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Unawareness in Dynamic Psychological Games*

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Abstract

Building on Battigalli and Dufwenberg (2009)'s framework of dynamic psychological games and the recent progress in the modeling of dynamic unawareness, we provide a general framework that allows for 'unawareness' in the strategic interaction of players motivated by belief-dependent psychological preferences like reciprocity and guilt. We show that unawareness has a pervasive impact on the strategic interaction of psychologically motivated players. Intuitively, unawareness influences players' beliefs concerning, for example, the intentions and expectations of others which in turn impacts their behavior. Moreover, we highlight the strategic role of communication concerning feasible paths of play in these environments.

Keywords: Unawareness; Extensive-form games; Communication; Belief-dependent preferences; Sequential equilibrium.

JEL-Classifications: C72, C73, D80

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1 Introduction

Recent lab and field evidence suggests that people not only care about the monetary consequences of their actions, but that their behavior is also driven by belief-dependent psychological preferences [see e.g. Fehr *et al.* (1993), Charness and Dufwenberg (2006), Falk *et al.* (2008), Bellemare *et al.* (2010)]. Two prominent examples of belief-dependent preferences in the hitherto existing literature are reciprocity [see e.g. Rabin (1993), Dufwenberg and Kirchsteiger (2004), Falk and Fischbacher (2006)] and guilt aversion [see e.g. Charness and Dufwenberg (2006), Battigalli and Dufwenberg (2007b)]. Departing from the strictly consequentialist tradition in economics e.g. Geanakoplos *et al.* (1989) and Battigalli and Dufwenberg (2009) present general frameworks for analyzing the strategic interaction of people with belief-dependent psychological preferences: 'psychological games'. Roughly speaking, psychological games are games in which players' preferences depend upon players' beliefs about the strategies that are being played, players' beliefs about the beliefs of others about the strategies that are being played ad infinitum.

A classical yet widely unspoken assumption that is underlying all psychological as well as standard (i.e. non-psychological) game-theoretic analyses is that players are aware of the complete structure of the strategic environment they are in. Bluntly speaking, it is assumed that players are aware of everything. In many real life situations this is not the case – people often have asymmetric awareness levels concerning their own as well as others' feasible choices although they are part of the same strategic environment. Players are frequently 'surprised' meaning they become aware of new strategic alternatives by e.g. observing actions they had previously been unaware of or through communication. In fact, it has been shown that any non-trivial notion of unawareness is precluded in standard models [see e.g. Dekel *et al.* (1998), Modica and Rustichini (1999)]. Standard models therefore also preclude 'surprise' by implicitly assuming that players know all states of the world (the axiom of awareness) and know all events they do not know (the axiom of wisdom) [see Samuelson (2004)]. In other words, players can assign probabilities to all states of the world and, hence, cannot be truly 'surprised'.

However, it is not only in standard non-psychological games that unawareness is important. As we will show in our analysis here, asymmetric awareness also has a profound impact on the strategic interaction of players with belief-dependent psychological preferences. To see this consider the following intuitive examples: Imagine two friends, *Ann* and *Bob*. Assume it is *Bob*'s birthday, he is planning a party and would be very happy, if *Ann* could come. Unfortunately, however, *Ann* has an important exam the next day and therefore cannot make it. Obviously *Ann* knows that *Bob* would feel let down, if she were to cancel his party without having a very good excuse.[1] Quite intuitively, in this situation *Ann* does not experience any guilt towards *Bob* for not coming to his party. She knows that the important exam is a good excuse and that *Bob* is not let down as he does not expect her to come. In contrast, consider now the following variant of the same example: *Ann* is aware of the fact that the exam is postponed, meaning that it is feasible for her to attend *Bob*'s party. However, she has studied so hard for days and nights that she feels too tired to go to *Bob*'s party. Quite intuitively, in this situation *Ann* does not feel guilty towards *Bob* as long as she believes that *Bob* is unaware of the fact that the exam is postponed. As long as she believes that *Bob* is unaware of the fact that she actually has the possibility/time to come, she might not feel guilty towards him as she believes that he does not expect her to come and, hence, is not let down. In fact, if she were sure that *Bob* would never become aware of the fact that her exam is postponed, she probably had a strong emotional incentive in this situation to stick to the original story and leave him unaware in order not to raise his expectations. In other words, she had a strong incentive not to make him aware of the fact that she actually has the time to come to his party, but is too tired. Interestingly, if *Ann* were only interested in her own payoff in this strategic situation with unawareness, she would not care whether *Bob* is or will become aware of the postponement. She would simply not attend his party irrespective of his awareness. Only her belief-dependent feeling of guilt towards *Bob* creates the strong emotional incentive not to make him aware.

Bob's unawareness concerning *Ann*'s possibility to come to his party and, connectedly, *Ann*'s incentive not to tell him about the postponement of her exam intuitively highlight the focus of our analysis here. We analyze the influence and importance of asymmetric awareness and communication concerning feasible paths of play for the strategic interaction of agents with belief-dependent preferences. This means, building on Battigalli and Dufwenberg (2009)'s framework of dynamic psychological games and the recent progress in the modeling of dynamic unawareness, we first provide a general framework that allows for unawareness and communication in the strategic interaction of players motivated by belief-dependent psychological preferences like reciprocity and guilt. Second, we provide a solution concept which can be used in our class of games with unawareness, communication and belief-dependent preferences and, third, we discuss an application to exemplify the influence of unawareness and communication using a specific type of belief-dependent preferences: reciprocity.

[1]We will throughout the paper apply a Bayesian view of knowledge, i.e. we assume knowing is equivalent to being certain (believing with probability 1) [see e.g. Brandenburger and Dekel (1987)]

More specifically, to allow for unawareness we extend the 'standard' multi-stage framework along two dimensions. First, we partition the 'standard' extensive form into subtrees consisting of paths of play, each describing a level of awareness. A player confined to a given subtree is aware of other subtrees which can be embedded in the one he is confined to, and unaware of all other subtrees. Second, as our setting is dynamic, players may become aware of more by learning from actions taken by others. However, as our analysis concentrates on the influence of asymmetric awareness on the strategic interactions of players with belief-dependent preferences, we abstract from the question how players become aware of new paths of play. We simply assume that whenever they observe an action that they had previously been unaware of they become aware of some 'more expressive' subtree which is consistent with the observed actions. Many different ways of modeling unawareness have been suggested in recent years both from a logic, an epistemic and a game theoretic perspective [see e.g. Fagin and Halpern (1988), Modica and Rustichini (1999), Halpern (2001), Heifetz *et al.* (2006), Halpern and Rêgo (2008), Heifetz *et al.* (2008), Li (2009) and Mengel *et al.* (2009)]. Closely related to our class of extensive forms with unawareness are the dynamic models by Halpern and Rêgo (2008), Feinberg (2009) and Heifetz *et al.* (2010). In fact, we also use the ordered structure proposed by Heifetz *et al.* (2010) in our formulation of unawareness in extensive forms and show in Appendix A that in every stage our multi-stage framework adheres to the unawareness properties of both Heifetz *et al.* (2006) and Dekel *et al.* (1998). However, different to Heifetz *et al.* (2010), the focus of our paper is not in proposing another framework to model unawareness, but rather to analyze the influence of unawareness on the strategic interaction of agents with belief-dependent preferences.

In the spirit of our example above, we also allow for communication in our framework. We model such communication by assuming that players can choose to send 'awareness messages' containing feasible paths of play (i.e. subtrees) they are aware of, or they can choose not to communicate. Note that this is different to the communication allowed for in the experimental setting of Charness and Dufwenberg (2006). In their setting players are aware of everything and can send messages e.g. concerning intended play. In contrast, a message in our setting is an information concerning a set of feasible paths of play. Communicating feasible paths of play is obviously meaningless in strategic environments without unawareness. This means, it is the asymmetric awareness of players which makes communication an important integral part of the strategic environment with unawareness. If a player observes a messages containing information about paths of play that he was previously unaware of, he will update his level of awareness by taking this information into account.

Having defined our class of extensive forms with unawareness and communication, we formally characterize belief-dependent preferences in this structure. In synthesis, for each player confined to a certain awareness level, his pure strategy is defined on the extensive form he is confined to and the other players' strategies are defined on each of the extensive forms induced by all subtrees he is aware of. A behavioral strategy profile is thus an independent probability distribution over these pure strategies each specifying a definite choice. Beliefs about others' pure strategies (first-order beliefs), beliefs about their beliefs about others' pure strategies (second-order beliefs), and so on, are shown to exists for all possible hierarchies. We use these hierarchies of beliefs for the general specification of the belief-dependent psychological preferences. As mentioned above, specific types of belief-dependent preferences that can be embedded in our general setting with unawareness and communication are among others reciprocity and guilt aversion. In both of these examples belief-dependent psychological preferences depend on first- and second order beliefs. In contrast to Battigalli and Dufwenberg (2009), in our setting such psychological preferences will be limited by the awareness of each player who play 'partial games'. A partial game is an extensive form with unawareness and communication augmented by the psychological preferences at a certain level of awareness. As players may be aware of different paths of play at different stages of the game, we define a dynamic psychological game with unawareness and communication as an ordered set of partial games which are relevant for the strategic situation.

Given the characterization of dynamic psychological games with unawareness and communication, we propose a sequential psychological equilibrium solution concept and prove its existence. We assume that a profile of conjectures (first-order beliefs) in a partial game is derived from a behavioral strategy profile in the same game. This implies, that any two players confined to the same partial game will independently hold the same conjectures about any third player. An assessment in our structure, a behavioral strategy profile and a profile of infinite hierarchies of beliefs, is consistent if the profile of first-order beliefs is derived from the behavioral strategy profile and each higher-order belief assigns probability one to lower-order beliefs. Intuitively, players aware of the same must in equilibrium hold common, correct beliefs about each others infinite belief hierarchies. A consistent assessment together with sequential rationality (based on belief-dependent preferences) induce a sequential psychological equilibrium in the partial game. As players are unaware of any situation in which other players are aware of more than themselves, they believe that the game they are confined to is the most expressive. This implies that there exists an equilibrium strategy in which players confined to a partial game fix the equilibrium strategies of other players, whom they believe are confined to 'less expressive' partial games, and then choose an equilibrium strategy based on this belief.

After defining our class of extended psychological games and characterizing our solution concept, we use an application to demonstrate the influence and importance of unawareness on the strategic interaction of agents with belief-dependent preferences. That is, we use the sequential prisoners dilemma also analyzed by Dufwenberg and Kirchsteiger (2004) to show the impact of unawareness and communication on the strategic interaction of reciprocal agents. As a benchmark we start from their results and subsequently discuss two scenarios in which players have asymmetric awareness levels. Importantly, the application shows how asymmetric awareness levels of players concerning the feasible choices can give rise to equilibrium prediction that are distinct from predictions using Dufwenberg and Kirchsteiger (2004) setting without unawareness and a standard setting in which people are only concerned about the monetary consequences of their actions.

The organization of the paper is as follows: In section 2 we introduce a class of extensive forms with unawareness and communication. Following this, in section 3 we define hierarchies of conditional beliefs and belief-dependent preferences in our class of extensive forms. Section 4 contains the definition of our equilibrium concept: psychological sequential equilibrium. In section 5 we discuss a specific application. Sections 6 and 7 contains extensions and discussion of some of our assumptions and a conclusion, respectively.

2 Framework

In this section we introduce a class of extensive forms with unawareness and communication. We first define an extensive form without communication (2.1) and consider how players learn from actions taken by others (2.2). Following this we augment the extensive form with unawareness to include strategic messages (2.3) and (2.4), and show how player also learn from messages sent by others (2.5).

2.1 Extensive forms with unawareness

We extend Battigalli and Dufwenberg (2009)'s setting of finite extensive forms with observable actions, no chance moves, and complete information, to include the possibility that players are unaware of parts of the extensive form. We assume that players simultaneously move in every history. Note that simultaneous moves do not exclude games where players move in alternation, as we allow for the possibility that players have singleton action sets meaning they are 'passive'. The restrictions made by observable actions, no chance moves, and complete information can be removed, at the cost of additional notational complexity.[2]

[2] Different extensions of our general framework are discussed in section 6

A finite extensive form with unawareness can be summarized by three building blocks:

(i) a finite extensive form $\langle N, \mathscr{T} \rangle$ with observable actions, where N is the set of players and $\mathscr{T}$ is an 'omniscient' game tree consisting of all feasible histories h,

(ii) a family $\boldsymbol{T}$ of subtrees,

(iii) a correspondence $\varphi_i : \mathscr{T} \to \boldsymbol{T}$ which assigns to each player $i \in N$ their level of unawareness at each history h.

In the following we will consider the three building blocks separately.

(Ad i) Finite extensive forms with observable actions: A finite extensive form with observable actions, no chance moves, and complete information is a tuple $\langle N, \mathscr{T} \rangle$ where $N = \{1, ..., n\}$ is the set of players, and $\mathscr{T}$ is the finite set of histories. A history of length l is a sequence $h = (a^1, \ldots, a^l)$ where each $a^t = (a_1^t, \ldots, a_n^t)$ represents the profile of actions taken at stage t $(1 \leq t \leq l)$. History $\tilde{h} = (\tilde{a}^1, \ldots, \tilde{a}^k)$ precedes $h = (a^1, \ldots, a^l)$, written $\tilde{h} \prec h$, if $\tilde{h}$ is a prefix of h (i.e., $k < l$) and $(\tilde{a}^1, \ldots, \tilde{a}^k) = (a^1, \ldots, a^k)$. The initial empty history, denoted h^0, is an element of $\mathscr{T}$. We denote finite set of feasible actions for player i at history h by $A_{i,h}$. $A_{i,h}$ is empty if and only if h is a terminal history. Let $\mathscr{Z}$ denote the set of terminal histories. Finally, we assume that there is perfect recall.

(Ad ii) Subtrees: Consider now a family $\boldsymbol{T}$ of subtrees of $\mathscr{T}$, ordered by the inclusion of histories. Each subtree $T \in \boldsymbol{T}$ represents a set of feasible paths of play. The most 'expressive' of these trees is equivalent to the set of feasible paths of play in the tree $\mathscr{T}$.

Definition 1 (Subtrees)**.** A subtree $T \in \boldsymbol{T}$ is a set of histories such that

$$T = \{h \in \mathscr{T} : h \leq z \text{ for all } z \in Z', Z' \subseteq \mathscr{Z}\},$$

where $h \leq z$ means that h is z or a predecessor of z.

Note that such a construction of subtrees ensures that any $T \in \boldsymbol{T}$ starts at the root h^0, that it is naturally ordered by proper subhistories, and implies that each terminal history of each subtree $z \in Z'$ is associated with a well defined terminal history in $\mathscr{Z}$.

Example 1: The construction of the family $\boldsymbol{T}$ can be demonstrated by a simple example. Consider the extensive form underlying the sequential prisoners dilemma also analyzed by Dufwenberg and Kirchsteiger (2004).

[Figures 1]

It is an extensive form without communication $\langle N, \mathscr{T} \rangle$ with $N = \{Ann, Bob\}$ and $\mathscr{T} = \{h^0, h^1, h^2, h^3, h^4, h^5, h^6\}$.[3] In the initial history h^0 *Ann* can choose between *cooperate* (C) and *defect* (D) and *Bob* is passive. In histories h^1 and h^2 *Bob* can respectively choose between *cooperate* (c) and defect (d) and *Ann* is passive. Histories h^3, h^4, h^5 and h^6 are terminal histories.

The family of subtrees $\boldsymbol{T}$ of $\mathscr{T}$ consists, for example, of the following elements:

[Figures 2]

(Ad iii) Awareness: To model that players may have different views on the set of feasible paths of play at different histories h we assume that there exists an 'awareness correspondence':

Definition 2 (Awareness)**.** For each player $i \in N$ there exists a correspondence

$$\varphi_i : \mathscr{T} \to \boldsymbol{T},$$

which assigns to player i his level of awareness at history h.

A given player i with awareness correspondence $\varphi_i(h) = T$ at h has 'awareness level' T. A player may however also believe that others players are aware of less than himself, and therefore has to consider the histories that he believes they are aware of. For any two trees $T, T'' \in \boldsymbol{T}$ denote by $\hookrightarrow$ the transitive closure. That is, $T \hookrightarrow T''$ if and only if there is a sequence of subtrees $T, T', \ldots, T'' \in \boldsymbol{T}$ satisfying $T \supset T' \supset \cdots \supset T''$. Hence, $\boldsymbol{T} = \{T'\}_{\mathscr{T} \hookrightarrow T'}$ is an ordered set of subtrees. At some T, players considers possible subtrees for which $T \hookrightarrow T'$. If $T \not\hookrightarrow T'$, then players may be interpreted as being unaware of the subtree T'. Such a characterization of unawareness can be shown to comply with the properties that any non-trivial concept of unawareness should satisfy, as suggested by Dekel *et al.* (1998). That is, attractive properties of unawareness obtain in our framework, including; strong plausibility, KU introspection, AU introspection, and weak necessitation [see Appendix (A) for the formal proof]. Especially, it can be shown that the state space implied by the proposed structure is not standard – players need not to know all tautologies. For players confined to any $T \in \boldsymbol{T}$, the extensive form is given by $\langle N, T \hookrightarrow T' \rangle$.

