Discussion Papers
Department of Economics
University of Copenhagen

No. 17-07

Using a Theory-Consistent CVAR Scenario to Test an Exchange Rate Model Based on Imperfect Knowledge

Katarina Juselius

Øster Farimagsgade 5, Building 26, DK-1353 Copenhagen K., Denmark
Tel.: +45 35 32 30 01 – Fax: +45 35 32 30 00
http://www.econ.ku.dk

ISSN: 1601-2461 (E)

Using a Theory-Consistent CVAR Scenario to Test an Exchange Rate Model Based on Imperfect Knowledge *

Katarina Juselius

April 10, 2017

Abstract

A theory-consistent CVAR scenario describes a set of testable regularieties one should expect to see in the data if the basic assumptions of the theoretical model are empirically valid. Using this method, the paper demonstrates that all basic assumptions about the shock structure and steady-state behavior of an an imperfect knowledge based model for exchange rate determination can be formulated as testable hypotheses on common stochastic trends and cointegration. This model obtained a remarkable support for almost every testable hypothesis and was able to adeqautely account for the long persistent swings in the real exchange rate.

Keywords: Theory-Consistent CVAR, Imperfect Knowledge, Theory-Based Expectations, International Puzzles, Long Swings, Persistence.

JEL Classification: F31, F41, G15, G17

1 Introduction

International macroeconomics is known for its many pricing puzzles, including the purchasing power parity (PPP) puzzle, the exchange rate disconnect puzzle, and the forward rate puzzle (Engel 2014). The basic problem stems

*Valuable comments from Roman Frydman, Michael Goldberg, Kevin Hoover, Søren Johansen, and Mikael Juselius are gratefully acknowledged.

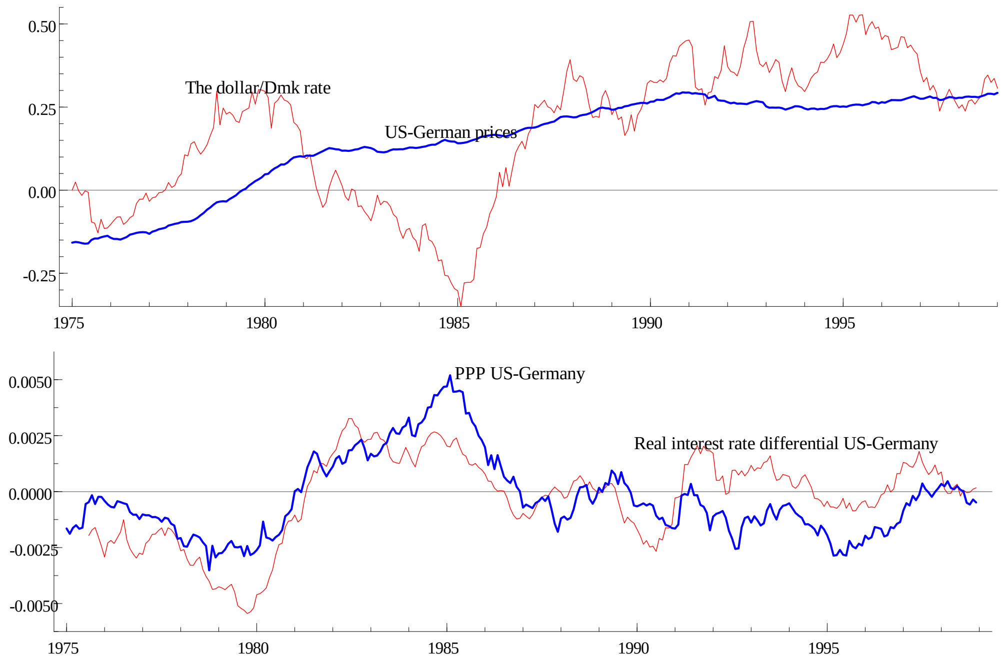

Figure 1: The graphs of the (mean and range adjusted) German-US price differential, pp, and the nominal exchange rate, s_{12} (upper panel), and the $ppp = pp - s_{12}$ and the real bond rate differential(lower panel).

from an inability of standard models based on the rational expectations hypothesis (REH) to account for highly persistent deviations from PPP and uncovered interest parity (UIP). See Engel (2014) and references therein for studies on REH and behavioral models.

Figure 1 illustrates the long swings in the nominal and the real exchange rate that have puzzled economists for such a long time. The upper panel shows relative prices for USA and Germany together with the nominal Deutshemark/Dollar rate for the post-Bretton Woods - pre-EMU period. While both series exhibit a similar upward sloping trend defining the long-run fundamental value of the nominal exchange rate, the nominal exchange rate fluctuates around the relative price with long persistent swings. The lower panel shows that the persistent long swings in the real exchange rate (deviation from the PPP) seem to almost coincide with similar long swings in the real interest rate differential.

The theory of imperfect-knowledge-based economics developed in Frydman and Goldberg (2007 and 2011) shows that the pronounced persistence in the data may stem from forecasting behavior of rational individuals who must cope with imperfect knowledge. Frydman et al. (2008, 2012) argue that the persistent swings in the exchange rate around long-run benchmark values are consistent with such forcasting behavior.

Hommes (2005) and Hommes *et al.* (2005a, 2005b) developed models for a financial market populated by fundamentalists and chartists where fundamentalists use long-term expectations based on economic fundamentals and chartists are trend-followers using short-term expectations. For a detailed overview, see Hommes (2006). Positive feedback prevails when the latter dominate the market. In these models agents switch endogenously between a mean-reverting fundamentalist and a trend-following chartist strategy.

As the above theories predict, Figure 1 shows that there are two very persistent trends in the data, the upward sloping trend in relative prices and the long persistent swings in the nominal exchange rate. It also suggests that the long swings in the real exchange rate and the real interest rate differential are related. Juselius (2009) showed empirically that it was not possible to control for the persistence in the PPP without bringing the interest rates into the analysis.

But, while a graphical analysis can support intuition, it cannot replace hypotheses testing. To be convincing, testing needs to be carried out in the context of a fully specified statistical model. Juselius (2006, 2015) argues that a well-specified Cointegrated Vector AutoRegression (CVAR) model is an approximate description of the data generating process and, therefore, an obvious candidate for such a model. Hoover et al. (2009) argues that the CVAR allows the data to speak freely about the mechanisms that have generated the data. But, since the empirical and the theoretical model represent two different entities a bridging principle is needed. A so called theory-consistent CVAR scenario (Juselius, 2006, Juselius and Franchi, 2007, Møller, 2008) offers such a principle. It does so by translating basic assumptions underlying the theoretical model into testable hypotheses on the pulling and pushing forces of a CVAR model. One may say that such a scenario describes a specified set of testable empirical regularities one should expect to see in the data if the basic assumptions of the theoretical model were empirically valid. A theoretical model that passes the first check of such basic properties is potentially an empirically relevant model. M. Juselius (2010) demonstrated this for a new Keynesian Phillips curve model. Hoover and

Juselius (2014) argue that it may represent a designed experiment for data obtained by passive observations in the sense of Haavelmo (1944).

When linking a theoretical model with the CVAR, a major problem to be solved is how to associate the (mostly unobserved) expectations in the theoretical model with the observed data without forcing prior assumptions onto the empirical relationships. Juselius (2017) proposed a simple rule for how to handle unobserved expectations in a CVAR model and applied the idea to a standard monetary model for exchange rate determination. All basic hypotheses were translated into a CVAR scenario for data on prices, interest rates, and the nominal exchange rate for Germany and USA in the post Bretton Woods era. The theoretical hypotheses were tested and strongly rejected.

The purpose of this paper is to derive a CVAR scenario for a similar model for exchange rate determination using the same data but now assuming expectations are formed in the context of imperfect knowledge as a means of explaining the pronounced persistence from long-run equilibrium states. The empirical results provide a remarkable support for essentially every single testable hypothesis of the imperfect knowledge based scenario.

The paper is organized as follows. Section 2 discusses principles underlying a theory-consistent CVAR scenario, Section 3 introduces an imperfect knowledge based monetary model for exchange rate determination, Section 4 discusses how to anchor expectations to observable variables and derives their time-series properties. Section 5 derives a theory-consistent CVAR scenario, Section 6 introduces the empirical CVAR model, Section 7 tests the order of integration of individual variables/relations and Section 8 reports an identified structure of long-run relations strongly supporting the empirical relevance of imperfect knowledge and self-reinforcing feed-back behavior. Section 9 concludes.

2 Formulating a theory-consistent CVAR scenario[1]

The basic idea is to derive theoretically consistent persistency properties of variables and relations and compare these with observed magnitudes mea-

[1] This section is an adaptation of section 2 in Juselius (2017) where it was used to discuss a rational-expectations-based monetary model.

sured by the order of integration, such as $I(0)$ for a highly stationary process, $I(1)$ or near $I(1)$ for a first order persistent process, and $I(2)$ or near $I(2)$ for a second order persistent process.[2] One may argue that it is implausible that economic variables move away from their equilibrium values for infinite times and, hence, that most economic relations should be classified as either stationary, *near* $I(1)$ or *near* $I(2)$. But this does not exclude the possibility that over finite samples they exhibit a persistence that is indistinguishable from a unit root or a double unit root process. In this sense the classification of variables into single or double unit roots should be seen as a useful way of classifying the data into more homogeneous groups. For a detailed discussion, see Juselius (2012).

