Our choice of ε guarantees that the right side is strictly positive so

$$(p, 0) \cdot (z^k, \bar{\mu}^k) > 0 = (p^k, q^k) \cdot (z^k, \bar{\mu}^k)$$

which again contradicts maximality. We conclude that $p_n^k > \varepsilon$ for each n.

Step 5.4 We show that $z^k = 0$. Notice that $(p^k, q^k) \cdot (z^k, \bar{\mu}^k) = 0$ and $q^k \cdot \bar{\mu}^k = 0$ so $p^k \cdot z^k = 0$. Hence, if $z^k \neq 0$ there are indices i, j such that $z^k_i < 0$ and $z^k_j > 0$. Define $\hat{p}$ by

$$\begin{array}{rcll} \hat{p}_i &=& p^k_i - \frac{1}{2}(p^k_i - \varepsilon) & \\ \hat{p}_j &=& p^k_j + \frac{1}{2}(p^k_i - \varepsilon) & \\ \hat{p}_n &=& p^k_n & n \neq i, j \end{array}$$

Because $p^k_i > \varepsilon$, it follows that $\hat{p} \in \Delta_\varepsilon$. Because $p^k \cdot z^k = 0$, it follows that $\hat{p} \cdot z^k > 0$, a contradiction to maximality. We conclude that $z^k = 0$.

Step 6 By definition, there is a selection $(z^k_a, \bar{\mu}^k_a)$ from the individual excess demand sets which integrate to $(z^k, \bar{\mu}^k)$. Set $x^k_a = z^k_a + e_a - \tau(\bar{\mu}_a)$ so that $(x^k, \bar{\mu}^k)$ is a state of the economy $\mathcal{E}^k$. We have just shown that excess demand z^k is 0, and that $\bar{\mu}^k \in \mathbf{Cons}$, so we conclude that $(x^k, \bar{\mu}^k), (p^k, q^k)$ constitute a pure transfer equilibrium for $\mathcal{E}^k$.

Step 7 By construction, club membership prices q^k are bounded by R, but R depends on k. We now replace the sequence of (q^k) by a bounded sequence $(\bar{q}^k)$ which leads to the same demands.

Passing to a subsequence if necessary, we may assume without loss that for each $\ell \in \mathbf{Lists}_M$ the sequence $(q^k \cdot \ell)$ converges to a limit G_ℓ, which may be finite or infinite. Write:

$$\begin{array}{rcl} L &=& \{\ell \in \mathbf{Lists}_M : q^k \cdot \ell \to G_\ell \in \mathbf{R}\} \\ L_+ &=& \{\ell \in \mathbf{Lists}_M : q^k \cdot \ell \to +\infty\} \\ L_- &=& \{\ell \in \mathbf{Lists}_M : q^k \cdot \ell \to -\infty\} \end{array}$$

Choose $\bar{G} \in \mathbf{R}$ so large that $|q^k \cdot \ell| \leq \bar{G}$ for each k, each $\ell \in L$.

Define the linear transformation $T : \mathbf{Trans} \to \mathbf{R}^L$ by $T(q)_\ell = q \cdot \ell$. Write $\operatorname{ran} T = T(\mathbf{Trans}) \subset \mathbf{R}^L$ for the range of T and $\ker T = T^{-1}(0) \subset \mathbf{Trans}$ for the kernel (null space) of T. The fundamental theorem of linear algebra implies that we can choose a subspace $H \subset \mathbf{Trans}$ so that $H \cap \ker T = \{0\}$ and $H + \ker T = \mathbf{Trans}$. Write $T_{|H}$ for the restriction of T to H. Note that $T_{|H} : H \to \operatorname{ran} T$ is a one-to-one and onto linear transformation, so it has an inverse $S : \operatorname{ran} T \to H$. Because S is a linear transformation, it is continuous, so there is a constant K such that $|S(x)| \le K|x|$ for each $x \in \text{ran } T$.

Let R^* be the constant constructed in Lemma 7.2. Choose k_0 so large that $k \ge k_0$ implies

$$\begin{aligned} q^k \cdot \ell &> +2K\bar{G}M + W \qquad \text{if } \ell \in L_+ \\ q^k \cdot \ell &< -2K\bar{G}M - \frac{W}{R^*} \quad \text{if } \ell \in L_- \end{aligned}$$

Write ST for the composition of S with T. For each $k \ge k_0$ set

$$\bar{q}^k = ST(q^k) - ST(q^{k_0}) + q^{k_0} \in \textbf{Trans}$$

Because $S, T_{|H}$ are inverses, the composition TS is the identity, so

$$T(\bar{q}^k) = TST(q^k) - TST(q^{k_0}) + T(q^{k_0}) = T(q^k)$$

We assert that for $k > k_0$, $\bar{\mu}_a^k \notin L_- \cup L_+$ for any $a \in A^k$. If $a \in A^k$ then $q^k \cdot \bar{\mu}_a^k \le W$ (because the value of endowment is bounded by W) so $\bar{\mu}_a^k \notin L_+$, by construction of L_+. Since $\{\bar{\mu}_a^k\}$ are strictly balanced and $q^k \in \textbf{Trans}$, it follows from Lemma 7.2 that $\min_{a \in A^k}\{q^k \cdot \bar{\mu}_a^k\} \ge -\frac{1}{R^*} \max_{a \in A^k}\{q^k \cdot \bar{\mu}_a^k\} \ge -\frac{W}{R^*}$, and hence $\bar{\mu}_a^k \notin L_-$ by the construction of L_-.