[3] We will draw on this example in the subsequent sections and develop it further along the lines of our analysis.

2.2 Learning from others' actions

Each player may become aware of more by learning from actions taken by others. Let $\varphi_i(h) = T$ be the subtree player i is confined to at the initial history h^0. At any succeeding history h' ($h^0 \preceq h$), each player i only learns if the actions $a_{-i,h} \in \prod_{j \neq i} A_{j,h}$ taken by others contain something he was previously unaware of. If a player observes that the actions taken by others are different from what he had foreseen, he will have an 'enlightening' moment and discover some subtree T' which allows for the actions just taken. That is

$$T' : a_{-i,h} \in \prod_{j \neq i} A_{j,h} \text{ for } h \in T', T' \in \boldsymbol{T},$$

otherwise $T' = \varnothing$. Note that this property is not very restrictive, it just says that observing some actions that a player was not previously aware of, makes him aware of some subtree that is consistent with the actions just taken.

Given the actions others made at h, player i will learn by updating his view of the game with the information contained in these actions.

Definition 3 (Learning from actions)**.** Player i's view of the game after observing the profile of actions a_h at h is

$$\varphi_i(h, a_h) = \{\varphi_i(h) \cup T'\}.$$

This characterization of learning implies that a player cannot become unaware during the game. More formally:

Remark 1. Awareness may only increase along the path: if there is a path $h, \dots, h' \in \mathscr{T}$, and $\varphi_i(h) = T$ while $\varphi_i(h') = T'$ then $T \subset T'$.

As argued in the introduction, when players have asymmetric awareness levels communication concerning the feasible paths of play becomes an integral part of the strategic environment. Therefore we next define the set of messages that players can send concerning the feasible paths of play and then augment our extensive form with unawareness to allow for communication.

2.3 Messages about feasible paths of play

Assume that players can either choose to communicate some set of feasible paths of play, i.e. a subtree, or choose not to communicate which we denote by sending the empty message $\{\varnothing\}$.

This means, each player i confined to some subtree T can send a message $m_i : T \to \{T \hookrightarrow T'\}$. Thus, the set of possible messages of a player i confined to T is

$$M_T = \{\{T'\}_{T \hookrightarrow T'} \cup \{\varnothing\}\}.$$

Note that the set of messages is the same for players confined to the same subtree. For notational ease, we therefore omit the player subscript.

Each of these messages only reveals information about the structure of the game, i.e. the feasible paths of play. Therefore, our messages are irrelevant in settings with full awareness since they contain no new information. However, in settings with asymmetric awareness such messages become an important part of the strategic interaction. Furthermore, by construction our messages can only be informative.

2.4 Extensive forms with unawareness and communication

We are now ready to define an extensive form with unawareness and communication. The message that a player chooses to sent will be treated as a deliberate strategic choice, such that a history now defines an action taken and a message sent. We will therefore assume that a history, augmented by messages $m_{\mathscr{T}}$, is a sequence $h_{\mathscr{T}} = (c^1, \ldots, c^l)$ of length l where each $c^t = (c_1^t, \ldots, c_n^t)$ representing the profile of choices taken at stage t $(1 \leq t \leq l)$. Let the set of such histories be denoted by $H_{\mathscr{T}}$.

Example 2: Consider again the extensive form in Figure 1. Let's concentrate on the initial history h^0. In our extensive form with communication Ann's set of feasible choices in the initial history $h^0 \in H_{\mathscr{T}}$ is, $C_{A,h^0} = \{(C, T_1), \cdots, (C, T_{15}), (C, \{\varnothing\}), (R, T_1), \cdots, (R, T_{15}), (R, \{\varnothing\})\}$. On the other hand, Bob who is passive, P, in h^0 can only communicate, i.e. $C_{B,h^0} = \{(P, T_1), \cdots, (P, T_{15}), (P, \{\varnothing\})\}$.

To model that players may have asymmetric awareness in our extensive setting with communication we also have to augment Definition 2 by assuming that φ_i is defined on $H_{\mathscr{T}}$ instead of $\mathscr{T}$. This is

Definition 4 (Augmented Awareness)**.** For each player $i \in N$ in our extensive form with communication there exists a correspondence

$$\varphi_i : H_{\mathscr{T}} \to \boldsymbol{T},$$

which assigns to player i his level of awareness at history $h_{\mathscr{T}} \in H_{\mathscr{T}}$.

This implies, each player i with $\varphi_i(h_{\mathscr{T}}) = T$ believes that the set of histories in the extensive form with communication is $\langle N, H_T \rangle$ where a history $h_T \in H_T$ is a sequence representing a profile of choices taken at each stage. More precisely, each player i thinking he is in $h_T \in H_T$ takes an action a_{i,h_T} from the set of actions $A_{i,h_T} \subseteq A_{i,h}$, and sends a messages m_T from the set of messages M_T. Hence, the set of choices of player i at every h_T is defined by $C_{i,h_T} = A_{i,h_T} \times M_T$. We denote a typical element of C_{i,h_T} by c_{i,h_T} to highlight that it is the choice made at h_T. Note that our definition of H_T implies that if one assumed that players can only send empty messages, then the set of histories H_T is equivalent to the subtree T. Let Z_T denote the set of terminal histories in H_T.

A player may think that the other players are aware of less than himself, and therefore also has to consider the histories that he thinks they are aware of. The possible histories that a player is aware of and the histories that he believes his opponents could be aware of, given the subtree T he is confined to, is a set of disjoint spaces $\boldsymbol{H}_T = \{H_{T'}\}_{T \hookrightarrow T'}$ partially ordered by inclusion. The sequence of histories consisting of the history h_T a player confined to T believes to be in and all copies thereof in the subtrees $T \hookrightarrow T'$ is denoted by $h_T \hookrightarrow h_{T'}$. Furthermore, the set of terminal histories in $\boldsymbol{H}_T$ is denoted by $\boldsymbol{Z}_T$. For any $T \in \boldsymbol{T}$, the extensive form with communication that a player confined to T is aware of is given by $\langle N, \boldsymbol{H}_T \rangle$.

2.5 Learning from others' messages

Besides learning from actions taken by other players, a player may also learn from messages sent by others containing new information about possible paths of play. When player i observes the messages send, he will aggregate the information $\bigcup_{j \neq i} m_j$ and compare it to his awareness level at $h_{\mathscr{T}} \in H_{\mathscr{T}}$

Definition 5 (Learning from messages)**.** Player i's awareness after observing messages $\bigcup_{j \neq i} m_j$ at $h_{\mathscr{T}}$ is

$$\varphi_i(h_{\mathscr{T}}, c_{h_{\mathscr{T}}}) = \left\{ \varphi_i(h_{\mathscr{T}}, a_{h_{\mathscr{T}}}) \cup \left\{ \bigcup_{j \neq i} m_j \right\} \right\}.$$

Note that Remark 1 also holds in our setting with communication.

To sum up, learning implies that player i updates his current awareness with information in either the unforeseen actions taken by others (Definition 2.2), or messages containing new information (Definition 2.5). This concludes the definition of our class of extensive forms with observable actions, messages and unawareness. In the next section we define dynamic psychological games in the context of our class of extensive forms.

3 Dynamic psychological games with unawareness

We now develop our notion of dynamic psychological games with unawareness. We start by considering some mathematical preliminaries (3.1) and the definition of pure and behavioral strategies (3.2). The notion of a hierarchies of conditional beliefs and coherency are then given (3.3) and (3.4), and games with unawareness and belief-dependent preferences are defined (3.5).

3.1 Mathematical preliminaries

A topological space is deemed Polish if it is separable and completely measurable. The countable product of Polish spaces, endowed with the product topology, is Polish. For a given Polish space X and associated Borel sigma-algebra $\mathcal{B}$, let $\Delta(X)$ be the set of Borel probability measures $\mu : \mathcal{B} \to [0,1]$ on $(X, \mathcal{B})$. A class $\mathcal{B}$ of subsets of X is a Borel sigma-algebra if it contains X itself and is closed under the formation of complements and countable unions. An element $\mu \in \Delta(X)$ satisfies $\mu(\varnothing) = 0$, $\mu(X) = 1$, $\mu(E) \in [0,1]$ for $E \in \mathcal{B}$. If the topology on X is Polish, then the weak topology is also Polish. A sequence $\{\mu^k\}_{k=1}^{\infty}$ in $\Delta(X)$ converges in a weak sense to a measure $\mu \in \Delta(X)$, written $\mu^k \xrightarrow{w} \mu$, if and only if, for every bounded, continuous function $\psi : X \to \mathbb{R}$, $\int_X \psi(x) d\mu^k = \int_X \psi(x) d\mu$. Finally, if μ is a measure on some product space $X \times Y$, the marginal of μ on X is denoted $\text{marg}_X \mu$.

Consider players who are uncertain about which element in a set X is true. Assume X is a compact Polish space. Players assign probabilities to events $E, F, \dots$ in the Borel sigma-algebra $\mathcal{B}$ of X according to some (countably additive) probability measure. Let $\Delta(X)$ denote the set of all probability measures on $(X, \mathcal{B})$. As events unfold players update their beliefs. The actual and/or potential beliefs of a player are described by a conditional probability system. Let $\mathcal{C} \subseteq \mathcal{B}$ denote the collection of potentially observable events (or conditioning events). The player holds probabilistic beliefs conditional on each event $F \in \mathcal{C}$.

Definition 6. A conditional probability system (*cps*) is a function $\mu(\cdot|\cdot) : \mathcal{B} \times \mathcal{C} \to [0,1]$ defined on $(X, \mathcal{B}, \mathcal{C})$ such that for all $E \in \mathcal{B}$ and $F', F \in \mathcal{C}$:

1. $\mu(\cdot|\cdot) \in \Delta(X)$,
2. $\mu(F|F) = 1$,
3. $E \subseteq F' \subseteq F$ implies $\mu(E|F) = \mu(E|F')\mu(F'|F)$.[4]

[4] The tuple $(X, \mathcal{B}, \mathcal{C}, \mu)$ is called a conditional probability space by Renyi (1955). When $\mathcal{B} = 2^X$, $\mathcal{C} = 2^X \setminus \{\varnothing\}$ and X is finite we obtain Myerson (1986)'s cps'. Battigalli and Siniscalchi (1999) shows how to construct cps' when X is σ-additive.

The set of cps' on $(X, \mathcal{B}, \mathcal{C})$ is a subset of the topological space $[\Delta(X)]^{\mathcal{C}}$ (the set of mappings from $\mathcal{C}$ to $\Delta(X)$) and it is denoted $\Delta^{\mathcal{C}}(X)$. Accordingly, we often write $\mu = (\mu(\cdot|F))_{F\in\mathcal{C}} \in \Delta^{\mathcal{C}}(X)$. The topology on X and $\mathcal{B}$ are alway understood and need not be explicit in our notation. Thus we simply say 'conditional probability system (or cps) on $(X, \mathcal{C})$.' We endow $\Delta(X)$ with the topology of weak convergence of measures, and $[\Delta(X)]^{\mathcal{C}}$ with the product topology. The set $\Delta^{\mathcal{C}}(X)$ of cps' on $(X, \mathcal{C})$ is a closed subset of $[\Delta(X)]^{\mathcal{C}}$. Therefore $\Delta^{\mathcal{C}}(X)$ (endowed with the relative topology inherited from $[\Delta(X)]^{\mathcal{C}}$) and $X \times \Delta^{\mathcal{C}}(X)$ (endowed with the product topology) are compact Polish spaces.

3.2 Pure and behavioral strategies

In an augmented extensive form $\langle N, \boldsymbol{H}_T\rangle$ a pure strategy is a complete plan which includes 'instructions' contingent on every history in H_T he is confined to. More formally, the set of own pure strategies that player i is aware of in h_T is $S_i^{H_T}$ and the set of pure strategies that he is aware of and that allow for h_T is $S_i^{H_T}(h_T)$. A typical strategy is denoted by $s_i^T = (s_{i,h_T}^T)_{h_T\in H_T\setminus Z_T}$, where s_{i,h_T}^T is the choice that would be selected by s_i^T if history h_T obtains. Furthermore, as already noted a player confined to some subtree T believes that others may be aware of less than him $\{H_{T'}\}_{T\to T'}$. That is, in h_T he considers the set of pure strategies $S_{-i}^{\boldsymbol{H}_T} := \bigcup_{H_{T'}\in\boldsymbol{H}_T} S_{-i}^{H_{T'}}$. For a pure strategy profile $(s_i^T, s_{-i}^{T'}) \in S_i^{H_T} \times S_{-i}^{\boldsymbol{H}_T}$, let $\zeta(s_i^T, s_{-i}^{T'}) \in Z_T$ denote the terminal history induced by $(s_i^T, s_{-i}^{T'})$.[5]

We will assume that players consider behavioral strategies. A behavioral strategy is an independent probability distribution over pure strategies each specifying a definite choice at each history the player is aware of. Formally, let $\sigma_{i,T}(\cdot|h_{T'}) \in \Delta(C_{i,h_{T'}})$ be the profile of behavioral strategies. We denote the set of behavioral strategies by $\sigma_{i,T} = (\sigma_{i,T}(\cdot|h_{T'}))_{h_{T'}\in\boldsymbol{H}_T}$, and the set of behavioral strategy profiles confined to the same subtree T by $\sigma_T = (\sigma_{i,T})_{i\in N}$. The notion reflects that a player plan a collection of randomizations, one for each of the points at which he has to make a choice. However, in our interpretation we exclude actual randomizations. Rather, we assume that players do not know the pure strategies of others, and the randomization of these players represents their uncertainty, their conjecture (independent first-order cps) about others pure strategies (Aumann and Brandenburger, 1995).

[5] The path function $\zeta : S_i^{H_T} \times S_{-i}^{\boldsymbol{H}_T} \to Z$ is defined such that $z = (c^1, \ldots, c^L) = \zeta(s_i^T, s_{-i}^{T'})$ if and only if $c^1 = (s_{i,h_T^0}^T, s_{-i,h_{T'}^0}^{T'})$ and $c^{t+1} = (s_{i,(c^1,\ldots,c^t)}^T, s_{-i,(c^1,\ldots,c^t)}^{T'})$ for all $t \in \{1, \ldots, L-1\}$.

3.3 Hierarchies of conditional beliefs

Let $(X, \mathcal{B}, \mathcal{C})$ be defined by $X = S_{-i}^{\boldsymbol{H}_T}$ (a finite set) or $X = S_{-i}^{\boldsymbol{H}_T} \times Y$ where Y is a compact Polish space representing a set of other players beliefs. For every partial history $h_{T'}$, $S_{-i}^{\boldsymbol{H}_T}$ is a representation of i's information about other players' strategies at $h_{T'}$. The Borel sigma-algebra $\mathcal{B}$ on $S_{-i}^{\boldsymbol{H}_T}$ is implicitly understood[6], and conditioning events corresponds to histories, i.e., $\mathcal{C} = \{F \subseteq S_{-i}^{\boldsymbol{H}_T} \times Y : F = S_{-i}^{\boldsymbol{H}_T}(h_{T'}) \times Y, h_{T'} \in \boldsymbol{H}_T\}$ (or $\mathcal{C} = \{F \subseteq S_{-i}^{\boldsymbol{H}_T} : F = S_{-i}^{\boldsymbol{H}_T}(h_{T'}), h_{T'} \in \boldsymbol{H}_T\}$). Since each element of $\mathcal{C}$ represents the event that some history $h_{T'} \in \boldsymbol{H}_T$ obtains, we simplify our notation for cps' and replace $\mathcal{C}$ with $\boldsymbol{H}_T$. The set of cps' is denoted $\Delta^{\boldsymbol{H}_T}(S_{-i}^{\boldsymbol{H}_T} \times Y)$ a subset of $[\Delta(S_{-i}^{\boldsymbol{H}_T} \times Y)]^{\boldsymbol{H}_T}$. Indeed, we shall denote events $F = S_{-i}^{\boldsymbol{H}_T}(h_{T'}) \times Y$ by $h_{T'}$ when needed. Finally, we will let $[h_{T \hookrightarrow T'}]$ denote the event that $F = \bigcup_{h_T \hookrightarrow h_{T'}} S_{-i}^{\boldsymbol{H}_T}(h_{T'}) \times Y$ (or $F = \bigcup_{h_T \hookrightarrow h_{T'}} S_{-i}^{\boldsymbol{H}_T}(h_{T'})$).

Each player knows the strategies he is aware of, and holds cps' about others' strategies given this awareness. A player's conditional first-order cps is then an element of $\Delta^{\boldsymbol{H}_T}(S_{-i}^{\boldsymbol{H}_T})$. Since a player may not know the cps' of other players, he must have second-order beliefs. That is, a player's second-order cps is an element of $\Delta^{\boldsymbol{H}_T}(S_{-i}^{\boldsymbol{H}_T} \times \prod_{j \neq i} \Delta^{\boldsymbol{H}_T}(S_{-j}^{\boldsymbol{H}_T}))$, and so on. Similar for the other players. Formally, we define beliefs inductively for all $i \in N$ by the spaces:

$$
\begin{aligned}
&X_{-i}^0 = S_{-i}^{\boldsymbol{H}_T};\\
&\text{for all } k \geq 1,\\
&X_{-i}^k = X_{-i}^{k-1} \times \prod_{j \neq i} \Delta^{\boldsymbol{H}_T}(X_{-j}^{k-1}).
\end{aligned}
$$

An element $\mu_{i,T}^k \in \Delta^{\boldsymbol{H}_T}(X_{-i}^{k-1})$ is a k-order cps. That is, player i confined to H_T evaluates opponents strategies conditioned on his awareness $\{H_{T'}\}_{T \hookrightarrow T'}$.