Unobservable expectations are often a crucial part of a theoretical model, whereas the empirical regularities to be uncovered by a CVAR analysis are based on the observed data. Therefore, we need a rule for how to associate the persistency properties of expectations with the ones of the observed variables. Rather than deriving a model-based solution to the unobserved expectations, we assume here that agents form expectations that are broadly consistent with the underlying theory. This means that they know the theory-consistent order of integration of the forecast variable, for example $x_t \sim I(1)$ or $x_t \sim I(2)$ and that their forecasts reflect this. If the theory assumes that Δx_t is unpredictable, then the forecast of x_t could be based on a random walk model. The idea is illustrated below:

1. $x_t \sim I(1)$, for example, $x_t = x_{t-1} + w_t$ where w_t is a stationary uncorrelated error. A theory-consistent data-based forecasting rule assuming imperfect knowledge can, for example, be

$$x^e_{t+1|t} = x_t + f_t, \tag{1}$$

where $x^e_{t+1|t}$ denotes the expected value at time t of the variable x_{t+1} and f_t is a forecast shock that might be uncorrelated or correlated over time. f_t accounts for the fact that agents in an imperfect knowledge world do not know for sure whether the postulated model is correctly specified, nor whether it will remain stable in the future. In such a world expectations are likely to influence outcomes, so that

$$x_{t+1} = x^e_{t+1|t} + \varepsilon_{t+1}. \tag{2}$$

[2] A highly persistent process is one for which a characteristic root is either close to or on the unit circle. See for example Elliot (1198) and Franchi and Johansen (2017).

If ε_t is white noise then (2) is consistent with economic agents which are rational by not making systematic forecast errors. However, in an imperfect knowledge world forecast errors might be correlated, but nonetheless stationary.

Inserting (1) in (2) leads to

$$\begin{aligned} x_{t+1} &= x_t + f_t + \varepsilon_{t+1} \\ &= x_t + w_{t+1}. \end{aligned}$$

If the forecast shock, f_t, is a random noise, then x_t is a random walk and the presumed time-series model remains the same. However, the data generating process for x_t will contain both the intrinsic shocks to the process and the forecast shocks and, therefore, the variance of the process will be affected. But if the forecast shock, f_t, is persistent, for example, as a result of technical trading, the process x_t will become more persistent and may ultimately result in $x_t \sim$ *near* $I(2)$. For example, the variable x_t may be a random walk in a period of regulation when speculative behavior along the trend is not a dominant feature and become a near $I(2)$ process after deregulation.

1. $x_t \sim I(2)$, for example $x_t = x_{t-1} + \Delta x_{t-1} + w_t$. A consistent forecasting rule under imperfect knowledge would be $x^e_{t+1|t} = x_t + \Delta x_t + f_t$ where f_t is a forecast shock due to imperfect knowledge. In this case

$$v_t = x^e_{t+1|t} - x_t = \Delta x_t + f_t$$

showing that the difference between the forecast and the observed value is an $I(1)$ process. Under the assumption that the forecast error is a white noise process we get:

$$\begin{aligned} x_{t+1} &= x^e_{t+1|t} + \varepsilon_{t+1} \\ &= x_t + \Delta x_t + f_t + \varepsilon_{t+1} \\ &= x_{t-1} + \Delta x_{t-1} + w_{t+1}. \end{aligned}$$

or

$$\Delta x_t = \Delta x_{t-1} + w_t$$

where $w_t = f_{t-1} + \varepsilon_t$. If f_t is an uncorrelated forecast error, Δx_t remains a random walk process. Again, economic agents are rational in the sense of not making systematic forecast errors, x_t will be affected by the intrinsic shocks to the process as well as the forecast shocks.

Assumption A exploits these simple ideas:

Assumption A When $x_t \sim I(1)$, $(x^e_{t+1|t} - x_t) = v_t$ is assumed to be $I(0)$ and when $x_t \sim I(2)$ it is assumed to be $I(1)$.

Note that Assumption A disregards $x_t \sim I(3)$, as it is considered empirically implausible, and $x_t \sim I(0)$, as it defines a non-persistent process for which cointegration and stochastic trends have no informational value.

Note also that $x_t \sim I(1)$ implies that $\Delta x_t \sim I(0)$, whereas $x_t \sim I(2)$ implies that $\Delta x_t \sim I(1)$ and $\Delta^2 x_t \sim I(0)$. Given Assumption A, we have that:

Corollary When $x_t \sim I(1)$, x_t, x_{t+1} and $x^e_{t+1|t}$ share the same common stochastic trend of order $I(1)$, i.e. they have the same persistency property. When $x_t \sim I(2)$, Δx_t, Δx_{t+1} and Δx^e_{t+1} share the same common stochastic $I(1)$ trend, i.e. they have the same persistency property.

Consequently, when $x_t \sim I(1)$, $\beta' x_t$, has the same persistency property as $\beta' x^e_{t+1|t}$ or $\beta' x_{t+1}$. When $x_t \sim I(2)$, $\beta' x_t + d' \Delta x_t$ has the same order of integration as $\beta' x_t + d' \Delta x^e_{t+1|t}$ and $\tau' \Delta x_t$ has the same order of integration as $\tau' \Delta x^e_{t+1|t}$ and $\tau' \Delta x_{t+1}$.[3] Thus, Assumption A allows us to make valid inference about a long-run equilibrium relation in a theoretical model even though the postulated behavior is a function of expected rather than observed outcomes.

Based on the above, the steps behind a theory-consistent CVAR scenario can be formulated as follows:

1. Express the expectations variable(s) as a function of observed variables. For example, according to Uncovered Interest Rate Parity (UIP), the expected change in the nominal exchange rate is equal to the interest rate differential. Hence, the persistency property of the latter is also a measure of the persistency property of the unobservable expected change in nominal exchange rate and can, therefore, be empirically tested.

2. For a given order of integration of the unobserved expectations variable and of the forecast shocks, v_t, derive the theory-consistent order of integration for all remaining variables and for the postulated behavioral relations of the system.

[3] Section 6 provides a definition of β, d and τ.

3. Translate the stochastically formulated theoretical model into a theory-consistent CVAR scenario by formulating the basic assumptions underlying the theoretical model as a set of testable hypotheses on cointegration relations and common trends.

4. Estimate a well-specified VAR model and check the empirical adequacy of the derived theory-consistent CVAR scenario.

When formulating a theory-consistent scenario one has to consider errors in the statistical model as well as postulated shocks in the theoretical model. The following notation will be used to discriminate between these different types of shocks: $\varepsilon_t \sim Niid(0, \sigma_\varepsilon^2)$ is a white noise process; e_t is a stationary deviation from a long-run equilibrium relation; $v_t = x_{t+1}^e - x_t$ is a forecast shock; and $u_t = f(\varepsilon_t)$ is an unobserved 'structural' shock assumed to be a linear function of the shocks to the system.

3 Imperfect knowledge and the nominal exchange rate

While essentially all asset price models assume that today's price depends on expected future prices, models based on rational expectations versus imperfect knowledge differ with respect to how agents are assumed to make forecasts and how they react on forecasting errors. In REH-based models agents are adjusting back toward the equilibrium value of the theoretical model after having made a forecast error, implying that expectations are endogenously adjusting to the proposed true model. However, when "perfect knowledge" is replaced by imperfect knowledge, the role of expectations changes.

In IKE-based models, individuals recognize that they do not know the (or may not believe in the existence of a) "true" model. They also revise their forecasting strategies as changes in policy, institutions, and other factors cause the process to undergo structural change at times and in ways that cannot be foreseen. Frydman and Goldberg, 2007) show that these revisions (or expectational shocks) have a permanent effect on market outcomes and thus act as an exogenous force in the model.

If PPP prevails in the goods market, one would expect the nominal exchange rate to appoximately follow relative prices and the real exchange rate,

q_t,

$$q_t = s_t - (p_{d,t} - p_{f,t}) \sim I(0) \tag{3}$$

to be stationary.[4]

If uncovered interest rate parity prevails in the foreign currency market, the interest rate differential, $(i_{d,t} - i_{f,t})$, should reflect the expected change in the exchange rate, $s^e_{t+1|t} - s_t$. But, interest rate differentials tend to move in long persistent swings whereas the change in nominal exchange rates are characterized by a pronounced short-run variability. The excess return puzzle describes the empirical fact that the excess return, exr_t, defined as

$$exr_t = (i_d - i_f)_t - (s^e_{t+1|t} - s_t), \tag{4}$$

often behaves like a nonstationary process. To solve the puzzle it has been customary to add a risk premium, rp_t, to (4), which usually is a measure of the volatility in the foreign currency market. But, although a risk premium can account for exchange rate volatility, it cannot account for the persistent swings in the interest rate differential. To control for the latter, Frydman and Goldberg (2007) proposed an uncertainty premium, up_t, measuring agents' *loss* aversion due to imperfect knowledge to be added to the UIP.[5] The Uncertainty Adjusted UIP (UA-UIP) is defined as

$$(i_d - i_f)_t = (s^e_{t+1|t} - s_t) + rp_t + up_t, \tag{5}$$

describing an economy in which loss averse financial actors require a minimum return—an uncertainty premium—to speculate in the foreign exchange market. When the exchange rate moves away from its long-run value, the uncertainty premium starts increasing until the direction of the exchange rate reverses towards equilibrium. Frydman and Goldberg (2007) argue that the uncertainty premium is likely to be closely associated with the PPP gap, but that other gaps, for example the current account balance, could play a role as well. Using the PPP gap, the UA-UIP is formulated as

$$(i_d - i_f)_t = (s^e_{t+1} - s_t) + rp_t + f(p_d - p_f - s)_t. \tag{6}$$

[4] The PPP puzzle describes the empirical fact that the real exchange rate often tends to move in long persistent swings and that the volatility of the nominal exchange rate, s_t, is much larger than the one of relative prices.