Chose $k_1 \ge k_0$ so that $q^k \cdot \ell < q^{k_0} \cdot \ell - 2KGM$ for all $\ell \in L_-$ and all $k > k_1$. We claim that for $k > k_1$, $(x^k, \bar{\mu}^k), (p^k, \bar{q}^k)$ is a pure transfer equilibrium for $\mathcal{E}^k$. Because $(x^k, \bar{\mu}^k), (p^k, q^k)$ is a pure transfer equilibrium, it suffices to show that, for almost all $a \in A^k$ the choice $(x_a^k, \bar{\mu}_a^k)$ is budget feasible and optimal at $(p^k, \bar{q}^k)$. We have shown above that $\bar{\mu}_a^k \in L$ for almost all a; by construction $\bar{q}^k \cdot \ell = q^k \cdot \ell$ for all $\ell \in L$ because $T(\bar{q}^k) = T(q^k)$. Hence choices are budget feasible. Suppose then that (y, ν) is budget feasible for a at prices $(p^k, \bar{q}^k)$ and preferred to $(x_a^k, \bar{\mu}_a^k)$. Budget feasibility of (y, ν) at prices $(p^k, \bar{q}^k)$ implies that $\bar{q}^k \cdot \nu \le W$ and hence $q^{k_0} \cdot \nu \le W + 2K\bar{G}M$ because $|ST(q^k)| \le K\bar{G}$ and $|ST(q^{k_0})| \le K\bar{G}$. Thus $\nu \notin L_+$. For $\ell \in L_-$ and $k > k_1$, we similarly obtain $\bar{q}^k \cdot \ell > q^{k_0} \cdot \ell - 2K\bar{G}M > q^k \ell$. Thus, $\bar{q}^k \cdot \ell \ge q^k \cdot \ell$ for $\ell \in L_-$. Hence, $\bar{q}^k \cdot \ell \ge q^k \cdot \ell$ for $\ell \in L_- \cup L$. Thus, budget feasibility of (y, ν) at prices $(p^k, \bar{q}^k)$ implies budget feasibility of (y, ν) at prices (p^k, q^k), so $(x^k_a, \bar{\mu}^k_a)$ is not optimal prices (p^k, q^k). It follows that $(x^k, \bar{\mu}^k), (p^k, \bar{q}^k)$ is a pure transfer equilibrium for $\mathcal{E}^k$.

Step 8 By construction, $|\bar{q}^k \cdot \ell| \leq 2K\bar{G}M + |q^{k_0} \cdot \ell|$ for $k > k_0$ and all lists ℓ, so the prices of lists are bounded. Because singleton memberships are themselves lists, it follows that $(\bar{q}^k)$ is also a bounded sequence in **Trans**. We thus have bounded sequences (p^k), $(\bar{q}^k)$, $(\bar{\mu}^k)$. Passing to a subsequence if necessary, we may assume that $p^k \to p^* \in \Delta$, $\bar{q}^k \to q^* \in$ **Trans**, $\bar{\mu}^k \to \bar{\mu}^* \in$ **Cons**. The sequence $(\bar{\mu}^k)$ is uniformly bounded, hence uniformly integrable, so Schmeidler's version of Fatou's lemma (see Hildenbrand (1974, p. 225)) provides a measurable mapping $(x^*, \mu^*) : A \to \mathbf{R}^N_+ \times \mathbf{R}^{\mathcal{M}}$ such that i) for almost all $a \in A$: $(x^*_a, \mu^*_a) \in B(a, p^*, q^*)$; ii) for almost all $a \in A$: (x^*_a, μ^*_a) belongs to agent a's quasi-demand set (that is, there does not exist $(x', \ell') \in X_a$ such that $u_a(x', \ell') > u_a(x^*_a, \mu^*_a)$ and $(p^*, q^*) \cdot (x', \ell') + p^* \cdot \tau(\ell') < p^* \cdot e_a$; iii) $\int_A [x^*_a + \tau(\mu^*_a)]\, d\lambda \leq \bar{e}$; iv) $\int_A \mu^*_a \, d\lambda = \bar{\mu}^*$. Conditions i) and ii) together imply that $(p^*, q^*) \cdot (x^*_a, \mu^*_a) + p^* \cdot \tau(\mu^*_a) = p^* \cdot e_a$ for almost all a, so

$$p^* \cdot \left[\bar{e} - \int_A [x^*_a + \tau(\mu^*_a)]\, d\lambda \right] = 0$$

That is, left over goods (if any) are free. Distributing these free goods arbitrarily yields a pure transfer quasi-equilibrium $(x^{**}_a, \mu^*_a), (p^*, q^*)$ for $\mathcal{E}$. Club irreducibility implies that $(x^{**}_a, \mu^*_a), (p^*, q^*)$ is a pure transfer equilibrium for $\mathcal{E}$, so the proof is complete.[11] ■

[11] Because utility functions are strictly monotone in private goods, no goods are free at equilibrium, so in fact there are no leftover goods to distribute.

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