The assumed topology implies that for all $k \geq 1$, X_{-i}^k and $\Delta^{\boldsymbol{H}_T}(X_{-i}^k)$ are compact Polish spaces. Since each X_{-i}^k is a cross-product of compact Polish spaces, it is compact Polish itself. Player i's hierarchy of cps' is an infinite sequence of cps' $\mu_{i,T} = (\mu_{i,T}^1, \mu_{i,T}^2, \dots) \in \prod_{k=0}^{\infty} \Delta^{\boldsymbol{H}_T}(X_{-i}^k)$.

Intuitively, player i's cps $\mu_{i,T}$ defines his conditional belief about the set of others' strategies he is aware of (first-order cps), his belief about each of his opponents beliefs (second-order cps), his belief about each of her opponents beliefs about others' beliefs (third-order cps), and so on.

[6] $\mathcal{B}$ obtains from the product of the discrete topology on $S_{-i}^{\boldsymbol{H}_T}$

3.4 Coherent hierarchies of conditional beliefs

The conditional belief system may not be meaningful; a player's belief may fail to uniquely specify her own cps. For example, for i's first-order cps $\mu^1_{i,T} \in \Delta^{\boldsymbol{H}_T}(X^0_{-i})$ and her second-order cps $\mu^2_{i,T} \in \Delta^{\boldsymbol{H}_T}(X^1_{-i})$ to be meaningful beliefs, the marginal distribution of $\mu^2_{i,T}$ on X^0_{-i} must coincide with $\mu^1_{i,T}$. We therefore impose that the various levels of beliefs of a player cannot contradict each other. In other words beliefs should be coherent, i.e.,

$$\mu^k_{i,T}(\cdot|h_{T'}) = \text{ marg}_{X^{k-1}}\mu^{k+1}_{i,T}(\cdot|h_{T'}) \text{ for all } k \geq 1 \text{ and } h_{T'} \in \boldsymbol{H}_T.$$[7]

This does however not imply that a player is certain that opponents' conditional beliefs about her beliefs are coherent. In particular, a player might belief one or more of her opponents beliefs are incoherent, or that they may belief that one or more of their opponents may have incoherent beliefs, and so on. That is, coherency should be known with common knowledge. The set of cps' $\mu_{i,T}$ for player i in which she is certain that coherency is common knowledge is denoted $B_{i,T}$.[8] We let $B^k_{i,T}$ denote the set of k-order beliefs consistent with collective coherency, i.e., the projection of $B_{i,T}$ on $\Delta^{\boldsymbol{H}_T}(X^{k-1}_{-i})$, and let $B^k_{-i,T} = \prod_{j\neq i} B^k_{j,T}$, $B_{-i,T} = \prod_{j\neq i} B_{j,T}$, and $B_T = \prod_{i\in N} B_{i,T}$.

We have seen that i's k-order conditional beliefs induces beliefs about the set $S^{\boldsymbol{H}_T}_{-i}$ and opponents $(k-1)$-order beliefs, but that does not guarantee that there exists a hierarchy wherein i has beliefs about opponents beliefs in the limit (i.e., $k \to \infty$). Thus, a model that specifies only finite k-order beliefs is not closed. The following Lemma states that i's coherent infinite hierarchy induces beliefs about $S^{\boldsymbol{H}_T}_{-i}$ and the infinite hierarchy of her opponents.

Lemma 1. For each $i \in N$ there is a homeomorphism

$$f_i = (f_{i,h_{T'}})_{h_{T'}\in\boldsymbol{H}_T} : B_{i,T} \to \Delta^{\boldsymbol{H}_T}(S^{\boldsymbol{H}_T}_{-i} \times B_{-i,T}).$$[9]

[7] It can be show by a version of Kolmogorov's Existence Theorem [see e.g. Brandenburger and Dekel (1993) and Battigalli and Siniscalchi (1999) for the proof] that coherent infinite cps' exists and are unique measures in the limit. If we let $\bar{B}_{i,T}$ be the set of i's coherent infinite cps' $\mu_{i,T}$, then there is a 'canonical' homeomorphism

$$g_i = (g_{i,h_{T'}})_{h_{T'}\in\boldsymbol{H}_T} : \bar{B}_{i,T} \to \Delta^{\boldsymbol{H}_T}(S^{\boldsymbol{H}_T}_{-i} \times \textstyle\prod_{j\neq i}\prod_{k=0}^{\infty}\Delta^{\boldsymbol{H}_T}(X^k_{-j})).$$

[8] Collective coherency can be defined as follows: Again let $\bar{B}_{i,T}$ be the set of i's coherent infinite cps' $\mu_{i,T}$. We will say that i knows some event $E \subseteq S^{\boldsymbol{H}_T}_{-i} \times \prod_{j\neq i}\bar{B}_{j,T}$ at $h_{T'} \in \boldsymbol{H}_T$ if $g_{i,h_{T'}}(\mu_{i,T})(E) = 1$. For every $i \in N$ and $h_{T'} \in \boldsymbol{H}_T$ inductively define, for $m = 1, 2, \ldots$, the sets

$\bar{B}_{i,T}(1) = \bar{B}_{i,T}$, and $\forall m \geq 2$

$\bar{B}_{i,T}(m) = \{\mu_{i,T} \in \bar{B}_{i,T}(m-1) :$ for all $h_{T'} \in \boldsymbol{H}_T$ and $g_{i,h_{T'}}(\mu_{i,T})(S^{\boldsymbol{H}_T}_{-i} \times \prod_{j\neq i}\bar{B}_{j,T}(m-1)) = 1\}$

Then $B_{i,T} = \bigcap_{m=1}^{\infty}\bar{B}_{i,T}(m)$ is i's set of collectively coherent cps'.

[9] See proof in Battigalli and Siniscalchi (1999, Proposition 2).

One might be concerned as to why the homeomorphism g is 'natural'. The reason is that the marginal probability assigned by each $f_{i,h_{T'}}(\mu^1_{i,T}, \mu^2_{i,T}, \dots)$ to a given event in X^{k-1}_{-i} is equal to the probability that $\mu^k_{i,T}$ assigns to that same event. That is, in deriving probabilities on the product space $S^{\boldsymbol{H}_T}_{-i} \times B_{-i,T} = X^0_{-i} \times \prod_{j \neq i} \Delta^{\boldsymbol{H}_T}(X^0_{-j}) \times \prod_{j \neq i} \Delta^{\boldsymbol{H}_T}(X^1_{-j}) \times \cdots$ from $(\mu^1_{i,T}, \mu^2_{i,T}, \dots)$, the function $f_{i,h_{T'}}$ preserves the probabilities specified by $\mu^k_{i,T}$ on each X^{k-1}_{-i}.

Definition 7. A k-order cps of a player $i \in N$ at $h_{T'} \in \boldsymbol{H}_T$ is such that for all $\mu_{i,T} = (\mu^1_{i,T}, \mu^2_{i,T}, \dots) \in B_{i,T}$, $k \geq 1$,

$$\mu^k_{i,T}(\cdot | h_{T'}) = \operatorname{marg}_{S^{\boldsymbol{H}_T}_{-i} \times B^1_{-i,T} \times \cdots \times B^{k-1}_{-i,T}} f_{i,h_{T'}}(\mu_{i,T}).$$

This definition of a general (conditional) belief system with unawareness is slightly different from a standard belief system with full awareness. We assume that players confined to H_T are capable of forming infinite hierarchies of cps' in any $h_{T'} \in \boldsymbol{H}_T$ (i.e., histories he is aware of). Player i may however not know which copy $h_{T'}$ that is being played by others. This uncertainty is given by $\Pr_{\mu_{i,T}} \in \Delta^{\boldsymbol{H}_T}(S^{\boldsymbol{H}_T}_{-i} \times B_{-i,T})$, where the events $E \in \mathcal{B}$ are defined by $E \subseteq S^{\boldsymbol{H}_T}_{-i} \times B_{-i,T'} : E = S^{\boldsymbol{H}_T}_{-i}(h_{T'}) \times B_{-i,T'}, h_{T'} \in \boldsymbol{H}_T$ (or $E \subseteq S^{\boldsymbol{H}_T}_{-i} : E = S^{\boldsymbol{H}_T}_{-i}(h_{T'}), h_{T'} \in \boldsymbol{H}_T$) such that they correspond to histories $h_{T'} \in \boldsymbol{H}_T$. That is, the induced belief $f_i(\mu_{i,T})(E) = \Pr_{\mu_{i,T}}(E)$. The special event that all histories $\boldsymbol{H}_T$ obtain is given by $E = \bigcup_{h_{T'} \in \boldsymbol{H}_T} S^{\boldsymbol{H}_T}_{-i}(h_{T'}) \times B_{-i,T'}$ and is equal to $X = S^{\boldsymbol{H}_T}_{-i} \times B_{-i,T'} \in \mathcal{B}$. This implies, that the probability that player i assigns to others playing at the copy $h_{T'}$, conditional on him playing at h_T, is $\Pr_{\mu_{i,T}}(\cdot | [h_{T \hookrightarrow T'}]) \in \Delta(\boldsymbol{H}_T)$ an by Bayes' rule we have that:

$$\Pr_{\mu_{i,T}}(h_{T'} | [h_{T \hookrightarrow T'}]) = \begin{cases} \frac{\Pr_{\mu_{i,T}}(h_{T'})}{\sum_{h_T \to h_{T'}} \Pr_{\mu_{i,T}}(h_{T'})} & \text{if } \{h_{T'} \cap [h_{T \hookrightarrow T'}]\} \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

The structure collapses to Battigalli and Dufwenberg (2009)'s model if all players are confined to the same subtree T, they believe that the only histories in H_T are those in which no messages is sent, and there is common knowledge about this. Formally, define a subset of 'Battigalli-Dufwenberg strategies' $S^{\text{BD}}_{-i} \subset S^{\boldsymbol{H}_T}_{-i}$ for which a typical element is a profile of others' BD strategies $s^{\text{BD}}_{-i} = (a_{-i,h_T}, \{\varnothing\})_{h_T \in H_T}$. Consider now m inductive steps of reasoning about a players knowledge about S^{BD}_{-i}, knowledge about others knowledge about S^{BD}_{-i}, and so on. To do so, define a sequence of sets $\{\hat{B}_{i,T}(m)\}_{m \geq 1}$ by $\hat{B}_{i,T}(1) = \Delta^{\boldsymbol{H}_T}(S^{\text{BD}}_{-i} \times \prod_{j \neq i} \Delta^{\boldsymbol{H}_T}(S^{\text{BD}}_{-j}) \times \cdots)$ and for all $m \geq 2$

$$\hat{B}_{i,T}(m) = \{\mu_i \in \hat{B}_{i,T}(1) : (f_{i,h_T})_{h_T \in H_T}(\mu_i)(S^{\text{BD}}_{-i} \times \hat{B}_{-i,T}(m-1)) = 1\}.$$

Let $B^{\text{BD}}_i = \bigcap_{m=1}^{\infty} \hat{B}_{i,T}(m)$ be player i's 'Battigalli–Dufwenberg belief set'. The set B^{BD}_i is the subset of $B_{i,T}$ obtained by requiring the following states to hold: (1) i knows (believes with probability one) others' BD strategies ; (2) others know others' BD strategies; (3) i knows others know his BD strategies; and so on. That is, B_i^{BD} is the set of beliefs which satisfy common knowledge of S_{-i}^{BD}.

3.5 Games with unawareness and belief-dependent preferences

We are now ready to state our definition of a $\boldsymbol{H}_T$-partial game:

Definition 8. A $\boldsymbol{H}_T$-partial game with belief-dependent preferences based on the augmented extensive form $\langle N, \boldsymbol{H}_T \rangle$ is a structure $\Gamma_T = \langle N, \boldsymbol{H}_T, (u_{i,T})_{i \in N} \rangle$ where $u_{i,T} : Z_T \times B_{i,T} \to \mathbb{R}$ is i's psychological payoff function.

A dynamic psychological game with unawareness is an ordered set of $\boldsymbol{H}_T$-partial games satisfying $T \in \boldsymbol{T}$ (including $\mathscr{T}$ which is the most expressive subtree).

Definition 9. A dynamic psychological game with unawareness is an ordered set of $\boldsymbol{H}_T$-partial games $\Gamma = \{\Gamma_T\}_{\mathscr{T} \hookrightarrow T}$.

This game is not a game in the traditional sense since: (i) players need not to be aware of all histories; (ii) players does not need to believe that they play the same game; and (iii) players may become aware of more as the game proceeds.

4 Psychological sequential equilibrium

In the following we will propose a version of Kreps and Wilson (1982)'s sequential equilibrium concept for $\boldsymbol{H}_T$-partial games. We will define and interpret consistent assessments in a $\boldsymbol{H}_T$-partial game (4.1), give the main equilibrium definition and provide an existence theorem (4.2). Lengthy mathematical proofs are relegated to Appendix (B).

4.1 Consistent assessments

Let $\Pr_{\sigma_{j,T}}(s_j^{T'} | \hat{h_{T'}}) \in \Delta(S_j^{\boldsymbol{H}_T}(\hat{h}_{T'}))$ denote the probability measure over j's strategies that i is aware of, conditional on $\hat{h}_{T'} \in \boldsymbol{H}_T$ and derived from behavioral strategy $\sigma_{j,T}$ under the assumption of independence across histories such that for all $s_j^{T'} \in S_j^{\boldsymbol{H}_T}(\hat{h}_{T'})$:

$$\Pr_{\sigma_{j,T}}(s_j^{T'} | \hat{h}_{T'}) := \prod_{h_{T'} \in \boldsymbol{H}_T \setminus \boldsymbol{Z}_T : h_{T'} \nless \hat{h}_{T'}} \sigma_{j,T}(s_{j,h_{T'}}^{T'} | h_{T'})$$

where $h_{T'} \nprec \hat{h}_{T'}$ means that $h_{T'}$ does not precede $\hat{h}_{T'}$. Intuitively, player i who is confined to Γ_T evaluates the conditional probability of some other player j making a certain choice if the path of play reaches history $h_{T'}$.

A profile of conjectures (first-order cps') $\mu^1_T = (\mu^1_{i,T})_{i \in N}$ is derived from a behavioral strategy profile $\sigma_T = (\sigma_{i,T})_{i \in N}$ if for all $i \in N$, $s^{T'}_{-i} \in S^{\boldsymbol{H}_T}_{-i}$, $h_{T'} \in \boldsymbol{H}_T$

$$\mu^1_{i,T}(s^{T'}_{-i}|h_{T'}) = \prod_{j \neq i} \mathrm{Pr}_{\sigma_{j,T}}(s^{T'}_j|h_{T'}).$$

This implies that for any three players i, j, k confined to Γ_T, the conjectures of i and j about k coincide. That is, for all $h_{T'} \in \boldsymbol{H}_T$:

$$\mathrm{marg}_{S^{\boldsymbol{H}_T}_k} \mu^1_{i,T}(\cdot|h_{T'}) = \mathrm{Pr}_{\sigma_{k,T}}(\cdot|h_{T'}) = \mathrm{marg}_{S^{\boldsymbol{H}_T}_k} \mu^1_{j,T}(\cdot|h_{T'}).$$

Since we assume that the behavioral strategies are independent, the conjectures will also be independent. That is, uncertainty in the minds of the other players, as to how a player will act, is independent. It should be noted that there is nothing in the interpretation of conjectures that compels us to assume that the beliefs of others over a players strategies should exhibit independence. The independence of conjectures involves an additional assumption. We are now ready to define consistent assessments:

Definition 10. An assessment (σ_T, μ_T) in Γ_T is consistent if

(i) μ^1_T is derived from σ_T,

(ii) and higher order beliefs in μ_T assign probability 1 to the lower order beliefs, such that for all $i \in N$, $k > 1$, $h_{T'} \in \boldsymbol{H}_T$

$$\mu^k_{i,T}(\cdot|h_{T'}) = \mu^{k-1}_{i,T}(\cdot|h_{T'}) \times \delta_{\mu^{k-1}_{-i,T}}$$

where δ_x is the Dirac measure which assigns probability 1 to singleton $\{x\}$.

Intuitively, players with common awareness must hold common, correct beliefs about each others belief hierarchies $\mu_{i,T} = (\mu^1_{i,T}, \mu^2_{i,T}, \dots)$. The interpretation is that beliefs are the end-product of transparent reasoning by intelligent players, which implies that any two players with the same awareness must share the same initial first-order cps about any other player, and every player comes to a correct conclusion about the belief hierarchies of other players he is aware of because he is able to replicate their reasoning.

Consistency captures the assumption that each player regards the others' choices at different histories as stochastically independent, and any two players have the same (prior and conditional) beliefs about any third player. These assumptions have a nice interpretation. Initially (at the empty history $\{h^0_{T'}\}_{T\hookrightarrow T'}$) each player, confined to Γ_T, has common beliefs about others' awareness which is independent of their own awareness and satisfy independence across other players. Players with the same awareness thus share a common independent conjecture μ^1_T of the probability of being matched with some a player j. As the game proceeds and players observe choices they (Bayesian) update their believes in a consistent manner, even if they become aware of something they were previous unaware of. The reason for this is that he is unaware of the possibility that his current awareness and beliefs may be wrong.