[5] The assumption that agents are loss averse, rather than risk averse, builds on the prospect theory by Kahneman and Tversky (1979).

where rp_t is associated with short-term changes in interest rates, inflation rates, nominal exchange rates, etc. Thus, the expected change in the nominal exchange rate is not directly associated with the observed interest rate differential but with the interest rate differential corrected for the PPP gap and the risk premium.

4 The persistence of the PPP gap

That agents have diverse forecasting strategies is a defining feature of imperfect knowledge based models - bulls hold long positions of foreign exchange and bet on appreciation while bears hold short positions and bet on depreciation. Speculators are likely to change their forecasting strategies depending on how far away the price is from the long-run benchmark value. For example, Hommes (2006) assumes that the proportion of chartists relative to fundamentalists decreases as the PPP gap grows. When the exchange rate is not too far from its fundamental value, the proportion of chartists is high and the rate behaves roughly as a random walk. When it has moved to a far-from-equilibrium region, the proportion of fundamentalists is high and the real exchange rate becomes mean-reverting.

Frydman and Goldberg (2007, 2011) explain the persistence of the PPP gap by non-constant parameters due to forecasting under imperfect knowledge. Following this idea, financial actors are assumed to know that, in the long run, the nominal exchange rate follows the relative price of the two countries whereas in the short run it reacts on a number of other determinants, z_t, which may include, for example, changes in interest rates, relative incomes and consumption, etc. Therefore, financial actors attach time-varying weights, B_t, to relative prices depending on how far away the nominal exchange rate is from its fundamental PPP value, i.e.,

$$s_t = A + B_t(p_{d,t} - p_{f,t}) + z_t. \tag{7}$$

The change in the nominal exchange rate can then be expressed as:

$$\Delta s_t = B_t \Delta(p_{d,t} - p_{f,t}) + \Delta B_t(p_{d,t} - p_{f,t}) + \Delta z_t.$$

Frydman and Goldberg (2007) make the assumption that $|\Delta B_t(p_{d,t} - p_{f,t})| \ll |B_t \Delta(p_{d,t} - p_{f,t})|$. This is backed up by simulations showing that a change in ΔB_t has to be implausibly large for $\Delta B_t(p_{d,t} - p_{f,t})$ to have a noticeable effect on Δs_t. Therefore, we assume that

$$\Delta s_t \simeq B_t \Delta(p_{d,t} - p_{f,t}) + \Delta z_t. \tag{8}$$

To study the properties of this type of time-varying parameter model, Tabor (2017) considers the model:

$$\begin{aligned} \Delta Y_t &= \alpha(Y_{t-1} - \beta_t X_{t-1}) + \varepsilon_{y,t} \\ \Delta X_t &= \varepsilon_{x,t}. \end{aligned} \tag{9}$$

He generates the data with $\alpha = -1$ and $\beta_t = \beta_0 + \rho\beta_{t-1} + \varepsilon_{\beta,t}$, so that $E(\beta_t) = \frac{\beta_0}{1-\rho} = \beta$ for $\rho = \{0.0, 0.5, 0, 95, 1.0\}$. $\alpha = -1$ implies that the adjustment of Y_t back to $\beta_t' X_t$ is immediate. Instead of estimating a time-varying parameter model, Tabor fits a *constant* parameter CVAR model to the simulated data, so that $(\beta_t - \beta)X_t$ becomes part of the CVAR residual. It corresponds approximately to the forecast shock f_t in the previous section. The simulation results show that the closer ρ is to 1, the more persistent is the estimated gap term, $Y_t - \hat{\beta}' X_t$, and the smaller is the estimated adjustment coefficient α (while still highly significant). As long as $\rho < 1$, the mean of the estimated $\hat{\beta}$ approximately equals its true value β.

Thus, the pronounced persistence that often characterizes constant-parameter asset price models can potentially be a result of time-varying coefficients due to forecasting under imperfect knowledge.

Assume now that agents are forecasting the change in the nominal exchange rate by using (8), i.e. by relating Δs_t to relative inflation rates with a time-varying coefficient β_t,

$$\Delta s_t = \beta_t \Delta(p_1 - p_2)_t + \varepsilon_{s,t}, \tag{10}$$

where $\beta_t = b_0 + \rho\beta_{t-1} + \varepsilon_{\beta,t}$ and $E(\beta_t) = (b_0/1 - \rho) = 1$. If ρ is close, but not equal to one, the Tabor results imply that $\Delta s_t - \Delta(p_1 - p_2)_t = \Delta q_t$ is likely to be a persistent near $I(1)$ process and, hence, that q_t is a near $I(2)$ process.

The near $I(2)$ approximation is useful as it allows for a linear VAR representation and, hence can make use of a vast econometric literature on estimation and testing. Another option is to use a non-linear adjustment model, for example proposed by Bec and Rahbek (2004).

5 Associating expectations with observables in an imperfect knowledge based model

The first step of a theory-consistent CVAR scenario is to formulate a consistent description of the time series properties of the data given some basic assumption of agents' expectations formation. In the foreign currency market, expectations are primarily feeding into the model through the UA-UIP condition (6). It states that the expected change in the nominal exchange rate is given by the interest rate differential corrected for an uncertainty and a risk premium.

The latter is assumed to be associated with a volatility measure and, hence, to be stationary and the former with persistent gap effects often found to be near $I(2)$. If interest rate differentials are affected by a risk and an uncertainty premium, then so are the individual interest rates:

$$i_{j,t} = i_{j,t-1} + \omega_{j,t} + \Delta rp_{j,t} + \varepsilon_{j,t} \qquad j = d, f \tag{11}$$

where $\varepsilon_{j,t}$ stands for white noise, $\omega_{j,t} = \Delta up_{j,t}$ stands for the change in the uncertainty premium. A very persistent uncertainty premium will always dominate a stationary risk premium and the latter will, for notational simplicity, hereafter be part of the error term $\varepsilon_{j,t}$.

In the foreign exchange market, the uncertainty premium is often measured by the PPP gap which has often been found to be near $I(2)$ (Johansen et al. (2010), Juselius and Assenmacher (2017)). Therefore, the change in the uncertainty premium, ω_t, is assumed to follow a persistent $AR(1)$ process:

$$\omega_{j,t} = \rho_j \omega_{j,t-1} + \varepsilon_{\omega,j,t}, \ \text{ and } \varepsilon_{\omega,j,t} \sim Niid(0, \sigma^2_{\varepsilon_{\omega},,j}) \qquad j = d, f.$$

The autoregressive coefficient $\rho_{j,t}$ is considered to be approximately 1.0 in periods when the real exchange rate is in the neighborhood of its long-run benchmark value and $\ll 1.0$ when it is far away from this value. Since the periods when $\rho_{j,t} \ll 1.0$ are likely to be short compared to the ones when $\rho_{j,t}$ ≈ 1.0, the average $\bar{\rho}_j$ is likely to be close to 1.0 so that

$$\omega_{j,t} = \sum_{i=1}^{t} \bar{\rho}_j^{t-i} \varepsilon_{\omega,j,i} + \bar{\rho}_j^t \omega_0$$

is a near $I(1)$ process. Integrating (11) over t gives:

$$
\begin{aligned}
i_{j,t} &= i_{j,0} + \sum_{i=1}^{t} \varepsilon_{j,i} + \sum_{i=1}^{t} \omega_{j,i}, \\
&= i_{j,0} + \sum_{i=1}^{t} \varepsilon_{j,i} + up_{j,t}, \qquad j = d, f
\end{aligned}
\tag{12}
$$

where

$$
up_{j,t} = \sum_{s=1}^{t} \sum_{i=1}^{s} \bar{\rho}_j^{s-i} \varepsilon_{\omega,j,i} + \bar{\rho}_j \omega_0 \sum_{i=1}^{t} \bar{\rho}_j^{s} + up_{j,0}.
$$

Thus, under the near $I(1)$ assumption of $\omega_{j,t}$, $\sum_{i=1}^{t} \omega_{j,i}$ is near $I(2)$ and so are nominal interest rates. Note, however, that the shocks to the uncertainty premium, while persistent, are likely to be tiny compared to the interest rate shocks, capturing the empirical fact that the variance of the process is usually much larger than the variance of the drift term (for more details, see Juselius, 2014). The process (12) is consistent with persistent swings of shorter and longer durations typical of observed interest rates. The interest rate differential can be expressed as:

$$
(i_{d,t} - i_{f,t}) = (i_{d,0} - i_{f,0}) + up_t + \sum_{i=1}^{t} (\varepsilon_{d,i} - \varepsilon_{f,i}). \tag{13}
$$

where $up_t = up_{d,t} - up_{f,t}$. The term $\sum_{i=1}^{t}(\varepsilon_{d,i} - \varepsilon_{f,i})$ implies a first order stochastic trend in the interest rate differential, unless $\sum_{i=1}^{t} \varepsilon_{d,i} = \sum_{i=1}^{t} \varepsilon_{f,i}$, which would be highly unlikely in an imperfect knowledge world. As the uncertainty premium, $up_{j,t}$, is assumed to be near $I(2)$, the differential up_t is also near $I(2)$, unless $up_{d,j} - up_{f,j} = 0$. Equality would, however, imply no uncertainty premium in the foreign currency market, which again violates the imperfect knowledge assumption.