4.2 Equilibrium concept

Let some player i be confined to Γ_T fix a hierarchy of cps' $\mu_{i,T}$ and a strategy $s^T_i \in S^{\boldsymbol{H}_T}_i(h_T)$. The expectation of $u_{i,T}$ conditional on a non terminal history $h_T \in H_T \backslash Z_T$ and $\mu_{i,T}$ is

$$\mathbb{E}_{s^T_i,\mu_{i,T}}[u_{i,T}|h_T] := \sum_{h_T \hookrightarrow h_{T'}} \Pr_{\mu_{i,T}}(h_{T'}|[h_{T\hookrightarrow T'}]) \times \sum_{s^{T'}_{-i} \in S^{\boldsymbol{H}_T}_{-i}(h_{T'})} \prod_{j \neq i} \Pr_{\sigma_{j,T}}(s^{T'}_{-i}|h_{T'}) u_{i,T}(\zeta(s^T_i, s^{T'}_{-i}), \mu_{i,T}), \quad (1)$$

where $\Pr_{\mu_{i,T}}(h_{T'}|[h_{T\hookrightarrow T'}])$ is i beliefs about $h_{T'}$ obtaining, and $\Pr_{\sigma_{j,T}}(s^{T'}_{-i}|h_{T'})$ is the probability measure over j's strategy conditional on $h_{T'}$ derived from behavioral strategy $\sigma_{j,T}$ at H_T.

Definition 11. An assessment (σ_T, μ_T) in Γ_T is a sequential equilibrium if it is consistent and for all $i \in N$, $h_T \in H_T$, $s^{T,*}_i \in S^{\boldsymbol{H}_T}_i(h_T)$

$$\Pr_{\sigma_{i,T}}(s^{T,*}_i|h_T) > 0 \Rightarrow s^{T,*}_i \in \arg \max_{s^T_i \in S^{\boldsymbol{H}_T}_i(h_T)} \mathbb{E}_{s^T_i,\mu_{i,T}}[u_{i,T}|h_T]$$

We can also take the point of view of an 'agent' (i, h_T) confined to a $\boldsymbol{H}_T$-partial game and in charge of the move at history h_T, who seeks to maximize i's conditional expected utility given the consistent assessment (σ_T, μ_T). The expected utility of i conditional on h_T and $c_{i,h_T} \in C_{i,h_T}$ given (σ_T, μ_T) can be expressed as

$$\mathbb{E}_{\sigma_T,\mu_T}[u_{i,T}|h_T, c_{i,h_T}] \coloneqq \sum_{h_T \hookrightarrow h_{T'}} \Pr_{\mu_{i,T}}(h_{T'}|[h_{T\hookrightarrow T'}])\times$$
$$\sum_{s_{-i}^{T'} \in S_{-i}^{\boldsymbol{H}_T}(h_{T'})} \prod_{j \neq i} \Pr_{\sigma_{j,T}}(s_{-i}^{T'}|h_{T'}) \sum_{s_i^T \in S_i^{\boldsymbol{H}_T}(h_T, c_{i,h_T})} \Pr_{\sigma_{i,T}}(s_i^T|h_T, c_{i,h_T}) u_{i,T}(\zeta(s_i^T, s_{-i}^{T'}), \mu_{i,T}),$$

where $\Pr_{\sigma_{i,T}}(s_i^T|h_T, c_{i,h_T}) \coloneqq \prod_{h'_T \in H_T \setminus Z_T : h'_T \npreceq h_T} \sigma_{i,T}(s_{i,h_T}^T|h'_T)$ ($h'_T \npreceq h_T$ means that h'_T is not h_T nor a predecessor of h_T) is the probability measure over strategy s_i^T conditional on the history which follows his choice.

Definition 12. An assessment (σ_T, μ_T) in Γ_T is a sequential equilibrium if it is consistent and for all $i \in N$, $h_T \in H_T$,

$$\operatorname{supp}(\sigma_{i,T}(\cdot|h_T) \subseteq \arg \max_{c_{i,h_T} \in C_{i,h_T}} \mathbb{E}_{\sigma_T,\mu_T}[u_{i,T}|h_T, c_{i,h_T}]$$

A version of the One-Shot-Deviation principle holds in our framework. Intuitively, the One-Shot-Deviation principle says that a strategy profile $s^{T,*}$ in the $\boldsymbol{H}_T$-partial game induces an equilibrium in every subgame if no player i, whenever it is his turn to move, has an incentive to make a choice different from the one prescribed by $s_i^{T,*}$ in the continuation (assuming he believes that the other players confined to Γ_T stick to $s_{-i}^{T,*}$ in the current stage and in the continuation).

Proposition 1. An optimal strategy of player $i \in N$ confined to Γ_T satisfies the One-Shot-Deviation property since it holds for all $i \in N$, $h_T \in H_T \setminus Z_T$ that

$$\max_{c_{i,h_T} \in C_{i,h_T}} \mathbb{E}_{\sigma_T,\mu_T}[u_{i,T}|h_T, c_{i,h_T}] = \max_{s_i^T \in S_i^{\boldsymbol{H}_T}(h_T)} \mathbb{E}_{s_i^T,\mu_{i,T}}[u_{i,T}|h_T].$$

Proof. See Appendix (B). ∎

The following existence theorem obtains:

Theorem 1. There exists at least one sequential equilibrium assessment if the belief-dependent utilities are continuous.

Proof. See Appendix (B). ∎

The proof of existence basically relies on the trembling-hand perfect equilibrium concept [due to Selten (1975)]. The idea is no matter how close to rational players are, they will never be perfectly rational. There will always be some chance that a player will make a mistake. This idea can be used to approximate a candidate equilibrium behavioral strategy profile by a nearby completely mixed strategy profile (tremble) and require that any deliberately made choices, i.e., those given positive probability in the candidate strategy profile, be optimal, not only against the candidate strategy profile, but also against the nearby mixed strategy profile. More formally, a profile of behavioral strategies σ_T is a perfect equilibrium if there is a sequence of completely mixed strategy profiles $\{\epsilon^k\}$ such that at each history $h_T \in H_T$ and for each ϵ^k, the behavior of σ_T at the history is optimal against ϵ^k, i.e., is optimal when behavior at all other histories is given by ϵ^k. It is shown by Kakutani's fixed point theorem that there in each ϵ^k-perturbed game exists at least one ϵ^k-equilibrium strategy profile σ_T^k, implying that there exist an assessment $(\sigma_T^k, \beta(\sigma_T^k))$ where $\beta(\sigma_T^k)) = \mu_T$.[10] As $\epsilon^k \to 0$ the corresponding strategy σ_T^k has an accumulation point σ_T^*, such that $(\sigma_T^*, \beta(\sigma_T^*))$. For each 'agent' (i, h_T), $\sigma_{i,T}^*(\cdot|h_T)$ assigns positive probability only to choices that are best responses to $(\sigma_T^*, \beta(\sigma_T^*))$ at h_T. By Definition 12, $(\sigma_T^*, \beta(\sigma_T^*))$ is a sequential equilibrium assessment.

Corollary 1. For all embeddable subtrees $T \hookrightarrow T'$ suppose equilibrium in each corresponding $\Gamma_{T'}$-partial game. Then for each of the equilibria of $\Gamma_{T'}$, there is an equilibrium of Γ_T in which players confined to $\Gamma_{T'}$ (due to unawareness) play their equilibrium assessments in $\Gamma_{T'}$.

Proof. See Appendix (B). ∎

This proposition suggests a procedure for constructing equilibria in a dynamic psychological game with unawareness. First, fix the $\boldsymbol{H}_T$-partial game to which some player is initially confined to. Then, start from the last stage: any $h_T \hookrightarrow h_{T'}$ such that all feasible choices at each $h_{T'}$ terminates the game. If there are more than one player who can move at each $h_{T'}$ then we look for an equilibrium with unawareness in each subgame, by: (i) calculating the best response choices of other players at the copy in the least expressive game, and (ii) extend the equilibrium step-by-step to more expressive copies by finding a fixed point taking the choice of the other players in the respective less expressive copies as give. If there is one

[10]Let $\beta^1(\sigma_T) = (\beta^1(\sigma_T))_{i\in N}$ denote the profile of first-order beliefs derived from σ_T according to condition (i) in Definition 10. The profile of infinite belief hierarchies $\mu_T = \beta(\sigma_T)$ is obtained by applying condition (ii) in the same definition.

player in charge of the move at each $h_{T'}$, then we only need to fix his best response at each copy. Now we go backwards and look at histories and copies thereof in the second-to-last stage. The best responses has already been calculated for all histories $(h_{T'}, c_{h_{T'}}) \hookrightarrow (h_{T'}, c_{h_{T'}})$, because such histories correspond to the last stage of the game. We assume that each active player at the second-to-last stage makes feasible choices that maximizes his expected utility given the best responses in the last stage, because he expects that the other players will also best response in the last stage. We continue to go backwards in this ways until we reach the first stage. A sequential equilibrium in the $\boldsymbol{H}_T$-partial game rules out any profitable deviations given the player's awareness level. However the definition does not exclude the possibility that deviations at successive stages might increase his belief-dependent utility as he may become aware of more paths of play. This implies that in analyzing a dynamic psychological game with unawareness, one has to consider each player's awareness at each stage. If a player history becomes aware of more, he reevaluates the strategic situation and starts over by backwards inducting until the initial stage.

5 Application

In the following we will use a sequential prisoners dilemma to highlight the impact and importance of unawareness in strategic interactions of agents with belief-dependent preferences. The specific belief-dependent motivation that we concentrate on is a modified version of Dufwenberg and Kirchsteiger (2004)'s 'theory of sequential reciprocity' (5.1). A full description of the strategic interaction with all possible awareness levels and equilibria is beyond the scope of this paper. Therefore, we limit the analysis to two different awareness scenarios and the respective characterization of only one equilibrium (5.2). Results and intuitions are presented in this section, lengthy mathematical proofs are relegated to the Appendix (C).

5.1 A sequential prisoners dilemma with reciprocity

Consider the following sequential prisoners dilemma also analyzed by Dufwenberg and Kirchsteiger (2004):

[Figure 3]

Figure 3 is an extensive form game without communication with $N = \{Ann, Bob\}$, $\mathscr{T} = \{h^0, h^1, h^2, h^3, h^4, h^5, h^6\}$, and material payoffs associated with each joint strategy profile. In the initial history h^0 Ann can choose between *cooperate* (C) and *defect* (D) and Bob is passive. On the other hand, in history h^1 and h^2 Bob can respectively choose between

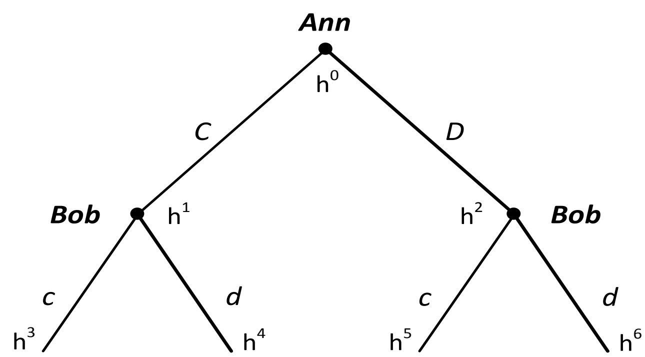

Figure 1: An extensive form without communication

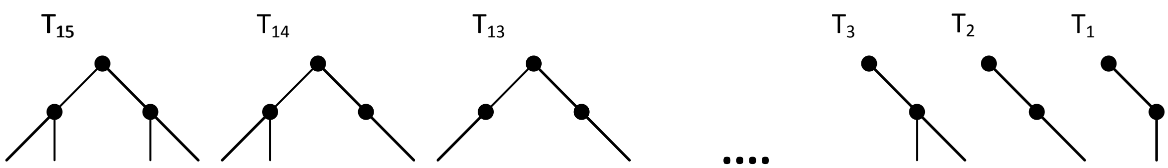

Figure 2: The family of subtrees $\boldsymbol{T}$

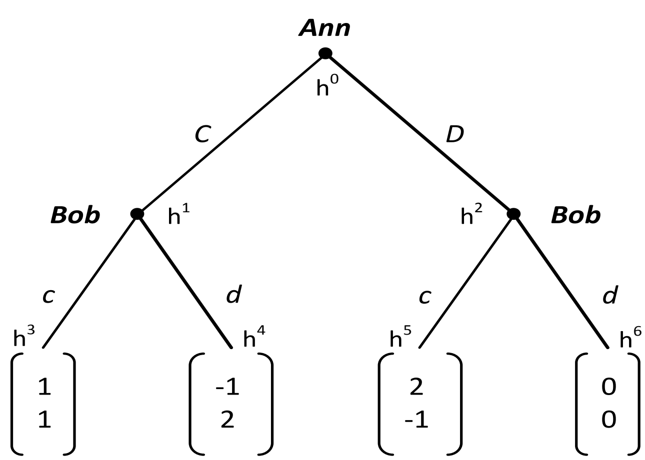

Figure 3: 'Sequential Prisoners Dilemma' without communication

cooperate (c) and *defect* (d) and *Ann* is passive. In case, (C,c) is chosen both players get a material payoff of 1, if (C,d) is chosen *Ann* gets −1 and *Bob* gets 2. Furthermore, in case (D,d) is chosen both players get a material payoff of 0 and if (D,c) is chosen *Ann* gets 2 and *Bob* gets −1.

Following section (2), we consider now the extensive form with communication $\langle \mathcal{N}, \boldsymbol{H}_{\mathscr{T}} \rangle$ associated with the extensive form without communication $\langle N, \mathscr{T} \rangle$.

For simplicity we assume that only *Bob* is motivated by belief-dependent reciprocity.[11] More specifically, *Bob*'s utility is:

$$u_B(\zeta(s_B^T, s_A), \mu_B) = \pi_B(\cdot) + Y \times \kappa_{BA}(\cdot) \times \lambda_{BAB}(\cdot),$$

where $\pi_B(\cdot)$ is *Bob*'s expected monetary payoff which depends on his first-order belief concerning *Ann*'s strategy $(\mu_B^1(s_a))$ and *Bob*'s strategy (s_B^T) and $Y > 0$ is a constant that captures his sensitivity to reciprocity towards *Ann*. *Bob*'s belief about his kindness towards *Ann* is $\kappa_{BA}(\cdot)$ and *Bob*'s perception of *Ann*'s kindness towards him is $\lambda_{BAB}(\cdot)$.

Formally, *Bob*'s perception of *Ann*'s kindness towards him in a history h_T in which he is confined to H_T is:

$$\lambda_{BAB}(\cdot) = \pi_B\left(\mu_{B,T}^1\left(s_A^T | h_T\right), \mu_{B,T}^2\left(\cdot | h_T\right)\right) - \pi_B^{e_A}\left(\mu_{B,T}^1\left(s_A^T | h_T\right), \mu_{B,T}^2\left(\cdot | h_T\right)\right),$$

where $\mu_{B,T}^1\left(s_A^T | h_T\right)$ and $\mu_{B,T}^2\left(\cdot | h_T\right)$ respectively are *Bob*'s (updated) first- and second-order beliefs conditional on h_T. Intuitively, these beliefs describe what *Bob* believes *Ann* would do and believe had she the same awareness level as him. Given this, $\pi_B(\cdot)$ and $\pi_B^{e_A}(\cdot)$ respectively describe what *Bob* believes *Ann* would intend for him and the average that *Ann* would be able to give had she the same awareness level as *Bob*. The equitable payoff is formally defined as follows:

$$\begin{aligned}\pi_B^{e_A}(\cdot) = \frac{1}{2}\Big[& \max\left\{\pi_B\left(\mu_{B,T}^1\left(s_A^T | h_T\right), \mu_{B,T}^2\left(\cdot | h_T\right)\right), s_A^T \in S_A^{H_T}\right\} \\ & + \min\left\{\pi_B\left(\mu_{B,T}^1\left(s_A^T | h_T\right), \mu_{B,T}^2\left(\cdot | h_T\right)\right), s_A^T \in S_A^{H_T}\right\}\Big],\end{aligned} \tag{2}$$

where $S_A^{H_T}$ is the set of strategies of *Ann* in the partial game that Bob is confined to in history h_T. The first term in the brackets, $\max\{\pi_B(\mu_{B,T}^1(s_A^T | h_T), \mu_{B,T}^2(\cdot | h_T))\}$, describes *Bob*'s belief about *Ann*'s belief about the maximum that she could have given to him. On the other hand, $\min\{\pi_B(\mu_{B,T}^1(s_A^T | h_T), \mu_{B,T}^2(\cdot | h_T))\}$ describes *Bob*'s belief about *Ann*'s belief

[11]It is assumed that *Ann* is only interested in her own monetary payoff.

concerning the minimum she could have given to him. Intuitively *Bob* does not blame *Ann* for being unaware of some paths of play. He just forms a belief about what *Ann* would and could do were she of the same awareness level as he is.

Note that in Dufwenberg and Kirchsteiger (2004) the set of joint strategy profiles is commonly known. However, in our setting with unawareness kindness perceptions take into account the fact that others might be aware of less. Furthermore, full awareness implies, that the basis upon which the others' kindness is evaluated remains unchanged. In contrast, in our setting the basis upon which the own as well as the kindness of others is judged changes as players become aware of more feasible paths of play.

Bob's kindness towards *Ann* in history h_T in which he is confined to H_T can be described as:

$$\kappa_{BA}(\cdot) = \pi_A\left(\mu^1_{B,T}\left(s_A|h_T\right), s^T_B\right) - \pi^{e_B}_A\left(\mu^1_{B,T}\left(s_A|h_T\right), s^T_B\right),$$

where $s_A \in S^{\boldsymbol{H}_T}_A$ and $\pi^{e_B}_A(\cdot)$ is defined in an analogous fashion to equation 2.

Ann's expected material payoff $\pi_A(\cdot)$ describes what *Bob* believes *Ann* gets, given his beliefs concerning her strategy s_A and his own strategy $s^T_B \in S^{H_T}_B$ where $S^{H_T}_B$ is the set of own strategies that *Bob* is aware of in history h_T. Furthermore, $\pi^{e_B}_A(\cdot)$ is *Bob*'s belief about the average that he can give to *Ann*.

This concludes the definition of our sequential prisoners dilemma with reciprocity.

5.2 Three Different Awareness Scenarios

In our context with unawareness every application has to start with a description of what people are initially aware of, what they become aware of when actions are chosen that they were previously unaware of and what they belief concerning the awareness level of others.

Scenario 1: As a first awareness scenario consider the benchmark case in which *Ann* and *Bob* are aware of everything. That is, there is no unawareness. Obviously, in such an environment messages that contain feasible paths of play are irrelevant because everyone is aware of all feasible paths of play:

Remark 2. If every player is aware of everything, messages about feasible paths of play are irrelevant.

Given this we can abstract from messages in our benchmark case and concentrate on the actions of *Ann* and *Bob*. From Dufwenberg and Kirchsteiger (2004) we know that:

Result 1. If *Ann* defects by choosing D, *Bob* also defects by choosing d in equilibrium independent of his sensitivity to reciprocity Y. Furthermore, if *Ann* cooperates by choosing C, *Bob* cooperates by choosing c in equilibrium if his sensitivity to reciprocity is $Y \geq 1$.

Proof. See Dufwenberg and Kirchsteiger (2004), p. 293.

Given *Bob*'s behavior following *Ann*'s action, it also holds in our benchmark case that:

Result 2. If *Bob*'s sensitivity to reciprocity is $Y \geq 1$, *Ann* cooperates by choosing C in equilibrium.

Proof. See Dufwenberg and Kirchsteiger (2004), p. 293.

This shows that without unawareness and *Bob* who acts reciprocally ($Y \geq 1$), *Ann* can trigger a cooperative reaction from *Bob* by choosing to cooperate. Note that this very intuitive result stands in contrast to the result we would obtain with traditional assumptions about human behavior, i.e. egoistic preferences. If both players are only interested in their own monetary payoff, then *Ann* and *Bob* choosing defect would be the only pure strategy sequential equilibrium.

Scenario 2: As a second simple awareness scenario consider now the following:

- *Bob* is aware of everything, i.e. $T_{15} = \{h^0, h^1, h^2, h^3, h^4, h^5, h^6\}$.[12]
- *Bob* correctly believes that *Ann* is initially only aware of the subtree $T_3 = \{h^0, h^2, h^5, h^6\}$.
- *Bob* correctly believes that, wherever *Ann* finds herself, she will believe that *Bob* has the same awareness level.

Note that in scenario 2 we for simplicity assume that *Bob* and *Ann* believe with probability 1 that the other player is of a certain awareness level and believes with probability 1 that they are of a certain awareness level etc. However, different to the previous scenario without unawareness, in this scenario *Ann* is initially unaware of her action C and *Bob*'s actions c and d following it. As before, we start by looking at the optimal behavior of *Bob*. This is we start to look at all possible partial games *Bob* can find himself in after *Ann*'s choice. We fix his optimal behavior in these worlds and then go one step back to analyze *Ann*'s optimal choice given the optimal choice of *Bob*.

[12] Note that subtrees in our application are indexed in line with the subtrees in Figure 2.

Result 3. If *Ann* defects by choosing D, than *Bob* choosing c and sending any message is the optimal choice of *Bob* if his sensitivity to reciprocity is $Y \geq 1$.

Proof. See Appendix (C). ■

The intuitive reason for why *Bob* nevertheless cooperates even after the seemingly unkind action D of *Ann* is the following: *Bob* is aware of the fact that *Ann* is not aware of her action C and his actions c and d following it. However, *Bob* evaluates *Ann*'s kindness on the basis of what he is aware of. *Bob* holds the equilibrium belief that *Ann* would have cooperated had she been aware of what he is aware of. This means, in equilibrium *Bob* believes that *Ann* would have played C and, hence, would have acted kind, had she been aware of what he is aware of. As he is the last to choose in this situation, his choice is independent of the specific message that he sends, i.e. any of his messages is part of this equilibrium.

Concerning the behavior of *Ann* it is easy to see that her equilibrium behavior is

Result 4. In all sequential equilibria *Ann* chooses D and sends any message.

Obviously *Ann* chooses D in scenario 2 because this is the only feasible action that she is initially aware of. Furthermore, as she believes that *Bob* is aware of what she is aware of, any message is part of this sequential equilibrium. This completes the second awareness scenario.

Different to the setting without unawareness by Dufwenberg and Kirchsteiger (2004), *Bob* in our setting with unawareness still cooperates by choosing c even after the seemingly unkind action D. *Bob* simply takes into account that *Ann* was unaware of her action C and his subsequent actions d and c and, hence, evaluates her kindness on what she would have done had she been aware of what he is aware of. Importantly, (D, c) is neither part of an equilibrium given classical assumptions about human behavior, nor is it part of an equilibrium given reciprocal preferences and full awareness. It is the asymmetric awareness of *Bob* and *Ann* that produces this prediction. This demonstrates how allowing for asymmetric awareness influences our equilibrium predictions.

Furthermore, this scenario practically demonstrates how one can solve for sequential equilibira in our class of psychological games with unawareness and communication. One first has to look at the optimal behavior of all players active in the last non-terminal histories in all their partial games and then go backward history by history repeating the same procedure until the initial history.

Scenario 3: To furthermore see the importance of messages assume now the following awareness scenario:

- *Ann* is aware of everything, i.e. $T_{15} = \{h^0, h^1, h^2, h^3, h^4, h^5, h^6\}$.
- *Ann* knows that *Bob* is initially only aware of the subtree $T_4 = \{h^0, h^1, h^3, h^4\}$.
- *Ann* correctly believes that, wherever *Bob* finds himself, he will believe that *Ann* has the same awareness level.
- *Ann* correctly believes that *Bob* will become aware of everything, if she chooses to defect D.

We start again by analyzing this situation by looking at *Bob*'s choices in all the partial games that he can be in.

Result 5. If *Ann* chooses D and any message, *Bob* defects by choosing d and sending any message in all sequential equilibria.

Proof. See Appendix (C). ■

To see this, remember that if *Ann* chooses D, *Bob* becomes aware of everything independent of the message that *Ann* sends in addition to her action. This means, in any history following *Ann*'s action D *Bob* re-evaluates *Ann*'s kindness towards him on the basis of $T_{15} = \{h^0, h^1, h^2, h^3, h^4, h^5, h^6\}$. Doing this, *Bob* perceives *Ann*'s choice as unkind independent of the message that she sends. Therefore, *Bob* chooses d out reciprocity as well as own monetary considerations. Note, our result 5 is analog to Dufwenberg and Kirchsteiger (2004, p. 282)'s observation 1 in the context of their sequential prisoners dilemma.

Next, consider *Bob*'s behavior following *Ann*'s action C:

Result 6. If *Ann* chooses action C and sends

(i) a message that does not contain any new information on the feasible paths of play, then *Bob* defects in equilibrium by choosing d and sending any message independent of his sensitivity to reciprocity.

(ii) a message which contains $T_3 = \{h^0, h^2, h^5, h^6\}$ as new information, then *Bob* cooperates in equilibrium by choosing c and sending any message, if his sensitivity to reciprocity is $Y \geq 1$.

(iii) a message which contains only $T_2 = \{h^0, h^2, h^6\}$ as new information, then *Bob* cooperates in equilibrium by choosing c and sending any message, if his sensitivity to reciprocity is $Y \geq 1$.

(iv) a message which contains only $T_1 = \{h^0, h^2, h^5\}$ as new information, then *Bob* cooperates in equilibrium by choosing c and sending any message, if his sensitivity to reciprocity is $Y \geq \frac{1}{2}$.

Proof. See Appendix (C). ∎

Result 6 gives a first impression of how messages about feasible paths of play influence the strategic interaction of reciprocal players. Different to Dufwenberg and Kirchsteiger (2004, p. 282) in the context of their sequential prisoners dilemma with full awareness, our result 6 depends on *Ann*'s message to *Bob*. By sending a message *Ann* can influence the basis on which *Bob* evaluates her kindness. That is, she can influence the partial game that *Bob* will find himself in. If unaware of *Ann*'s action D and all of his own subsequent actions, *Bob* evaluates the kindness of *Ann* following her choice C on the basis of $T_4 = \{h^0, h^1, h^3, h^4\}$. This implies that he perceives a kindness $\lambda_{BAB} = 0$. This in turn means that *Bob* only takes into account his own monetary payoff when optimizing his choice. Only when *Ann* sends a message that contains some new information, i.e. a subtree consistent with her action D, *Bob*'s awareness and, hence, the partial game he plays as well as the basis upon which he evaluates *Ann*'s kindness changes.

By sending a message which contains $T_1 = \{h^0, h^2, h^5\}$ as new information, *Bob* becomes aware of $T_{12} = \{h^0, h^1, h^2, h^3, h^4, h^5\}$ (case (iv) of result 6). Hence, *Bob* finds himself in a new partial game and has a new basis upon which he evaluates the kindness of *Ann*. Now *Bob* is aware of the fact that *Ann* could have chosen D which would have implied (according to his awareness) a material payoff of -1 for him. Given this, he perceives *Ann*'s choice C as kind because independent of his choice following *Ann*'s choice C, his material payoff is higher than -1. He reciprocates this kindness in equilibrium if his sensitivity to reciprocity is $Y \geq \frac{1}{2}$. Following the same kind of reasoning in cases (*ii*) and (*iii*) implies that *Bob* reciprocates by choosing c, if his sensitivity to reciprocity is $Y \leq 1$.

As can easily be seen, if *Ann* had no possibility to send a message to *Bob*, i.e. to make *Bob* aware of what else she could have done, *Ann* would be unable to induce *Bob* to cooperate. *Bob* would simply remain aware of what he was aware of before and continue to evaluate *Ann*'s kindness on this basis.

This brings us to the equilibrium behavior of *Ann*

Result 7. *Ann*'s equilibrium behavior depends on *Bob*'s sensitivity to reciprocity Y:

(i) If *Bob*'s sensitivity to reciprocity is $Y < \frac{1}{2}$, *Ann* defects by choosing D in equilibrium and sends any message.

(ii) If *Bob*'s sensitivity to reciprocity is $\frac{1}{2} \leq Y \leq 1$, *Ann* cooperates by choosing C in equilibrium and sends a message which contains *only* $T_1 = \{h^0, h^2, h^5\}$ as new information.

(iii) If *Bob*'s sensitivity to reciprocity is $Y \geq 1$, *Ann* cooperates by choosing C in equilibrium and sends a message which contains *at least* $T_1 = \{h^0, h^2, h^5\}$ as new information.

Proof. See Appendix (C). ■

Intuitively, if *Bob*'s sensitivity to reciprocity is low, i.e. $Y < \frac{1}{2}$, *Ann* knows that what ever she makes *Bob* aware of, he will always choose d. Given this, she prefers to choose D to get 0 in monetary payoffs, rather than C which would give her -1. Now, if *Bob* has a sensitivity to reciprocity $Y > \frac{1}{2}$, *Ann* can induce *Bob* to cooperate by choosing C making him aware of her action D and *Bob*'s subsequent possibility c (case (ii) of result 7). Making *Bob* aware changes the basis on which he evaluates the kindness of *Ann* towards him. Aware of *Ann*'s action D and *Bob*'s action c, *Bob* realizes that *Ann*'s action C was actually kind. This is something he would not have realized had he remained unaware of D and his subsequent action c. By choosing action C and communicating either $T_3 = \{h^0, h^2, h^5, h^6\}$ or $T_2 = \{h^0, h^2, h^6\}$ *Ann* also induces a positive perception of her action, but less than in case (ii). Hence, *Ann* only chooses C and one of these messages in equilibrium if *Bob*'s sensitivity is higher $Y \geq 1$.

The bottom line: awareness messages are important in the interaction of players with reciprocal preferences as they influence their perceptions about their own as well as others' kindness.

These three simple awareness scenarios demonstrate how unawareness influences the strategic interaction of players with belief-dependent preferences. Furthermore, they show the important role of awareness messages through which players can influence other players' awareness. By influencing awareness levels players influence equilibrium behavior. To put it differently, taking into account asymmetric awareness levels of players when analyzing strategic interactions leads to new and intuitive equilibrium predictions.

6 Extensions and discussion

In this section we first consider relevant extensions of our model, namely guilt aversion (6.1), moves by nature (6.2), initial asymmetric information (6.3), and strategic information transmission (6.4). We then go on to discuss how to interpret hierarchies of beliefs (6.5), whether unawareness in any meaningful way can be modeled as zero probability events (6.6), and finally consider the relevance of non-equilibrium solution concepts in out setting (6.7).

6.1 Guilt aversion and unawareness

In Section (5) we focused on reciprocity, however our framework is general meaning that it can be used to analyze how unawareness affect other forms of belief-dependent motivation such as guilt and regret. In the following we will consider a simple two player example highlighting how unawareness might influence guilt aversion.

We will say that *Ann* 'lets down' *Bob* if his actual material payoff from *Ann*'s strategy, denoted $\pi_B(s_A)$, is lower than the payoff *Ann* belief he expects to get, $\pi_B(\mu_A^2(\cdot|h_T), \mu_A^1(s_B|h_T))$. This can be measured by the following expression:

$$\max\{0, (\pi_B(\mu_A^2(\cdot|h_T), \mu_A^1(s_B|h_T)) - \pi_B(s_A))\}.$$

Taking *Ann*'s belief concerning *Bob*'s disappointment into account, we obtain the following utility function exhibiting guilt aversion:

$$u_A(\zeta(s_A, s_B), \mu_A) = \pi_A(z) - Y \times \max\{0, (\pi_B(\mu_A^2(\cdot|h_T), \mu_A^1(s_B|h_T)) - \pi_B(s_A))\},$$

where $Y \geq 0$ is some psychological sensitivity parameter of *Ann*.

Now consider the example considered in the introduction, in which *Ann*'s exam is postponed and she could go to *Bob*'s party. Remember, *Ann* would rather not go to the party because she is tired. Now imagine that *Ann* correctly believes that *Bob* is unaware of the postponement: *Ann* will in equilibrium be certain that *Bob* will be certain that she cannot come, and *Ann* will therefore feel no guilt if she stays away. In a game with full awareness this would however not be an unique equilibrium. *Ann* could also be certain that *Bob* expects her to come because her exam was canceled. If *Ann*'s sensitivity to disappointing *Bob* in this situation is high enough, she will come to his party.

The two forms of belief-dependent motivation we have considered up to now (reciprocity and guilt) has relied on first- and second-order beliefs. However, our model is not restricted to only looking at these forms of beliefs – Definition 7 allows also for higher-order belief-dependence. An example involving dependence on third-order beliefs is Battigalli and Dufwenberg (2007b)'s 'guilt from blame,' which assumes that a player cares about the other player's inferences regarding the extend to which he is willing to let him down. Intuitively, *Ann* experience guilt to the extent that *Bob*'s beliefs indicate that *Ann* intended to disappoint him.

6.2 Moves by nature

Moves by nature is an important extensions for applications. For example, Sebald (2010) shows that the strategic interactions of reciprocal players may be influenced by the possibility of material payoffs are influenced by moves of nature rather than players. One could easily imagine that such considerations might be amplified, or mitigated, by unawareness.

Let $N^0 = \{0, 1, \ldots, n\}$ where index 0 denotes nature, and $\sigma_{0,T} := \sigma_{0,T}(\cdot|h_{T'}) \in \prod_{h_{T'} \in \boldsymbol{H}_T \setminus \boldsymbol{Z}_T} \Delta^0(A_{0,h_{T'}} \times \{\varnothing\})$ be the strictly positive objective plan of moves by nature. Note that given some awareness level, a 'real' player could never imagine that nature would send messages from which he could learn some new paths of play. We do therefore not in our model consider messages send by nature.