Approximating up_t with a fraction, ϕ, of the PPP gap, $(p_d - p_f - s)_t$ gives:

$$
(i_{d,t} - i_{f,t}) - \phi(p_{d,t} - p_{f,t} - s_t) = (i_{d,0} - i_{f,0}) + \sum_{i=1}^{t} (\varepsilon_{d,i} - \varepsilon_{f,i}), \tag{14}
$$

showing that the interest rate differential corrected for the uncertainty premium is $I(1)$.

The Fisher parity defines the real interest rate as $r_{j,t} = i_{j,t} - \Delta p^e_{j,t+1|t}$. Using $\Delta p^e_{j,t+1|t} = \Delta p_{j,t} + v_{p_j,t}$ we get

$$r_{j,t} = i_{j,t} - \Delta p_{j,t} - \nu_{p_j,t}, \qquad j = d, f \tag{15}$$

Alternatively, (15) can be expressed for the inflation rate:

$$\Delta p_{j,t} = i_{j,t} - r_{j,t} + v_{p_j,t}, \qquad j = d, f. \tag{16}$$

Inserting (12) in (16) gives:

$$\Delta p_{j,t} = i_{j,0} + \sum_{s=1}^{t} \varepsilon_{j,s} + up_{j,t} - r_{j,t} + \nu_{p_j,t} \qquad j = d, f. \tag{17}$$

It appears that the inflation rate would be near $I(2)$ (which is implausible) unless $r_{j,t}$ and up_t cointegrate. Goods prices are generally determined by demand and supply in competitive international goods markets and only exceptionally affected by speculation. If nominal interest rates exhibit persistent swings but consumer price inflation does not, then the real interest rate will also exhibit persistent swings. Thus, the uncertainty premium, $\omega_{j,t}$, should affect nominal interest rates, but not the price of goods, implying that up_t is part of $r_{j,t}$ rather than the inflation rate. Therefore, $up_{j,t} - r_{j,t}$ is $I(0)$, the inflation rate is $I(1)$ and the real interest rate is near $I(2)$. This implies a delinking of the inflation rate and the nominal interest rate as a stationary Fisher parity relationship (see Frydman and Goldberg, 2006 and Frydman et al., 2008).

Integrating (17) over t gives an expression for prices:

$$p_{j,t} = i_{j,0} \times t + \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_{j,i} + \sum_{i=1}^{t} \nu_{p_j,i} + p_{j,0}, \qquad j = d, f \tag{18}$$

showing that prices are $I(2)$ around a linear trend.

The inflation spread between the two countries can be expressed as

$$(\Delta p_{d,t} - \Delta p_{f,t}) = (i_{d,0} - i_{f,0}) + \sum_{s=1}^{t} (\varepsilon_{d,s} - \varepsilon_{f,s}) + (\nu_{p_d,t} - \nu_{p_f,t}), \tag{19}$$

showing that the inflation spread is $I(1)$. Integrating (19) over t gives an expression for the relative price:

$$p_{d,t} - p_{f,t} = p_{d,0} - p_{f,0} + (i_{d,0} - i_{f,0})t + \sum_{i=1}^{t} \sum_{s=1}^{i} (\varepsilon_{d,s} - \varepsilon_{f,s}) + \sum_{i=1}^{t} (\nu_{p_d,i} - \nu_{p_f,i}), \tag{20}$$

showing that the relative price is $I(2)$ with a linear trend.

An expression for the change in nominal exchange rates can be found from the uncertainty adjusted UIP:

$$\Delta s_t = (i_d - i_f)_{t-1} - up_{t-1}. \tag{21}$$

Inserting the expression for $(i_d - i_f)_t$ from (13) gives:

$$\begin{aligned} \Delta s_t &= (i_{d,0} - i_{f,0}) + \sum_{i=1}^{t-1} (\varepsilon_{d,i} - \varepsilon_{f,i}) + up_t - up_{t-1}, \\ &= (i_{d,0} - i_{f,0}) + \sum_{i=1}^{t-1} (\varepsilon_{d,i} - \varepsilon_{f,i}) + \Delta up_t \end{aligned}$$

Summing over t gives an expression for the nominal exchange rate:

$$s_t = s_0 + (i_{d,0} - i_{f,0})t + \sum_{i=1}^{t-1} \sum_{s=1}^{i} (\varepsilon_{d,s} - \varepsilon_{f,s}) + up_t. \tag{22}$$

Thus, the nominal exchange rate contains a local linear trend originating from the initial value of the interest rate differential, an $I(2)$ trend describing the stochastic trend in the relative price and a near $I(2)$ trend describing the long swings due to the uncertainty premium.

An expression for the real exchange rate can now be obtained by subtracting (20) from (22):

$$s_t - p_{d,t} + p_{f,t} = (s_0 - p_{d,0} - p_{f,0}) - (\varepsilon_{d,t} - \varepsilon_{f,t}) - up_t - \sum_{i=1}^{t} (\nu_{p_{d,i}} - \nu_{p_{f,i}}), \tag{23}$$

showing that the real exchange rate is a near $I(2)$ process due to the uncertainty premium. Thus, under imperfect knowledge both the nominal and the real exchange rate will show a tendency to move in similar long swings. Figure 1 illustrates that this has been the case for Germany and USA in the post-Bretton Woods - pre-EMU period.

Finally, inserting the expression for (19) in (14) gives:

$$(i_{d,t} - i_{f,t}) - \phi(p_{d,t} - p_{f,t} - s_t) = \Delta p_{d,t} - \Delta p_{f,t} + (\nu_{p_{d,t}} - \nu_{p_{d,t}}), \tag{24}$$

showing that the real interest rate differential cointegrates with the PPP gap to a stationary relation as assumed in Frydman and Goldberg (2007).

Thus, imperfect knowledge predicts that both exchange rates and interest rates in nominal and real values are integrated of the same order and that the Fisher parity does not hold as a stationary condition.

6 A theory-consistent CVAR scenario for imperfect knowledge

The first step in a scenario describes how the underlying stochastic trends are assumed to load into the data provided the theory model is empirically correct. The results of the previous section showed that the data vector $x_t = [p_{d,t}, p_{f,t}, s_t, i_{d,t}, i_{f,t}]$ should be integrated of order two and be affected by two stochastic trends, one originating from twice cumulated interest rate shocks, $\sum_{s=1}^{t}\sum_{i=1}^{s}\varepsilon_{j,i}$, and the other from the uncertainty premium being near $I(2)$. Two stochastic $I(2)$ trends that load into five variables implies three relations which are cointegrated $CI(2,1)$. These relations can be decomposed into r relations, $\beta' x_t$, that can become stationary by adding a linear combination of the growth rates, $d'\Delta x_t$, and s_1 linear combinations $\beta'_{\perp 1}x_t$ which can only become stationary by differencing. Thus, stationarity can be achieved by r polynomially cointegrated relations $(\beta' x_t + d'\Delta x_t) \sim I(0)$ and s_1 medium-run relations among the differences $\beta'_{\perp 1}\Delta x_t$. For a more detailed exposition see, for example, Juselius (2006, Chapter 17).

The three $CI(2,1)$ relations are consistent with different choices of r and s_1 as long as $r + s_1 = p - s_2 = 3$ where s_2 is the number of $I(2)$ trends. Theoretically, (14) predicts that $(i_{d,t} - i_{f,t})$ and $(s_t - p_{d,t} + p_{f,t})$ are cointegrated $CI(2,1)$ and (24) that $(i_{d,t} - i_{f,t})$, $(s_t - p_{d,t} + p_{f,t})$ and $(\Delta p_{d,t} + \Delta p_{f,t})$ are cointegrated $CI(2,2)$, so $\{3 \geq r \geq 1\}$. The following two cases satisfy this condition: $\{r = 2, s_1 = 1, s_2 = 2\}$, and $\{r = 3, s_1 = 0, s_2 = 2\}$. Juselius (2017) finds that the trace test supports $\{r = 2, s_1 = 1, s_2 = 2\}$ and the scenario will be formulated for this case. The pushing force of this scenario comprises three autonomous shocks, $u_{1,t}$, $u_{2,t}$ and $u_{3,t}$, two of which cumulate twice to produce the two $I(2)$ trends, while the third shock cumulates only once to produce an $I(1)$ trend. The pulling force consists of two polynomially cointegrated relations and one medium-run relation between growth rates.