An assessment $(\sigma_T, \mu_T) = (\sigma_{i,T}, \mu_{i,T})_{i \in N^0}$ is consistent if there is a sequence of strictly positive behavioral strategy profiles $\sigma^k \to \sigma$ such that for all $i \in N$, $s_{-i}^{T'} \in S_{-i}^{\boldsymbol{H}_T}$, $h_{T'} \in \boldsymbol{H}_T$

$$\mu_{i,T}^1(s_{-i}^{T'}|h_{T'}) = \lim_{k \to \infty} \frac{\Pr_{\sigma_{0,T}}(s_0^{T'}) \prod_{j \neq 0,i} \Pr_{\sigma_{j,T}^k}(s_j^{T'})}{\sum_{s_{-i}'^{T'} \in S_{-i}^{T'}(h_{T'})} \Pr_{\sigma_{0,T}}(s_0'^{T'}) \prod_{j \neq 0,i} \Pr_{\sigma_{j,T}^k}(s_j'^{T'})}$$

Kreps and Wilson (1982, Section 5) have a similar condition that refers to cps' of histories (or nodes), and further more for all $l > 1$, $\mu_{i,T}^l$ assigns probability 1 to $\mu_{-i,T}^{l-1}$. (σ_T, μ_T) is a sequential equilibrium if it is consistent and for all $i \in N$, $h_T \in H_T$, $s_i^{T,*} \in S_i^{H_T}(h_T)$

$$\Pr_{\sigma_{i,T}}(s_i^T|h_T) > 0 \Rightarrow s_i^{T,*} \in \arg \max_{s_i^T \in S_i^{H_T}(h_T)} \mathbb{E}_{s_i^T, \mu_{i,T}}[u_{i,T}|h_T],$$

where $\mathbb{E}_{s_i^T, \mu_{i,T}}[u_{i,T}|h_T]$ is the obvious modification of Equation 1. It can easily be proven that that the existence theorem also holds when we add nature as a player (if the utility functions are continuous).

6.3 Initial asymmetric information

In addition, one might well argue that it is unrealistic to assume that players know one another's psychological propensities, unless one models interaction within a family or amongst friends. This observation motivate the following extension.

If we want to model asymmetric information about initial moves by nature, we should assume that at the initial history h_T^0 or copy thereof the only active player is 0 (nature), $A_{0,h_T^0} = \Theta$, where $\Theta \subseteq \Theta_1 \times \cdots \times \Theta_n$ is some exogenous parameter, each player i observes only coordinate θ_i of $\theta = (\theta_1, \ldots, \theta_n)$; θ may affect payoffs, or choice sets, or the probability of future moves by nature. Note that by defining asymmetric information in this way one introduces fictitious ex ante beliefs.

A full blown generalization of information in our model would also include imperfectly observable choices. However, such an extension is non-trivial: the information sets of players $i \neq 0$ need to apply to some consistency requirements. For example, an information set may not be such that at some histories are indistinguishable at some subtrees while not at others. A full characterization of imperfect information in our model is beyond the scope of this paper.

6.4 Strategic information transmission

Strategic information transmission has been studied in economic theory for over a quarter of a century. Traditionally this has been done via signalling, whereby a player can influence the beliefs of other players by his actions (e.g., choice of education). To highlight the difference between influencing players perceptions through signals or 'awareness messages', we will focus solely on the updating of players' beliefs. The discussion is therefore relevant for, among others, costly market signalling [Spence (1973), Rothschild and Stiglitz (1976), Wilson (1977)], cheap talk [Crawford and Sobel (1982), Farrell (1993)], and observational learning [Banerjee (1992), Bikhchandani *et al.* (1992), Smith and Sørensen (2000)].

The canonical signalling game for our class of unawareness games is basically a Bayesian extensive form with observable actions. We will say that nature selects types independently for the players and refer to player i after he receives information θ_i as type θ_i and $\theta = (\theta_1 \times \cdots \times \theta_n)$ as the state of nature. We assume that there exists a common prior over states of nature $p \in \Delta(\Theta)$ with the properties that for all i, θ_i and θ_{-i}, $p(\{\theta_i\} \times \Theta_{-i}) > 0$ (type θ_i has positive 'prior' probability) and $p(\theta_{-i}|\theta_i) = p((\theta_i, \theta_{-i})|\{\theta_i\} \times \Theta_{-i})$ (i.e., $p(\theta_{-i}|\theta_i)$ is the conditional probability of θ_{-i} given θ_i). Since types are independent we have that the product measures $p = (p_1 \times \cdots \times p_n)$ is a common prior, where $p_i \in \Delta(\Theta_i)$ is the marginal probability on $\Theta_1 \times \cdots \Theta_n$ for some $i \in N$; equivalently, $p(\theta_{-i}|\theta_i) = \prod_{j \neq i} p_j(\theta_j)$ for all i and θ. We can now associate a signalling game with the set of histories $\boldsymbol{H}_T \times \Theta$ and each information set of each player j takes the form $I(h_{T'}, \theta_j) = \{(h_{T'}, (\theta_j, \theta'_{-j})) : \theta'_{-j} \in \Theta_{-j}\}$ for $\theta_j \in \Theta_j$. We the set of player j's behavioral strategies by $\sigma_{j,T'}(\cdot|(h_{T'}, \theta_j)) \in [\Delta(A_j(h_{T'}))]^{\Theta}$, interpreting $\sigma_{j,T'}$ as a common array of common conditional first-order beliefs $\mu^1_{-j,T'}$ held by j's opponents. As is standard in signalling we assume that beliefs are determined by actions, which implies that: (i) if player j does not have to move then the actions taken do not affect the other players' belief about player j's type and (ii) if player j is one of the players who takes an action then the other players' beliefs about j's type depend only on the action taken by j, not on the the other players' actions. (This is consistent with behavioral strategies being independent.) If $p_j(\theta_j|h^0_{T'}) = p_j(\theta_j)$ and $a_{j,h_{T'}}$ is in the support of $\mu^1_{-j,T'}(\cdot|(h_{T'}, \theta_j))$ then for any $\theta_j \in \Theta_j$ we have

$$p_j(\theta_j|h_{T'}, a_{h_{T'}}) = \frac{\mu^1_{-j,T'}(a_{j,h_{T'}}|(h_{T'}, \theta_j)) \cdot p_j(\theta_j|h_{T'})}{\sum_{\theta'_j \in \Theta_j} \mu^1_{-j,T'}(a_{j,h_{T'}}|(h_{T'}, \theta'_j) \cdot p_j(\theta'_j|h_{T'})}.$$

Upon observing the signal from player j the other players update their beliefs about player j's exogenous type using Bayes' rule until his behavior contradicts other player common belief $\mu^1_{-j,T'}$, at which point they form a new conjecture about player i's type that is the basis for future Bayesian updating until there is another conflict with $\mu^1_{-j,T'}$. Such influencing of others' beliefs through signalling does not exists when there is complete information (i.e., Θ is a singleton).

Taking actions, or sending messages, that other players are unaware of can in our class of games (with complete information) also be interpreted as strategic information transmission. Since each of these actions/messages only reveals information about the structure of the game, and not about the probability of other players being of certain exogenous types, the information transmission we allow for is somehow different from that known from signalling. Remember, in equilibrium player i confined to some subtree forms beliefs about some other player j's equilibrium beliefs at each subtree he might be confined to (which can be embedded in the subtree i is confined to). By strategically revealing paths of play, player i can exclude the subtrees player j can be confined to which does not allow for the revealed paths. This means that our information revealing actions/messages are irrelevant in settings in which all players are of the same awareness. However, in games with asymmetric awareness such information transmission becomes an important part of the strategic interaction.

6.5 Hierarchy representation of beliefs

The hierarchy representation of beliefs plays a prominent role in belief-dependent preferences. The interpretation of such a representation has been discussed a great deal in the literature, and it is therefore important to clarify of how one should interpret such hierarchies in our framework. By using a hierarchy representation, we implicitly assume that the game is analyzed at a 'point in time' subsequent to the player knowing his beliefs. That is, there exist no beliefs at a 'prior' point in time, nor is there any information about what the players would have believed had their information been 'less' or 'more' than what it in fact is. The hierarchy of beliefs therefore offers no meaningful argument for identifying beliefs at a prior point in time. When considering unawareness, any interpretation of beliefs at a prior point in time becomes nonsensical: one would have to imagine that each player had been aware of all relevant paths of play at some prior point, and then become unaware of some of the paths ex-ante, while nevertheless having received more information about the paths they are aware of. Finally, insisting that priors be common does in this setting not reflect where differences in beliefs may come from, but rather constitutes a complex and unintuitive restriction on each hierarchy of beliefs. Even if we were to impose common priors, this would not render a prior point in time relevant, nor would it render the prior distribution meaningful.[13]

6.6 Unawareness as zero probability events

One may also wonder to what extend unawareness of paths of play can be modeled as zero probability events. First, assigning probability zero to an event is still compatible with realizing what could happen if the probability zero event were nevertheless to obtain. This is conceptually different from being completely unaware of the event. Second, if one nevertheless wants to model unawareness as zero probability events, then it is impossible in the standard framework Savage (1954). According to Dekel *et al.* (1998), a player should be unaware of an event if and only if he is unaware of being aware of it. So a player being unaware of an event would have to assign probability zero both to the event and its negation. Because of additivity, a probability measure in the standard framework can never assign both zero to an event and its complement.

6.7 Non-equilibrium solution concepts

Our solution concept ideally involves interpreting hierarchies beliefs as a rest-point of a transparent reasoning process, one could argue that it is difficult to carry over such interpre-

[13]The plausibility and justification of the ex-ante versus the interim view of beliefs has been extensively discussed in the literature, see Harsanyi (1967–68), Dekel and Gul (1997), Gul (1998), and Aumann (1998).

tations to a setting in which every increase of awareness is by definition a shock or surprise. Once the player's view of the game itself is challenged in the course of play, some may find it difficult to justify the idea that a new set of equilibrium hierarchies beliefs for the continuation of the game are readily available. One could, for example, consider some version of extensive-form rationalizability (Battigalli, 1997) because it embodies forward inductive reasoning. If somebody makes a player aware of some relevant paths of play, it seems like a strong assumption to dismiss the increased level of awareness as an unintended consequence of others behavior. Rather, the player should try to infer from others' choices, re-interpret others' past behavior, and try to infer from it their future moves. In psychological games payoffs are affected by hierarchical beliefs, so rationalizability has to be defined as a property of the whole structure the player is aware of rather than of strategies, and one therefore has to consider players' belief revision processes (Battigalli and Siniscalchi, 2002).

However, in order to facilitate comparison, and highlight common features, with the existing literature on psychological games with sequential moves, we have chosen to adopt Kreps and Wilson (1982)'s sequential equilibrium concept which has become a benchmark for the analysis of such games (see for example, Dufwenberg and Kirchsteiger, 2004 and Battigalli and Dufwenberg, 2007b).

7 Conclusion

In our analysis we have shown that unawareness has a profound impact on the strategic interaction of agents with belief-dependent preferences. That means, taking account of asymmetric awareness levels leads to intuitive and distinct equilibrium predictions. Furthermore, we have demonstrated that communication concerning feasible paths of play is an important integral part of the strategic environment when players have asymmetric awareness levels – a type of communication that is meaningless in environments without unawareness. In our analysis we have first formalized a general framework with unawareness, communication and belief-dependent psychological preferences. Second, we have presented a solution concept and shown that all dynamic psychological games with continuous utility functions have at least one sequential psychological equilibrium. Third, we have analyzed a specific application to demonstrate the impact of unawareness and communication in a specific context with reciprocal agents. The application has highlighted the fact that any analysis of strategic interactions with asymmetric awareness levels has to start with a description of what players are aware of and what they become aware of when play unravels. Furthermore, the application has also practically demonstrated how sequential psychological equilibria can be found in specific strategic settings.

Summarizing, unawareness has a profound impact on the strategic interaction of players with belief-dependent psychological preferences. Thus, it should not be neglected and assumed away, but rather taken into account as an integral part of strategic environments.

A Appendix

A.1 Static unawareness properties of Heifetz *et al.* (2006)

We will in this subsection show that our unawareness structure comply to the interactive unawareness properties of Heifetz *et al.* (2006, p. 83) (or Heifetz *et al.* (2010, p. 47)). For each subtree $T \in \boldsymbol{T}$ denote by h_T the copy of the history $h \in \mathscr{T}$ whenever the copy is a part of the tree T. We let $h_T \hookrightarrow h_{T'}$ denote the sequence of copies a player confined to T is aware of. The correspondence φ_i has the following static properties:

1. Confined awareness: If $h \in \mathscr{T}$ then $\varphi_i(h) = T$ with $T \subseteq \mathscr{T}$.
2. Generalized reflexivity: If $T \subseteq \mathscr{T}$, $h \in \mathscr{T}$, $\varphi_i(h) = T$ and T contains a copy h_T of h, then $h_T \in \varphi_i(h)$.
3. Introspection: If $h_T \in \varphi_i(h)$ then $\varphi_i(h) = T$.
4. Subtrees preserve awareness: If $T' \subset T \subseteq \mathscr{T}$, $h \in \mathscr{T}$, and $\varphi_i(h) = T$ and T' contains copies h_T, $h_{T'}$ of h, then $h_T \hookrightarrow h_{T'}$.
5. Subtrees preserve ignorance: If $T' \subset T \subseteq \mathscr{T}$, $h \in \mathscr{T}$, and $\varphi_i(h) = T'$ and T contains copies $h_{T'}$, h_T of h, then $h_{T'} \not\hookrightarrow h_T$.
6. Subtrees preserve knowledge: If $T' \subset T \subseteq \mathscr{T}$, $h \in \mathscr{T}$, and $\varphi_i(h) = T'$ and T contains copies $h_{T'}$, h_T of h, then T' consists of copies $h_{T'} \hookleftarrow h_T$ of the histories in T.

The first three properties (1)–(3) follow trivially from the characterization of subtrees, however we mention them here for completeness. Confined awareness says that the histories a player considers possible in a given history h are all 'expressible' by the player. Generalized reflexivity yields the truth property–that what a player knows indeed obtains; and the introspection property guarantees that a player knows what he knows. Properties (4)–(6) ensures the coherence of knowledge and awareness of players when they have subjective views. The properties guarantee that after creating his subjective view a player learns nothing he did not know before, does not forget anything he knew, and does not become aware of new histories, or unaware of histories of which he was aware.

A.2 Static unawareness properties of Dekel *et al.* (1998)

We will in this section show that the form of unawareness we are considering is non-trivial in the sense of Dekel *et al.* (1998). To do so, we first need to develop a formal (epistemic) language of events and operators.

- Let Ω be the set of states, elements $\omega \in \Omega$ corresponds to a complete description of all the relevant aspects of the strategic situation, including what each player beliefs. The state of a player is therefore given by is strategy and his hierarchy of cps (s_i, μ_i). The set of states for player i is $\Omega_i = S_i^{\boldsymbol{H}_{\mathscr{T}}} \times B_{i,\mathscr{T}}$, and the set of states of the world is $\Omega = \prod_{i \in N} \Omega_i$. We let $\Omega_{-i} = \prod_{j \neq i} \Omega_j$ and with a slight abuse of notation we also write $\omega = (\omega_i, \omega_{-i}) \in \Omega = \Omega_i \times \Omega_{-i}$.

- Let $\mathcal{B}_\Omega$ denote the Borel sigma-algebra on Ω. Each element $E \in \mathcal{B}_\Omega$ is an event; its negation is denoted $\neg E = \Omega \backslash E$. An event about i is any $E = E_i \times \Omega_{-i}$, where $E_i \subseteq \Omega_i$ is a Borel set. $\mathcal{E}_i$ is the family of events about i. Events about other players are similarly defined; the collection of such events is denoted $\mathcal{E}_{-i}$.

As is standard in most epistemology, we disregard players' beliefs about themselves. A state $\omega = (s_i, \mu_i, \omega_{-i})$, player i would believe event $E = E_i \times \Omega_{-i} \in \mathcal{E}_{-i}$ conditional on history h_T in the set H_T with probability $f_{i,h_T}(\mu_{i,T})(E_{-i})$. Thus $\{(s_i, \mu_i, \omega_{-i}) : f_{i,h_T}(\mu_{i,T})(E_{-i}) = 1\}$ is the event 'i would know E conditional on h_T.' E may concern the beliefs of the other players. Note that player i's induced believes are confined to this awareness level T.

- We define a belief operator as a mapping $\boldsymbol{B}_{i,h_T} : \mathcal{E}_{-i} \to \mathcal{E}_i$ defined as follows for all $h_T \in H_T$, $E = \Omega_i \times E_{-i} \in \mathcal{E}_{-i}$:

$$\boldsymbol{B}_{i,h_T}(E) = \{(s_i, \mu_i, \omega_{-i}) : f_{i,h_T}(\mu_{i,T})(E_{-i}) = 1\}. \tag{3}$$

  That is, $\boldsymbol{B}_{i,h_T}$ contains events that player i, given his confined awareness T, would know to obtain.

- An awareness operator is an mapping $\boldsymbol{A}_{i,h_T} : \mathcal{E}_{-i} \to \mathcal{E}_i$ such that for all $h_T \in H_T$, $p \in [0,1]$, $E = \Omega_i \times E_{-i} \in \mathcal{E}_{-i}$:

$$\boldsymbol{A}_{i,h_T}(E) = \{(s_i, \mu_i, \omega_{-i}) : f_{i,h_T}(\mu_{i,T})(E_{-i}) = p\}. \tag{4}$$

  $\boldsymbol{A}_{i,h_T}$ is thus defined in the spirit of Monderer and Samet (1989)'s 'p-belief' operator. Events in $\boldsymbol{A}_{i,h_T}$ are those to which player i can assign some probability to.

- The unawareness operator is naturally defined as the negation of awareness:

$$\boldsymbol{U}_{i,h_T}(E) = \neg \boldsymbol{A}_{i,h_T}(E). \tag{5}$$

$\boldsymbol{U}_{i,h_T}(E)$ contains events which the player can assign no probability to. For player i, confined to subtree T, these events $[E = \Omega_i \times E_{-i} \in \mathcal{E}_{-i}]$ are those for which $E_{-i} \subseteq \Omega_{-i} \backslash (S_{-i}^{\boldsymbol{H}_T} \times B_{-i,T})$. That is, a player need not to be aware of all tautologies. This is a violation of the 'axiom of wisdom' and our model is therefore not a standard state space model.

- With slight abuse of notation we write $\boldsymbol{B}_{i,h_T}(E_{-i})$, $\boldsymbol{A}_{i,h_T}(E_{-i})$ and $\boldsymbol{U}_{i,h_T}(E_{-i})$ for the events $\mathcal{E}_i$ which corresponds to events $\boldsymbol{B}_{i,h_T}(E)$, $\boldsymbol{A}_{i,h_T}(E)$ and $\boldsymbol{U}_{i,h_T}(E)$ in $\mathcal{E}$, respectively. For example, $\boldsymbol{B}_{i,h_T}(E) = \Omega_i \times \boldsymbol{B}_{i,h_T}(E_{-i})$ in $\mathcal{E}_i$.

By showing that the unawareness operator complies with the properties that any appealing concept, as suggested by Dekel *et al.* (1998), the following proposition proves that unawareness in our model is non-trivial.

Proposition 2. Let $E = \Omega_i \times E_{-i} \in \mathcal{E}_{-i}$ be an event. In stepwise thinking the following properties of unawareness obtains for all $h_T \in H_T$:

1. Plausibility: $\boldsymbol{U}_{i,h_T}(E) \subseteq \neg \boldsymbol{B}_{i,h_T}(E) \cap \neg \boldsymbol{B}_{i,h_T} \neg \boldsymbol{B}_{i,h_T}(E)$,

2. *BU* introspection: $\boldsymbol{B}_{i,h_T}\boldsymbol{U}_{i,h_T}(E) = \varnothing$,

3. *AU* introspection: $\boldsymbol{U}_{i,h_T}(E) \subseteq \boldsymbol{U}_{i,h_T}\boldsymbol{U}_{i,h_T}(E)$.

4. Weak necessitation: $\neg \boldsymbol{U}_{i,h_T}(E) \subseteq \boldsymbol{B}_{i,h_T}(\Omega)$.

Proof. Proof of each of the propositions follows:

1. *Plausibility*: This property is equivalent to $\boldsymbol{B}_{i,h_T}(E) \cup \boldsymbol{B}_{i,h_T}\neg \boldsymbol{B}_{i,h_T}(E) \subseteq \boldsymbol{A}_{i,h_T}(E)$. By Equation 3 and 4 we have that $\boldsymbol{B}_{i,h_T}(E) \subseteq \boldsymbol{A}_{i,h_T}(E)$. To see that $\boldsymbol{B}_{i,h_T}\neg \boldsymbol{B}_{i,h_T}(E) \subseteq \boldsymbol{A}_{i,h_T}(E)$, note that $\omega \in \boldsymbol{B}_{i,h_T}\neg \boldsymbol{B}_{i,h_T}(E)$ *iff* $f_{i,h_T}(\mu_{i,T})(\neg \boldsymbol{B}_{i,h_T}(E_{-i})) = 1$. This implies that $\neg \boldsymbol{B}_{i,h_T}(E) \subseteq \boldsymbol{A}_{i,h_T}(E)$. Hence $\omega \in \boldsymbol{A}_{i,h_T}(E)$.

2. *BU introspection*: $\boldsymbol{B}_{i,h_T}\boldsymbol{U}_{i,h_T}(E) = \varnothing$. To see that this is true consider that some $\omega \in \boldsymbol{B}_{i,h_T}\boldsymbol{U}_{i,h_T}(E)$ *iff* $f_{i,h_T}(\mu_{i,T})(\boldsymbol{U}_{i,h_T}(E_{-i})) = 1$, which can only be true if $\boldsymbol{U}_{i,h_T}(E) \subseteq \boldsymbol{A}_{i,h_T}(E)$. By Equation 5 this is impossible and $\omega \notin \boldsymbol{B}_{i,h_T}\boldsymbol{U}_{i,h_T}(E)$.

3. *AU introspection*: $\boldsymbol{U}_{i,h_T}(E) \subseteq \boldsymbol{U}_{i,h_T}\boldsymbol{U}_{i,h_T}(E)$ is equivalent to $\boldsymbol{A}_{i,h_T}\boldsymbol{U}_{i,h_T}(E) = \boldsymbol{A}_{i,h_T}(E)$. Then $\omega \in \boldsymbol{A}_{i,h_T}\boldsymbol{U}_i(E)$ *iff* $f_{i,h_T}(\mu_{i,T})(\boldsymbol{U}_{i,h_T}(E_{-i})) \geq \pi$. Hence $\omega \in \boldsymbol{A}_{i,h_T}\boldsymbol{U}_{i,h_T}(E)$ *iff* $\omega \in \boldsymbol{A}_{i,h_T}(E)$ by Equation 4.

4. *Weak necessitation*: $\neg \boldsymbol{U}_{i,h_T}(E) \subseteq \boldsymbol{B}_{i,h_T}(\Omega)$ is equivalent to $\boldsymbol{A}_{i,h_T}(E) \subseteq \boldsymbol{B}_{i,h_T}(\Omega)$. $\omega \in \boldsymbol{A}_{i,h_T}(E)$ *iff* $f_{i,h_T}(\mu_{i,T})(E_{-i}) \geq p$ (Equation 4), and $\omega \in \boldsymbol{B}_{i,h_T}(\Omega)$ *iff* $f_{i,h_T}(\mu_{i,T})(\Omega_{-i}) = 1$ (Equation 3). Since $E_{-i} \subseteq \Omega_{-i}$ and $p \leq 1$ (awareness if a weaker condition than belief) then it hold true that $\omega \in \boldsymbol{A}_{i,h_T}(E)$ *iff* $\omega \in \boldsymbol{B}_{i,h_T}(\Omega)$.

■

Plausibility implies that a player is unaware of E if he does not have any beliefs about E, and does not have any beliefs about not having any beliefs about E. *BU* introspection states that a player cannot have any beliefs about her own unawareness. *AU* introspection is the property that if a player is unaware of an event E, then she must be unaware of being unaware. Finally, weak necessitation says that if a player is not unaware of E, then he knows any tautology involving E. The four properties together preclude unawareness in any standard state space model.

B Appendix

B.1 Proof of Proposition 1

The Proof follows naturally from the following Lemma, which itself is essentially an adaptation of the dynamic programming approach due to Battigalli and Dufwenberg (2007a, Section 3). We want to relate the problem $\max_{s_i^T \in S_i^{\boldsymbol{H}_T}(h_T)} \mathbb{E}_{s_i^T, \mu_{i,T}}[u_{i,T}|h_T]$ to dynamic programming on a multidimensional decision tree induced by $\mu_{i,T}$. First we develop some notation needed for the Lemma:

For any fixed hierarchy of cps' $\mu_{i,T}$, we obtain a well defined multidimensional decision tree that can be solved by backward induction at each dimension. Define value functions $V_{\mu_{i,T}} : H_T \to \mathbb{R}$ and $\overline{V}_{\mu_{i,T}} : (H_T \backslash Z_T) \times C_{i,h_T} \to \mathbb{R}$ as follows

- For terminal histories $z_T \in Z_T$, let

$$V_{\mu_{i,T}}(z_T) = u_{i,T}(z_T, \mu_{i,T})$$

- Assume that $V_{\mu_{i,T}}(h_T, c_{h_T})$ has been defined for all the immediate successors (h_T, c_{h_T}) of history $h_T \in H_T \backslash Z_T$,

$$\overline{V}_{\mu_{i,T}}(h_T, c_{i,h_T}) = \sum_{h_T \hookrightarrow h_{T'}} \Pr_{\mu_{i,T}}(h_{T'}|[h_{T \hookrightarrow T'}]) \times \sum_{c_{-i,h_{T'}} \in C_{-i,h_{T'}}} \sum_{s_{-i}^{T'} \in S_{-i}^{\boldsymbol{H}_T}(h_{T'}, c_{-i,h_{T'}})} \Pr_{\sigma_{-i,T}}(s_{-i}^{T'}|h_{T'}, c_{-i,h_{T'}}) V_{\mu_{i,T}}(h_T, c_{h_T})$$

  for each $c_{i,h_T} \in C_{i,h_T}$; then $V_{\mu_{i,T}}(h_T)$ is defined as

$$V_{\mu_{i,T}}(h_T) = \max_{c_{i,h_T} \in C_{i,h_T}} \overline{V}_{\mu_{i,T}}(h_T, c_{i,h_T}).$$

For any given strategy s_i^T and history $h_T \in H_T \backslash Z_T$, we use the following notation:

- For each k with $0 \leq k \leq l(h_T)$ (recall that $l(h_T)$ denotes the length of history h_T), c_{i,h_T}^k is the choice made by i in h_T at the prefix of h_T of length k. Thus, by definition $h_T = (c_{h_T}^0, c_{h_T}^1, \ldots, c_{h_T}^{l(h_T)-1})$, where $c_{h_T}^k = (c_{1,h_T}^k, \ldots, c_{n,h_T}^k)$.
- $(s_i^T|h_T)$ denotes the strategy that takes all the actions of player i in history h_T and behaves as s_i^T otherwise:

$$(s_i^T|h_T)_{h'_T} = \begin{cases} s^T_{i,h'_T} & \text{if } h'_T \nprec h_T, \\ c^{l(h')}_{i,h_T} & \text{if } h'_T \prec h_T. \end{cases}$$

Intuitively, $(s_i^T|h_T)$ is a strategy that takes on the observed choices made prior to the history h, and then agrees with strategy s_i at h and in what follows.

- Now change $(s_i^T|h_T)$ at h_T so that it is the strategy obtained from $(s_i^T|h_T)$ by replacing s^T_{i,h'_T} with $c_{i,h_T} \in C_{i,h_T}$. The resulting strategy is denoted $(s_i^T|h_T, c_{i,h_T})$. That is,

$$(s_i^T|h_T, c_{i,h_T})_{h'_T} = \begin{cases} (s_i^T|h_T)_{h'_T} & \text{if } h'_T \neq h_T, \\ c_{i,h_T} & \text{if } h'_T = h_T. \end{cases}$$

  In words, $(s_i^T|h_T, c_{i,h_T})$ is the strategy consistent with h_T that chooses c_{i,h_T} at h_T and behaves as $(s_i^T|h_T)$ in all other histories h'_T. If history (h_T, c_{i,h_T}) is consistent with s_i^T, $s_i^T \in S_i^{\boldsymbol{H}_T}(h_T, c_{i,h_T})$, then $(s_i^T|h_T, c_{i,h_T}) = s_i^T$. That is, $(s_i^T|h_T)$ takes an ex ante (before player i makes his choice at h_T) point of view of the strategy $s_i^T \in S_i^{\boldsymbol{H}_T}(h_T)$ which is consistent with h_T, while $(s_i^T|h_T, c_{i,h_T})$ takes on an ex post (after player i makes his choice at h_T) view of the strategy $s_i^T \in S_i^{\boldsymbol{H}_T}(h_T, c_{i,h_T})$ which is consistent with h_T and the choice c_{i,h_T} he is about to make.

- Finally, let $d(h_T) = \max_{h_T \preceq z_T}[l(z_T) - l(h_T)]$ denote the depth of the subtree with root h_T.

Lemma 2 (Dynamic Programming)**.** Suppose that for all $h_T \in H_T \backslash Z_T$,

$$s^{T,*}_{i,h_T} \in \arg\max_{c_{i,h_T} \in C_{i,h_T}} \overline{V}_{\mu_{i,T}}(h_T, c_{i,h_T}).$$

Then for all $h_T \in H_T \backslash Z_T$,

$$\mathbb{E}_{(s_i^{T,*}|h_T),\mu_{i,T}}[u_{i,T}|h_T] = V_{\mu_{i,T}}(h_T) = \max_{s_i^T \in S_i^{\boldsymbol{H}_T}(h_T)} \mathbb{E}_{s_i^T,\mu_{i,T}}[u_{i,T}|h_T]. \tag{DP}$$

Proof. The proof is by induction on $d(h_T)$.

Basic step: We start from the last stage of the T-partial game: h_T is such that all feasible choices at h_T terminate the game, i.e. $d(h_T) = 1$. Clearly DP holds for all h_T for which $d(h_T) = 1$ because strategies and actions coincides.

Inductive step: We now fix some stage $k \geq 1$, which is not the last stage, and look at the stage just preceding it. Suppose DP holds for all h_T such that $1 \leq d(h_T) \leq k$. Let $d(h_T) = k+1$. By the law of iterated expectations for all $c_{i,h_T} \in C_{i,h_T}$

$$\mathbb{E}_{(s_i^{T,*}|h_T,c_{i,h_T}),\mu_{i,T}}[u_{i,T}|h_T] = \sum_{h_T \hookrightarrow h_{T'}} \Pr_{\mu_{i,T}}(h_{T'}|[h_{T \hookrightarrow T'}]) \times \sum_{c_{-i,h_T} \in C_{-i,h_T}} \sum_{s_{-i}^{T'} \in S_{-i}^{\boldsymbol{H}_T}(h_{T'},c_{-i,h_{T'}})} \Pr_{\sigma_{-i,T}}(s_{-i}^{T'}|h_{T'},c_{-i,h_{T'}}) \times \mathbb{E}_{(s_i^{T,*}|h_T,c_{i,h_T}),\mu_{i,T}}[u_{i,T}|h_T,(c_{i,h_T},c_{-i,h_T})] \tag{6}$$

By the inductive hypothesis, for all $c_{i,h_T} \in C_{i,h_T}$ and $c_{-i,h_T} \in C_{-i,h_T}$

$$\mathbb{E}_{(s_i^{T,*}|h_T),\mu_{i,T}}[u_{i,T}|h_T,(c_{i,h_T},c_{-i,h_T})] = V_{\mu_{i,T}}(h_T,(c_{i,h_T},c_{-i,h_T})) = \max_{s_i^T \in S_i^{\boldsymbol{H}_T}(h_T,(c_{i,h_T},c_{-i,h_T}))} \mathbb{E}_{s_i^T,\mu_{i,T}}[u_{i,T}|h_T,(c_{i,h_T},c_{-i,h_T})] \tag{7}$$

Taking expectations w.r.t. c_{-i,h_T} (plug eq. 7 into Eq. 6):

$$\mathbb{E}_{(s_i^{T,*}|h_T,c_{i,h_T}),\mu_{i,T}}[u_{i,T}|h_T] = \overline{V}_{\mu_{i,T}}(h_T,c_{i,h_T}).$$

Therefore

$$\mathbb{E}_{(s_i^{T,*}|h_T),\mu_{i,T}}[u_{i,T}|h_T] = V_{\mu_{i,T}}(h_T) = \max_{s_i^T \in S_i^{\boldsymbol{H}_T}(h_T)} \mathbb{E}_{s_i^T,\mu_{i,T}}[u_{i,T}|h_T]$$

$$\textit{if and only if}$$

$$s_{i,h_T}^{T,*} \in \arg\max_{c_{i,h_T} \in C_{i,h_T}} \mathbb{E}_{(s_i^{T,*}|h_T,c_{i,h_T}),\mu_{i,T}}[u_{i,T}|h_T]$$

$$\textit{if and only if}$$

$$s_{i,h_T}^{T,*} \in \arg\max_{c_{i,h_T} \in C_{i,h_T}} \overline{V}_{\mu_{i,T}}(h_T,c_{i,h_T}).$$

The latter condition holds by assumption and the inductive step is hereby proven. ■

Player i expected utility at (h_T, c_{h_T}) given he has made choice c_{i,h_T} in the optimal strategy, but not knowing $j \neq i$'s choices is $\mathbb{E}_{(s_i^{T,*}|h_T,c_{i,h_T}),\mu_{i,T}}[u_{i,T}|h_T,(c_{i,h_T},c_{-i,h_T})] = \sum_{s_i^T \in S_i^{\boldsymbol{H}_T}(h_T,c_{i,h_T})} \Pr_{\sigma_{i,T}}(s_i^T|h_T,c_{i,h_T}) u_{i,T}(\zeta(s_i^T,s_{-i}^{T'}),\mu_{i,T})$. Hence, we have that $\overline{V}_{\mu_{i,T}} = \mathbb{E}_{\sigma_T,\mu_T}[u_{i,T}|h_T,c_{i,h_T}]$. It therefore follows from Lemma 2, that:

$$V_{\mu_{i,T}} = \max_{c_{i,h_T} \in C_{i,h_T}} \mathbb{E}_{\sigma_T,\mu_T}[u_{i,T}|h_T, c_{i,h_T}] = \max_{s_i^T \in S_i^{\boldsymbol{H}_T}(h_T)} \mathbb{E}_{s_i^T,\mu_{i,T}}[u_{i,T}|h_T].$$

B.2 Proof of Theorem 1

First let $\beta^1(\sigma_T) = (\beta^1(\sigma_T))_{i\in N}$ denote the profile of first-order beliefs derived from σ_T according to condition (i) in Definition 10. The profile of infinite belief hierarchies $\mu_T = \beta(\sigma_T)$ is obtained by condition (ii) in Definition 10. By construction, the assessment $(\sigma_T, \beta(\sigma_T))$ is consistent. It follows from the construction the $\beta(\cdot)$ is a continuous function.