Based on the derivations in the previous section, it is possible to impose testable restrictions on some of the coefficients in the scenario. For example, relation (18) assumes that the uncertainty premium does not affect goods prices so that $(c_{21}, c_{22}) = 0$. Relation (23) assumes that the long-run stochastic trend in relative prices and nominal exchange rate, $\sum_{i=1}^{t-1}\sum_{s=1}^{i}(\varepsilon_{d,s} - \varepsilon_{f,s})$, cancels in $(p_d - p_f - s)$, so that $(c_{11} - c_{12}) = c_{13}$. Relation (12) assumes that the relative price trend does not load into the two interest rates, so that

($c_{14} = c_{15} = 0$). Based on these restrictions, the imperfect knowledge scenario is formulated as:

$$\begin{bmatrix} p_d \\ p_f \\ s \\ i_d \\ i_f \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{12} & 0 \\ c_{11} - c_{12} & c_{23} \\ 0 & c_{24} \\ 0 & c_{25} \end{bmatrix} \begin{bmatrix} \Sigma\Sigma u_1 \\ \Sigma\Sigma u_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \\ b_{14} & b_{24} & b_{34} \\ b_{15} & b_{25} & b_{35} \end{bmatrix} \begin{bmatrix} \Sigma u_1 \\ \Sigma u_2 \\ \Sigma u_3 \end{bmatrix} + Z_t, \tag{25}$$

where u_1 is a relative price shock and u_2 a shock to the uncertainty premium. Section 2 showed that forecasting under imperfect knowledge is able to generate an additional trend in the data. Hence u_3 is considered a forecast shock and Σu_3 can be interpreted as a medium-run trend originating from forecast shocks. Consistent with the derivations in the previous section all variables are assumed to be $I(2)$. Since the two prices and the exchange rate share two stochastic $I(2)$ trends, there exists just one relation, $(p_d - w_1 p_f - w_2 s) \sim I(1)$ with $(w_1, w_2) \neq 1$.

Based on (25) the following three $CI(2,1)$ cointegration relations, $\tau' x_t$, can be found:

1. $\{(p_d - p_f - s) - a_1(i_d - i_f) - \gamma_1 t\} \sim I(1)$ if $c_{23} + a_1(c_{24} - c_{25}) = 0$
2. $(i_d - a_2 p_d - a_3 s - \gamma_2 t) \sim I(1)$, if $c_{24} - a_3 c_{23} = 0$ and $a_2 c_{11} + a_3(c_{11} - c_{12}) = 0$
3. $(i_f - a_4 p_f - a_5 s - \gamma_3 t) \sim I(1)$, if $c_{25} - a_5 c_{23} = 0$ and $a_4 c_{12} + a_5(c_{11} - c_{21}) = 0$

The first relation corresponds to (14), whereas the two remaining relations, while not explicitly discussed above, are consistent with the theoretical model set-up which the next section will demonstrate. Note also that the inclusion of a linear trend in the relation means that trend-adjusted price/nominal exchange rate rather than the price itself is the relevant measure. Any linear combination of the three relations are of course also $CI(2,1)$.

The case ($r = 2, s_1 = 1, s_2 = 2$) implies two multicointegrating relations, $\beta' x_t + d\Delta x_t$, and one medium-run relation between the differences, $\beta_{\perp,1}\Delta x_t$. To obtain stationarity, two of the $CI(2,1)$ relations need to be combined with growth rates in a way that cancels the $I(1)$ trends. As an illustration, the scenario restrictions consistent with stationarity are given below for the first polynomially cointegrated relation given by (24). The corresponding scenario restrictions on the remaining two relations can be similarly derived. The second polynomially cointegrated relation is similar to (15). It will be further discussed in the next section.

1. $\{ppp - a_1(i_d - i_f) - a_6(\Delta p_d - \Delta p_f) - \gamma_1 t\} \sim I(0)$, if $c_{23} + a_1(c_{24} - c_{25}) = 0$, $\{(b_{11} - b_{12} - b_{13}) - a_1(b_{14} - b_{15}) - a_6 c_{11}\} = 0$,
$\{(b_{21} - b_{22} - b_{23}) - a_1(b_{24} - b_{25}) + a_6 c_{21}\} = 0$, and
$\{(b_{31} - b_{32} - b_{33}) + a_6(b_{34} - b_{35})\} = 0$

2. $(i_d - a_2 p_d - a_3 s - a_7 \Delta p_d - \gamma_2 t) \sim I(0)$,

and one medium-run relation, $\beta'_{\perp 1}\Delta x_t$:

1. $(\Delta p_d + d_1 \Delta p_f + d_2 \Delta s + d_3 \Delta i_f) \sim I(0)$.

Note that linear combinations of the proposed stationary relations are, of course, also stationary.

7 The empirical specification of the *CVAR* model

The empirical analysis is based on German-US data for the post-Bretton Woods, pre-EMU period[6]. The sample starts in 1975:8 and ends in 1998:12 when the Deutshmark was replaced by the Euro. The empirical VAR correponds to the one in Juselius (2017) and has two lags and contains a few dummy variables, D_t:

$$\begin{aligned}\Delta^2 x_t &= \Gamma \Delta x_{t-1} + \Pi x_{t-1} + \mu_0 + \mu_{01} Ds_{91.1,t} + \mu_1 t + \mu_1 t_{91.1} \\ &\quad + \phi_1 D_{tax,t} + \phi_2 Dp_{86.2,t} + \phi_3 Dp_{91.2,t} + \varepsilon_t,\end{aligned} \tag{26}$$

where $x_t = [p_{d,t}, p_{f,t}, s_t, i_{d,t}, i_{f,t}]$ and p_t stands for CPI prices, s_t for the Dmk/dollar exchange rate, i_t for long-term bond rates, a subscript d for Germany and a subscript f for USA, t is a linear trend starting in 1975:3,

[6] All estimates are based on a recent Beta version of CATS 3 in OxMetrics (Doornik et al., 2017).

$t_{91:1,t}$ allows the linear trend to have a different slope from 1991:1 onwards, and $Ds_{91:1,t}$ is a step dummy also starting in 1991:1, controlling for the reunification of East and West Germany. $D_{tax,t}$ is an impulse dummy accounting for three different excise taxes introduced to pay for the German reunification, $Dp_{86.2}$ $Dp_{86.2}$ is controlling for a large shock to the US price and bond rate in connection with the Plaza Accord, and $Dp_{91.2}$ accounts for a large shock to the exchange rate after the reunification.

The hypothesis that x_t is $I(1)$ is formulated as a reduced rank hypothesis, $\Pi = \alpha\beta'$, where α is $p \times r$ and β is $p_1 \times r$ with $p_1 = p+2$. The hypothesis that x_t is $I(2)$ is formulated as an additional reduced rank hypothesis, $\alpha'_\perp \Gamma \beta_\perp = \xi\eta'$, where ξ, η are $(p-r) \times s_1$ and $\alpha_\perp, \beta_\perp$ are the orthogonal complements of α, β respectively (Johansen, 1992, 1995). The first reduced rank condition is associated with the levels of the variables and the second with the differenced variables. The intuition is that the differenced process also contains unit roots when data are $I(2)$. Juselius (2017) finds that the maximum likelihood trace test (Nielsen and Rahbek, 2007) support the case $\{r = 2, s_1 = 1, s_2 = 2\}$.

Since the $I(2)$ condition is formulated as a reduced rank on the transformed $\mathbf{\Gamma}$ matrix, the latter is no longer unrestricted as in the $I(1)$ model. To circumvent this problem we use the following parameterization (see Johansen, 1997, 2006, Doornik et al., 2017):

$$\Delta^2 x_t = \alpha \left[\begin{pmatrix} \beta \\ \tau_{01} \\ \tau_0 \end{pmatrix}' \begin{pmatrix} x_{t-1} \\ t_{91:1,t-1} \\ t-1 \end{pmatrix} + \begin{pmatrix} d \\ d_{01} \\ d_0 \end{pmatrix}' \begin{pmatrix} \Delta x_{t-1} \\ D_{s91:1,t-1} \\ 1 \end{pmatrix} \right]$$

$$+ \zeta \begin{pmatrix} \tau \\ \tau_{01} \\ \tau_0 \end{pmatrix}' \begin{pmatrix} \Delta x_{t-1} \\ D_{s91:1,t-1} \\ 1 \end{pmatrix} + \Phi_1 D_{tax,t} + \Phi_2 Dp_{86.2,t} + \phi_3 Dp_{91.2,t} + \varepsilon_t,$$

$$t = 1975.09 - 1998.12 \tag{27}$$

where $\tau = [\beta, \beta_{\perp 1}]$ and d is proportional to $\tau_\perp$. In (26) an unrestricted constant (and step dummy) will cumulate twice to a quadratic trend, and a linear (broken) trend to a cubic trend. By specifying the broken trend to be restricted to the β part and the differenced broken trend to the d part of model (27) these undesirable effects are avoided. For more details, see Doornik et al. (2017), this volume, Kongsted et al. (1999), Juselius (2006, Chapter 17).