Suppose that each player i is subject to a slight imperfection of rationality (tremble) of the following kind. At every history h_T there is a small positive probability ϵ_{i,h_T} for the breakdown of rationality. Whenever rationality breaks down, every choice c_{i,h_T} will be selected with some positive probability $\sigma_{i,T}(c_{i,h_T}|h_T) = \epsilon_{i,h_T}(c_{i,h_T})$. Formally, fix a strictly positive vector $\epsilon = (\epsilon_{i,h_T}(c_{i,h_T})_{c_{i,h_T}\in C_{i,h_T}})_{i\in N, h_T\in H_T\setminus Z_T}$ s.t. for all $h_T \in H_T\setminus Z_T$, $\sum_{c_{i,h_T}\in C_{i,h_T}} \epsilon_{i,h_T}(c_{i,h_T}) < 1$. Now define an (agent-form, psychological) ϵ-constrained equilibrium in the T-partial game:

Definition 13 (ϵ-constrained equilibrium)**.** An ϵ-constrained equilibrium in the T-partial game is a behavioral strategy profile σ_T s.t. for all $i \in N$, $h_T \in H_T$, $c_{i,h_T} \in C_{i,h_T}$,

(i) $\sigma_{i,T}(c_{i,h_T}|h_T) \geq \epsilon_{i,h_T}(c_{i,h_T})$,

(ii) $c_{i,h_T} \notin \arg\max_{c_{i,h_T}\in C_{i,h_T}} \mathbb{E}_{\sigma_T,\beta(\sigma_T)}[u_{i,T}|h_T, c_{i,h_T}] \Rightarrow \sigma_{i,T}(c_{i,h_T}|h_T) = \epsilon_{i,h_T}(c_{i,h_T})$.

Let $\Sigma_\epsilon = \prod_{i\in N} \Sigma_{\epsilon,i}$ be the set of behavioral strategy profiles satisfying condition (i) in Definition 13, and let $\mathrm{BR}_\epsilon : \Sigma_\epsilon \to \Sigma_\epsilon$ be the ϵ-best response correspondence that assigns to each profile σ_T the subset og profiles in Σ_ϵ satisfying condition (ii) of the definition, such that

$$\mathrm{BR}_{\epsilon,i}(\sigma_T) = \{\sigma_{i,T} \in \Sigma_{\epsilon,i} : c_{i,h_T} \notin \arg\max_{c_{i,h_T}\in C_{i,h_T}} \mathbb{E}_{\sigma_T,\beta(\sigma_T)}[u_{i,T}|h_T, c_{i,h_T}] \Rightarrow \sigma_{i,T}(c_{i,h_T}|h_T) = \epsilon_{i,h_T}(c_{i,h_T}) \text{ for all } h_T \in H_T, c_{i,h_T} \in C_{i,h_T}\},$$

$$\mathrm{BR}_\epsilon(\sigma_T) = \prod_{i\in N} \mathrm{BR}_{\epsilon,i}(\sigma_T).$$

$\mathrm{BR}_{\epsilon,i}(\sigma_T)$ is a nonempty convex subset of Euclidean space $\Delta(C_{i,h_T})$. Since $\mathbb{E}_{\sigma_T,\mu_T}[u_{i,T}|h_T, c_{i,h_T}]$ is continuous in (σ_T, μ_T) and $\mu_T = \beta(\sigma_T)$ is a continuous function, $\mathbb{E}_{\sigma_T,\beta(\sigma_T)}[u_{i,T}|h_T, c_{i,h_T}]$ is continuous in σ_T.

We now have structure to apply Kakutani's fixed point theorem to the best response correspondence. $\mathrm{BR}_\epsilon(\sigma_T)$ is upper hemicontinuous because $\mathbb{E}_{\sigma_T,\beta(\sigma_T)}[u_{i,T}|h_T, c_{i,h_T}]$ is continuous for each (finite) $h_T \in H_T$ and $c_{i,h_T} \in C_{i,h_T}$, nonempty since each $\mathbb{E}_{\sigma_T,\beta(\sigma_T)}[u_{i,T}|h_T, c_{i,h_T}]$ is continuous and Σ_ϵ is compact, and convex valued because each $\mathbb{E}_{\sigma_T,\beta(\sigma_T)}[u_{i,T}|h_T, c_{i,h_T}]$ is quasi-concave on Σ_ϵ. Therefore $\mathrm{BR}_\epsilon(\sigma_T)$ has a fixed point, which is an ϵ-constrained equilibrium.

Fix a sequence $\epsilon^k \to 0$ and a corresponding sequence of ϵ^k-constraint equilibrium strategies σ_T^k. By compactness, the sequence (σ_T^k) has a limit point σ_T^*. A trembling-hand perfect equilibrium is any limit of ϵ-constraint equilibria as $\epsilon^k \to 0$. We will now prove that the trembling-hand perfect equilibrium $(\sigma_T^*, \beta(\sigma_T^*))$ is a sequential equilibrium.

Assessment $(\sigma_T^*, \beta(\sigma_T^*))$ is continuous: to see this note that, by continuity, $\beta(\sigma_T^*)$ is a limit point of $\beta(\sigma_T^k)$, and that the set of consistent assessment is closed. By continuity of $\mathbb{E}_{\sigma_T,\beta(\sigma_T)}[u_{i,T}|h_T, c_{i,h_T}]$ in σ_T (and fitness of C_{i,h_T}), for k sufficiently large

$$\arg\max_{c_{i,h_T}\in C_{i,h_T}} \mathbb{E}_{\sigma_T^*,\beta(\sigma_T^*)}[u_{i,T}|h_T, c_{i,h_T}] = \arg\max_{c_{i,h_T}\in C_{i,h_T}} \mathbb{E}_{\sigma_T^k,\beta(\sigma_T^k)}[u_{i,T}|h_T, c_{i,h_T}].$$

This implies that

$$\mathrm{supp}(\sigma_{i,T}^*(\cdot|h_T) \subseteq \arg\max_{c_{i,h_T}\in C_{i,h_T}} \mathbb{E}_{\sigma_T^*,\beta(\sigma_T^*)}[u_{i,T}|h_T, c_{i,h_T}].$$

By Definition 12, $(\sigma_T^*, \beta(\sigma_T^*))$ is a sequential equilibrium.

B.3 Proof of Corollary 1

First note that the existence of an equilibrium assessment in each $\Gamma_{T'}$-partial game is ensured by Theorem 1.

Now imagine an equilibrium assessment $(\sigma_T^*, \beta(\sigma_T^*))$ for players confined to Γ_T, and assume that players confined to $\Gamma_{T'}$ play their component in the equilibrium assessment $(\sigma_{T'}^*, \beta(\sigma_{T'}^*))$.

We need to show that $(\sigma_T^*, \beta(\sigma_T^*))$ is an equilibrium assessment in Γ_T. Suppose not, then there would be a profitable deviation

$$\mathbb{E}_{((\sigma'_{i,X},\sigma^*_{-i,X}),\beta(\sigma'_{i,X},\sigma^*_{-i,X}))}[u_{i,X}|h_X, c_{i,h_X}] > \mathbb{E}_{(\sigma^*_X,\beta(\sigma^*_X))}[u_{i,X}|h_X, c_{i,h_X}]$$

for some $i \in N$, $h_X \in H_X$, $c_{i,h_X} \in C_{i,h_X}$ and $X = \{T, T'\}$.

1. For players confined to the Γ_T-partial game ($X = T$): a player's assessment $((\sigma'_{i,T}, \sigma^*_{-i,T}), \beta(\sigma'_{i,T}, \sigma^*_{-i,T}))$ is not an equilibrium assessment in Γ_T by Definition 12–a contradiction.

2. For players confined to the $\Gamma_{T'}$-partial game ($X = T'$): a player's expected payoff is (due to unawareness) identical in $\Gamma_{T'}$ and Γ_T, thus his assessment $((\sigma'_{i,T'}, \sigma^*_{-i,T'}), \beta(\sigma'_{i,T'}, \sigma^*_{-i,T'}))$ is not an equilibrium strategy in $\Gamma_{T'}$ by Definition 12–a contradiction.

Hence $(\sigma_T^*, \beta(\sigma_T^*))$ must be an equilibrium assessment in Γ_T.

C Appendix

C.1 Proof of Result 3

Remember in scenario 2 *Bob* is aware of everything. Hence, if *Ann* chooses D, *Bob* evaluates *Ann*'s kindness on the basis of $T_{15} = \{h^0, h^1, h^2, h^3, h^4, h^5, h^6\}$ in the history that he finds himself in. In result 1 we have shown that full awareness would imply that *Bob* chooses d out of monetary and reciprocity reasons. Although *Bob* is aware of everything and observes *Ann*'s choice D, he knows that *Ann* is unaware of her action C and his subsequent actions. *Bob*, hence, forms an equilibrium belief about what *Ann* would have down had she been of the same awareness level as he is. From scenario 1 we know that the only sequential equilibrium given full awareness and $Y \geq 1$ involves *Ann* playing C and *Bob* playing c. This means, *Bob* holds the equilibrium belief given his awareness level that $(C, (c, d))$ would have been the actions in the joint equilibrium strategy, if *Ann* had been of the same awareness level as he is. Given this, *Bob*'s evaluation of *Ann*'s kindness even following *Ann*'s choice of D is:

$$\lambda_{BAB} = 1 - \frac{1}{2}[1 + 0] = 0.5.$$

Note $\lambda_{BAB} = 0.5$ is Bob's perception about *Ann*'s kindness after *Ann*'s action C in the equilibrium they would have played had both been aware of everything. As *Bob* does not hold her responsible for being unaware, this is also his perception concerning *Ann*'s kindness following her choice D and awareness levels $T_{15} = \{h^0, h^2, h^5, h^6\}$. In other words, this is *Bob*'s equilibrium belief about *Ann*'s kindness given her awareness level $T = \{h^0, h^2, h^5, h^6\}$ and following her choice of action D. On the other hand, the kindness that *Bob* can show to *Ann* is given by

$$\kappa_{BA} = 2 - \frac{1}{2}(2 + 0) = 1$$

by choosing c and

$$\kappa_{21} = 0 - \frac{1}{2}(2 + 0) = -1$$

by choosing d. Bringing things together, *Bob* chooses c if the utility from choosing c, i.e. $-1 + Y \cdot (0.5) \cdot (1)$, is higher than the utility from choosing d, i.e. $0 + Y \cdot (0.5) \cdot (-1)$. This is the case when $Y \geq 1$. In other words, *Bob* chooses to accept -1 in order not to be unkind to *Ann* who he believes would have been kind to him if she had been aware of everything that he is aware of.

C.2 Proof of Result 5

To understand result 5 it is important to see that whatever *Ann* beliefs about *Bob*'s strategy following her choice of D, *Bob* is worse of than if she would chose C (see also result 1 and the proof to observation 1 in Dufwenberg and Kirchsteiger (2004)). This means it is sure that *Bob* who becomes aware of everything when *Ann* chooses D considers D as an unkind choice. Given this, his belief-dependent reciprocity preferences plus his own monetary payoff makes him to choose his action d. Furthermore, as *Bob* correctly believes that *Ann* is also aware of everything he chooses any message.

C.3 Proof of Result 6

Consider first part (i): By sending a message which does not contain any new information *Bob* does not become aware of any new feasible path of play. This implies that *Bob* will continue to evaluate *Ann*'s kindness on the basis of $T_5 = \{h^0, h^1, h^3, h^4\}$. As $T_5 = \{h^0, h^1, h^3, h^4\}$ only entails one action for *Ann*, *Bob*'s belief about the intentions of *Ann* towards him as well as *Bob*'s belief about the maximum and minimum that *Ann* could have given to him coincide. Hence, $\lambda_{BAB} = 0$. Given this, *Bob*'s psychological utility from reciprocity is $Y \cdot \kappa_{BA} \cdot \lambda_{BAB} = 0$ and he consequently maximizes his own monetary payoff, i.e. *Bob* chooses action d. Consider now part (ii) and (iii): if *Ann* chooses C and a message that contains at least $T_2 = \{h^0, h^2, h^6\}$ as new information, then *Bob* evaluates *Ann*'s kindness either on $T_{15} = \{h^0, h^1, h^2, h^3, h^4, h^5, h^6\}$ or $T_{13} = \{h^0, h^1, h^2, h^3, h^4, h^6\}$ depending on *Ann*'s message. To evaluate *Bob*'s perception concerning *Ann*'s kindness in this case we have to specify his belief concerning *Ann*'s belief regarding his choice following *Ann*'s action C. Denote *Bob*'s belief concerning *Ann*'s belief concerning the likelihood with which he plays c following her action C by β. This implies that he believes that *Ann* believes that he plays d following her choice of C with probability $(1-\beta)$. Furthermore, note that in this situation *Bob* believes that in equilibrium he would have chosen d following *Ann*'s choice D giving him a payoff of 0. Given this, *Bob* perceives *Ann*'s choice C and the message which contains *at least* $T_2 = \{h^0, h^2, h^6\}$ as

$$\lambda_{BAB} = \beta + (1-\beta)2 - \frac{1}{2}[\beta + (1-\beta)2 + 0]$$

where $\frac{1}{2}[\beta + (1-\beta)2 + 0]$ is *Bob*'s perception given his awareness level concerning the average that *Ann* could have given him. λ_{BAB} reduces to $1 - \frac{1}{2}\beta$. In equilibrium beliefs have to be correct! Hence, *Bob*'s perception of *Ann*'s kindness in an equilibrium involving his choice c following *Ann*'s choice C ($\beta = 1$) is $\frac{1}{2}$. On the other hand, in this situation *Bob*'s kindness towards *Ann* by choosing c and any message is $\kappa_{BA} = 1 - \frac{1}{2}[1 + (-1)] = 1$ and his kindness from choosing d and any message is $\kappa_{BA} = 1 - \frac{1}{2}[1 + (-1)] = 1$. This means he chooses c in equilibrium if:

$$1 + Y(\frac{1}{2})(1) \geq 2 + Y(\frac{1}{2})(-1)$$

which holds if $Y \geq 1$. Consider now part (iv). We follow the same reasoning as before: if *Ann* chooses C and a message that contains only $T_1 = \{h^0, h^2, h^5\}$ as new information, then *Bob* evaluates *Ann*'s kindness on $T_{12} = \{h^0, h^1, h^2, h^3, h^4, h^5\}$. In this case *Bob* believes that he would have chosen c following *Ann*'s choice D as this is the only of his actions following *Ann*'s choice D that he has become aware of by *Ann*'s message. Again, denote *Bob*'s belief concerning *Ann*'s belief concerning the likelihood with which he plays c following *Ann*'s action C by β. This means that *Bob* perceives *Ann*'s choice C and the message which contains only $T_1 = \{h^0, h^2, h^5\}$ as new information as:

$$\lambda_{BAB} = \beta + (1 - \beta)2 - \frac{1}{2}[\beta + (1 - \beta)2 + (-1)]$$

which reduces to $1\frac{1}{2} - \frac{1}{2}\beta$. As before, in equilibrium beliefs have to be correct. Hence, *Bob*'s perception of *Ann*'s kindness in an equilibrium involving his choice c following *Ann*'s choice C ($\beta = 1$) is 1. As in the cases (ii) and (iii), in this situation *Bob*'s kindness towards *Ann* by choosing c and any message is $\kappa_{BA} = 1 - \frac{1}{2}[1 + (-1)] = 1$ and his kindness from choosing d and any message is $\kappa_{BA} = 1 - \frac{1}{2}[1 + (-1)] = 1$. This means he chooses c in equilibrium if:

$$1 + Y(1)(1) \geq 2 + Y(1)(-1)$$

which holds if $Y \geq \frac{1}{2}$.

C.4 Proof of Result 7

Case (i): If *Bob*'s sensitivity to reciprocity is $Y < \frac{1}{2}$, *Ann* knows that *Bob* will defect no matter what she does and which messages she sends. Hence, she chooses D to get in equilibrium 0, rather than -1 which she would get by choosing C. Case (ii): If *Bob*'s sensitivity to reciprocity is $\frac{1}{2} \leq Y \leq 1$, *Ann* knows that *Bob* will cooperate when she chooses C and a message which contains only $T_1 = \{h^0, h^2, h^5\}$ as new information. As this gives her 1 in monetary payoffs which is more than with any of her other actions and messages, she chooses to cooperate and send a message which contains only $T_1 = \{h^0, h^2, h^5\}$ as new information. Case (iii): In case (iii) we can apply the same reasoning as in case (ii). But, as a message that contains either $T_3 = \{h^0, h^2, h^5, h^6\}$ or $T_2 = \{h^0, h^2, h^6\}$ implies a lower kindness perception in *Bob*'s eyes about *Ann*'s action C, *Bob* chooses to collaborate c in equilibrium only if $Y \geq 1$.

Hence, *Ann* chooses this action and message only if $Y \geq 1$.

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