8 Testable hypotheses on integration and cointegration

Section 4 showed that all the five variables are individually (near) $I(2)$ under imperfect knowledge. Relevant linear combinations of the variables are characterized by the following testable hypotheses:

- $(p_{d,t} - p_{f,t}) \sim near\ I(2)$,
- $s_t \sim near\ I(2)$,
- $(i_{d,t} - i_{f,t}) \sim near\ I(2)$,
- $(s_t - p_d - p_f)_t \sim near\ I(2)$,
- $\{(i_{d,t} - i_{f,t}) - b_1(s_t - p_d - p_f)_t\} \sim near\ I(1)$

The above hypotheses can be formulated as a known vector b_1 in τ, i.e. $\tau = (b_1, b_{1\perp}\varphi)$ where $b_{1\perp}\varphi$ defines the remaining vectors to be in the orthogonal space of b_1.[7] For example $b_1' = [0, 0, 0, 1, 0, 0, 0]$ tests whether the German bond rate is a unit vector in τ. If not rejected, $b_{d,t}$ can be considered $I(1)$, if rejected $I(2)$. Note, however, that Section 4 found that prices and the nominal exchange rate contain both deterministic and stochastic trends and the tests have to take this into account. For example, $H_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ tests whether trend-adjusted German price is $I(1)$. To allow for a deterministic trend is important as it would otherwise bias the tests towards a rejection of $I(1)$.

Table 1 reports the test results. Except for the German bond rate, the results are supporting the imperfect knowledge hypothesis that all variables are near $I(2)$. Even though, the $I(1)$ hypothesis of the nominal and the real exchange rate is borderline acceptable, the low p-value is more in line with *near* $I(2)$ than $I(1)$.[8] That the German bond rate could be rejected as (near) $I(2)$ with a p-value of 0.45 may indicate that the German bond rate was less affected by speculative movements than the US rate. Similar results have

[7] Note that in the $I(1)$ model this type of hypothesis is testing whether a variable/relation is $I(0)$, whereas in the $I(2)$ model whether it is $I(1)$.

[8] Juselius (2017) shows that the rational expectations hypothesis predict that the real exchange rate should behave like an $I(0)$ or possibly a near-$I(1)$ process.

Table 1: Testing hypotheses of I(1) versus I(2)

		p_d	p_f	s	b_1	b_2	t_{91}	t	$\chi^2(v)$	$p-val$
$\mathcal{H}_1$	τ_1'	1	–	–	–	–	$*$	$*$	31.9(2)	0.00
$\mathcal{H}_2$	τ_1'	–	1	–	–	–	$*$	$*$	32.5(2)	0.00
$\mathcal{H}_3$	τ_1'	1	-1	–	–	–	$*$	$*$	24.7(2)	0.00
$\mathcal{H}_4$	τ_1'	–	–	1	–	–	$*$	$*$	5.3(2)	0.07
$\mathcal{H}_5$	τ_1'	1	-1	-1	–	–	–	–	8.4(4)	0.07
$\mathcal{H}_6$	τ_1'	–	–	–	1	–	–	–	3.7(4)	0.45
$\mathcal{H}_7$	τ_1'	–	–	–	–	1	–	–	12.6(4)	0.01
$\mathcal{H}_8$	τ_1'	–	–	–	1	-1	–	–	12.4(4)	0.01
$\mathcal{H}_9$	τ_1'	1	-1	-1	a	$-a$	–	–	3.0(5)	0.70

been found in Juselius and Assenmacher (2017). Hypothesis $\mathcal{H}_9$ corresponds to (14) and support the results in Section 4 that the PPP gap and the interest rate differential should cointegrate from $I(2)$ to $I(1)$.

9 The pulling forces

The long and persistent swings away from long-run equilibrium values visible in Figure 1 suggest the presence of self-reinforcing feedback mechanisms in the system. Such behavior is likely show up as a combination of equilibrium error increasing (positive feedback) and error correcting behavior (negative feedback) either in the adjustment of the two polynomially cointegrating relations, $\alpha(\beta^{'}x_t + d'\Delta x_t)$, or in the adjustment to the changes in the τ relations, $\zeta\tau'\Delta x_t$. Juselius and Assenmacher (2016) argue that the adjustment dynamics in the $I(2)$ model, given by α and d, can be interpreted as two levels of equilibrium correction: the d adjustment describing how the growth rates, Δx_t, adjust to the long-run equilibrium errors, $\beta^{'}x_t$ and the α adjustment describing how the acceleration rates, $\Delta^2 x_t$, adjust to the dynamic equilibrium relations, $\beta^{'}x_t + d'\Delta x_t$. The interpretation of d as a medium-run adjustment is, however, conditional on $\alpha \neq 0$.

The adjustment dynamics are illustrated for the variable $x_{i,t}$:

$$\Delta^2 x_{i,t} = \cdots \sum_{j=1}^{r} \alpha_{ij} \sum_{i=1}^{p} (\beta_{ij} x_{i,t-1} + d_{ij} \Delta x_{i,t-1}) + \sum_{j=1}^{r} \zeta_{ij} \sum_{i=1}^{p} (\beta_{ij} \Delta x_{i,t-1}) + \varepsilon_{i,t},$$
$$i = 1, ..., p$$

The signs of β, d, and α determine whether the variable $x_{i,t}$ is error increasing or error correcting in the medium and the long run. If $\alpha_{ij}\beta_{ij} < 0$ or/and $\alpha_{ij}d_{ij} < 0$, then the acceleration rate, $\Delta^2 x_{i,t}$, is equilibrium correcting to $(\beta_j^{'} x_t + d_j^{\prime} \Delta x_t)$; if $d_{ij}\beta_{ij} > 0$ (given $\alpha_{ij} \neq 0$), then $\Delta x_{i,t}$, is equilibrium error correcting to $\beta_j^{'} x_t$; if $\zeta_{ij}\tau_{ij} < 0$ then $\Delta^2 x_{i,t}$ is equilibrium correcting to $\tau_j^{\prime} \Delta x_{t-1}$. In all other cases the system is equilibrium error increasing.

The two stationary polynomially cointegrating relations, $\beta_i^{'} x_t + d_i^{\prime} \Delta x_t$, $i = 1, 2$ can be interpreted as dynamic equilibrium relations in the following sense: When data are $I(2)$, $\beta' x_t$ is in general $I(1)$ describing a very persistent static equilibrium error. In a market economy, a movement away from equilibrium would trigger off a compensating movement elsewhere else in the system. The $I(2)$ structure tells us that it is the changes of the system, $d' \Delta x_t$, that adjust to the static equilibrium error either in an error-correcting manner bringing $\beta^{'} x_t$ back towards equilibrium, or in an error-increasing manner, pushing $\beta^{'} x_t$ further away from equilibrium.

However, as long as all characteristic roots of the model are inside the unit circle, any equilibrium error increasing behavior is compensated by error correcting behavior somewhere else in the system. For example, speculative behavior may push the real exchange rate away from equilibrium but an increasing uncertainty premium will eventually pull it back toward equilibrium. The largest unrestricted root in our model is 0.48, so the system is stable and all persistent movements in the data are properly accounted for.

Table 2 reports an overidentified structure of $\beta x_t + d' \Delta x_t$ and an unrestricted estimate of $\beta_{\perp 1}$. For a given identified β, the d parameters are uniquely determined as long as d is proportional to $\tau_{\perp}$. See Doornik (2017) in this special issue. The standard errors of β are derived in Johansen (1997)[9] and those of d by the delta method in Doornik (2017). To facilitate interpretation, statistically insignificant adjustment coefficients (with a t-ratio $<$

[9] Note that all β coefficients have t ratios that are sufficiently large to be statistically significant also after a near unit root correction. See Johansen and Franchi (2017) and Elliot (1998).

Table 2: An identified long-run structure in β

$\tilde{\beta} = (h_1 + H_1\varphi_1, \ldots, h_r + H_r\varphi_r)$, $\chi^2(6) = 4.60[0.72]$							
	$p_{d,t}$	$p_{f,t}$	s_t	$i_{d,t}$	$i_{f,t}$	$t_{91.1}$	$t^{1)}$
$\tilde{\beta}'_1$	-0.013 [−16.9]	0.031 [16.9]	0.013 [16.9]	1.00	-1.00	−	−
d'_1	**0.16** [3.2]	1.11 [5.3]	**−0.48** [−6.9]	**−0.0005** [−3.4]	**0.0054** [3.2]	0.006 [3.8]	-0.013 [−8.5]
α'_1	0.45 [15.3]	-0.13 [−4.4]	**1.51** [3.1]	-0.01 [−3.8]	0.02 [4.0]		
$\tilde{\beta}'_2$	−	-0.009 [−15.1]	-0.009 [−15.1]	−	1.00	0.002 [2.5]	0.52 [1.5]
d'_2	-0.22 [−1.7]	-0.89 [−5.5]	$*$	0.0007 [12.8]	-0.0072 [−12.8]	$*$	0.038 [14.1]
α'_2	0.67 [10.3]	0.40 [6.5]	3.25 [3.0]	-0.03 [−5.0]	$*$	−	−
$\tilde{\beta}'_{\perp,1}$	1.00	-0.14	-0.30	0.33	-0.49	0.0020	-0.0026

1) The trend estimate has been multiplied by 1000.

Error-increasing coefficients in bold face. A * means an insignificant coefficient (<|1.4|)

|1.6|) are replaced by an asterisk (*). Error-increasing coefficients are shown in bold face. As discussed above, the α, d and ζ coefficients allow us to investigate how the variables have responded to imbalances in the system.

The β structure contain altogether 6 overidentifying restrictions which are tested with the likelihood ratio test described in Johansen et al. (2010) and accepted with a p-value of 0.72.

The first polynomially cointegrated relation corresponds closely to the Uncertainty Adjusted UIP relation (24):

$$\begin{aligned}(i_{d,t} - i_{f,t}) \quad &= \quad 0.01(p_{d,t} - p_{f,t} - s_t) - 0.16\Delta p_{d,t} - 1.1\Delta p_{f,t} + 0.48\Delta s_t + \\ &\quad\ 0.0005\Delta i_{d,t} - 0.006\Delta i_{f,t} + 0.013 - 0.006Ds_{91,t} + e_{1,t}\end{aligned}$$

where the PPP gap is a proxy for the uncertainty premium and $d'\Delta x_t \approx (0.16\Delta p_{d,t} + 1.1\Delta p_{f,t} - 0.48\Delta s_t - 0.0005\Delta i_{d,t} + 0.006\Delta i_{f,t})$ can be thought of as a proxy for Δs^e_{t+1} and a risk premium measuring short-term variability in the market. While the coefficient to the PPP is tiny, describing a very slow adjustment to the long-run PPP, the adjustment to the combined (excess return) relation is very fast as measured by the α_1 coefficients. The latter show that in the long run all variables, except for the nominal exchange rate, adjust in an error correcting manner to the disequilibrium $e_{1,t}$. In the medium run, German inflation and the nominal exchange rate are error increasing ($d_{11}\beta_{11}, d_{13}\beta_{13} < 0$) and so are the two interest rates ($d_{14}\beta_{14}, d_{15}\beta_{15} < 0$). Since an increasing PPP gap is likely to cause imbalances in the real economy and such imbalances have to be financed, the level of interest rates is likely to respond, which can explain their error-increasing behavior in the medium run.

The first cointegration relation seems to tell the following story: The PPP gap moves in long persistent swings as a result of error-increasing behavior of the nominal exchange rate and the interest rate differential follows suit. As long as the PPP gap and the interest rate differential move in tandem, the long-run equilibrium error, $\beta_1' x_t$, is small and the response of the system is moderate. But when the disequilibrium starts increasing, all variables, except for the nominal exchange rate, will react in an error-correcting manner so as to restore equilibrium.

The second polynomially cointegrated relation corresponds approximately to the relation (15) in Section 4. :

$$\begin{aligned} b_{f,t} - 0.89\Delta p_{f,t} &= 0.01(\widehat{p_{f,t} + s_t}) + 0.22\Delta p_{d,t} - 0.0007\Delta i_{d,t} + \\ &\quad 0.0072\Delta i_{f,t} - 0.04 + e_{2,t}. \end{aligned}$$

where $\widehat{x_t}$ stands for "trend-adjusted". An increase/decrease in the US bond rate relative to the US inflation rate (i.e. the real bond rate in (15)) is associated with an increase/decrease in the trend-adjusted US price denominated in Dmk.[10] Each of the d and α coefficients represents error-correcting adjustment, even the nominal exchange rate is error-correcting in α.

The medium-run stationary relation between growth rates, $\beta_{\perp 1}' \Delta x_t$, is given by

$$\begin{aligned} \Delta p_{d,t} &\simeq 0.14\Delta p_{f,t} + 0.30\Delta s_t - 0.33\Delta i_{d,t} + 0.49\Delta i_{f,t} + \\ &\quad 0.0026 - 0.002 Ds_{91,t} + e_{3,t} \end{aligned} \tag{28}$$

showing that German inflation rate has been co-moving with US inflation rate, with the change in the nominal exchange rate and with the change in the interest rate differential. The relation resembles relation (24) in differences, except that the coefficients are not consistent with proportional effects. Thus, in the medium run, German price inflation has not fully reacted to changes

[10] A similar relationship was found in Juselius and Assenmacher (2016).

in the US price and the nominal exchange rate. As a consequence it has contributed to the long swings in the real exchange rate visible in Figure 1. The estimates of ζ_3 in Table 3 show that the German and the US inflation rates are primarily adjusting to this relation, supporting the interpretation of (28) as a medium-run secular trend relationship between inflation rates.

Table 3 also reports the estimated adjustment coefficients ζ of $\beta'\Delta x_t$ where β is given by the estimates of Table 2. It appears that the changes of the two disequilibria have had a very significant effect on both interest rates: $\beta_1'\Delta x_t$ in an error increasing manner and $\beta_2'\Delta x_t$ in an error correcting manner. Interestingly, the nominal exchange rate does not adjust very significantly to any of the three equilibrium errors. Thus, in the medium run speculative movements in the exchange rate seems to have been the main driver in the Dollar-Deutshemark market.[11] Since both bond rates are equilibrium-error increasing in d_1 and ζ_1, the results may tentatively suggest that it is the interest rates that respond to the speculative movements in the nominal exchange. It is also notable that the coefficients on the exchange rate are much larger in absolute value than those on the price levels, suggesting that the changes in the inflation rates were too small to compensate the movements away from long-run equilibrium PPP values caused by financial speculators (trend followers/chartists)[12]. This supports the imperfect knowledge hypothesis that in the medium run the nominal exchange rate tends to move away from its long-run equilibrium values, while in the long run it moves back towards equilibrium.

10 A plausible story?

The results generally confirm the hypothetical scenario in Juselius (2009) where prices of tradable goods are assumed to be determined in very competitive customer markets (Phelps, 1994). Hence, prices are not much affected by speculation[13] and, therefore, do not exhibit persistent speculative swings around benchmark values.

A shock to the long-term interest rate (for example, as a result of a domestic increase in sovereign debt) without a corresponding increase in the inflation rate, is likely to increase the amount of speculative capital moving

[11] The latter result is also found in Juselius and Stillwaggon (2016).
[12] Similar results were found in Juselius and Assenmacher (2017).
[13] Energy, precious metals and, recently, grain may be exceptions in this respect.

Table 3: The adjustment coefficients ζ

	$\zeta_1(\beta_1'\Delta x_t)$	$\zeta_2(\beta_2'\Delta x)$	$\zeta_3(\beta_{\perp 1}'\Delta x_t)$
$\Delta\Delta p_{d,t}$	$*$	$*$	-0.82 [-16.5]
$\Delta\Delta p_{f,t}$	$*$	$*$	0.23 [4.7]
$\Delta\Delta s_t$	**13.9** [1.9]	$*$	$*$
$\Delta\Delta i_{d,t}$	**0.35** [9.2]	-0.71 [-13.9]	$*$
$\Delta\Delta i_{f,t}$	**-0.73** [-10.6]	-0.37 [-4.0]	**-0.02** [-2.0]

Coefficients with a $|t - value| < 1.3$ is replaced with an *

into the economy. The exchange rate appreciates, jeopardizing competitiveness in the tradable sector, the trade balance worsens, and the pressure on the interest rate increases. Under this scenario, the interest rate is likely to keep increasing as long as the structural imbalances are growing, thus generating persistent movements in real interest rates and real exchange rates. The estimates of $\beta' x_t + d'\Delta x_t$ and the error-increasing behavior of the interest rates in d_1 and ζ_1 support this interpretation.

The tendency of the domestic real interest rate to increase and the real exchange rate to appreciate at the same time reduces domestic competitiveness in the tradable sector. In an IKE economy in which the nominal exchange rate is determined by speculation, firms cannot in general count on exchange rates to restore competitiveness after a permanent shock to relative costs. Unless firms are prepared to loose market shares, they cannot use constant mark-up pricing as their pricing strategy. See, for example, Krugman (1986), Phelps (1994), Feenstra (). To preserve market shares, they would have to adjust productivity or profits rather than increasing the product price. Therefore, in an IKE economy we would expect customer market pricing (Phelps, 1994) to replace constant mark-up pricing, implying that profits are squeezed in periods of persistent appreciation and increased during periods of depreciation. Evidence of a nonstationary profit share comoving with the real exchange rate has for instance been found in Juselius (2006).

The results showed that German prices have been equilibrium error-increasing ($d_{11}\beta_{11} < 0$) in the medium-run at the same time as the nominal exchange has moved away from its long-run equilibrium value. Thus, Germany's reaction to the long swings in the real exchange rate has been to suppress price changes as a means to preserve competitiveness. US prices, on the other hand, have been error correcting ($d_{12}\beta_{12} > 0$) to the PPP gap, albeit very slowly so, indicating that USA's reaction has been more prone to letting prices follow the swings in the dollar rate as a result of speculative flows.[14] Judging from the accumulating US trade deficits in this period, US enterprises might have lost market shares accordingly.

To conclude: the IKE behavior of interest rates and the nominal exchange rate seem key for understanding the long swings in the currency market.

11 Conclusions

The paper demonstrates how basic assumptions underlying a theory model can be translated into testable hypotheses on the order of integration and cointegration of key variables and their relationships. The idea is formalized as a theory-consistent CVAR scenario describing the empirical regularities we expect to see in the data if the long-run properties of a theory model are empirically relevant. The procedure is illustrated for a monetary model of real exchange rate determination based on imperfect knowledge expectations.

The empirical results provide overwhelmingly strong support for the informationally less demanding imperfect knowledge type of model. In particular, this model seems able to explain the long and persistent swings in the nominal and the real exchange rate that have puzzled economists for long. The key for understanding these long swings in exchange rates and interest rates (both real and nominal) is to recognize the importance of imperfect knowledge, reflexivity, and positive and negative feedback mechanisms (Soros, Frydman and Goldberg (2013), Hands (2013) and Hommes (2013).

As the real exchange rate and the real interest rate are among the most important determinants for the real economy, the results point to the importance of understanding the underlying causes of the long persistent movements with which they typically evolve over time. The failure of extant models to foresee the recent financial and economic crisis and to propose adequate policy measures in its aftermath gives a strong argument for this. Without such an understanding financial behavior in the foreign currency

[14] Based on the tiny the speed of adjustment coefficient (0.01) it would take on average 6-8 years for the real exchange rate to return to its long-run value if the US inflation rate alone were to adjust.

and the stock market is likely to continue to generate bubbles and crises with serious implications for the macroeconomy and subsequent political turmoil.

12 References

Bec, F. and A. Rahbek (2004). "Vector Equilibrium Correction Models with Non-Linear Discontinuous Adjustments." *Econometrics Journal* 7, 628–651.

Doornik, J. (2017). "Estimation of the $I2$ Model." Preprint, Institute of Economic Modelling, Oxford Martin School, and Economics Department, University of Oxford.

Doornik, J. and K. Juselius (2017). *Cointegration Analysis of Time Series Using CATS 3 for OxMetrics.* Timberlake Consultants Ltd.

Elliot, G. (1998). "The Robustness of Cointegration Methods when Regressors Almost Have Unit Roots", *Econometrica*, 66, pp. 149-58.

Engle, C. (2014). "Exchange Rate and Interest Rate Parity", Chapter 8, Handbook of International Economics, Vol. 4, Elsevier. pp 453-522

Feenstra, R (2015), *Advanced International Trade: Theory and Evidence*, 2^{nd} edition. Princeton University Press, Princeton.

Franchi, M. and S. Johansen (2017), "The Behavior of Cointegration Methods in a Vector Autoregression With near Unit Roots, Under Review for *Econometrics*

Frydman, R. and M. Goldberg (2007). *Imperfect Knowledge Economics: Exchange rates and Risk.* Princeton University Press, Princeton.

Frydman, R. and M. Goldberg (2011). *Beyond Mechanical Markets: Risk and the Role of Asset Price Swings*, Princeton University Press, Princeton.

Frydman, R. and M. Goldberg (2013). "Change and Expectations in Macroeconomic Models: Recognizing the Limits to Knowability", *Journal of Economic Methodology* 20, 118-138.

Frydman, R., M. Goldberg, K. Juselius, and S. Johansen (2008). "Reexamining the Purchasing Power Parity Puzzle with Imperfect Knowledge Economics", Working Paper, University of Copenhagen.

Frydman, R., M. Goldberg, K. Juselius, and S. Johansen (2012). "Long Swings in Currency Markets: Imperfect Knowledge and I(2) Trends", Preprint, University of Copenhagen.

Haavelmo, T. (1944). "The Probability Approach to Econometrics." *Econometrica*, 12 (Supplement), 1-118.

Hands, D. W. (2013). "Introduction to Symposium on Reflexivity and Economics: George Soros's Theory of Reflexivity and the Methodology of Economic Science," *Journal of Economic Methodology* 20, 303-308.

Hommes, C. H. (2006). "Heterogeneous Agent Models in Economics and Finance," in: L. Tesfation and K.L. Judd (Eds.), *Handbook of Computational Economics* 2, Chapter 23, 1109-1186, Elsevier.

Hommes, C. H. (2013). "Reflexivity, Expectations Feedback and Almost Self-Fulfilling Equilibria: Economic Theory, Empirical Evidence and Laboratory Experiments," *Journal of Economic Methodology* 20, 406-419.

Hommes C.H., H. Huang, D. Wang (2005a). "A Robust Rational Route to Randomness in a Simple Asset Pricing Model". *Journal of Economic Dynamics and Control* **29**: 1043-1072. DOI: 10.1016/j.jedc.2004.08.003

Hommes C. H., J. Sonnemans, J. Tuinstra, H. van de Velden (2005b). "Coordination of Expectations in Asset Pricing Experiments". *Review of Financial Studies* **18**: 955-980. DOI: 10.1093/rfs/hhi003

Hoover, K. and K. Juselius (2015). "Trygve Haavelmo's Experimental Methodology and Scenario Analysis in a Cointegrated Vector Autoregression", *Econometric Theory,* 31, 2, pp. 249-274.

Hoover, K., S. Johansen, and K. Juselius (2008). "Allowing the Data to Speak Freely: The Macroeconometrics of the Cointegrated VAR", *American Economic Review*, 98, pp. 251-55.

Johansen, S. (1992). "A Representation of Vector Autoregressive Processes Integrated of Order 2." *Econometric Theory* 8, 188–202.

Johansen, S. (1995), *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models.* 2.ed. 1996 Oxford University Press, Oxford.

Johansen, S. (1997). "Likelihood Analysis of the $I(2)$ Model." *Scandinavian Journal of Statistics* 24, 433-462.

Johansen, S. (2006). "Statistical analysis of hypotheses on the cointegrating relations in the $I(2)$ model." *Journal of Econometrics* 132, 81–115.

Johansen, S., Juselius, K., Frydman, R. and M. Goldberg, (2010). "Testing Hypotheses in an $I(2)$ Model With Piecewise Linear Trends. An Analysis of the Persistent Long Swings in the Dmk/$ Rate," *Journal of Econometrics* 158, 117-129.

Juselius, K. (2006). *The Cointegrated VAR Model: Methodology and Applications.* Oxford University Press, Oxford.

Juselius, K. (2009). "The Long Swings Puzzle. What the data tell when allowed to speak freely" in T.C. Mills and K. Patterson (eds.) *The New Palgrave Handbook of Empirical Econometrics*, Chapter 8, MacMillan, London.

Juselius, K. (2012). "Imperfect Knowledge, Asset Price Swings and Structural Slumps: A Cointegrated VAR Analysis of Their Interdependence", in eds. E. Phelps and R. Frydman: *Rethinking Expectations: The Way Forward for Macroeconomics*, Princeton University Press, Princeton.

Juselius, K. (2014). "Testing for Near $I(2)$ Trends When the Signal to Noise Ratio Is Small," *Economics-The Open-Access, Open-Assessment E-Journal,* 21, pp. 1-30. http://dx.doi.org/10.5018/economics-ejournal.ja.2014-21.

Juselius, K. (2015). "Haavelmo's Probability Approach and the Cointegrated VAR model, *Econometric Theory,* 31, 2, pp 213 - 232.

Juselius, K. (2017). "A Theory-Consistent CVAR Scenario: Testing a Rational Expectations Based Monetary Model",

Juselius, K. and K. Assenmacher (2017). "Real Exchange Rate Persistence and the Excess Return Puzzle: the Case of Switzerland Versus the US," Forthcoming in the Journal of Applied Econometrics.

Juselius, K. and Franchi, M. (2007). "Taking a DSGE Model to the Data Meaningfully", *Economics*, 4, 1.

Juselius, M. (2010). "Testing Steady-State Restrictions of Linear Rational Expectations Models when Data are Highly Persistent", *Oxford Bulletin of Economics and Statistics,* 73, 3, pp. 315-334.

Kongsted, H. C., A. Rahbek and C. Jørgensen (1999). "Trend Stationarity in the $I(2)$ Cointegration Model." *Journal of Econometrics* 90, 265–289.

Krugman, P. (1986), "Pricing to Market When the Exchange Rate Changes", NBER Working Paper 1926

Nielsen, H. B. and A. Rahbek, (2007), The Likelihood Ratio Test for Cointegration Ranks in the $I(2)$ Model. *Econometric Theory* 23, 615–637.

Møller, F. N. (2008). "Bridging Economic Theory Models and the Cointegrated Vector Autoregressive Model". *Economics: The Open-Access, Open-Assessment E-Journal*, Vol. 2, 2008-36.

Phelps, E. (1994), *Structural Slumps*, Princeton University Press, Princeton.

Rogoff, K. (1996). "The Purchasing Power Parity Puzzle". *Journal of Economic Literature*, 34, 647-68.

Spanos, A. (2009). "The Pre-Eminence of Theory Versus the European CVAR Perspective in Macroeconometric Modeling". *Economics: The Open-Access, Open-Assessment E-Journal*, Vol. 3.

Soros, G. (1987). *The Alchemy of Finance.* John Wiley & Sons, New Jersey.

Tabor, M. N. (2017). "Time-Varying Cointegration Parameters as a Source of Slow Error-Correction in the Cointegrated VAR Model: A Simulation Study," Submitted to this special issue.